1

Day 2

Optimistic plan: today: 2d structures, BV actions

tomorrow: boundary chiral algs, line ops, local ops in A & B

BFN

HT

Thursday: 4d $\mathcal{N}=2$ HT twist

twists of 3d $\mathcal{N}=4$

B-twist of 2d $\mathcal{N}=(2,2)$

Simplest data: $\mathcal{X}$ (Kähler) assume smooth variety / $\mathbb{C}$

"sigma model w/ target $\mathcal{X}$"

Bulk local ops $\mathrm{Ops}^B_{\mathcal{X}} \simeq PV(\mathcal{X})$ polyvector fields

$$\simeq H^\bullet(\mathcal{X}, \Lambda^\bullet T_{\mathcal{X}})$$

alg.

total coh degree $= p+q$

$$\simeq \bigoplus_{p,q} H^{0,q}_{\bar\partial}(\mathcal{X}, \Lambda^p T^{(1,0)}_{\mathcal{X}})$$

diff geom

Locally, given alg^c coords on $\mathcal{X}$ $x^1, \dots, x^n$ vec fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$

$$\mathrm{Ops}^B_{\mathcal{X}} \simeq \bar\partial \text{ coh. of } \left\{ f(x,\bar x) \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_p}} d\bar x^{i_1} \wedge \dots \wedge d\bar x^{i_q} \right\}$$

From now on $\frac{\partial}{\partial x^i} =: \psi_i$ coh degrees: $|x| = 0$ $|\psi| = 1$

$|\bar\partial| = 1$ $|d\bar{x}| = 1$

In derived alg^c geom: $\mathrm{Ops}^B_{\mathcal{X}} = \mathbb{C}^{der}[T^*[1]\mathcal{X}]$

ψ_i x^i

product on $\mathrm{Ops}^B_{\mathcal{X}}$ $\leftrightarrow$ naive $\wedge$ product i.e. product of functions

Poisson bracket $\leftrightarrow$ S.N. bracket on PV fields

induced from $\underbrace{\{\overset{0}{x^i}, \overset{1}{\psi_j}\}}_{\text{Lie bracket}} = \overset{0}{\delta^i_j}$

Some physics (BV)

The B-model w/ target $\mathcal{X}$ (local coords x^i)

$\mathbb{R}^2$

has fields $X^i \in \Omega^\bullet(\mathbb{R}^2)$ $\Psi_i \in \Omega^\bullet(\mathbb{R}^2)[1]$

$$= \underset{0}{x^i(t,u)} + \underset{-1}{\chi^i_t}\,\underset{1}{dt} + \underset{-1}{\chi^i_u}\,\underset{1}{du} + \underset{-2}{a^i}\,\underbrace{dt\,du}_{2} = \underset{1}{\psi_i} + \underset{0}{b_{it}}\,dt + \underset{0}{b_{iu}}\,du + \underset{-1}{\eta_i}\,dt\,du$$

degree

3

Globally, $\text{Fields} \simeq \text{Maps}(\mathbb{R}^2_{dR}, T^*[1]\mathcal{X})$.

$$S = \int_{\mathbb{R}^2} \underbrace{\Psi_i \, dX^i}_{\text{take } dt\,du \text{ part}}$$

← Liouville 1-form on $T^*[1]\mathcal{X}$

$S : \text{Fields} \to \mathbb{C}$

Note: only complex/alg^c structure of $\mathcal{X}$ is used to define S (no metric on $\mathcal{X}$ or on $\mathbb{R}^2$)

BV data includes a Poisson bracket on Fields^*

defined by $\{ X^i(t,u), \Psi_j(t',u') \} = \delta^i_j \, \delta^{(2)}(t-t', u-u') \, dt\,du$

Q acts on Fields^* by $Q := \{S, -\}$

$$Q^2 = 0 \iff \{S, S\} = 0$$

Check: $\sim \int_{\mathbb{R}^2} d(\Psi_i X^i) = 0$

Local ops (at t,x) $\quad \mathrm{Ops}(t,x) \sim$ alg^e functions on jet space of Fields at t,x

i.e. polys in $X(t,x)$, $\psi(t,x)$, a, χ ...

and their derivs

Find (check!)

$$Q\, X(t,x) = d X(t,x)$$

$$Q\, \Psi(t,x) = d\Psi(t,x)$$

$$X = x + \underset{\text{1-form}}{\chi} + \underset{\text{2-form}}{a}$$

$$= x + x^{(1)} + x^{(2)}$$

$$\Psi = \psi + \psi^{(1)} + \psi^{(2)}$$

in ptic. $\quad Qx^i = 0 \qquad Q\psi_i = 0$

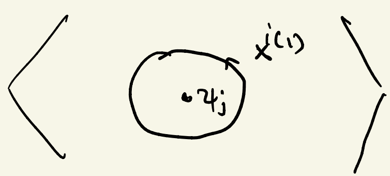
$$Q\chi = dx \qquad Qb = d\psi$$

Q-cohomology $\quad H^\bullet(\mathrm{Ops}, Q) \simeq \mathbb{C}[x, \psi]$

w/ naive product

and $\{x^i, \psi_j\} = \delta^i_j$

$\swarrow$ from

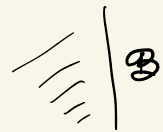


$$= \int_{\text{Fields}} \psi_j(0) \oint_{S^1} x^{i(1)}\, e^{-S} \sim \text{const} \cdot \delta^i_j$$

Boundary conditions

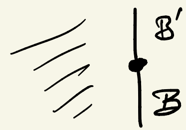
For B-model w/ target $\mathcal{X}$, $\quad \mathbb{B}\mathrm{dy} \simeq D^b \mathrm{Coh}(\mathcal{X})$

Q-preserving b.c. (up to equivalence)



$\xleftarrow{1-1}$ objects $\mathcal{B}$ in $D^b(\mathrm{Coh}(\mathcal{X}))$ up to q-iso

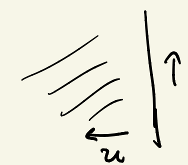
and



$\mathrm{Hom}_{\mathbb{B}\mathrm{dy}}(\mathcal{B},\mathcal{B}') \simeq \mathrm{Hom}_{D^b\mathrm{Coh}}(\mathcal{B},\mathcal{B}')$

Motivate locally for $\mathcal{X} = \mathbb{C}^n$

$$S = \int_{\mathbb{R}\times\mathbb{R}_+} \Psi_i \, dX^i$$



Need $Q^2 = 0$ i.e. $\{S, S\}_{BV} = \int_{\mathbb{R}\times\{0\}} \Psi_i X^i$

Simplest Q-preserving b.c. put constraints on X, Ψ s.t. vanishes.

Restrict (X, Ψ) to a Lagrangian $\mathcal{L} \subset T^*[1]\mathbb{C}^n$

get one for every subvariety $Y \subseteq \mathbb{C}^n$ $\qquad \mathcal{L} = N^*[1]Y$

E.g. b.c. $\mathcal{D}^{(m)}$ has $Y = \{x^1 = \cdots = x^m = 0\} \subseteq \mathbb{C}^n$

$$\mathcal{L} = N^*[1]Y = \{x^1 = \cdots = x^m = \psi_{m+1} = \cdots = \psi_n = 0\}$$

physical b.c. is $X^i|_\partial = 0 \quad i = 1, \dots, m$
$(x=0,\ \text{all } t)$

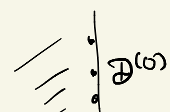
$$\Psi_i|_\partial = 0 \quad i = m+1, \dots, n$$

object in $\mathrm{Coh}(\mathbb{C}^n)$ is $\mathcal{O}_{\mathbb{C}^m}$.

Let's check (endo)morphisms for $\mathcal{X} = \mathbb{C}$ Consider $\mathcal{D}^{(0)}$ & $\mathcal{D}^{(1)}$

$\mathcal{D}^{(0)}$: $\Psi|_\partial = 0$ Note $\Psi\Phi|_\partial = 0$ ✓ At bdy, $QX = dX$
X "free"

just get



bdy local ops $\simeq \mathbb{C}[x]$

$= \mathrm{Hom}_{\mathrm{Bdy}}(\mathcal{D}^{(0)}, \mathcal{D}^{(0)})$

Happily matches $\mathrm{Hom}_{\mathrm{Coh}(\mathbb{C})}(\mathcal{O}, \mathcal{O}) = \mathbb{C}[x]$ $= \mathrm{End}_{\mathrm{Bdy}}(\mathcal{D}^{(0)}, \mathcal{D}^{(0)})$

$\mathcal{D}^{(1)}$ $\quad X|_\partial = 0$ $\quad$ same way, find $\quad \mathrm{End}_{\mathrm{Bdy}}(\mathcal{D}^{(1)}) = \mathbb{C}[\psi] \simeq \mathbb{C}^2$

Ψ "free" $\qquad\qquad \psi^2 = 0$

In $\mathrm{Coh}(\mathbb{C})$ $\quad \mathcal{D}^{(1)} \mapsto \mathcal{O}_{0\in\mathbb{C}}$ $\quad$ skyscraper

$$\mathrm{End}_{\mathrm{Coh}(\mathbb{C})}(\mathcal{O}_0) = \mathbb{C}$$ not enough!

<u>But</u>

$$\mathrm{End}_{D^b\mathrm{Coh}(\mathbb{C})}(\mathcal{O}_0) := \mathrm{Hom}^\bullet(\mathcal{O} \xrightarrow{x} \mathcal{O}, \mathcal{O}_0)$$

$$\simeq \mathbb{C} \oplus \mathbb{C}$$