Branes, Quivers and BPS algebras

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1. Motivation

1.1. Geometric engineering

- Our starting point is the ten-dimensional type IIA string theory together with its D0-, D2-, D4-, D6- and D8-branes.
- Studying string theory on $M_4 \times M_6$ and sending the volume of M_6 to zero, the system should have an effective description in terms of a theory on M_4.
- Supersymmetric (BPS) particles and line operators can be engineered from D0-branes sitting at a point, D2-branes wrapping a two-cycle, D4-branes wrapping a four-cycle or D6-branes wrapping the whole M_6.

1.2. Twisted theory in Ω-background

- As a toy model, we are going to look at the simplest example of $M_6 = \mathbb{C}^3$ compactified on Ω-background.
- $\mathbb{C}^3$ admits an action of a three torus $U(1)^3$ rotating the three coordinate lines $\mathbb{C}$ inside $\mathbb{C}^3$. We can introduce a deformation of the theory parametrized by $\epsilon_1, \epsilon_2, \epsilon_3$ associated to each generator of $U(1)^3$.
- Such Ω-deformation localizes the theory to the fixed-point locus and effectively compactifies the theory to M_4.
- Our discussion naturally extends to more complicated toric Calabi-Yau three-folds but we are going to restrict only to the example of $\mathbb{C}^3$.

1.3. Branes in Ω-background

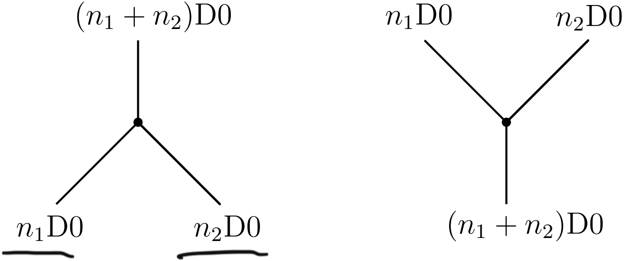
- Ω-background forces the support of D-branes to be along subvarieties fixed by the $U(1)^3$ action:

Branes	$\mathbb{R}^4$	$\mathbb{C}_{\epsilon_1}$	$\mathbb{C}_{\epsilon_2}$	$\mathbb{C}_{\epsilon_3}$
D0	$\mathbb{R}$	0	0	0
D2	$\mathbb{R}$	$\times$	0	0
D2	$\mathbb{R}$	0	$\times$	0
D2	$\mathbb{R}$	0	0	$\times$
D4	$\mathbb{R}$	0	$\times$	$\times$
D4	$\mathbb{R}$	$\times$	0	$\times$
D4	$\mathbb{R}$	$\times$	$\times$	0
D6	$\mathbb{R}$	$\times$	$\times$	$\times$

- Branes with compact support are going to be treated as light, dynamical objects (BPS particles) after compactification.
- Branes with non-compact support are going to be treated as heavy and non-dynamical leading to BPS line operators.

1.4. BPS algebra

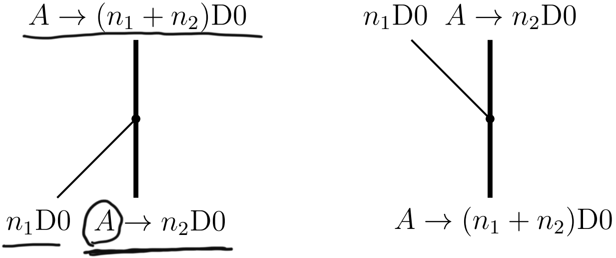
- More precisely, the spectrum of BPS particles are in correspondence with supersymmetric vacua of the quantum mechanics describing the low-energy behavior of D0-branes.
- BPS particles associated to compactly supported branes can mutually scatter:



- Scattering of such particles is captured by a BPS algebra. In our case of $\mathbb{C}^3$, the BPS algebra is known as the affine Yangian of $\mathfrak{gl}_1$.

1.5. Representations of BPS algebra

- Let us now fix a configuration of the non-compact branes (e.g. a stack of N D4-branes along $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$) with n D0 bound to it.
- Processes of bounding/removing D0-branes should lead to a representation of the BPS algebra:



- This leads to a conjecture that the $\mathfrak{gl}_1$ affine Yangian should admit a module for any configuration of noncompactly-supported branes.

1.6. Brane configuration $\to$ Quiver QM

- The low-energy dynamics of a system of branes is described by a quantum field theory living on their support.
- The low-energy dynamics of D0-branes bound to higher-dimensional branes is thus described by a quantum mechanics along $\mathbb{R}$.
- The field content (in our situation specified by a framed quiver diagram) together with the potential specifying such a quantum mechanics is usually determined by an analysis of the string spectra in a prescribed background of D-branes.
- Instead, we are going determine the relevant data by an analysis of the system within the context of derived category of coherent sheaves modeling our brane systems.

1.7. Quiver QM $\to$ BPS states

- The quiver quantum mechanics admits a continuum moduli of vacua. To compactify this moduli space, we can deform the system by introducing the Ω-background associated to the $U(1)^3$ action above.
- We are going to identify the space of vacua of such a deformed theory with the equivariant critical cohomology of the moduli space of quiver representations.
- Working equivariantly allows us to identify the cohomology with fixed points of the the corresponding moduli space, restricted to the critical locus of the potential.
- Counting such fixed points is going to lead to a rich combinatorics of melted crystals.

1.8. BPS states $\to$ Yangian module

- Let $M(n)$ be the space of vacua associated to n D0 branes bound to a fixed configuration of non-compact branes.
- There exists a correspondence $M(n+1,n) \subset M(n+1) \times M(n)$ with a map p to $M(n)$ and a map q to $M(n+1)$.
- Starting with an element in the equivariant critical cohomology $H^*(M(n))$, pulling it back by p^* and pushing forward by q_*, we are going to construct an action of rising operators of the desired BPS algebra.
- Analogously, pulling back by q_* and pushing forward by p_* gives rise to the action of lowering generators of the algebra.
- Different choices of non-compact branes then lead to modules of very different nature. As we are going to see, they give rise to Cherednik algebras (for D2-branes), corner vertex operator algebras (for D4-branes) and the MacMahon representation (for D6-branes).

2. Quivers from branes

2.1. Branes as coherent sheaves

- The first step in our construction is a derivation of a quantum mechanics (QM) describing a stack of D0-branes bound to different systems of D2-, D4- and D6-branes.
- The data specifying such a QM is going to consist of a framed quiver with potential.
- The most straightforward yet tedious derivation would rely on analysis of the spectrum and interactions of strings ending on involved branes. (See e.g. [Nekrasov-Prabhakar (2016)])
- Instead, we are going to use the language of derived categories of coherent sheaves as a model of our D-branes and derive the quivers by studying morphisms in such a category.
- In the rest of the lecture, we are going to argue that the derived categories of coherent sheaves provide a good model for branes. (See [Sharpe (2003)] for a nice review accessible to physicists.)

2.2. Sheaves

- A sheaf $\mathcal{S}$ on a space X is an assignment a module of sections $\mathcal{S}(U)$ to each open set U together with a collection of restriction maps $\rho_{U,V} : \mathcal{S}(U) \to \mathcal{S}(V)$ for any $V \subset U$. This data must satisfy some compatibility conditions that I am not going to spell out.

$\mathcal{O}$

- An example of a sheaf is the structure sheaf $\mathcal{O}_X$ of a complex variety X. The structure sheaf assigns the ring of holomorphic functions on U to each open set U.
- More generally, to any holomorphic bundle $E \to X$ of rank k, we can associate the ring of sections over U.
- Such a sheaf is obviously a module for the structure sheaf and the class of sheaves of this form are known as locally-free sheaves. The structure sheave itself is a free sheaf.

- Since a brane in string theory is specified by its support together with a (Chan-Paton) bundle over it, it is natural to identify locally-free sheaf with a stack of k D6-branes wrapping X with k being the rank of our bundle.
- In the simple case of $\mathbb{C}^3$, we associate to each open subset U the ring of holomorphic functions on U. In particular, we associate the coordinate ring

$$\mathbb{C}[x_1, x_2, x_3]$$

to the whole $U = \mathbb{C}^3$.

2.3. Coherent sheaves

- Coherent sheaves form a class of sheaves that can be locally defined by imposing a set of relations on a locally-free sheaves, i.e. they can be locally identified with the cokernel of

$$f : \mathcal{O}_X^l|_U \to \mathcal{O}_X^m|_U$$

- For a trivial map

$$f : 0 \to \mathbb{C}[x_1, x_2, x_3]^{\oplus k}$$

the sheaf is formed by k copies of the structure sheaf and can be identified with a stack of k D6-branes wrapping $\mathbb{C}^3$.

- The skyscraper sheaf can be expressed as a cokernel of

$$\mathbb{C}[x_1, x_2, x_3]^3 \xrightarrow{(x_1, x_2, x_3)} \mathbb{C}[x_1, x_2, x_3]$$

i.e.

$$\mathbb{C}[x_1, x_2, x_3]/(x_1, x_2, x_3)$$

- Away from the origin, the cokernel is trivial since we can write

$$f = x_1 \left(\frac{f}{x_1}\right) \leftarrow x_1 \neq 0$$

$$= \begin{pmatrix} f/x_1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}[x_1, x_2, x_3]^3$$

- The corresponding sheaf is thus supported at the origin and it is natural to associated it with the D0-brane.
- To get a sheaf associated to the stack of n D0-branes, we can simply take a direct sum of n such sheaves.

- Analogously, for

$$\mathbb{C}[x_1, x_2, x_3] \xrightarrow{x_1} \mathbb{C}[x_1, x_2, x_3]$$

the cokernel

$$\mathbb{C}[x_1, x_2, x_3]/(x_3)$$

can be associated with a D4-brane supported along $x_3 = 0$. A sheaf associated to a multiple of D4-branes is then just a direct sum of k copies of this sheaf.

- Similarly, the map

$$\mathbb{C}[x_1, x_2, x_3]^2 \xrightarrow{(x_1, x_2)} \mathbb{C}[x_1, x_2, x_3]$$

produces a sheaf associated to D2-branes along $x_1 = x_2 = 0$.

- We found a coherent sheaf modeling a D-brane of any support from the introduction.

2.4. Nilpotent Higgs vev

- The world of coherent sheaves is much richer.
- For example, one can easily see that the support of

0

$$\mathbb{C}[x_1, x_2, x_3]/(x_3^2) \ni f(x_1,x_2) + g(x_1,x_2)x_3$$

  is again along $x_3 = 0$ as in the case of D4-branes but the module structure is obviously different.
- As a module for $\mathbb{C}[x_1, x_2]$, it is isomorphic to the direct sum of two D4-brane sheaves but the action of x_3 is now twisted.
- We can think about such a sheaf in terms of a deformation of a pair of D4-branes by turning on a nilpotent vacuum expectation value for the Higgs field living on their support.

2.5. Derived category of coherent sheaves

- How to describe brane bound states?
- We need to extend the category of coherent sheaves to complexes of sheaves

  $$\dots \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \xrightarrow{d_3} \dots$$

  with differential squaring to zero $d_{i+1} \circ d_i = 0$.
- Intuitively, A_i are sheaves describing a system of branes and anti-branes and differentials d_i specify the exact form of the bound state.
- Complexes describe the same configuration if they are related by a quasi-isomorphisms.

- A quasi-isomorphism is a map between complexes

$$
\begin{tikzcd}
\dots \arrow[r, "d_0"] & A_1 \arrow[r, "d_1"] \arrow[d, "f_1"] \arrow[dr, phantom, "\circlearrowright"] & A_2 \arrow[r, "d_1"] \arrow[d, "f_2"] \arrow[dr, phantom, "\circlearrowright"] & A_3 \arrow[r, "d_2"] \arrow[d, "f_3"] & \dots \\
\dots \arrow[r, "d_0'"] & B_1 \arrow[r, "d_1'"] & B_2 \arrow[r, "d_2'"] & B_3 \arrow[r, "d_3'"] & \dots
\end{tikzcd}
$$

  satisfying $f_2 \circ d_1 = d_1' \circ f_1$ and inducing isomorphism on the cohomology.
- I will sometimes call the derived category of coherent sheaves simply the brane category.
- Let me give two examples of a quasi-isomorphism.

- First, we have an obvious exact sequence

$x_3 \to 0$

$$0 \longrightarrow \mathbb{C}[x_1, x_2, x_3] \xrightarrow{x_3} \mathbb{C}[x_1, x_2, x_3] \xrightarrow{d} \frac{\mathbb{C}[x_1,x_2,x_3]}{(x_3)} \longrightarrow 0$$

- Let me bend the complex as

$\ker d = \mathbb{C}[x_1, x_2, x_3]\, x_3 = \operatorname{Im} x_3$

$$\begin{tikzcd}
\mathbb{C}[x_1, x_2, x_3] \arrow[r, "x_3"] \arrow[d] & \mathbb{C}[x_1, x_2, x_3] \arrow[d, "d"] \\
0 \arrow[r] & \frac{\mathbb{C}[x_1,x_2,x_3]}{(x_3)}
\end{tikzcd}$$

- d obviously induces an isomorphism on the cohomology.
- Consequently, a D4-brane along $x_3 = 0$ is quasi-isomorphic to

$D6$ $\qquad$ $\overline{D6}$

$$\mathbb{C}[x_1, x_2, x_3] \xrightarrow{x_3} \mathbb{C}[x_1, x_2, x_3]$$

- Physically, this statement can be interpreted as a D4-brane arising from a tachyon condensation of a space-filling brane-anti-brane pair with a non-trivial tachyonic profile given by x_3. Quasi-isomorphisms thus model tachyon condensation.

- Let us instead consider

D6 D0

$$0 \longrightarrow \text{Ker } d \longrightarrow \mathbb{C}[x_1, x_2, x_3] \xrightarrow{d} \frac{\mathbb{C}[x_1,x_2,x_3]}{(x_1,x_2,x_3)} \longrightarrow 0$$

where the kernel of d is simply generated by elements vanishing at the origin

$$x_1 f_1(x_1, x_2, x_3) + x_2 f_2(x_1, x_2, x_3) + x_3 f_3(x_1, x_2, x_3)$$

- Such a sheaf is obviously isomorphic to $\mathbb{C}[x_1, x_2, x_3]$ at a generic point but it caries a non-trivial modification at 0.
- We can interpret this sheaf as describing a non-trivial bound state a D6-brane with a D0-brane.
- Bound states of this form together with their quiver descriptions will be the main object of interest in our discussion.

2.6. Morphisms in the brane category

- Morphisms $\mathrm{Hom}(A, B)$ in our category capture the information about the specturem of massless modes of a string stretched between A and B.
- The starting point in the calculation of $\mathrm{Hom}(A, B)$ is a projective resolution of A, B, i.e. an exact sequence of the form

$$\dots \xrightarrow{d_{-4}} A_{-3} \xrightarrow{d_{-3}} A_{-2} \xrightarrow{d_{-2}} A_{-1} \xrightarrow{d_{-1}} A$$

  with all A_i being projective.
- In our situation, we will be able to find a resolution of all the sheaves in terms of free sheaves $\mathbb{C}[x_1, x_2, x_3]^{\oplus n}$ that are automatically projective.

- Let us now find projective resolutions of our elementary sheaves.

$\mathcal{O} = \mathbb{C}[x_1, x_2, x_3]$

- The projective resolution of a D0 brane is given by

$$\mathcal{O} \xrightarrow{\begin{pmatrix} -x_1 \\ x_2 \\ -x_3 \end{pmatrix}} \mathcal{O}^3 \xrightarrow{\begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x & 0 \end{pmatrix}} \mathcal{O}^3 \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} \mathcal{O} \to \frac{\mathbb{C}[x_1, x_2, x_3]}{(x_1, x_2, x_3)}$$

- The projective resolution of a D2-brane supported along $x_1 = x_2 = 0$ is

$\ker(x_1, x_2) \ni \begin{pmatrix} f \\ g \end{pmatrix}$ s.t. $f = -\frac{x_2}{x_1} g$

$g = x_1 h$

$\parallel$

$\boxed{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} h}$

$$\mathcal{O} \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} \boxed{\mathcal{O}^2} \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} \mathcal{O} \to \frac{\mathbb{C}[x_1, x_2, x_3]}{(x_1, x_2)}$$

and analogously for D2-brane of other orientations.

- The projective resolution of a D4-brane supported along $x_3 = 0$ is

$$\mathcal{O} \xrightarrow{x_3} \mathcal{O} \to \frac{\mathbb{C}[x_1, x_2, x_3]}{(x_3)}$$

and analogously for D4 branes along $x_2 = 0$ and $x_3 = 0$.

- $\mathrm{Hom}^n(A, B)$ can be now identified with the chain maps between the two projective resolutions modulo chain homotopies.
- Let

$$\cdots \xrightarrow{d_{-4}} A_{-3} \xrightarrow{d_{-3}} A_{-2} \xrightarrow{d_{-2}} A_{-1}$$

$$\cdots \xrightarrow{d'_{-4}} B_{-3} \xrightarrow{d'_{-3}} B_{-2} \xrightarrow{d'_{-2}} B_{-1}$$

  be projective resolutions of A and B.
- For a fixed n and any $m < 0$, let $f_{n,m} \colon A_m \to B_{m+n}$ be a set of maps between the entries of the two complexes.
- For example, $n = 1$ would correspond to a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d_{-5}"] & A_{-4} \arrow[r, "d_{-4}"] \arrow[d, "f_{1,-4}"] & A_{-3} \arrow[r, "d_{-3}"] \arrow[d, "f_{1,-3}"] & A_{-2} \arrow[r, "d_{-2}"] \arrow[d, "f_{1,-2}"] & A_{-1} \\
\cdots \arrow[r, "d_{-4}"] & B_{-3} \arrow[r, "d_{-3}"] & B_{-2} \arrow[r, "d_{-2}"] & B_{-1} &
\end{tikzcd}$$

- Let us now define a differential $\partial : f_{n,m} \to f_{n+1,m}$ increasing the first index of $f_{n,m}$ by formula

$$\partial f_{n,m} = d_{m+n} \circ f_{n,m} - (-1)^n f_{n,m+1} \circ d_m$$

- Collection of $f_{n,m}$ for fixed n is called a chain map if it lies in the kernel of this map (this is equivalent to all the squares above commuting or anti-commuting).
- Chain homotopies then correspond to the image of ∂.
- The spectrum of strings $\mathrm{Hom}^n(A, B)$ can be thus identified with the cohomology of ∂ acting on the collection of maps $f_{n,m}$ for $m < 0$.
- The integer n is called the ghost number.