QUANTUM MATTER FROM ALGEBRAIC GEOMETRY AND NUMBER THEORY

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Abstract. The exciting and rapidly-growing field of topological materials has brought with it unexpected new connections between physics and pure mathematics. Algebraic topology in particular has played a significant role in classifying topological materials. In this brief note, I report on joint work with J. Maciejko and offer a brief look at another emerging chapter in this story in which algebraic geometry and number theory anticipate new forms of quantum matter associated to hyperbolic lattices.

The purpose of this brief note is to report on some recent interactions between complex algebraic geometry, number theory, and the burgeoning field of quantum matter[1]. The propagation of waves in periodic media under a potential respecting the symmetry of the medium is governed by Bloch's theorem [1]. A direct result is that the admissible momenta of the wave are captured by a topological space called the *Brillouin zone*. When the underlying medium is $\mathbb{R}^2$ with the standard Euclidean inner product, and when the periodicity is given by a parallelogram or hexagon lattice Λ, the Brillouin zone is a torus. Multiple interpretations are available for this torus. Most immediately, it is the space of $U(1)$-representations of the translation group of the lattice. Equivalently, it is the space of $U(1)$-representations of the fundamental group of $\mathbb{R}^2/\Lambda$. Note here that $\mathbb{R}^2/\Lambda$ is itself a torus, but this is not the same as the Brillouin zone. The quotient is the configuration space, or position space, of the wave while the Brillouin zone is obtained from it as the so-called reciprocal space via a Fourier transform. By applying the Riemann-Hilbert correspondence to the Brillouin zone as a space of $U(1)$-representations, one can interpret it as a space of smooth data, namely flat unitary line bundles on the torus. From here, one can furthermore apply the Narasimhan-Seshadri correspondence [6] and interpret the Brillouin zone as a space of holomorphic data: holomorphic line bundles on the elliptic curve $E = \mathbb{R}^2/\Lambda$. In this latter interpretation, the Brillouin zone becomes the Jacobian of E, which is the moduli space of holomorphic line bundles on E.

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[1]This report is the summary of an invited talk that I delivered on August 12, 2021 at the *Nankai Symposium on Mathematical Dialogues*, held at the Chern Institute of Mathematics in celebration of the 110th anniversary of Prof. S.-S. Chern and the 90th birthday of Sir R. Penrose. I express my gratitude to the orgnaizers, Profs. Yang-Hui He, Cheng-Ming Bai, and Mo-Lin Ge, for the kind invitation and to Jiakang Bao, Ed Hirst, Hong-Qin Li, and Suvajit Majumder for all of their support in organizing this conference. For posterity, I would like to add that, due to the Covid-19 pandemic, the conference was held in a hybrid in-person / virtual format, and I was one of many virtual speakers and attendees. The conference was extremely lively and conversational, capturing exactly the spirit of an in-person conference. I commend the organizers for crafting what has been one of the finest virtual events I have had the privilege of attending.

While the considerations above apply to propagation in any periodic medium, it has been particularly fruitful to apply Bloch's theorem in condensed matter to electrical conductivity and resistivity. Here, the wave phenomenon is the motion of electrons in a highly crystalline material that, in two dimensions, can be modelled by the lattice Λ above. Here, the symmetric potential is incorporated into the Hamiltonian operator of Schrödinger's equation (with the correct periodic boundary conditions), which are the natural equations of motion for the electrons. From this point of view, the picture becomes one of *quantum* condensed matter. By studying the eigenvalues of the Hamiltonian as functions over the Brillouin zone, we initiate the celebrated electronic band theory: namely, the topology of the spectral surface of the Hamiltonian can be used to classify the material as an insulator or (semi)metal. This also anticipates the so-called *topological materials*, which have robust, topologically-protected conductivity properties. The 2016 Nobel Prize in Physics was awarded in recognition of both the theoretical prediction and the experimental realization of such materials.

For the purposes of most condensed matter and solid-state physics discourse, the interpretation of the Brillouin zone as a space of $U(1)$-representations or "phase factors" is entirely sufficient. The interpretations in our first paragraph above regarding flat bundles and holomorphic bundles, respectively, make no appearance in any quantum condensed matter theory literature — or more broadly any literature about periodic waves — that we are aware of. Thinking in these terms, however, allows us to generalize the theory of periodic quantum condensed matter to new geometries, thereby anticipating new forms of quantum matter. This is the foundation of what we term *hyperbolic band theory*, developed in [4].

Persisting in two dimensions, we replace the kinetic part of the Hamiltonian with the Laplace-Beltrami operator for the Poincaré metric. In other words, we replace the standard complex plane $\mathbb{C} \cong \mathbb{R}^2$ with the hyperbolic plane $\mathbb{H}$. The potential part is chosen to be a suitable function that is invariant with respect to the action of a (co-compact, strictly hyperbolic) Fuchsian group Γ, which defines a tessellation of $\mathbb{H}$ — a periodic, crystalline geometry but with a generally noncommutative translation group now. We may also generalize the periodic boundary conditions appropriately. With this in hand, we show via direct construction [4] that there exist wave solutions ψ that satisfy a generalized equivariance property:

$$\psi \circ \gamma \ = \ \chi(\gamma)\psi,$$

where $\gamma \in \Gamma$ and $\chi : \Gamma \to U(1)$ is a representation. In other words, there exist wavefunctions that are weight-0 factors of automorphy. It is not so surprising that these staples of number theory should appear here as, on the one hand, they naturally generalize the quasi-periodicity of Euclidean Bloch waves and, on the other hand, a natural source of Γ-periodic potentials is given by generalizations of the types of infinite series (e.g. Eisenstein series) that typically occur in the theory of modular forms.

This program replaces the Brillouin zone with the space of representations of a higher-genus position space, namely $X = \mathbb{H}/\Gamma$. Specifically, when X has genus $g \geq 2$ (which can be achieved, for instance, when Γ is the translation group of a $4g$-gonal tessellation), the Brillouin zone is a $2g$-dimensional real, compact torus. The Fourier duality between position and momentum is now less obvious, especially considering the dimensional difference between the position surface and the momentum torus. By interpreting the Brillouin zone as the Jacobian of the Riemann surface X (which takes advantage of the natural complex structure on X coming from the quotient), we can appeal to the Abel-Jacobi map as a natural map from the symmetric product of X with itself g-many times to the Jacobian. Albeit non-canonical and only birational in a global sense, the map is almost everywhere a complex analytic isomorphism. In this way, complex algebraic geometry provides a wave-particle duality that completes the theory.

A natural question concerns our ability to calculate the band structure and make predictions in specific examples. One such example is explored at length in our article [4] for the Bolza surface, a genus-2 Riemann surface arising from a highly-symmetric octogonal tessellation, which we equip with both the zero potential (the "empty lattice") and with a generalized Eisenstein series. Here, numerical calculations verify expectations around degeneracies and branching of the spectrum. Another query concerns the classification of hyperbolic lattices in this context — in other words, their crystallography — for which a partial catalogue in two dimensions is attempted in [2]. A question concerning the extent to which Bloch's theorem holds for two-dimensional hyperbolic lattices, rather than simply the existence of *some* automorphic wavefunctions, is also natural. This is resolved in [5], which notably appeals to higher-rank representations of Γ and hence, by Narasimhan-Seshadri, to the moduli space of stable holomorphic bundles of arbitrary rank on X. Finally, one may ask a very compelling question about if and how hyperbolic matter might be experimentally realized. This actually predates our mathematical and theoretical investigations by a very short interval in time: in [3], success in constructing artificial photonic hyperbolic circuits is described. The authors lament the lack of a hyperbolic band theory, which [4] now provides.

We encourage interested readers to interact with the various articles listed above, which are by no means exhaustive as further articles emerge in this new subject of hyperbolic quantum matter. On the one hand, such matter may lead to novel models of topoelectric circuits and qubits for quantum computing, to name just one potential disruptive application. At a foundational level, the study of it is leading to new interactions between mathematics and physics by breathing algebraic geometry and number theory into condensed matter theory, subjects that have interacted only marginally if at all with this branch of physics.

References

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