HYPERPOLYGONS AND HIGGS BUNDLES

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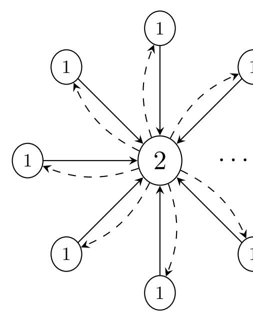
The moduli space of semistable Higgs bundles on a fixed algebraic curve is an infinite-dimensional hyperkähler quotient that has been investigated from various points of view. These include: (1) its topology, (2) its integrable system, (3) its geometry (captured by the natural hyperkähler metric), and (4) its mirror symmetry. Nakajima quiver varieties are finite-dimensional hyperkähler quotients that share much in common with Higgs bundle moduli spaces. Hyperpolygon spaces, in particular, come closest to bridging the gap between Nakajima quiver varieties and Higgs bundles. Themes 1 and 2 above are studied for hyperpolygons in [2]. We report on 3 and 4 in joint work with H. Weiß and L. Schaposnik, respectively[1].

For us, a *hyperpolygon* is a representation of the *star-shaped* quiver:



The quiver has $n+1 \geq 4$ vertices in total. A representation of a solid (ingoing) arrow is a linear map $x_i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)$. Once x_i is chosen, a representation of a dashed (outgoing) arrow involving the same nodes is a linear map

$$y_i \in T^*_{x_i}\mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)^* \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}).$$

We denote a representation by $[x|y]$, where x is an n-tuple of column vectors x_i in $\mathbb{C}^2$ and y is an n-tuple of row vectors y_i in $(\mathbb{C}^2)^*$. We now choose a sufficiently generic vector $\alpha \in \mathbb{R}^n$ with positive entries and define the *hyperpolygon equations*:

$$\begin{aligned} \sum_{i=1}^{n} (x_i x_i^* - y_i^* y_i)_0 &= 0, \quad |x_i|^2 - |y_i|^2 = \alpha_i, \text{ for each } i \in \{1, \ldots, n\}, \\ \sum_{i=1}^{n} (x_i y_i)_0 &= 0, \quad y_i x_i = 0, \text{ for each } i \in \{1, \ldots, n\}, \end{aligned}$$

where the subscript 0 is an instruction to remove the trace, and norms $|x_i|$ and $|y_i|$ are the standard Euclidean ones. The left-hand sides of these equations can

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be interpreted as moment maps. The first $n+1$ equations are (rescaled) moment maps for the action of $G = (SU(2) \times U(1)^n)/\pm 1$ on the representation data (with the action encoded by the quiver) and the latter $n+1$ equations are moment maps for the corresponding $G^{\mathbb{C}}$-action. We define *hyperpolygon space* $\mathcal{X}_n(\alpha)$ to be the solution set of the hyperpolygon equations modulo G. The name "hyperpolygon" is motivated by the fact that, when we restrict to the level set $y = 0$, we obtain a space parametrizing equivalence classes of polygons in $\mathbb{R}^3$.

The quotient $\mathcal{X}_n(\alpha)$ is a smooth quasiprojective variety of dimension $2(n-3)$ and its hyperkähler metric is complete whenever α is sufficiently generic [12, 8, 4, 2]. As with the moduli space of Higgs bundles, the space $\mathcal{X}_n(\alpha)$ comes equipped with a Hamiltonian $U(1)$-action that acts through the rotation $[x|y] \mapsto [x|\exp(i\theta)y]$ [8, 2]. Regarding cohomology, in [2] we show that a class of Nakajima quiver varieties that includes $\mathcal{X}_n(\alpha)$ has the hyperkähler Kirwan surjectivity property.[2]

Now, choose an affine coordinate z on the complex projective line $\mathbb{P}^1$ and a divisor $D = \sum_{i=1}^n z_i$ of pairwise distinct points $z_i \neq \infty$. The map

$$\Phi(z) = \sum_{i=1}^{n} \frac{(x_i y_i)_0}{z - z_i} dz$$

defines from $[x|y]$ a parabolic Higgs field for the trivial bundle $E = \mathbb{P}^1 \times \mathbb{C}^2$. The map respects stability (for sufficiently generic α) and notions of equivalence, and so we obtain an embedding of moduli spaces [2]. The target moduli space is that of β-semistable strongly parabolic Higgs bundles of rank 2 and degree 0 on $\mathbb{P}^1$ punctured along D, for some choice of parabolic weights β at the punctures (cf. [4]). The embedding map is not surjective, as only parabolic Higgs bundles with the trivial underlying bundle are obtained. The map is also not hyperkähler, as the Nakajima hyperkähler metric on $\mathcal{X}_n(\alpha)$ is complete while the Higgs bundle one pulled back to $\mathcal{X}_n(\alpha)$ is not.

Geometry. A sequence of hyperpolygons $[x^k|y^k]$ that escapes to infinity under the L^2-norm $\mu([x|y]) = \sum_{i=1}^n |y_i|^2$ will satisfy a rescaled version of the hyperpolygon equations with each α_i replaced by $\alpha_i/\sqrt{\mu([x^k|y^k])}$. The limit will thus satisfy the equations with $\alpha_i = 0$. We call these objects *limiting hyperpolygons*, which are analogous to the limiting Higgs bundles of [10]. The limiting hyperpolygons are parametrized, up to G-isomorphism, by the singular hyperkähler variety $\mathcal{X}_n(0)$. This can be regarded as the "tangent cone at infinity" to $\mathcal{X}_n(\alpha)$ with α generic. For $n = 4$, i.e. the affine D_4 quiver, the tangent cone $\mathcal{X}_4(0)$ is classically known to be $\mathbb{C}^2/\Gamma$, where $\Gamma = Q_8$ is a quaternion subgroup of order 8 in $SU(2)$. This fits neatly into the classification of ALE gravitational instantons, which can be regarded as a geometrization of the McKay correspondence. Here, a moduli space of gravitational instantons is determined by their geometry at infinity, given by the tangent cone. This is essentially the result of [9]. The geometry at infinity is a Du Val / Kleinian singularity produced by the action on $\mathbb{C}^2$ of a finite group $\Gamma < SU(2)$. This group in turn determines an (affine) ADE Dynkin type, via McKay. Taking us back from the Dynkin quiver to a gravitational instanton in the original moduli space is the Nakajima quiver variety construction [12].

For $n = 5$, we are no longer in a Dynkin type and the quotient is now an 8-manifold. However, we do know there is a stratification of $\mathcal{X}_5(0)$ by "edge collapse", as pairs (x_i, y_i) are allowed to tend to 0 now. Hence, there are 5 lower-dimensional

[2]This has been extended recently to all Nakajima quiver varieties in [11].

strata corresponding to embeddings of $\mathcal{X}_4(0)$. Using this information, can we realize $\mathcal{X}_5(0)$ as $\mathbb{C}^4/\Gamma$ for some finite subgroup $\Gamma < SL(4,\mathbb{C})$? How about for general n? A positive answer to these questions will establish the decay rate of Nakajima's hyperkähler metric to the Euclidean metric as being quasi-ALE, in the sense of [6]. This is joint work in progress with H. Weiß.

Mirror Symmetry. Because $\mathcal{X}_n(\alpha)$ is a smooth, noncompact Calabi-Yau manifold for generic α, and because the Calabi-Yau structure arises from a hyperkähler structure, we can ask about the existence of different types of triple branes, as motivated by [7]. For example, a (B, A, A) brane is one that is a complex submanifold with regards to the I complex structure and Lagrangian with regards to the ω_J and ω_K symplectic forms. We note that constructions of triple branes in Nakajima quiver varieties appear in [5, 3]. As expected, they arise generally from holomorphic and anti-holomorphic involutions on the $[x|y]$ data that descend to the quotient, consistent with the picture for Higgs bundles in, for instance, [1]. For $\mathcal{X}_n(\alpha)$, we aim to characterize these branes explicitly as subvarieties containing hyperpolygons of special type (e.g. polygons with no y data). At the same time, we want to understand how mirror symmetry interacts with various kinds of hyperpolygon branes. This is joint work in progress with L. Schaposnik.

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