

THE MEROMORPHIC R -MATRIX OF THE YANGIAN

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To Kolya Reshetikhin, on his 60th birthday.

ABSTRACT. Let \mathfrak{g} be a complex semisimple Lie algebra and $Y_{\hbar}(\mathfrak{g})$ its Yangian. Drinfeld proved that the universal R -matrix $\mathcal{R}(s)$ of $Y_{\hbar}(\mathfrak{g})$ gives rise to rational solutions of the QYBE on irreducible, finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$. This result was recently extended by Maulik–Okounkov to symmetric Kac–Moody algebras, and representations arising from geometry. We show that rationality ceases to hold on arbitrary finite-dimensional representations, if one requires such solutions to be natural and compatible with tensor products. Equivalently, the tensor category of finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$ does not admit rational commutativity constraints. We construct instead two meromorphic commutativity constraints, which are related by a unitarity condition. Each possesses an asymptotic expansion in s which has the same formal properties as $\mathcal{R}(s)$, and therefore coincides with it by uniqueness. In particular, we give a constructive proof of the existence of $\mathcal{R}(s)$. Our construction relies on the Gauss decomposition $\mathcal{R}^+(s) \cdot \mathcal{R}^0(s) \cdot \mathcal{R}^-(s)$ of $\mathcal{R}(s)$. The divergent abelian term \mathcal{R}^0 was resummed on finite-dimensional representations by the first two authors in [12]. In the present paper, we construct $\mathcal{R}^{\pm}(s)$, prove that they are rational on finite-dimensional representations, and that they intertwine the standard coproduct of $Y_{\hbar}(\mathfrak{g})$ and the deformed Drinfeld coproduct introduced in [12].

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1. INTRODUCTION

1.1. Let \mathfrak{g} be a complex, semisimple Lie algebra with an invariant inner product (\cdot, \cdot) , and $Y_{\hbar}(\mathfrak{g})$ the corresponding Yangian, which is a Hopf algebra deforming the current algebra $U(\mathfrak{g}[z])$ introduced by Drinfeld [4]. We assume that $\hbar \in \mathbb{C}^\times$ is fixed throughout. Drinfeld proved that $Y_{\hbar}(\mathfrak{g})$ possesses a unique universal R -matrix. Specifically, let $\Delta : Y_{\hbar}(\mathfrak{g}) \rightarrow Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})$ be the coproduct of $Y_{\hbar}(\mathfrak{g})$, and $\tau_s : Y_{\hbar}(\mathfrak{g}) \rightarrow Y_{\hbar}(\mathfrak{g})$ the one-parameter group of automorphisms which quantizes the shift automorphism $z \mapsto z + s$ of $U(\mathfrak{g}[z])$. Note that, if s is considered as a variable, τ_s may be regarded as a homomorphism $Y_{\hbar}(\mathfrak{g}) \rightarrow Y_{\hbar}(\mathfrak{g})[s]$. Then, the following holds.

Theorem. [4, Thm. 3]

(1) *There is a unique formal series*

$$\mathcal{R}(s) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k s^{-k} \in Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]]$$

such that the following holds¹ in $Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s; s^{-1}]]$

$$\tau_s \otimes \mathbf{1} \circ \Delta^{\text{op}}(a) = \mathcal{R}(s) \cdot \tau_s \otimes \mathbf{1} \circ \Delta(a) \cdot \mathcal{R}(s)^{-1} \quad (1.1)$$

for any $a \in Y_{\hbar}(\mathfrak{g})$, and

$$\Delta \otimes \mathbf{1}(\mathcal{R}(s)) = \mathcal{R}_{13}(s) \cdot \mathcal{R}_{23}(s) \quad (1.2)$$

$$\mathbf{1} \otimes \Delta(\mathcal{R}(s)) = \mathcal{R}_{13}(s) \cdot \mathcal{R}_{12}(s) \quad (1.3)$$

(2) \mathcal{R} satisfies the following identities

- *1-jet:* $\mathcal{R}(s) = 1 + \hbar s^{-1} \Omega_{\mathfrak{g}} + O(s^{-2})$
- *Unitarity:* $\mathcal{R}(s)^{-1} = \mathcal{R}_{21}(-s)$
- *Translation:* $\tau_a \otimes \tau_b(\mathcal{R}(s)) = \mathcal{R}(s + a - b)$

where $\Omega_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir tensor corresponding to (\cdot, \cdot) .

(3) \mathcal{R} is a solution of the quantum Yang–Baxter equation (QYBE)²

$$\mathcal{R}_{12}(s_1) \mathcal{R}_{13}(s_1 + s_2) \mathcal{R}_{23}(s_2) = \mathcal{R}_{23}(s_2) \mathcal{R}_{13}(s_1 + s_2) \mathcal{R}_{12}(s_1) \quad (1.4)$$

1.2. One of the main goals of this paper is to clarify the analytic nature of the formal power series $\mathcal{R}(s)$, and of the solutions of the QYBE obtained from it. Let V_1, V_2 be two finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$, and $\mathcal{R}_{V_1, V_2}(s) \in \text{End}(V_1 \otimes V_2)[[s^{-1}]]$ the corresponding evaluation of $\mathcal{R}(s)$. Drinfeld proved that $\mathcal{R}_{V_1, V_2}(s)$ has a zero radius of convergence in general [4, Examples 1,2], but nevertheless gives rise to a rational solution of the QYBE as follows.

Theorem. [4, Thm. 4] *Assume that V_1 and V_2 are irreducible with highest weight vectors v_1, v_2 , and let $\rho_{V_1, V_2}(s) \in 1 + s^{-1}\mathbb{C}[[s^{-1}]]$ be the matrix element of $\mathcal{R}_{V_1, V_2}(s)$ given by*

$$\mathcal{R}_{V_1, V_2}(s) v_1 \otimes v_2 = \rho_{V_1, V_2}(s) v_1 \otimes v_2$$

¹Our conventions differ slightly from those of [4], where the intertwining equation (1.1) is written as $\mathbf{1} \otimes \tau_s \circ \Delta^{\text{op}}(a) = \mathcal{R}(s)^{-1} \cdot \mathbf{1} \otimes \tau_s \circ \Delta(a) \cdot \mathcal{R}(s)$. Thus, our $\mathcal{R}(s)$ is Drinfeld's $\mathcal{R}(-s)^{-1}$.

²The QYBE may be viewed as an equation in $Y_{\hbar}(\mathfrak{g})^{\otimes 3}[[s_1; s_1^{-1}]][[s_2^{-1}]]$ by expanding it as if $|s_2| \gg |s_1|$, that is setting $(s_1 + s_2)^{-1} = \sum_{k \geq 0} (-1)^k s_1^k s_2^{-k-1}$, or as an equation in $Y_{\hbar}(\mathfrak{g})^{\otimes 3}[[s_2; s_2^{-1}]][[s_1^{-1}]]$, by setting $(s_1 + s_2)^{-1} = \sum_{k \geq 0} (-1)^k s_2^k s_1^{-k-1}$. The precise statement of (3) above is that (1.4) holds in either of these cases.

Then,

- (1) $\mathbf{R}_{V_1, V_2}(s) = \mathcal{R}_{V_1, V_2}(s) \cdot \rho_{V_1, V_2}(s)^{-1}$ is a rational function of s .
(2) If $V_1 = V = V_2$, (1.4) together with the factorisation

$$\mathcal{R}_{V_1, V_2}(s) = \mathbf{R}_{V_1, V_2}(s) \cdot \rho_{V_1, V_2}(s) \quad (1.5)$$

imply that $\mathbf{R}_{V, V}(s)$ is a rational solution of the QYBE.

More recently, a geometric construction of R -matrices corresponding to the (extended) Yangian of a symmetric Kac–Moody algebra was given by Maulik–Okounkov [20], which provides in particular an alternative construction of rational solutions of the QYBE on the equivariant cohomology of Nakajima quiver varieties.

1.3. One of the byproducts of this paper is to extend the factorisation (1.5) to an arbitrary pair of (not necessarily irreducible) finite-dimensional representations. In this case, the divergent factor $\rho_{V_1, V_2}(s)$ takes values in $\text{End}(V_1 \otimes V_2)[[s^{-1}]]$, and intertwines the action of $Y_{\hbar}(\mathfrak{g})$ given by $\Delta_s = \tau_s \otimes \mathbf{1} \circ \Delta$, whereas the rational factor $\mathbf{R}_{V_1, V_2}(s)$ intertwines Δ_s and $\Delta_s^{\text{op}} = \tau_s \otimes \mathbf{1} \circ \Delta^{\text{op}}$. However, since $\rho_{V_1, V_2}(s)$ is not scalar-valued in general, it is not clear whether $\mathbf{R}_{V_1, V_2}(s)$ satisfies the QYBE when $V_1 = V_2$.

We prove in fact that, even for $\mathfrak{g} = \mathfrak{sl}_2$, no rational intertwiner $\mathbf{R}_{V_1, V_2}(s) \in \text{End}(V_1 \otimes V_2)$ exists which is defined for any $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, is natural in V_1 and V_2 , and satisfies the cabling identities (1.2)–(1.3). Equivalently, the tensor category of finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$ does not admit rational commutativity constraints. In particular, this raises the question of whether one can consistently define rational solutions of the QYBE on all finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$.³

1.4. In the present paper, we propose an alternative solution to this issue, by constructing *meromorphic* commutativity constraints on $\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, and in particular consistent meromorphic solutions of the QYBE on all $V \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$. Namely, we prove that the universal R -matrix of $Y_{\hbar}(\mathfrak{g})$, while generally divergent on a pair of finite-dimensional representations V_1, V_2 , can be canonically *resummed*, in two distinct ways. This yields a *pair* of meromorphic functions

$$\mathcal{R}_{V_1, V_2}^{\uparrow}(s), \mathcal{R}_{V_1, V_2}^{\downarrow}(s) : \mathbb{C} \rightarrow \text{End}(V_1 \otimes V_2)$$

which are natural with respect to V_1, V_2 , satisfy the intertwining relation (1.1), the cabling identities (1.2)–(1.3), as well as the translation property. The function $\mathcal{R}_{V_1, V_2}^{\uparrow}(s)$ (resp. $\mathcal{R}_{V_1, V_2}^{\downarrow}(s)$) is asymptotic to $\mathcal{R}_{V_1, V_2}(s)$ as $s \rightarrow \infty$ with $\text{Re}(s/\hbar) > 0$ (resp. $\text{Re}(s/\hbar) < 0$), and is related to $\mathcal{R}_{V_1, V_2}^{\downarrow}(s)$ by the unitarity relation

$$\mathcal{R}_{V_1, V_2}^{\uparrow}(s)^{-1} = \mathcal{R}_{V_2, V_1}^{\downarrow}(-s)^{21}$$

The situation is somewhat analogous to the case of the quantum loop algebra $U_q(L\mathfrak{g})$. In that case, if $\mathcal{R} \in U_q(L\mathfrak{g}) \widehat{\otimes} U_q(L\mathfrak{g})$ is the universal R -matrix, then

$$\mathcal{R}^{\infty}(z) = \tau_z \otimes \mathbf{1}(\mathcal{R}) \in U_q(L\mathfrak{g})^{\otimes 2}[[z^{-1}]] \quad \text{and} \quad \mathcal{R}^0(z) = \mathbf{1} \otimes \tau_z(\mathcal{R}) \in U_q(L\mathfrak{g})^{\otimes 2}[[z]]$$

converge, near $z = \infty$ and $z = 0$ respectively, to meromorphic functions of $z \in \mathbb{C}^{\times}$ on the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_2$ of any two finite-dimensional representations [7, 16],

³The Maulik–Okounkov construction mentioned in 1.2 provides a partial solution to this question, since an arbitrary representation of $Y_{\hbar}(\mathfrak{g})$ may not have a geometric realisation.

and are related by $\mathcal{R}_{V_1, V_2}^\infty(z)^{-1} = \mathcal{R}_{V_2, V_1}^0(z^{-1})^{21}$. In the case of $U_q(L\mathfrak{g})$, however, $\mathcal{R}^\infty(z)$ and $\mathcal{R}^0(z)$ are convergent as is, and do not need to be resummed.

1.5. Our approach does not rely on Drinfeld's cohomological construction of $\mathcal{R}(s)$ to carry out the resummation. It produces the functions $\mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s)$ through a direct, explicit construction, which shows in particular that they have an asymptotic expansion as $s \rightarrow \infty$. The fact that the latter coincides with $\mathcal{R}_{V_1, V_2}(s)$, and therefore *a posteriori* that $\mathcal{R}_{V_1, V_2}(s)$ can be resummed, follows from the fact that the asymptotic expansion can be lifted to $Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]]$, and shown to have the properties which uniquely determine $\mathcal{R}(s)$ by Theorem 1.1. In particular, our construction yields an independent, and constructive proof of the existence of $\mathcal{R}(s)$.

1.6. Our construction can be motivated by the following considerations. The R -matrix of $Y_{\hbar}(\mathfrak{g})$ is expected to arise as the canonical element in $DY_{\hbar}(\mathfrak{g}) \widehat{\otimes} DY_{\hbar}(\mathfrak{g})$, where $DY_{\hbar}(\mathfrak{g}) \supset Y_{\hbar}(\mathfrak{g})$ is the double Yangian of \mathfrak{g} , which is a quantisation of the graded Drinfeld double

$$(\mathfrak{g}[z^{\pm 1}], \mathfrak{g}[z], z^{-1}\mathfrak{g}[z^{-1}])$$

of $\mathfrak{g}[z]$. Although a detailed understanding of $DY_{\hbar}(\mathfrak{g})$ is still lacking at present (see, however, [17] and [24]), this suggests that, given a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ of \mathfrak{g} , $\mathcal{R}(s)$ should have a corresponding Gauss decomposition

$$\mathcal{R}(s) = \mathcal{R}^+(s) \cdot \mathcal{R}^0(s) \cdot \mathcal{R}^-(s) \quad (1.6)$$

where $\mathcal{R}^0(s)$ quantises the canonical element in $\mathfrak{h}[z] \widehat{\otimes} z^{-1}\mathfrak{h}[z^{-1}]$, and $\mathcal{R}^\pm(s)$ those in $\mathfrak{n}_\pm[z] \widehat{\otimes} z^{-1}\mathfrak{n}_\mp[z^{-1}]$ respectively. Moreover, the unitarity of $\mathcal{R}(s)$ suggests that

$$\mathcal{R}^0(s)^{-1} = \mathcal{R}^0(-s)^{21} \quad \text{and} \quad \mathcal{R}^+(s)^{-1} = \mathcal{R}^-(-s)^{21}$$

Accordingly, we construct each factor $\mathcal{R}^0(s), \mathcal{R}^-(s), \mathcal{R}^+(s) = (\mathcal{R}^-(-s)^{21})^{-1}$, and their resummation on finite-dimensional representations separately.

1.7. Khoroshkin–Tolstoy gave a heuristic formula for \mathcal{R}^0 [17], as the exponential of an infinite sum in the abelian subalgebra of $DY_{\hbar}(\mathfrak{g})$ which quantises $\mathfrak{h}[z, z^{-1}]$. In [12], the first two named authors gave a precise version of this formula, where the exponent takes values in the abelian subalgebra $Y_{\hbar}^0(\mathfrak{g})$ of $Y_{\hbar}(\mathfrak{g})$ which quantises $\mathfrak{h}[z]$. We showed moreover that this expression can be resummed on a tensor product $V_1 \otimes V_2$ of finite-dimensional representations in two different ways. This yields two meromorphic functions $\mathcal{R}_{V_1, V_2}^{0, \uparrow}(s), \mathcal{R}_{V_1, V_2}^{0, \downarrow}(s)$, which have the same asymptotic expansion on $\pm \operatorname{Re}(s/\hbar) > 0$, and are related by $\mathcal{R}_{V_1, V_2}^{0, \uparrow}(s)^{-1} = \mathcal{R}_{V_2, V_1}^{0, \downarrow}(-s)^{21}$.

1.8. An important discovery of [12] is that these abelian R -matrices play a similar role to that of the full R -matrix of $Y_{\hbar}(\mathfrak{g})$, but with respect to the *deformed Drinfeld tensor product*. The latter was introduced in [12] by degenerating the Drinfeld tensor product of the quantum loop algebra introduced by Hernandez [15]. It gives rise to a family of actions of $Y_{\hbar}(\mathfrak{g})$ on the vector space $V_1 \otimes V_2$, which is denoted by $V_1 \otimes_{D, s} V_2$ and is a rational function of a parameter $s \in \mathbb{C}$. The tensor product $\otimes_{D, s}$ is associative, in that the identification of vector spaces

$$(V_1 \otimes_{D, s_1} V_2) \otimes_{D, s_2} V_3 = V_1 \otimes_{D, s_1 + s_2} (V_2 \otimes_{D, s_2} V_3)$$

intertwines the action of $Y_{\hbar}(\mathfrak{g})$ for any $s_1, s_2 \in \mathbb{C}$, and endows $\operatorname{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ with the structure of a meromorphic tensor category in the sense of [22, 23].

The endomorphisms $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}$ are meromorphic commutativity constraints with respect to $\otimes_{D, s}$, that is they satisfy the representation theoretic version of the identities (1) of Theorem 1.1. In the present paper, we complement the results of [12] by lifting $\otimes_{D, s}$ to a *deformed Drinfeld coproduct*

$$\Delta_{D, s} : Y_{\hbar}(\mathfrak{g}) \rightarrow (Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))[[s; s^{-1}]]$$

and the common asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ to an element

$$\mathcal{R}^0(s) \in (Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))[[s^{-1}]]$$

which satisfy the identities (1) of Theorem 1.1, with $\tau_s \otimes \mathbf{1} \circ \Delta$ replaced by $\Delta_{D, s}$, and $\mathcal{R}(s)$ by $\mathcal{R}^0(s)$.

1.9. The central ingredient of the present paper is the construction of $\mathcal{R}^{\pm}(s)$, which is based on the following. The fact that $\mathcal{R}(s)$ (resp. $\mathcal{R}^0(s)$) conjugates the standard coproduct $\Delta_s = \tau_s \otimes \mathbf{1} \circ \Delta$ (resp. the deformed Drinfeld coproduct $\Delta_{D, s}$) to its opposite, together with the Gauss decomposition (1.6), suggest that $\mathcal{R}^{-}(s)$ should conjugate the standard coproduct Δ_s to the deformed Drinfeld coproduct $\Delta_{D, s}$. This is consistent with the fact that an analogous statement holds for the quantum loop algebra, and the related construction of twists conjugating quantum coproducts corresponding to different polarisations of a Manin triple given in [6]. In this case, the standard (resp. Drinfeld) coproducts on $U_q(L\mathfrak{g})$ correspond, respectively, to the polarisations

$$\mathfrak{g}[z] \oplus z^{-1}\mathfrak{g}[z^{-1}] = \mathfrak{g}[z^{\pm 1}] = (\mathfrak{n}_-[z^{\pm 1}] \oplus \mathfrak{h}[z]) \oplus (z^{-1}\mathfrak{h}[z^{-1}] \oplus \mathfrak{n}_+[z^{\pm 1}])$$

1.10. We prove that this intertwining property uniquely determines an element $\mathcal{R}^{-}(s)$, provided it is required to lie in $(Y_{\hbar}^{-}(\mathfrak{g}) \otimes Y_{\hbar}^{+}(\mathfrak{g}))[[s^{-1}]]$ and have constant term 1, where $Y_{\hbar}^{\pm}(\mathfrak{g}) \subset Y_{\hbar}(\mathfrak{g})$ is the subalgebra deforming $U(\mathfrak{n}_{\pm}[z])$. We show in fact that, under this triangularity assumption, $\mathcal{R}^{-}(s)$ is uniquely determined by the requirement that it intertwines the standard and Drinfeld coproducts of the loop generators $t_{i,0}, t_{i,1}$ of $Y_{\hbar}(\mathfrak{g})$ which deform $\mathfrak{h} \oplus \mathfrak{h} \otimes z \subset \mathfrak{h}[z]$.

We then show that, for any $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, $\mathcal{R}_{V_1, V_2}^{-}(s)$ is a rational function of s . Moreover, the following cocycle identity holds for any $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$

$$\mathcal{R}_{V_1 \otimes_{D, s_1} V_2, V_3}^{-}(s_2) \cdot \mathcal{R}_{V_1, V_2}^{-}(s_1) = \mathcal{R}_{V_1, V_2 \otimes_{D, s_2} V_3}^{-}(s_1 + s_2) \cdot \mathcal{R}_{V_2, V_3}^{-}(s_2) \quad (1.7)$$

Together with the identities satisfied by $\mathcal{R}^0(s)$, this guarantees that the product $\mathcal{R}(s) = \mathcal{R}^{+}(s) \cdot \mathcal{R}^0(s) \cdot \mathcal{R}^{-}(s)$, where $\mathcal{R}^{+}(s) = (\mathcal{R}^{-}(-s)^{21})^{-1}$, satisfies the identities (1.1)–(1.3) on any pair of finite-dimensional representations. A separation of points argument then implies that $\mathcal{R}(s)$ satisfies Drinfeld's uniqueness criterion of the universal R -matrix of $Y_{\hbar}(\mathfrak{g})$, and in particular coincides with it.⁴

Finally, since $\mathcal{R}_{V_1, V_2}^{-}(s)$ is a rational function of s , the product

$$\mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) = \mathcal{R}_{V_1, V_2}^{+}(s) \cdot \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdot \mathcal{R}_{V_1, V_2}^{-}(s) \quad (1.8)$$

⁴The passage to finite-dimensional representations is dictated by the fact that the cocycle identity (1.7) does not appear to have a natural lift to $Y_{\hbar}(\mathfrak{g})$. Indeed, when lifted to $Y_{\hbar}(\mathfrak{g})$, the left-hand side lies in $Y_{\hbar}(\mathfrak{g})^{\otimes 3}[[s_1; s_1^{-1}]][[s_2^{-1}]]$, while the right-hand side lies in $Y_{\hbar}(\mathfrak{g})^{\otimes 3}[[s_2; s_2^{-1}]][[s_1^{-1}]]$.

is a resummation of $\mathcal{R}_{V_1, V_2}(s)$, as well a meromorphic commutativity constraint on $\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ with respect to the standard tensor product.

1.11. Our results may be rephrased as follows. As proved in [12], and mentioned above, finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$, together with the deformed Drinfeld tensor product $\otimes_{D, s}$ and one of the resummed abelian R -matrices $\mathcal{R}^{0, \uparrow/\downarrow}(s)$ is a meromorphic braided tensor category.

Similarly, $\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ endowed with the deformed standard tensor product $\otimes_s = \otimes \circ (\tau_s^* \otimes \mathbf{1})$ is a meromorphic (in fact, polynomial) tensor category. Our construction of the resummed R -matrices $\mathcal{R}^{\uparrow/\downarrow}(s)$ endows this category with a meromorphic braiding.⁵ Moreover, the element $\mathcal{R}^-(s)$ is a rational braided tensor structure on the identity functor

$$(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes_{D, s}, \mathcal{R}^{0, \uparrow/\downarrow}) \rightarrow (\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes_s, \mathcal{R}^{\uparrow/\downarrow})$$

That is, $\mathcal{R}^-(s)$ gives rise to a system of natural isomorphisms of $Y_{\hbar}(\mathfrak{g})$ -modules $\mathcal{R}_{V_1, V_2}^-(s) : V_1 \otimes_s V_2 \rightarrow V_1 \otimes V_2$, which is compatible with the (trivial) associativity constraints and the meromorphic braidings, *i.e.*, such that the following diagrams commute for any $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$

$$\begin{array}{ccc} (V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 & \xlongequal{\quad} & V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3) \\ \downarrow \mathcal{R}_{V_1, V_2}^-(s_1) \otimes \mathbf{1}_{V_3} & & \downarrow \mathbf{1}_{V_1} \otimes \mathcal{R}_{V_2, V_3}^-(s_2) \\ (V_1 \otimes_{D, s_1} V_2) \otimes_{s_2} V_3 & & V_1 \otimes_{s_1+s_2} (V_2 \otimes_{D, s_2} V_3) \\ \downarrow \mathcal{R}_{V_1 \otimes_{D, s_1} V_2, V_3}^-(s_2) & & \downarrow \mathcal{R}_{V_1, V_2 \otimes_{D, s_2} V_3}^-(s_1+s_2) \\ (V_1 \otimes_{D, s_1} V_2) \otimes_{D, s_2} V_3 & \xlongequal{\quad} & V_1 \otimes_{D, s_1+s_2} (V_2 \otimes_{D, s_2} V_3) \end{array}$$

as dictated by the cocycle equation (1.7), and

$$\begin{array}{ccc} V_1(s) \otimes V_2 & \xrightarrow{(12) \circ \mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s)} & V_2 \otimes V_1(s) \\ \downarrow \mathcal{R}_{V_1, V_2}^-(s) & & \downarrow \mathcal{R}_{V_2, V_1}^-(-s) \\ V_1(s) \otimes_{D, 0} V_2 & \xrightarrow{(12) \circ \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)} & V_2 \otimes_{D, 0} V_1(s) \end{array}$$

which follows from the Gauss decomposition (1.8), together with the fact that $(\mathcal{R}^-(-s)^{21})^{-1} = \mathcal{R}^+(s)$.

⁵An analogous statement was proved by Kazhdan–Soibelman for the quantum loop algebra in [16]. As pointed out in 1.4, however, in the case of $U_q(L\mathfrak{g})$ no resummation of the universal R -matrix of $U_q(L\mathfrak{g})$ is needed.

1.12. Outline of the paper. We review the definition of $Y_{\hbar}(\mathfrak{g})$ in Section 2, and that of the standard and Drinfeld coproducts in Section 3. In Section 4, we prove the existence and uniqueness of $\mathcal{R}^-(s)$, and establish its various properties. In Section 5, we give an explicit expressions for $\mathcal{R}^-(s)$ when $\mathfrak{g} = \mathfrak{sl}_2$. In Section 6, we review the construction of $\mathcal{R}^{0,\uparrow/\downarrow}(s)$ given in [12]. We then explicitly lift its asymptotic expansion to $Y_{\hbar}^0(\mathfrak{g})^{\otimes 2}[[s^{-1}]]$, and prove that it satisfies properties analogous to those of Drinfeld's R -matrix, but with respect to the deformed Drinfeld coproduct. We also prove that there is no rational commutativity constraint on $\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$. Combined with the results of Section 4, we obtain the same assertions for the standard tensor product in Section 7. We give a proof of the uniqueness of the universal R -matrix of the Yangian in Appendix B, thus completing the proof that our construction gives rise to Drinfeld's R -matrix. In Section 8, we restate our results in the language of meromorphic tensor categories. In the final Section 9, we discuss the analogous case of the quantum loop algebra, and relate the two by means of the meromorphic tensor functor constructed in [12]. Appendix A contains a proof due to Drinfeld that finite-dimensional representations separate points of $Y_{\hbar}(\mathfrak{g})$.

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2. THE YANGIAN $Y_{\hbar}(\mathfrak{g})$

2.1. Let \mathfrak{g} be a complex, semisimple Lie algebra and (\cdot, \cdot) an invariant, symmetric, non-degenerate bilinear form on \mathfrak{g} . Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of \mathfrak{g} , $\{\alpha_i\}_{i \in \mathbf{I}} \subset \mathfrak{h}^*$ a basis of simple roots of \mathfrak{g} relative to \mathfrak{h} and $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ the entries of the corresponding Cartan matrix \mathbf{A} . Let $\Phi_+ \subset \mathfrak{h}^*$ be the corresponding set of positive roots, and $\mathbf{Q} = \mathbb{Z}\Phi_+ = \bigoplus_{i \in \mathbf{I}} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ the root lattice. We assume that (\cdot, \cdot) is normalised so that the square length of short roots is 2. Set $d_i = (\alpha_i, \alpha_i)/2 \in \{1, 2, 3\}$, so that $d_i a_{ij} = d_j a_{ji}$ for any $i, j \in \mathbf{I}$. In addition, we set $h_i = \nu^{-1}(\alpha_i)/d_i$ and choose root vectors $x_i^{\pm} \in \mathfrak{g}_{\pm\alpha_i}$ such that $[x_i^+, x_i^-] = d_i h_i$, where $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ is the isomorphism determined by (\cdot, \cdot) .

2.2. The Yangian $Y_{\hbar}(\mathfrak{g})$ [5]. Let $\hbar \in \mathbb{C}$. The Yangian $Y_{\hbar}(\mathfrak{g})$ is the \mathbb{C} -algebra generated by elements $\{x_{i,r}^{\pm}, \xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}_{\geq 0}}$, subject to the following relations.

$$(Y1) \text{ For any } i, j \in \mathbf{I}, r, s \in \mathbb{Z}_{\geq 0}: [\xi_{i,r}, \xi_{j,s}] = 0$$

$$(Y2) \text{ For } i, j \in \mathbf{I} \text{ and } s \in \mathbb{Z}_{\geq 0}: [\xi_{i,0}, x_{j,s}^{\pm}] = \pm d_i a_{ij} x_{j,s}^{\pm}$$

$$(Y3) \text{ For } i, j \in \mathbf{I} \text{ and } r, s \in \mathbb{Z}_{\geq 0}:$$

$$[\xi_{i,r+1}, x_{j,s}^{\pm}] - [\xi_{i,r}, x_{j,s+1}^{\pm}] = \pm \hbar \frac{d_i a_{ij}}{2} (\xi_{i,r} x_{j,s}^{\pm} + x_{j,s}^{\pm} \xi_{i,r})$$

$$(Y4) \text{ For } i, j \in \mathbf{I} \text{ and } r, s \in \mathbb{Z}_{\geq 0}:$$

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm \hbar \frac{d_i a_{ij}}{2} (x_{i,r}^{\pm} x_{j,s}^{\pm} + x_{j,s}^{\pm} x_{i,r}^{\pm})$$

(Y5) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{Z}_{\geq 0}$: $[x_{i,r}^+, x_{j,s}^-] = \delta_{ij} \xi_{i,r+s}$

(Y6) Let $i \neq j \in \mathbf{I}$ and set $m = 1 - a_{ij}$. For any $r_1, \dots, r_m \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$

$$\sum_{\pi \in \mathfrak{S}_m} \left[x_{i,r_{\pi(1)}}^{\pm}, \left[x_{i,r_{\pi(2)}}^{\pm}, \left[\dots, \left[x_{i,r_{\pi(m)}}^{\pm}, x_{j,s}^{\pm} \right] \dots \right] \right] \right] = 0$$

We denote by $Y_{\hbar}^0(\mathfrak{g})$ and $Y_{\hbar}^{\pm}(\mathfrak{g})$ the unital subalgebras of $Y_{\hbar}(\mathfrak{g})$ generated by $\{\xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}_{\geq 0}}$ and $\{x_{i,r}^{\pm}\}_{i \in \mathbf{I}, r \in \mathbb{Z}_{\geq 0}}$, respectively.

2.3. Assume henceforth that $\hbar \neq 0$, and define $\xi_i(u), x_i^{\pm}(u) \in Y_{\hbar}(\mathfrak{g})[[u^{-1}]]$ by

$$\xi_i(u) = 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \quad \text{and} \quad x_i^{\pm}(u) = \hbar \sum_{r \geq 0} x_{i,r}^{\pm} u^{-r-1}$$

Proposition. [11, Prop. 2.3] *The relations (Y1), (Y2)–(Y3), (Y4), (Y5) and (Y6) are respectively equivalent to the following identities in $Y_{\hbar}(\mathfrak{g})[[u, v; u^{-1}, v^{-1}]]$.*

(Y1) For any $i, j \in \mathbf{I}$, $[\xi_i(u), \xi_j(v)] = 0$

(Y2) For any $i, j \in \mathbf{I}$, $[\xi_{i,0}, x_j^{\pm}(u)] = \pm d_i a_{ij} x_j^{\pm}(u)$

(Y3) For any $i, j \in \mathbf{I}$, and $a = \hbar d_i a_{ij} / 2$:

$$(u - v \mp a) \xi_i(u) x_j^{\pm}(v) = (u - v \pm a) x_j^{\pm}(v) \xi_i(u) \mp 2a x_j^{\pm}(u \mp a) \xi_i(u)$$

(Y4) For any $i, j \in \mathbf{I}$, and $a = \hbar d_i a_{ij} / 2$:

$$\begin{aligned} (u - v \mp a) x_i^{\pm}(u) x_j^{\pm}(v) \\ = (u - v \pm a) x_j^{\pm}(v) x_i^{\pm}(u) + \hbar ([x_{i,0}^{\pm}, x_j^{\pm}(v)] - [x_i^{\pm}(u), x_{j,0}^{\pm}]) \end{aligned}$$

(Y5) For any $i, j \in \mathbf{I}$:

$$(u - v) [x_i^+(u), x_j^-(v)] = -\delta_{ij} \hbar (\xi_i(u) - \xi_i(v))$$

(Y6) For any $i \neq j \in \mathbf{I}$, $m = 1 - a_{ij}$, $r_1, \dots, r_m \in \mathbb{Z}_{\geq 0}$, and $s \in \mathbb{Z}_{\geq 0}$:

$$\sum_{\pi \in \mathfrak{S}_m} \left[x_i^{\pm}(u_{\pi_1}), \left[x_i^{\pm}(u_{\pi(2)}), \left[\dots, \left[x_i^{\pm}(u_{\pi(m)}), x_j^{\pm}(v) \right] \dots \right] \right] \right] = 0$$

Remark. When $\mathfrak{g} = \mathfrak{sl}_2$, we will write $\xi_r, x_r^{\pm}, \xi(u)$ and $x^{\pm}(u)$ in place of $\xi_{i,r}, x_{i,r}^{\pm}, \xi_i(u)$ and $x_i^{\pm}(u)$, respectively.

2.4. **Alternative generators of $Y_{\hbar}^0(\mathfrak{g})$.** Let $\{t_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}_{\geq 0}} \subset Y_{\hbar}^0(\mathfrak{g})$ be the generators defined by

$$t_i(u) = \hbar \sum_{r \geq 0} t_{i,r} u^{-r-1} := \log(\xi_i(u))$$

In particular, $t_{i,0} = \xi_{i,0}$ and

$$t_{i,1} = \xi_{i,1} - \frac{\hbar}{2} \xi_{i,0}^2 \tag{2.1}$$

The relations (Y2)–(Y3) of $Y_{\hbar}(\mathfrak{g})$ imply that for any $i, j \in \mathbf{I}$ and $r \in \mathbb{Z}_{\geq 0}$,

$$[t_{i,1}, x_{j,r}^{\pm}] = \pm d_i a_{ij} x_{j,r+1}^{\pm} \tag{2.2}$$

so that $t_{i,1}$ act as shift operators on the generators $x_{j,r}^{\pm}$.

The relation (2.2) also implies that $\{\xi_{i,0}, x_{i,0}^\pm, t_{i,1}\}_{i \in \mathbf{I}}$ generate $Y_{\hbar}(\mathfrak{g})$ as an algebra. We refer the reader to [18, Thm. 1.2] for a presentation of $Y_{\hbar}(\mathfrak{g})$ given in terms of these generators, and to [13, Thm. 2.13] for a refinement of this result.

2.5. Shift automorphism. The group of translations of the complex plane acts on $Y_{\hbar}(\mathfrak{g})$ by

$$\tau_a(y_r) = \sum_{s=0}^r \binom{r}{s} a^{r-s} y_s$$

where $a \in \mathbb{C}$ and y is one of ξ_i, t_i, x_i^\pm . In terms of the generating series introduced in 2.3 and 2.4, we have

$$\tau_a(y(u)) = y(u - a)$$

Given a representation V of $Y_{\hbar}(\mathfrak{g})$ and $a \in \mathbb{C}$, set $V(a) = \tau_a^*(V)$.

2.6. PBW theorem. Consider the loop filtration $\mathcal{F}_\bullet(Y_{\hbar}(\mathfrak{g}))$ on $Y_{\hbar}(\mathfrak{g})$ defined by $\deg(y_r) = r$ for each of the generators $y = \xi_i, x_i^\pm$. Note that $\deg(t_{i,r}) = r$. The Hopf algebra structure on $Y_{\hbar}(\mathfrak{g})$ preserves this filtration, and endows $\text{gr}(Y_{\hbar}(\mathfrak{g}))$ with the structure of a graded Hopf algebra. The PBW Theorem for $Y_{\hbar}(\mathfrak{g})$ [19] (see also [8, Thm. B.6] and [14, Prop. 2.2]) is equivalent to the assertion that the assignments

$$x_i^\pm . z^r \mapsto \bar{x}_{i,r}^\pm \quad \text{and} \quad d_i h_i . z^r \mapsto \bar{\xi}_{i,r}$$

uniquely extend to an isomorphism of graded Hopf algebras

$$U(\mathfrak{g}[z]) \xrightarrow{\sim} \text{gr}(Y_{\hbar}(\mathfrak{g})), \quad (2.3)$$

where, for any fixed $k \in \mathbb{Z}_{\geq 0}$ and element $y_k \in \mathcal{F}_k(Y_{\hbar}(\mathfrak{g}))$,

$$\bar{y}_k \in \text{gr}_k(Y_{\hbar}(\mathfrak{g})) := \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})) / \mathcal{F}_{k-1}(Y_{\hbar}(\mathfrak{g}))$$

is defined to be the image of y_k in the k -th graded component $\text{gr}_k(Y_{\hbar}(\mathfrak{g}))$ of the associated graded algebra $\text{gr}(Y_{\hbar}(\mathfrak{g}))$.

Henceforth, we shall freely make use of the above identification without further comment. Similarly, we will exploit the fact that it allows us to identify $U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w])$ with $\text{gr}(Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})) \cong \text{gr}(Y_{\hbar}(\mathfrak{g})) \otimes \text{gr}(Y_{\hbar}(\mathfrak{g}))$, the associated graded algebra of $Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})$ with respect to the tensor product filtration $\mathcal{F}_\bullet(Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))$ induced by $\mathcal{F}_\bullet(Y_{\hbar}(\mathfrak{g}))$.

2.7. The embedding $U(\mathfrak{g}) \subset Y_{\hbar}(\mathfrak{g})$. Since $\text{gr}_0(Y_{\hbar}(\mathfrak{g})) = \mathcal{F}_0(Y_{\hbar}(\mathfrak{g})) \subset Y_{\hbar}(\mathfrak{g})$, the isomorphism (2.3) restricts to an embedding of $U(\mathfrak{g})$ into $Y_{\hbar}(\mathfrak{g})$, given by

$$x_i^\pm \mapsto x_{i,0}^\pm \quad \text{and} \quad d_i h_i \mapsto \xi_{i,0}$$

We shall henceforth identify $U(\mathfrak{g}) \subset Y_{\hbar}(\mathfrak{g})$, with the above embedding implicitly understood. Viewed as a module over $\mathfrak{h} \subset Y_{\hbar}(\mathfrak{g})$, we then have $Y_{\hbar}(\mathfrak{g}) = \bigoplus_{\beta \in \mathbb{Q}} Y_{\hbar}(\mathfrak{g})_\beta$

A second embedding $\mathbb{T} : \mathfrak{h} \rightarrow Y_{\hbar}(\mathfrak{g})$ is given by setting $\mathbb{T}(d_i h_i) = t_{i,1}$ for all $i \in \mathbf{I}$, where $t_{i,1}$ is defined by (2.1). The relation (2.2) then reads

$$[\mathbb{T}(h), x_{i,r}^\pm] = \pm \alpha_i(h) x_{i,r+1}^\pm \quad (2.4)$$

and implies in particular that, for any $h \in \alpha_i^\perp$ and $r \geq 0$,

$$[\mathbb{T}(h), x_{i,r}^\pm] = 0 \quad (2.5)$$

2.8. Formal series filtration. For any $k \in \mathbb{Z}$, set

$$\begin{aligned} \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}]) &= s^k \prod_{n \geq 0} \mathcal{F}_n(Y_{\hbar}(\mathfrak{g})^{\otimes 2})s^{-n} \\ &= \left\{ \sum_{m \leq M} y_m s^m \in Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}] \mid \deg(y_m) \leq k - m \right\} \end{aligned} \quad (2.6)$$

For $k \leq 0$, we shall write this as $\mathcal{F}_k(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$, for obvious reasons.

The above spaces generate a \mathbb{Z} -filtered algebra

$$\bigcup_{k \in \mathbb{Z}} \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}]) \subset Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}]$$

with associated graded algebra that can (and will) be identified with the $\mathbb{C}[s^{\pm 1}]$ -submodule of $(U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))[s; s^{-1}]$ generated by

$$\prod_{n \geq 0} (U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))_n s^{-n}$$

where $(U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))_n$ is the n -th graded component of $U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w])$.

If $\mathfrak{X}(s) \in \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}])$, we denote by

$$\begin{aligned} \overline{\mathfrak{X}(s)} &= \mathfrak{X}(s) \pmod{\mathcal{F}_{k-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}])} \\ &\in s^k \prod_{n \geq 0} (U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))_n s^{-n} \\ &\subset (U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))[s; s^{-1}] \end{aligned}$$

the image of $\mathfrak{X}(s)$ in the k -th graded component of the associated graded algebra. Note in passing that if $\mathfrak{X}(s) \in \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}])$, with $k < 0$, then $\exp(\mathfrak{X}(s)) - 1 \in \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}])$, and

$$\overline{\exp(\mathfrak{X}(s)) - 1} = \overline{\mathfrak{X}(s)}. \quad (2.7)$$

2.9. Rationality. The following rationality property is due to Beck–Kac [1] and Hernandez [15] for the analogous case of the quantum loop algebra, and to the first two authors for $Y_{\hbar}(\mathfrak{g})$. In the form below, the result appears in [11, Prop. 3.6].

Proposition. *Let V be a $Y_{\hbar}(\mathfrak{g})$ -module on which $\{\xi_{i,0}\}_{i \in \mathbf{I}}$ act semisimply with finite-dimensional weight spaces. Then, for every weight μ of V , the generating series*

$$\xi_i(u) \in \text{End}(V_{\mu})[[u^{-1}]] \quad \text{and} \quad x_i^{\pm}(u) \in \text{Hom}(V_{\mu}, V_{\mu \pm \alpha_i})[[u^{-1}]]$$

are the Laurent expansions at ∞ of rational functions of u . Specifically,

$$x_i^{\pm}(u) = \hbar u^{-1} \left(1 \mp \frac{\text{ad}(t_{i,1})}{2d_i u} \right)^{-1} x_{i,0}^{\pm}$$

and

$$\xi_i(u) = 1 + [x_i^+(u), x_{i,0}^-]$$

If V is a finite-dimensional $Y_{\hbar}(\mathfrak{g})$ -module, we define $\sigma(V) \subset \mathbb{C}$ to be the (finite) set of poles of the rational $\text{End}(V)$ -valued functions $\{\xi_i(u), x_i^{\pm}(u)\}_{i \in \mathbf{I}}$.

3. THE STANDARD AND DRINFELD COPRODUCTS

We review the definition of the standard coproduct on $Y_{\hbar}(\mathfrak{g})$ following [13], and the deformed Drinfeld tensor product on its finite-dimensional representations introduced in [12]. We then lift the latter to a deformed Drinfeld coproduct $\Delta_{D,s} : Y_{\hbar}(\mathfrak{g}) \rightarrow Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}]$.

3.1. Standard coproduct. Set

$$r = \sum_{\beta \in \Phi_+} x_{\beta,0}^- \otimes x_{\beta,0}^+ \quad (3.1)$$

where $x_{\beta,0}^{\pm} \in \mathfrak{g}_{\pm\beta} \subset Y_{\hbar}(\mathfrak{g})$ are root vectors such that $(x_{\beta,0}^-, x_{\beta,0}^+) = 1$. The coproduct $\Delta : Y_{\hbar}(\mathfrak{g}) \rightarrow Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})$ is defined by the following formulae

$$\begin{aligned} \Delta(\xi_{i,0}) &= \xi_{i,0} \otimes 1 + 1 \otimes \xi_{i,0} \\ \Delta(x_{i,0}^{\pm}) &= x_{i,0}^{\pm} \otimes 1 + 1 \otimes x_{i,0}^{\pm} \\ \Delta(t_{i,1}) &= t_{i,1} \otimes 1 + 1 \otimes t_{i,1} + \hbar \operatorname{ad}(\xi_{i,0} \otimes 1)r \\ &= t_{i,1} \otimes 1 + 1 \otimes t_{i,1} - \hbar \sum_{\beta \in \Phi_+} (\beta, \alpha_i) x_{\beta,0}^- \otimes x_{\beta,0}^+ \end{aligned}$$

We refer the reader to [13, §4.2] for a proof that Δ is an algebra homomorphism. It is immediate that Δ is coassociative (see [13, §4.5]).

3.2. Deformed Drinfeld tensor product. We review below the definition of the deformed Drinfeld tensor product introduced in [12, Section 4.4]. Let $V, W \in \operatorname{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, and $\sigma(V), \sigma(W) \subset \mathbb{C}$ their sets of poles. Let $s \in \mathbb{C}$ be such that $\sigma(V) + s$ and $\sigma(W)$ are disjoint, and define an action of the generators of $Y_{\hbar}(\mathfrak{g})$ on $V \otimes W$ by

$$\begin{aligned} \Delta_{D,s}(\xi_i(u)) &= \xi_i(u-s) \otimes \xi_i(u) \\ \Delta_{D,s}(x_i^+(u)) &= x_i^+(u-s) \otimes 1 + \oint_{C_2} \frac{1}{u-v} \xi_i(v-s) \otimes x_i^+(v) dv \\ \Delta_{D,s}(x_i^-(u)) &= \oint_{C_1} \frac{1}{u-v} x_i^-(v-s) \otimes \xi_i(v) dv + 1 \otimes x_i^-(u) \end{aligned}$$

where

- C_1 encloses $\sigma(V) + s$ and none of the points in $\sigma(W)$.
- C_2 encloses $\sigma(W)$ and none of the points in $\sigma(V) + s$.
- The integral \oint_{C_1} (resp. \oint_{C_2}) is understood to mean the holomorphic function of u it defines for u outside of C_1 (resp. C_2).

Note that in terms of the generators $t_{i,r}$ of $Y_{\hbar}^0(\mathfrak{g})$,

$$\Delta_{D,s}(t_i(u)) = t_i(u-s) \otimes 1 + 1 \otimes t_i(u)$$

Theorem. [12, Thm. 4.6]

- (1) The formulae above define an action of $Y_{\hbar}(\mathfrak{g})$ on $V \otimes W$. The corresponding representation is denoted by $V \otimes_{D,s} W$.
- (2) The action of $Y_{\hbar}(\mathfrak{g})$ on $V \otimes_{D,s} W$ is a rational function of s , with poles contained in $\sigma(W) - \sigma(V)$.

(3) *The identification of vector spaces*

$$(V_1 \otimes_{D,s_1} V_2) \otimes_{D,s_2} V_3 = V_1 \otimes_{D,s_1+s_2} (V_2 \otimes_{D,s_2} V_3)$$

intertwines the action of $Y_{\hbar}(\mathfrak{g})$.

(4) *If $V \cong \mathbb{C}$ is the trivial representation of $Y_{\hbar}(\mathfrak{g})$, then*

$$V \otimes_{D,s} W = W \quad \text{and} \quad W \otimes_{D,s} V = W(s)$$

(5) *The following holds for any $s, t \in \mathbb{C}$,*

$$V(t) \otimes_{D,s} W(t) = (V \otimes_{D,s} W)(t) \quad \text{and} \quad V(t) \otimes_{D,s} W = V \otimes_{D,s+t} W$$

In particular, $V \otimes_{D,s} W(t) = (V \otimes_{D,s-t} W)(t)$.

(6) *The following holds for any $s \in \mathbb{C}$,*

$$\sigma(V \otimes_{D,s} W) \subset (s + \sigma(V)) \cup \sigma(W)$$

3.3. Laurent expansion of the deformed Drinfeld tensor product.

Proposition. *The Laurent expansion at $s = \infty$ of the formulae of 3.2 is given by*

$$\Delta_{D,s}(t_{i,r}) = \tau_s(t_{i,r}) \otimes 1 + 1 \otimes t_{i,r}$$

and

$$\begin{aligned} \Delta_{D,s}(x_{i,r}^+) &= \tau_s(x_{i,r}^+) \otimes 1 + 1 \otimes x_{i,r}^+ \\ &\quad + \hbar \sum_{N \geq 0} \left(\sum_{n=0}^N (-1)^{n+1} \binom{N}{n} \xi_{i,n} \otimes x_{i,r+N-n}^+ \right) s^{-N-1} \end{aligned}$$

$$\begin{aligned} \Delta_{D,s}(x_{i,r}^-) &= \tau_s(x_{i,r}^-) \otimes 1 + 1 \otimes x_{i,r}^- \\ &\quad + \hbar \sum_{N \geq 0} \left(\sum_{n=0}^N (-1)^{n+1} \binom{N}{n} x_{i,r+n}^- \otimes \xi_{i,N-n} \right) s^{-N-1} \end{aligned}$$

PROOF. The expansion of $\Delta_{D,s}(t_{i,r})$ follows from $\Delta_{D,s}(t_i(u)) = \tau_s(t_i(u)) \otimes 1 + 1 \otimes t_i(u)$.

Expanding in u^{-1} yields

$$\Delta_{D,s}(x_{i,r}^+) = \tau_s(x_{i,r}^+) \otimes 1 + \frac{1}{\hbar} \oint_{C_2} v^r \xi_i(v-s) \otimes x_i^+(v) dv$$

Expanding now $\xi_i(v-s)$ with respect to s^{-1} by using $(1-x)^{-p-1} = \sum_{m \geq 0} \binom{p+m}{p} x^m$ yields:

$$\begin{aligned} \xi_i(v-s) &= 1 + \hbar \sum_{p \geq 0} \xi_{i,p}(v-s)^{-p-1} \\ &= 1 + \hbar \sum_{p \geq 0} \xi_{i,p}(-s)^{-p-1} \sum_{q \geq 0} \binom{p+q}{p} v^q s^{-q} \\ &= 1 + \hbar \sum_{m \geq 0} s^{-m-1} \sum_{\substack{p,q \geq 0 \\ p+q=m}} (-1)^{p+1} \binom{m}{p} v^q \xi_{i,p} \end{aligned}$$

Substituting gives

$$\begin{aligned} \Delta_{D,s}(x_{i,r}^+) &= \tau_s(x_{i,r}^+) \otimes 1 + \frac{1}{\hbar} \oint_{C_2} v^r 1 \otimes x_i^+(v) dv \\ &+ \sum_{m \geq 0} s^{-m-1} \sum_{\substack{p,q \geq 0 \\ p+q=m}} (-1)^{p+1} \binom{m}{p} \oint_{C_2} v^{r+q} \xi_{i,p} \otimes x_i^+(v) dv \end{aligned}$$

which is the claimed result since, for any $a \geq 0$, $\oint_{C_2} v^a x_i^+(v) dv = \hbar x_{i,a}^+$. The expansion of $\Delta_{D,s}(x_{i,r}^-)$ is obtained in the same way. \square

3.4. Deformed Drinfeld coproduct. We now lift the deformed Drinfeld tensor product to an algebra homomorphism

$$\Delta_{D,s} : Y_{\hbar}(\mathfrak{g}) \rightarrow (Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})) [s; s^{-1}].$$

Theorem. *The Laurent expansions of Section 3.3 give rise to an algebra homomorphism*

$$\Delta_{D,s} : Y_{\hbar}(\mathfrak{g}) \rightarrow (Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})) [s; s^{-1}]$$

The deformed Drinfeld coproduct $\Delta_{D,s}$ has the following properties.

- (1) *It is compatible with the counit ϵ , that is*

$$\epsilon \otimes \mathbf{1} \circ \Delta_{D,s} = \mathbf{1} \quad \text{and} \quad \mathbf{1} \otimes \epsilon \circ \Delta_{D,s} = \tau_s$$

- (2) *For every $x \in Y_{\hbar}(\mathfrak{g})$, the following holds in $(Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})) [a] [s; s^{-1}]$*

$$\tau_a \otimes \tau_a \circ \Delta_{D,s}(x) = \Delta_{D,s} \circ \tau_a(x) \quad \text{and} \quad \tau_a \otimes \mathbf{1} \circ \Delta_{D,s}(x) = \Delta_{D,s+a}(x)$$

- (3) *$\Delta_{D,s}$ is a filtered homomorphism, that is*

$$\Delta_{D,s}(\mathcal{F}_k(Y_{\hbar}(\mathfrak{g}))) \subset \mathcal{F}_k(Y_{\hbar}(\mathfrak{g}))^{\otimes 2} [s; s^{-1}]$$

for each $k \geq 0$, where $\mathcal{F}_{\bullet}(Y_{\hbar}(\mathfrak{g}))^{\otimes 2} [s; s^{-1}]$ is the filtration defined in (2.6).

PROOF. Using Theorem 4.1, and the fact that Δ_s is an algebra homomorphism, we conclude the same for $\Delta_{D,s}$. Properties (1) and (2) above are easy to verify directly from the definition. Property (3) follows immediately from the explicit formulas given in Proposition 3.3. \square

Remark. The coassociativity property of the Drinfeld tensor product $\otimes_{D,s}$ does not appear to have a natural lift to $\Delta_{D,s}$. The candidate identity

$$\Delta_{D,s_1} \otimes \mathbf{1} \circ \Delta_{D,s_2}(x) = \mathbf{1} \otimes \Delta_{D,s_2} \circ \Delta_{D,s_1+s_2}(x) \quad (3.2)$$

holds if x is one of the commuting generators $t_{i,r}$. However, if x is an arbitrary element of $Y_{\hbar}(\mathfrak{g})$, the left-hand side and right-hand side lie, respectively, in

$$Y_{\hbar}(\mathfrak{g})^{\otimes 3} [s_1; s_1^{-1}] [s_2; s_2^{-1}] \quad \text{and} \quad Y_{\hbar}(\mathfrak{g})^{\otimes 3} [s_2; s_2^{-1}] [s_1; s_1^{-1}]$$

and cannot therefore be directly compared. The coassociativity of $\otimes_{D,s}$ implies, however, that the evaluation of the left- and right-hand sides of (3.2) on a tensor product $V_1 \otimes V_2 \otimes V_3$ of finite-dimensional representations are, respectively, the expansions at $|s_2| \gg |s_1|$ and $|s_1| \gg |s_2|$ of the same rational function.

4. THE ELEMENT $\mathcal{R}^-(s)$

4.1. Set $\mathcal{Q}_+ = \bigoplus_{i \in \mathbf{I}} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathfrak{h}^*$. The following is one of the main result of this paper.

Theorem.

(1) *There is a unique element*

$$\mathcal{R}^-(s) \in (Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))[[s^{-1}]]$$

which is

- *strictly lower triangular, that is $\mathcal{R}^-(s) = \sum_{\beta, \gamma \in \mathcal{Q}_+} \mathcal{R}^-(s)_{\beta, \gamma}$, with*

$$\mathcal{R}^-(s)_{\beta, \gamma} \in (Y_{\hbar}(\mathfrak{g})_{-\beta} \otimes Y_{\hbar}(\mathfrak{g})_{\gamma})[[s^{-1}]] \quad \text{and} \quad \mathcal{R}^-(s)_0 = 1 \otimes 1$$

- *of weight zero*
- *such that, for any $i \in \mathbf{I}$,*

$$\mathcal{R}^-(s) \cdot \Delta_s(t_{i,1}) = \Delta_{D,s}(t_{i,1}) \cdot \mathcal{R}^-(s) \quad (4.1)$$

(2) *The element $\mathcal{R}^-(s)$ lies in $(Y_{\hbar}^-(\mathfrak{g}) \otimes Y_{\hbar}^+(\mathfrak{g}))[[s^{-1}]]$, and has the following additional properties.*

- *For any $i \in \mathbf{I}$,*

$$\mathcal{R}^-(s) \cdot \Delta_s(x_{i,0}^{\pm}) = \Delta_{D,s}(x_{i,0}^{\pm}) \cdot \mathcal{R}^-(s) \quad (4.2)$$

- *For any $a, b \in \mathbb{C}$,*

$$\tau_a \otimes \tau_b(\mathcal{R}^-(s)) = \mathcal{R}^-(s + a - b)$$

- *$\mathcal{R}^-(s) - 1 \in \mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$, with semiclassical limit given by*

$$\overline{\mathcal{R}^-(s) - 1} = \frac{\hbar r}{z + s - w} \in (U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))[[s^{-1}]]$$

In particular, $\mathcal{R}^-(s) = 1 + \hbar r/s + O(s^{-2})$.

(3) *Let $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, and $\mathcal{R}_{V_1, V_2}^-(s) \in \text{End}(V_1 \otimes V_2)[[s^{-1}]]$ the corresponding evaluation of $\mathcal{R}^-(s)$. Then,*

- *$\mathcal{R}_{V_1, V_2}^-(s)$ is the Taylor expansion at $s = \infty$ of a rational function.*
- *The following cocycle equation holds for any $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$:*

$$\mathcal{R}_{V_1 \otimes_{D, s_1} V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2}^-(s_1) = \mathcal{R}_{V_1, V_2 \otimes_{D, s_2} V_3}^-(s_1 + s_2) \cdot \mathcal{R}_{V_2, V_3}^-(s_2) \quad (4.3)$$

Remark. Analogously to Remark 3.4, the cocycle equation (4.3) only holds as an identity of rational functions with values in $\text{End}(V_1 \otimes V_2 \otimes V_3)$, and does not seem to possess a natural lift to $Y_{\hbar}(\mathfrak{g})$. Indeed, the candidate identity

$$\Delta_{D, s_1} \otimes \mathbf{1}(\mathcal{R}^-(s_2)) \cdot \mathcal{R}_{12}^-(s_1) = \mathbf{1} \otimes \Delta_{D, s_2}(\mathcal{R}^-(s_1 + s_2)) \cdot \mathcal{R}_{23}^-(s_2)$$

does not make sense since the left-hand side lies in $Y_{\hbar}(\mathfrak{g})^{\otimes 3}[s_1; s_1^{-1}][[s_2^{-1}]]$, while the right-hand side lies in $Y_{\hbar}(\mathfrak{g})^{\otimes 3}[s_2; s_2^{-1}][[s_1^{-1}]]$.

The rest of the section is devoted to the proof of Theorem 4.1.

4.2. Existence and uniqueness of $\mathcal{R}^-(s)$. The triangularity and zero-weight assumptions are equivalent to the requirement that $\mathcal{R}^-(s)$ have the form

$$\mathcal{R}^-(s) = \sum_{\beta \in \mathbb{Q}_+} \mathcal{R}^-(s)_\beta \quad \text{with} \quad \mathcal{R}^-(s)_\beta \in (Y_{\hbar}(\mathfrak{g})_{-\beta} \otimes Y_{\hbar}(\mathfrak{g})_\beta) \llbracket s^{-1} \rrbracket \quad (4.4)$$

and $\mathcal{R}^-(s)_0 = 1 \otimes 1$. We shall construct $\mathcal{R}^-(s)_\beta$ recursively in β , and prove that the sum over β converges in the s -adic topology. In fact, define $\nu : \mathbb{Q}_+ \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\nu(\beta) = \min \left\{ k \in \mathbb{Z}_{\geq 0} \mid \beta = \alpha^{(1)} + \dots + \alpha^{(k)}, \alpha^{(1)}, \dots, \alpha^{(k)} \in \Phi_+ \right\}$$

with $\nu(0) = 0$. Then, we shall prove that $\mathcal{R}^-(s)_\beta \in s^{-\nu(\beta)} Y_{\hbar}(\mathfrak{g})^{\otimes 2} \llbracket s^{-1} \rrbracket$.

Let r be given by (3.1) and, for any $h \in \mathfrak{h}$, set

$$r(h) = \text{ad}(h \otimes 1)(r) = - \sum_{\gamma \in \Phi_+} \gamma(h) x_{\gamma,0}^- \otimes x_{\gamma,0}^+.$$

By 3.1 and 3.3,

$$\begin{aligned} \Delta_{\mathbb{D},s}(\mathbb{T}(h)) &= \sum_{a=1}^2 \mathbb{T}(h)^{(a)} + sh^{(1)} \\ \Delta_s(\mathbb{T}(h)) &= \sum_{a=1}^2 \mathbb{T}(h)^{(a)} + sh^{(1)} + \hbar r(h), \end{aligned}$$

where we use the standard notation: $X^{(1)} = X \otimes 1$ and $X^{(2)} = 1 \otimes X$. The intertwining equation (4.1) therefore reads as $A(h)\mathcal{R}^-(s) = 0$ where, for any $h \in \mathfrak{h}$,

$$A(h) = \lambda(\Delta_{\mathbb{D},s}(\mathbb{T}(h))) - \rho(\Delta_s(\mathbb{T}(h))) = \text{ad} \left(\mathbb{T}(h)^{(1)} + \mathbb{T}(h)^{(2)} + sh^{(1)} \right) - \hbar \rho(r(h))$$

and λ, ρ denote left and right multiplication, respectively. In components, this reads:

$$(\mathcal{T}(h) - s\beta(h)) \mathcal{R}^-(s)_\beta = -\hbar \sum_{\alpha \in \Phi_+} \alpha(h) \mathcal{R}^-(s)_{\beta-\alpha} x_{\alpha,0}^- \otimes x_{\alpha,0}^+$$

where $\mathcal{T}(h) = \text{ad}(\mathbb{T}(h)^{(1)} + \mathbb{T}(h)^{(2)})$ and $\mathcal{R}^-(s)_\gamma = 0$ if $\gamma \notin \mathbb{Q}_+$.

If $h \in \mathfrak{h}$ is such that $\beta(h) \neq 0$ for any nonzero $\beta \in \mathbb{Q}_+$, this yields

$$\mathcal{R}^-(s)_\beta = \frac{\hbar}{s\beta(h)} \left(1 - \frac{\mathcal{T}(h)}{s\beta(h)} \right)^{-1} \sum_{\alpha \in \Phi_+} \alpha(h) \mathcal{R}^-(s)_{\beta-\alpha} x_{\alpha,0}^- \otimes x_{\alpha,0}^+ \quad (4.5)$$

$$= \hbar \sum_{k \geq 0} \frac{\mathcal{T}(h)^k}{(s\beta(h))^{k+1}} \sum_{\alpha \in \Phi_+} \alpha(h) \mathcal{R}^-(s)_{\beta-\alpha} x_{\alpha,0}^- \otimes x_{\alpha,0}^+ \quad (4.6)$$

This shows that $\mathcal{R}^-(s)$ is uniquely determined by $\mathcal{R}^-(s)_0$, and that it lies in $(Y_{\hbar}^-(\mathfrak{g}) \otimes Y_{\hbar}^+(\mathfrak{g})) \llbracket s^{-1} \rrbracket$ if $\mathcal{R}^-(s)_0$ does, since $Y_{\hbar}^-(\mathfrak{g}), Y_{\hbar}^+(\mathfrak{g})$ are invariant under $\text{ad } \mathbb{T}(h)$.

Fix now $h \in \mathfrak{h} \setminus \bigcup_{\beta \in (\mathbb{Q}_+ \setminus \{0\})} \text{Ker } \beta$. The above equations can be used to define elements $\mathcal{R}^-(s)_\beta$ recursively on the height of β , starting from $\mathcal{R}^-(s)_0 = 1$, which lie in $s^{-\nu(\beta)} Y_{\hbar}(\mathfrak{g})^{\otimes 2} \llbracket s^{-1} \rrbracket$. The corresponding sum $\mathcal{R}^-(s) = \sum_{\beta \in \mathbb{Q}_+} \mathcal{R}^-(s)_\beta$ is therefore well-defined, and satisfies $A(h)\mathcal{R}^-(s) = 0$. We claim that it also satisfies $A(h')\mathcal{R}^-(s) = 0$ for any $h' \in \mathfrak{h}$. Note that

$$[A(h), A(h')] = \lambda[\Delta_{\mathbb{D},s}(\mathbb{T}(h)), \Delta_{\mathbb{D},s}(\mathbb{T}(h'))] - \rho[\Delta_s(\mathbb{T}(h)), \Delta_s(\mathbb{T}(h'))]$$

which vanishes because Δ_s is an algebra homomorphism, and $\Delta_{D,s}(\mathbb{T}(h)), \Delta_{D,s}(\mathbb{T}(h')) \in Y_{\hbar}^0(\mathfrak{g})^{\otimes 2}$. Thus, $A(h')\mathcal{R}^-(s)$ satisfies

$$A(h)A(h')\mathcal{R}^-(s) = A(h')A(h)\mathcal{R}^-(s) = 0$$

Since $A(h')\mathcal{R}^-(s)$ is also triangular, with

$$(A(h')\mathcal{R}^-(s))_0 = \text{ad} \left(\mathbb{T}(h')^{(1)} + \mathbb{T}(h')^{(2)} + sh'^{(1)} \right) \mathcal{R}^-(s)_0 = 0$$

it follows by uniqueness that $A(h')\mathcal{R}^-(s) = 0$ as claimed.

4.3. Translation invariance. The identity $\tau_a \otimes \tau_b (\mathcal{R}^-(s)) = \mathcal{R}^-(s+a-b)$ follows by uniqueness, since both sides are strictly lower triangular, intertwine

$$\tau_a \otimes \tau_b \circ \Delta_s(t_{i,m}) = \tau_{a-b} \otimes \mathbf{1} \circ \Delta_s(\tau_b(t_{i,m})) = \Delta_{s+a-b}(\tau_b(t_{i,m}))$$

and $\tau_a \otimes \tau_b \circ \Delta_{D,s}(t_{i,m}) = \Delta_{D,s+a-b}(\tau_b(t_{i,m}))$ for $i \in \mathbf{I}$ and $m = 0, 1$, and the span of $\{t_{i,m}\}_{i \in \mathbf{I}, m=0,1}$ is invariant under τ_b .

4.4. Semiclassical limit. Since $\mathcal{T}(h)$ is a filtered operator of degree 1, the recursion (4.6) shows that $\mathcal{R}^-(s)_\beta \in \mathcal{F}_{-\nu(\beta)}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$ for any $\beta \in \mathbf{Q}_+$. In particular, $\mathcal{R}^-(s) - 1 \in \mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$ and, mod $\mathcal{F}_{-2}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$,

$$\mathcal{R}^-(s) - 1 = \sum_{\beta: \nu(\beta)=1} \mathcal{R}^-(s)_\beta = \frac{\hbar}{s} \sum_{\alpha \in \Phi_+} \sum_{k \geq 0} \frac{\mathcal{T}(h)^k}{(s\alpha(h))^k} x_{\alpha,0}^- \otimes x_{\alpha,0}^+$$

whose image in $\mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])/\mathcal{F}_{-2}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$ is $\hbar r/(z+s-w)$.

4.5. Rationality. The argument given in 4.2 can be carried out in $\text{End}(V_1 \otimes V_2)$ rather than $Y_{\hbar}(\mathfrak{g})^{\otimes 2}$, and shows the existence and uniqueness of an element

$$\mathcal{R}_{V_1, V_2}^-(s) = \sum_{\beta \in \mathbf{Q}_+} \mathcal{R}_{V_1, V_2}^-(s)_\beta \in \text{End}(V_1 \otimes V_2)[[s^{-1}]]$$

with

$$[h \otimes \mathbf{1}, \mathcal{R}_{V_1, V_2}^-(s)_\beta] = -\beta(h) \mathcal{R}_{V_1, V_2}^-(s)_\beta \quad [1 \otimes h, \mathcal{R}_{V_1, V_2}^-(s)_\beta] = \beta(h) \mathcal{R}_{V_1, V_2}^-(s)_\beta$$

for any $h \in \mathfrak{h}$, $\mathcal{R}_{V_1, V_2}^-(s)_0 = 1$, and

$$\mathcal{R}_{V_1, V_2}^-(s) \cdot \Delta_s(t_{i,1}) = \Delta_{D,s}(t_{i,1}) \cdot \mathcal{R}_{V_1, V_2}^-(s)$$

The recursive construction of $\mathcal{R}_{V_1, V_2}^-(s)$ given by (4.5) shows that each $\mathcal{R}_{V_1, V_2}^-(s)_\beta$ is a rational function of s , regular at $s = \infty$. Since only finitely many $\mathcal{R}_{V_1, V_2}^-(s)_\beta$ are non-zero by finite-dimensionality, $\mathcal{R}_{V_1, V_2}^-(s)$ is therefore rational. It follows by uniqueness that the Taylor expansion of $\mathcal{R}_{V_1, V_2}^-(s)$ is the evaluation of $\mathcal{R}^-(s)$ on $V_1 \otimes V_2$.

4.6. Cocycle equation. Let $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$. We obtain below an alternative version of the cocycle equation, namely

$$\mathcal{R}_{V_1, V_2}^-(s_1) \cdot \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^-(s_2) = \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2 \otimes_{s_2} V_3}^-(s_1 + s_2) \quad (4.7)$$

Note that (4.7) and (4.3) are equivalent, provided the intertwining equation (4.2) is established. The latter will be proved in 4.8, by relying in part on (4.7).⁶

The intertwining property of $\mathcal{R}^-(s)$ implies that

$$\begin{aligned} & \mathcal{R}_{V_1, V_2}^-(s_1) \cdot \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^-(s_2) \cdot \pi_{(V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3}(t_{i,1}) \\ &= \mathcal{R}_{V_1, V_2}^-(s_1) \cdot \pi_{(V_1 \otimes_{s_1} V_2) \otimes_{D, s_2} V_3}(t_{i,1}) \cdot \mathcal{R}_{V_1 \otimes_{D, s_1} V_2, V_3}^-(s_2) \\ &= \pi_{(V_1 \otimes_{D, s_1} V_2) \otimes_{D, s_2} V_3}(t_{i,1}) \cdot \mathcal{R}_{V_1, V_2}^-(s_1) \cdot \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^-(s_2), \end{aligned}$$

where the second equality stems from the fact that

$$\pi_{(V_1 \otimes_{s_1} V_2) \otimes_{D, s_2} V_3}(t_{i,1}) = \pi_{(V_1 \otimes_{s_1} V_2)}(\tau_{s_2}(t_{i,1})) + \pi_{V_3}(t_{i,1})$$

Similarly,

$$\begin{aligned} & \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2 \otimes_{s_2} V_3}^-(s_1 + s_2) \cdot \pi_{V_1 \otimes_{s_1 + s_2} (V \otimes_{s_2} V_3)}(t_{i,1}) \\ &= \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \pi_{V_1 \otimes_{D, s_1 + s_2} (V \otimes_{s_2} V_3)}(t_{i,1}) \cdot \mathcal{R}_{V_1, V_2 \otimes_{s_2} V_3}^-(s_1 + s_2) \\ &= \pi_{V_1 \otimes_{D, s_1 + s_2} (V \otimes_{D, s_2} V_3)}(t_{i,1}) \cdot \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2 \otimes_{s_2} V_3}^-(s_1 + s_2) \end{aligned}$$

Since $\otimes_{s, D, s}$ are coassociative, both sides of the cocycle equation (4.7) are therefore solutions of

$$X \cdot \Delta_{s_1} \otimes \mathbf{1} \circ \Delta_{s_2}(t_{i,1}) = \Delta_{D, s_1} \otimes \mathbf{1} \circ \Delta_{D, s_2}(t_{i,1}) \cdot X \quad (4.8)$$

By 3.1 and 3.3,

$$\begin{aligned} \Delta_{D, s_1} \otimes \mathbf{1} \circ \Delta_{D, s_2}(T(h)) &= \sum_{a=1}^3 T(h)^{(a)} + (s_1 + s_2) h^{(1)} + s_2 h^{(2)} \\ \Delta_{s_1} \otimes \mathbf{1} \circ \Delta_{s_2}(T(h)) &= \sum_{a=1}^3 T(h)^{(a)} + (s_1 + s_2) h^{(1)} + s_2 h^{(2)} + \hbar(r(h)_{13} + r(h)_{23}) \end{aligned}$$

The conclusion now follows by noting that, analogously to 4.2, (4.8) admits at most one rational solution $X(s_1, s_2)$ with values in $\text{End}(V_1 \otimes V_2 \otimes V_3)$, provided it is strictly lower triangular, that is of the form $X = \sum_{\beta, \gamma \in \mathbb{Q}_+} X_{\beta, \gamma}$, where

$$X_{\beta, \gamma} \in \text{End}(V_1)[- \beta] \otimes \text{End}(V_2)[\beta - \gamma] \otimes \text{End}(V_3)[\gamma]$$

and $X_{0,0} = \mathbf{1}$.

⁶The cocycle equation (4.3) arises from the definition of a tensor structure on a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ as a system of natural isomorphisms $J_{U,V} : F(U) \otimes_{\mathcal{D}} F(V) \rightarrow F(U \otimes_{\mathcal{C}} V)$, by interpreting $\mathcal{R}^-(s)$ as a tensor structure on the identity functor $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes) \rightarrow (\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes)$. Similarly, (4.7) arises by adopting the opposite convention of a tensor structure as a system of isomorphisms $K_{U,V} : F(U \otimes_{\mathcal{C}} V) \rightarrow F(U) \otimes_{\mathcal{D}} F(V)$, and interpreting $\mathcal{R}^-(s)$ as a tensor structure on $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes) \rightarrow (\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes)$.

4.7. Rank 1 reduction. Consider the intertwining identity

$$\mathcal{R}^-(s) \cdot \Delta_s(x_{i,0}^\pm) = \Delta_{D,s}(x_{i,0}^\pm) \cdot \mathcal{R}^-(s) \quad (4.9)$$

We claim that it holds for any \mathfrak{g} and $i \in \mathbf{I}$ if, and only if it holds for $\mathfrak{g} = \mathfrak{sl}_2$. Set $\mathbf{Q}^{(i)} = \mathbf{Q}/\mathbb{Z}\alpha_i \subset \mathfrak{h}^*/\mathbb{C}\alpha_i = (\alpha_i^\perp)^*$, and let $\mathbf{Q}_+^{(i)}$ be the image of \mathbf{Q}_+ in $\mathbf{Q}^{(i)}$. Both sides of (4.9) are lower triangular and of weight zero with respect to the adjoint action of $\alpha_i^\perp \subset \mathfrak{h}$ i.e., lie in $\bigoplus_{\beta \in \mathbf{Q}_+^{(i)}} (Y_{\hbar}(\mathfrak{g})_{-\beta}^{(i)} \otimes Y_{\hbar}(\mathfrak{g})_{\beta}^{(i)})[[s^{-1}]]$ where, for $\gamma \in \mathbf{Q}^{(i)}$,

$$Y_{\hbar}(\mathfrak{g})_{\gamma}^{(i)} = \{x \in Y_{\hbar}(\mathfrak{g}) \mid [h, x] = \gamma(h)x, h \in \alpha_i^\perp\}$$

Since both intertwine $\Delta_s(\mathbb{T}(h))$ and $\Delta_{D,s}(\mathbb{T}(h))$ for $h \in \alpha_i^\perp$, it follows by uniqueness that they are equal if, and only if, their projections on $(Y_{\hbar}(\mathfrak{g})_0^{(i)} \otimes Y_{\hbar}(\mathfrak{g})_0^{(i)})[[s^{-1}]]$ coincide.

Let $\pi_i \in \text{End}(Y_{\hbar}(\mathfrak{g}))$ be the projection onto $Y_{\hbar}(\mathfrak{g})_0^{(i)} = \bigoplus_{m \in \mathbb{Z}} Y_{\hbar}(\mathfrak{g})_{m\alpha_i}$. Then,

$$\begin{aligned} \pi_i \otimes \pi_i(\mathcal{R}^-(s) \cdot \Delta_s(x_{i,0}^\pm)) &= \pi_i \otimes \pi_i(\mathcal{R}^-(s)) \cdot \Delta_s(x_{i,0}^\pm) \\ \pi_i \otimes \pi_i(\Delta_{D,s}(x_{i,0}^\pm) \cdot \mathcal{R}^-(s)) &= \Delta_{D,s}(x_{i,0}^\pm) \cdot \pi_i \otimes \pi_i(\mathcal{R}^-(s)) \end{aligned}$$

Let $\varphi_i : Y_{d_i \hbar}(\mathfrak{sl}_2) \rightarrow Y_{\hbar}(\mathfrak{g})$ be the algebra homomorphism given by $x_r^\pm \mapsto d_i^{-1/2} x_{i,r}^\pm$ and $\xi_r \mapsto d_i^{-1} \xi_{i,r}$. Then,

$$\Delta_s(x_{i,0}^\pm) = \sqrt{d_i} \cdot \varphi_i^{\otimes 2} \circ \Delta_s(x_0^\pm) \quad \text{and} \quad \Delta_{D,s}(x_{i,0}^\pm) = \sqrt{d_i} \cdot \varphi_i^{\otimes 2} \circ \Delta_{D,s}(x_0^\pm).$$

We claim that $\pi_i \otimes \pi_i(\mathcal{R}^-(s)) = \varphi_i \otimes \varphi_i(\mathcal{R}_{(\mathfrak{sl}_2)}^-(s))$ so that, by the foregoing, (4.9) holds for any \mathfrak{g} and $i \in \mathbf{I}$ if, and only if it holds for $\mathfrak{g} = \mathfrak{sl}_2$. The claim follows by uniqueness, since both $\pi_i \otimes \pi_i(\mathcal{R}^-(s))$ and $\varphi_i \otimes \varphi_i(\mathcal{R}_{(\mathfrak{sl}_2)}^-(s))$ are strictly lower triangular and of weight zero with respect to the action of $\xi_{i,0}$, and both intertwine

$$\varphi_i^{\otimes 2} \circ \Delta_s(t_i) = d_i^{-1} \pi_i^{\otimes 2} \circ \Delta_s(t_{i,1}) \quad \text{and} \quad \varphi_i^{\otimes 2} \circ \Delta_{D,s}(t_i) = d_i^{-1} \Delta_{D,s}(t_{i,1}) = d_i^{-1} \pi_i^{\otimes 2} \circ \Delta_{D,s}(t_{i,1}).$$

4.8. Rank 1 intertwining relations. Assume $\mathfrak{g} = \mathfrak{sl}_2$, and consider the identity

$$\mathcal{R}^-(s) \cdot \Delta_s(x_0^-) = \Delta_{D,s}(x_0^-) \cdot \mathcal{R}^-(s) \quad (4.10)$$

The latter can be proved by a lengthy, direct calculation⁷. We give below an alternative proof, which relies on the cocycle identity (4.3) satisfied by $\mathcal{R}^-(s)$ to reduce it to the case when $\mathcal{R}^-(s)$ is acting on $\mathbb{C}^2 \otimes V$, where V is an arbitrary finite-dimensional representation.

By Appendix A, it is sufficient to prove that (4.10) holds on the tensor product $V_1 \otimes V_2$, where V_1 (resp. V_2) is chosen from a collection \mathcal{V}_1 (resp. \mathcal{V}_2) of finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$ which is stable under tensor product, contains the trivial representation, and a representation whose restriction to \mathfrak{g} is faithful. We choose \mathcal{V}_2 to consist of all finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$, while \mathcal{V}_1 consists of arbitrary tensor products $\mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_m)$ of evaluation representations.

⁷The calculation is carried out in §5 of the earlier version of this paper, arXiv:1907.03525 v1.

For any $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, the cocycle identity (4.7) implies that (4.10) holds on $(V_1(s_1) \otimes V_2) \otimes V_3$ if it holds on $V_1 \otimes V_2$, $V_2 \otimes V_3$, and $V_1 \otimes (V_2(s_2) \otimes V_3)$. Indeed,

$$\begin{aligned}
& \mathcal{R}_{V_1 \otimes V_2, V_3}^-(s_2) \cdot \pi_{(V_1 \otimes V_2) \otimes V_3}(x_0^-) \\
&= \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \cdot \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2 \otimes V_3}^-(s_1 + s_2) \cdot \pi_{V_1 \otimes (V_2 \otimes V_3)}(x_0^-) \\
&= \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \cdot \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \pi_{V_1 \otimes_{D, s_1+s_2} (V_2 \otimes V_3)}(x_0^-) \cdot \mathcal{R}_{V_1, V_2 \otimes V_3}^-(s_1 + s_2) \\
&= \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \cdot \pi_{V_1 \otimes_{D, s_1+s_2} (V_2 \otimes V_3)}(x_0^-) \cdot \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2 \otimes V_3}^-(s_1 + s_2) \\
&= \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \cdot \pi_{(V_1 \otimes_{D, s_1} V_2) \otimes_{D, s_2} V_3}(x_0^-) \cdot \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2 \otimes V_3}^-(s_1 + s_2) \\
&= \pi_{(V_1 \otimes V_2) \otimes_{D, s_2} V_3}(x_0^-) \cdot \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \cdot \mathcal{R}_{V_2, V_3}^-(s_2) \cdot \mathcal{R}_{V_1, V_2 \otimes V_3}^-(s_1 + s_2) \\
&= \pi_{(V_1 \otimes V_2) \otimes_{D, s_2} V_3}(x_0^-) \cdot \mathcal{R}_{V_1 \otimes V_2, V_3}^-(s_2)
\end{aligned}$$

It is therefore sufficient to check (4.10) on $\mathbb{C}^2(a) \otimes V$, where V is an arbitrary finite-dimensional representation of $Y_{\hbar}(\mathfrak{g})$, and $a \in \mathbb{C}$. Moreover, by using the translation invariance of $\mathcal{R}^-(s)$, \otimes , and $\otimes_{D, s}$, it is sufficient to consider the case $a = 0$.

Let now x^{\pm}, ξ be the standard nilpotent and semisimple generators of \mathfrak{sl}_2 acting on \mathbb{C}^2 . Then, the following define an action of $Y_{\hbar}(\mathfrak{sl}_2)$ on \mathbb{C}^2 :

$$x^{\pm}(u) = \hbar \frac{x^{\pm}}{u} \quad \text{and} \quad \xi(u) = 1 + \hbar \frac{\xi}{u}$$

By 3.1 and 3.2, $\pi_{\mathbb{C}^2 \otimes V}(x_0^-) = x^- \otimes 1 + 1 \otimes x_0^-$, and

$$\pi_{\mathbb{C}^2 \otimes V}(x_0^-) = \oint_{C_1} \frac{x^-}{v-s} \otimes \xi_i(v) dv + 1 \otimes x_0^- = x^- \otimes \xi(s) + 1 \otimes x_0^-$$

where C_1 encloses $\sigma(\mathbb{C}^2) + s = \{s\}$ and none of the points in $\sigma(V)$.

On the other hand, since $x^-(u)x^-(v) = 0$ on \mathbb{C}^2 , 5.4 and Theorem 5.5 yield

$$\mathcal{R}_{\mathbb{C}^2, V}^-(s) = 1 - \oint_{C_2} \frac{x^-}{u-s} \otimes x^+(u) du = 1 + \oint_{C_1} \frac{x^-}{u-s} \otimes x^+(u) du = 1 + x^- \otimes x^+(s)$$

where C_2 encloses $\sigma(V)$ and none of the points in $\sigma(\mathbb{C}^2) + s = s$, and the second equality follows because the residue of the integrand at infinity is 0. Therefore,

$$\begin{aligned}
& \mathcal{R}_{\mathbb{C}^2, V}^-(s) \cdot \pi_{\mathbb{C}^2 \otimes V}(x_0^-) = x^- \otimes 1 + 1 \otimes x_0^- + x^- \otimes x^+(s)x_0^- \\
& \pi_{\mathbb{C}^2 \otimes V}(x_0^-) \cdot \mathcal{R}_{\mathbb{C}^2, V}^-(s) = x^- \otimes \xi(s) + 1 \otimes x_0^- + x^- \otimes x_0^- x^+(s)
\end{aligned}$$

so that

$$\mathcal{R}_{\mathbb{C}^2, V}^-(s) \cdot \pi_{\mathbb{C}^2 \otimes V}(x_0^-) - \pi_{\mathbb{C}^2 \otimes V}(x_0^-) \cdot \mathcal{R}_{\mathbb{C}^2, V}^-(s) = x^- \otimes (1 + [x^+(s), x_0^-] - \xi(s))$$

which is equal to zero since $[x^+(s), x_0^-] = \xi(s) - 1$.

The identity $\mathcal{R}^-(s) \cdot \Delta_s(x_0^+) = \Delta_{D, s}(x_0^+) \cdot \mathcal{R}^-(s)$ is proved in a similar way, by taking \mathcal{V}_1 to consist of all finite-dimensional representations, \mathcal{V}_2 of tensor products $\mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_m)$, and using the cocycle identity to reduce this to a check on $V \otimes \mathbb{C}^2$, for an arbitrary $V \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$.

5. THE ELEMENT $\mathcal{R}^-(s)$ FOR \mathfrak{sl}_2

In this section, we give an explicit formula for the element $\mathcal{R}^-(s)$ when $\mathfrak{g} = \mathfrak{sl}_2$.

5.1. It will be convenient to consider the generating series

$$\mathcal{R}^-(s, z) = \sum_{n \geq 0} \mathcal{R}_{n\alpha}^-(s) z^n \in (Y_{\hbar}(\mathfrak{sl}_2) \otimes Y_{\hbar}(\mathfrak{sl}_2))[z][[s^{-1}]]$$

where α is the positive root of \mathfrak{g} . Let $G = PSL_2(\mathbb{C})$ be the complex Lie group of adjoint type corresponding to \mathfrak{g} , and $H \subset G$ its maximal torus with Lie algebra \mathfrak{h} . We identify H with \mathbb{C}^\times via the character corresponding to $-\alpha$.

In particular, $\mathcal{R}^-(s, z) = \text{Ad}(z^{(1)})\mathcal{R}^-(s)$. Moreover, $\mathcal{R}^-(s, z)$ satisfies

$$\mathcal{R}^-(s, z) \cdot \Delta_s^z(t_1) = \Delta_{D,s}^z(t_1) \cdot \mathcal{R}^-(s, z)$$

where $\Delta_s^z = \text{Ad}(z^{(1)}) \circ \Delta_s^z$ and $\Delta_{D,s} = \text{Ad}(z^{(1)}) \circ \Delta_{D,s}$, so that

$$\Delta_{D,s}^z(t_1) = \sum_{a=1}^2 t_1^{(a)} + sh^{(1)} \quad \text{and} \quad \Delta_s^z(t_1) = \sum_{a=1}^2 t_1^{(a)} + sh^{(1)} + \hbar zr$$

5.2. The intertwining equation for $\mathcal{R}^-(s, z)$ may be written as the following ODE, together with the initial condition $\mathcal{R}^-(s, 0) = 1 \otimes 1$

$$\left(sz\partial_z - \frac{1}{2} \text{ad}(t_1 \otimes 1 + 1 \otimes t_1) \right) \mathcal{R}^-(s, z) = \mathcal{R}^-(s, z) \cdot \hbar zr \quad (5.1)$$

where $r = x_0^- \otimes x_0^+$.

Lemma. *Set $\omega(s, z) = \mathcal{R}^-(s, z)^{-1} \cdot z\partial_z \mathcal{R}^-(s, z)$. Then, $\omega(s, z)$ satisfies*

$$(sz\partial_z - \mathcal{T} + \hbar z \text{ad}(r))\omega(s, z) = \hbar rz \quad (5.2)$$

where $\mathcal{T} = \frac{1}{2} \text{ad}(t_1 \otimes 1 + 1 \otimes t_1)$.

PROOF. Since $(sz\partial_z - \mathcal{T})$ is a derivation, we have

$$\begin{aligned} (sz\partial_z - \mathcal{T})\omega(s, z) &= -\hbar zr\omega(s, z) + \mathcal{R}^-(s, z)^{-1} z\partial_z(\mathcal{R}^-(s, z)\hbar zr) \\ &= -\hbar z[r, \omega(s, z)] + \hbar zr \end{aligned}$$

□

5.3. **Formula for $\omega(s, z)$.** Given a vector space \mathcal{H} , let

$$\oint -du : \mathcal{H}[u; u^{-1}] \rightarrow \mathcal{H}$$

be the formal residue, given by taking the coefficient of u^{-1} .

Proposition. *The series $\omega(s, z)$ is given by*

$$\omega(s, z) = \sum_{k \geq 1} z^k \frac{(-1)^k}{k\hbar} \oint x^-(u-s)^k \otimes x^+(u)^k du \quad (5.3)$$

PROOF. Write $\omega(s, z) = \sum_{k \geq 1} \omega_k(s) z^k$. For $k = 1$, (5.2) yields

$$\begin{aligned} \omega_1(s) &= (s - \mathcal{T})^{-1} \hbar r = \hbar \sum_{m \geq 0} s^{-m-1} \mathcal{T}^m r \\ &= \hbar \sum_{m \geq 0} s^{-m-1} \sum_{n=0}^m (-1)^n \binom{m}{n} x_n^- \otimes x_{m-n}^+ \end{aligned}$$

where the last equality follows from $[t_1/2, x_k^\pm] = \pm x_{k+1}^\pm$. On the other hand,

$$\begin{aligned} x^-(u-s) &= \hbar \sum_{n \geq 0} x_n^-(u-s)^{-n-1} \\ &= -\hbar \sum_{n \geq 0} (-1)^n x_n^- s^{-n-1} \sum_{a \geq 0} \binom{n+a}{n} u^a s^{-a} \\ &= -\hbar \sum_{m \geq 0} s^{-m-1} \sum_{n=0}^m (-1)^n \binom{m}{n} x_n^- u^{m-n} \end{aligned}$$

which shows that ω_1 is given by (5.3).

Let now $k \geq 2$, and set $I_k(v, s) = \frac{(-1)^k}{k\hbar} x^-(v-s)^k \otimes x^+(v)^k$. We claim that

$$(sk - \mathcal{T})I_k(v, s) = -[\hbar r, I_{k-1}(v, s)]$$

which proves in particular that (5.3) satisfies (5.2). We shall need the commutation relation $[x_0^\pm, x^\pm(u)] = \mp x^\pm(u)^2$. The latter follows from relation (Y4) of Proposition 2.3, namely

$$(u-v \mp \hbar)x^\pm(u)x^\pm(v) - (u-v \pm \hbar)x^\pm(u)x^\pm(v) = \hbar([x_0^\pm, x^\pm(v)] - [x^\pm(u), x_0^\pm]),$$

by taking $u = v$. Thus, for every $k \geq 1$, $\text{ad}(x_0^\pm) \cdot x^\pm(u)^k = \mp k x^\pm(u)^{k+1}$.

Using this, and $[\frac{t_1}{2}, x^\pm(u)] = \pm(ux^\pm(u) - \hbar x_0^\pm)$, yields

$$\begin{aligned} \left[\frac{t_1}{2}, x^\pm(u)^k \right] &= \pm \sum_{j=1}^k x^\pm(u)^{j-1} (ux^\pm(u) - \hbar x_0^\pm) x^\pm(u)^{k-j} \\ &= \pm k u x^\pm(u)^k \mp \hbar \sum_{j=1}^k x^\pm(u)^{j-1} x_0^\pm x^\pm(u)^{k-j} \\ &= \pm k u x^\pm(u)^k \mp \hbar r x^\pm(u)^{k-1} x_0^\pm + \hbar \frac{k(k-1)}{2} x^\pm(u)^k \end{aligned}$$

Thus,

$$\begin{aligned} (sr - \mathcal{T}) x^-(v-s)^k \otimes x^+(v)^k &= -\hbar k(k-1) x^-(v-s)^k \otimes x^+(v)^k \\ &\quad - \hbar k x^-(v-s)^{k-1} x_0^- \otimes x^+(v)^k + \hbar k x^-(v-s)^k \otimes x^+(v)^{k-1} x_0^+. \end{aligned}$$

Together with $[A \otimes B, C \otimes D] = [A, C] \otimes [B, D] + [A, C] \otimes DB + CA \otimes [B, D]$, this yields

$$\begin{aligned} [x_0^- \otimes x_0^+, x^-(u)^k \otimes x^+(u)^k] &= [x_0^-, x^-(u)^k] \otimes [x_0^+, x^+(u)^k] + [x_0^-, x^-(u)^k] \otimes x^+(u)^k x_0^+ \\ &\quad + x^-(u)^k x_0^- \otimes [x_0^+, x^+(u)^k] \\ &= -k^2 x^-(u)^{k+1} \otimes x^+(u)^{k+1} + k x^-(u)^{k+1} \otimes x^+(u)^k x_0^+ \\ &\quad - k x^-(u)^k x_0^- \otimes x^+(u)^{k+1} \end{aligned}$$

as claimed. \square

5.4. Formula for $\omega_{V_1, V_2}(s, z)$. If V_1, V_2 are finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$, Theorem 4.1 (3) implies that $\omega_{V_1, V_2}(s, z) = \pi_{V_1} \otimes \pi_{V_2}(\omega(s, z))$ is a rational function of s . It follows from the lemma below that

$$\omega_{V_1, V_2}(s, z) = \sum_{k \geq 1} \frac{(-1)^k z^k}{k \hbar} \oint_{C_2} x^-(u-s)^k \otimes x^+(u)^k du$$

where C_2 encloses $\sigma(V_2)$ and none of the points in $\sigma(V_1) + s$. Note that in this case the sum over k is finite since $x^\pm(u)$ are nilpotent on V_1, V_2 .

Lemma. *Let A be a finite-dimensional algebra over \mathbb{C} , and $f, g : \mathbb{C} \rightarrow A$ rational functions which are regular at ∞ . Consider the integral*

$$I(s) = \oint_{C_2} f(u-s)g(u) du$$

where $s \in \mathbb{C}$, and C_2 is a contour enclosing all poles of $g(u)$ and none of those of $f(u-s)$. Then, the Taylor expansion $\hat{I}(s)$ of $I(s)$ at $s = \infty$ is equal to

$$\hat{I}(s) = \oint \hat{f}(u-s)\hat{g}(u) du$$

where $\oint -du$ is the formal residue defined in 5.3, \hat{f}, \hat{g} are the Taylor series of f, g at ∞ , and $\hat{f}(u-s)$ is expanded in $A[[s^{-1}]]$.

PROOF. $\hat{I}(s)$ is equal to $\oint_{C_2} \hat{f}(u-s)g(u)du$. Since $\hat{f}(u-s) \in A[[s^{-1}]]$, it suffices

to prove that $\oint_{C_2} p(u)g(u)du = \oint p(u)\hat{g}(u)du$ for any $p \in A[[u]]$, which follows by deformation of contour. \square

5.5. Formula for $\mathcal{R}^-(s)$. Integrating $z\partial_z \mathcal{R}^-(s, z) = \mathcal{R}^-(s, z) \cdot \omega(s, z)$ and setting $z = 1$ yields the following corollary of Proposition 5.3.

Theorem. *The element $\mathcal{R}^-(s) \in Y_{\hbar}(\mathfrak{sl}_2)^{\otimes 2}[[s^{-1}]]$ is given by*

$$\mathcal{R}^-(s) = 1^{\otimes 2} + \sum_{n \geq 1} \sum_{\substack{k_1, \dots, k_r \in \mathbb{Z}_{\geq 1} \\ k_1 + \dots + k_r = n}} \frac{1}{k_1(k_1 + k_2) \cdots (k_1 + \dots + k_r)} \omega(s)_{k_1} \cdots \omega(s)_{k_r}$$

where

$$\omega(s)_k = \frac{(-1)^k}{k \hbar} \oint x^-(u-s)^k \otimes x^+(u)^k du$$

Remark. It is an interesting problem to give an explicit formula for $\mathcal{R}^-(s)$ for $\mathfrak{g} \not\cong \mathfrak{sl}_2$.

6. THE UNIVERSAL AND THE MEROMORPHIC ABELIAN R -MATRICES OF $Y_{\hbar}(\mathfrak{g})$

In this section, we review the construction of the meromorphic abelian R -matrix of $Y_{\hbar}(\mathfrak{g})$ given in [12]. We then show that it gives rise to rational intertwiners $V_1(s) \otimes_{\mathbb{D},0} V_2 \rightarrow V_2 \otimes_{\mathbb{D},0} V_1(s)$ for any finite-dimensional representations V_1, V_2 . We also prove that these cannot be chosen to be both natural and compatible with the Drinfeld tensor product. Finally, we lift the meromorphic abelian R -matrix to obtain a universal abelian R -matrix for the deformed Drinfeld coproduct.

6.1. The endomorphism $\mathcal{A}_{V_1, V_2}(s)$ [12]. Let $V \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$. For any $i \in \mathbf{I}$, let $\sigma_V(\xi_i)$ be the set of poles of $\xi_i(u)^{\pm 1}$ acting on V , and

$$X_V(\xi_i) = \bigcup_{a \in \sigma_V(\xi_i)} [0, a]$$

the union of straight line segments joining 0 to points in $\sigma_V(\xi_i)$. Then, the generating series $t_i(v) = \hbar \sum_{r \geq 0} t_{i,r} v^{-r-1}$ introduced in 2.4 converges to a holomorphic function $t_i(v) : \mathbb{C} \setminus X_V(\xi_i) \rightarrow \text{End}_{\mathbb{C}}(V)$, which is uniquely determined by $\exp(t_i(v)) = \xi_i(v)$ and $t_i(\infty) = 0$ [12, §5.4].

Let $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, $s \in \mathbb{C}$, and define $\mathcal{A}_{V_1, V_2}(s) \in \text{End}_{\mathbb{C}}(V_1 \otimes V_2)$ by

$$\mathcal{A}_{V_1, V_2}(s) = \exp \left(- \sum_{\substack{i, j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} \oint_{C_1} \frac{dt_i(v)}{dv} \otimes t_j \left(v + s + \frac{(\ell + r)\hbar}{2} \right) dv \right)$$

where

- C_1 is a contour enclosing $\sigma_{V_1}(\xi_i)$.
- $\ell = mh^\vee$, with h^\vee the dual Coxeter number of \mathfrak{g} and $m = \frac{(\theta, \theta)}{2}$ for $\theta \in \Phi_+$ the highest root.
- The non-negative integers $c_{ij}^{(r)}$ are the entries of the following matrix [12, Appendix A]⁸

$$\left(c_{ij}(q) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} q^r \right) = [\ell]_q \cdot ([d_i a_{ij}]_q)^{-1}$$

- s is large enough so that $t_j(v + s + \hbar(\ell + r)/2)$ is an analytic function of v within C_1 , for every $j \in \mathbf{I}$ and $r \in \mathbb{Z}$ such that $c_{ij}^{(r)} \neq 0$ for some $i \in \mathbf{I}$.

Then, $\mathcal{A}_{V_1, V_2}(s)$ is a rational function of s , regular at ∞ , with an expansion of the form $\mathbf{1} - \frac{\ell\hbar}{s^2} \Omega_{\mathfrak{h}} + O(s^{-3})$, where $\Omega_{\mathfrak{h}} \in \mathfrak{h} \otimes \mathfrak{h}$ is the Cartan part of the Casimir tensor [12, Thm. 5.5]. Moreover, $[\mathcal{A}_{V_1, V_2}(s), \mathcal{A}_{V_1, V_2}(s')] = 0$ for any $s, s' \in \mathbb{C}$.

⁸It was proved in [12, Appendix A] that $c_{ij}(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$. It is clear from the definition that $c_{ij}(q) = c_{ji}(q) = c_{ij}(q^{-1})$, and the matrix identity can be expanded as

$$\sum_{k \in \mathbf{I}} c_{ik}(q) [d_k a_{kj}]_q = \delta_{ij} [\ell]_q \quad \forall i, j \in \mathbf{I} \quad (6.1)$$

6.2. The meromorphic abelian R -matrix of $Y_{\hbar}(\mathfrak{g})$ [12]. Consider the additive difference equation determined by $\mathcal{A}_{V_1, V_2}(s)$

$$\mathcal{R}_{V_1, V_2}^0(s + \ell\hbar) = \mathcal{A}_{V_1, V_2}(s) \cdot \mathcal{R}_{V_1, V_2}^0(s) \quad (6.2)$$

It admits two meromorphic solutions $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$, which are uniquely determined by the requirement that $\mathcal{R}_{V_1, V_2}^{0, \uparrow}(s)$ (resp. $\mathcal{R}_{V_1, V_2}^{0, \downarrow}(s)$) is holomorphic and invertible for $\operatorname{Re}(s/\ell\hbar) \gg 0$ (resp. $\operatorname{Re}(s/\ell\hbar) \ll 0$), and possesses an asymptotic expansion of the form $1 + O(s^{-1})$ as $s \rightarrow \infty$ in any halfplane of the form $\operatorname{Re}(s/\ell\hbar) > m$ (resp. $\operatorname{Re}(s/\ell\hbar) < m$). These solutions are explicitly given by the infinite products

$$\mathcal{R}_{V_1, V_2}^{0, \uparrow}(s) = \prod_{n \geq 0}^{\rightarrow} \mathcal{A}_{V_1, V_2}(s + n\ell\hbar)^{-1} \quad \text{and} \quad \mathcal{R}_{V_1, V_2}^{0, \downarrow}(s) = \prod_{n \geq 1}^{\rightarrow} \mathcal{A}_{V_1, V_2}(s - n\ell\hbar)$$

The product defining $\mathcal{R}_{V_1, V_2}^{0, \uparrow}(s)$ (resp. $\mathcal{R}_{V_1, V_2}^{0, \downarrow}(s)$) converges uniformly on compact subsets of the complement of $\mathcal{Z} - \ell\hbar\mathbb{Z}_{\geq 0}$ (resp. $\mathcal{P} + \ell\hbar\mathbb{Z}_{> 0}$), where \mathcal{Z} (resp. \mathcal{P}) is the set of poles of $\mathcal{A}(s)^{-1}$ (resp. $\mathcal{A}(s)$).

Then, the following holds [12, Thm 5.9].

Theorem. Fix $\varepsilon \in \{\uparrow, \downarrow\}$. Then, the following holds for any $V_1, V_2, V_3 \in \operatorname{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$

(1) The map

$$(12) \circ \mathcal{R}_{V_1, V_2}^{0, \varepsilon}(s) : V_1(s) \otimes_{\mathbb{D}, 0} V_2 \rightarrow V_2 \otimes_{\mathbb{D}, 0} V_1(s)$$

is a morphism of $Y_{\hbar}(\mathfrak{g})$ -modules, which is natural in V_1 and V_2 .

(2) The following cabling identities hold:

$$\begin{aligned} \mathcal{R}_{V_1 \otimes_{\mathbb{D}, s_1} V_2, V_3}^{0, \varepsilon}(s_2) &= \mathcal{R}_{V_1, V_3}^{0, \varepsilon}(s_1 + s_2) \cdot \mathcal{R}_{V_2, V_3}^{0, \varepsilon}(s_2) \\ \mathcal{R}_{V_1, V_2 \otimes_{\mathbb{D}, s_2} V_3}^{0, \varepsilon}(s_1 + s_2) &= \mathcal{R}_{V_1, V_3}^{0, \varepsilon}(s_1 + s_2) \cdot \mathcal{R}_{V_1, V_2}^{0, \varepsilon}(s_1) \end{aligned}$$

(3) For any $a, b \in \mathbb{C}$,

$$\mathcal{R}_{V_1(a), V_2(b)}^{0, \varepsilon}(s) = \mathcal{R}_{V_1, V_2}^{0, \varepsilon}(s + a - b)$$

(4) The following unitary condition holds:

$$(12) \circ \mathcal{R}_{V_1, V_2}^{0, \uparrow}(-s) \circ (12)^{-1} = \mathcal{R}_{V_2, V_1}^{0, \downarrow}(s)^{-1}$$

(5) $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ have the same asymptotic expansion, as $s \rightarrow \infty$ in any halfplane of the form $\pm \operatorname{Re}(s/\ell\hbar) > m$, which is of the form

$$\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \sim 1 + \hbar\Omega_{\hbar}s^{-1} + O(s^{-2})$$

6.3. Existence of a rational intertwiner. Let $V_1, V_2 \in \operatorname{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$.

Theorem. There is a rational map $\mathbf{R}_{V_1, V_2}^0 : \mathbb{C} \rightarrow \operatorname{Aut}_{\mathbb{C}}(V_1 \otimes V_2)$, which is normalised by $\mathbf{R}_{V_1, V_2}^0(\infty) = \mathbf{1}$ and such that

$$(12) \circ \mathbf{R}_{V_1, V_2}^0(s) : V_1(s) \otimes_{\mathbb{D}, 0} V_2 \rightarrow V_2 \otimes_{\mathbb{D}, 0} V_1(s)$$

intertwines the action of $Y_{\hbar}(\mathfrak{g})$. In particular, $V_1(s) \otimes_{\mathbb{D}, 0} V_2$ and $V_2 \otimes_{\mathbb{D}, 0} V_1(s)$ are isomorphic as $Y_{\hbar}(\mathfrak{g})$ -modules for all but finitely many values of s .

PROOF. The map $R_{V_1, V_2}^0(s)$ will be obtained from the meromorphic intertwiners $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ as follows. Consider the difference equation (6.2) satisfied by $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$. Its monodromy is given by

$$\eta_{V_1, V_2}^0(s) = \mathcal{R}_{V_1, V_2}^{0, \uparrow}(s)^{-1} \cdot \mathcal{R}_{V_1, V_2}^{0, \downarrow}(s)$$

By construction, η_{V_1, V_2}^0 is an $\ell\hbar$ -periodic function of s , and in fact a rational function of $z = \exp\left(\frac{2\pi i s}{\ell\hbar}\right)$ which takes the value $\mathbf{1}$ at $z = 0, \infty$ [11, §4.8]. Moreover, by Theorem 6.2 (1), $\eta_{V_1, V_2}^0(s)$ commutes with the action of $Y_{\hbar}(\mathfrak{g})$ on $V_1 \otimes_{D, s} V_2$. In fact, the periodicity of η_{V_1, V_2}^0 implies that $\eta_{V_1, V_2}^0(s)$ commutes with the action of $Y_{\hbar}(\mathfrak{g})$ on $V_1 \otimes_{D, s'} V_2$ for any s' of the form $s + \ell\hbar m$, $m \in \mathbb{Z}$. Since that action is rational in s' , it follows that η_{V_1, V_2}^0 takes values in the subalgebra $\mathcal{Z} \subset \text{End}_{\mathbb{C}}(V_1 \otimes V_2)$ which consists of elements commuting with the action of $Y_{\hbar}(\mathfrak{g})$ on $V_1 \otimes_{D, s'} V_2$ for any $s' \in \mathbb{C}$.

Consider now the following factorisation problem. Find two meromorphic functions $\mathfrak{X}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) : \mathbb{C} \rightarrow \mathcal{Z}$ such that

- (1) $\mathfrak{X}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ is holomorphic and invertible for $\text{Re}(\pm s/\hbar) \gg 0$.
- (2) $\mathfrak{X}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ possesses an asymptotic expansion of the form $1 + O(s^{-1})$, valid in any half-plane of the form $\text{Re}(s/\ell\hbar) \geq m$.
- (3) $\eta_{V_1, V_2}^0(s) = \mathfrak{X}_{V_1, V_2}^{0, \uparrow}(s)^{-1} \cdot \mathfrak{X}_{V_1, V_2}^{0, \downarrow}(s)$.

Since $[\eta_{V_1, V_2}^0(s), \eta_{V_1, V_2}^0(s')] = 0$ for any s, s' , such a factorisation exists, and can be obtained explicitly, once a choice of representatives of poles of $\eta^0(s)$ modulo translations by $\ell\hbar\mathbb{Z}$ is made [11, §4.14]. We remark that the consistency equation, required upon such a choice in [11, §4.14], is vacuous in our case, since $\mathcal{A}_{V_1, V_2}(s) = 1 + O(s^{-2})$.

Summarising, $\eta_{V_1, V_2}^0(s)$ admits two factorisations in $\text{End}_{\mathbb{C}}(V_1 \otimes V_2)$, namely

$$\mathfrak{X}_{V_1, V_2}^{0, \uparrow}(s)^{-1} \cdot \mathfrak{X}_{V_1, V_2}^{0, \downarrow}(s) = \eta_{V_1, V_2}^0(s) = \mathcal{R}_{V_1, V_2}^{0, \uparrow}(s)^{-1} \cdot \mathcal{R}_{V_1, V_2}^{0, \downarrow}(s)$$

Set now

$$R_{V_1, V_2}^0(s) = \mathcal{R}_{V_1, V_2}^{0, \uparrow}(s) \cdot \mathfrak{X}_{V_1, V_2}^{0, \uparrow}(s)^{-1} = \mathcal{R}_{V_1, V_2}^{0, \downarrow}(s) \cdot \mathfrak{X}_{V_1, V_2}^{0, \downarrow}(s)^{-1}$$

Then, $(12) \circ R_{V_1, V_2}^0(s) : V_1(s) \otimes_{D, 0} V_2 \rightarrow V_2 \otimes_{D, 0} V_1(s)$ intertwines the action of $Y_{\hbar}(\mathfrak{g})$, and R_{V_1, V_2}^0 is a rational function of s equal to $\mathbf{1}$ at $s = \infty$ [11, §4.11]. \square

6.4. Non-existence of rational commutativity constraints. Theorem 6.3 raises the question of whether the rational factor $R_{V_1, V_2}^0(s)$ may be chosen consistently for any pair of representations V_1, V_2 so as to satisfy the cabling identities of Theorem 6.2. The following shows that not to be the case.

Theorem. *There is no function $R_{V_1, V_2}^0 : \mathbb{C} \rightarrow \text{Aut}_{\mathbb{C}}(V_1 \otimes V_2)$ which is rational, defined for any $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, and such that the following conditions hold.*

- (1) $(12) \circ R_{V_1, V_2}^0(s) : V_1(s) \otimes_{D, 0} V_2 \rightarrow V_2 \otimes_{D, 0} V_1(s)$ is $Y_{\hbar}(\mathfrak{g})$ -linear, and natural in V_1 and V_2 .

(2) For any $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$

$$\mathbf{R}_{V_1 \otimes_{\mathbb{D}, s_1} V_2, V_3}^0(s_2) = \mathbf{R}_{V_1, V_3}^0(s_1 + s_2) \cdot \mathbf{R}_{V_2, V_3}^0(s_2) \quad (6.3)$$

$$\mathbf{R}_{V_1, V_2 \otimes_{\mathbb{D}, s_2} V_3}^0(s_1 + s_2) = \mathbf{R}_{V_1, V_3}^0(s_1 + s_2) \cdot \mathbf{R}_{V_1, V_2}^0(s_2) \quad (6.4)$$

PROOF. Note first if V_1 is the trivial one-dimensional representation $\mathbf{1}$ of $Y_{\hbar}(\mathfrak{g})$, the first cabling identity (6.3) and part (4) of Theorem 3.2 imply that $\mathbf{R}_{\mathbf{1}, V_3}^0(s) = \text{Id}_{V_3}$. Setting $V_2 = \mathbf{1}$ in (6.3) then yields $\mathbf{R}_{V_1(a), V_3}^0(s) = \mathbf{R}_{V_1, V_3}^0(s + a)$. Similarly, upon setting $V_2 = \mathbf{1}$ first, and then $V_3 = \mathbf{1}$, the second cabling identity (6.4) implies that $\mathbf{R}_{V_1, \mathbf{1}}^0(s) = \text{Id}_V$, and that $\mathbf{R}_{V_1, V_2(b)}^0(s) = \mathbf{R}_{V_1, V_2}^0(s - b)$.

We now take $\mathfrak{g} = \mathfrak{sl}_2$ and proceed by contradiction, assuming the existence of a rational $\mathbf{R}_{V_1, V_2}^0(s)$ with the stated properties. We will use the following facts about $Y_{\hbar}(\mathfrak{sl}_2)$. There is a two-dimensional representation \mathbb{C}^2 of $Y_{\hbar}(\mathfrak{sl}_2)$ (see Section 4.8). Explicitly, in a fixed basis v_+, v_- , the action of $Y_{\hbar}(\mathfrak{sl}_2)$ is given by the following 2×2 matrices.

$$\xi(u) = \mathbf{1} + \frac{\hbar}{u} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad x^+(u) = \frac{\hbar}{u} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x^-(u)^T$$

Further, there is a $Y_{\hbar}(\mathfrak{sl}_2)$ -linear map $\mathbf{1} \rightarrow \mathbb{C}^2 \otimes_{\mathbb{D}, \hbar} \mathbb{C}^2$ given by $\mathbf{1} \mapsto v_+ \otimes v_-$.

Given $V \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, we consider $\mathbf{R}_{\mathbb{C}^2, V}^0(s) \in \text{End}_{\mathbb{C}}(\mathbb{C}^2 \otimes V)$ as a 2×2 -matrix with entries in $\text{End}(V)$ given by

$$\mathbf{R}_{\mathbb{C}^2, V}^0(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ \gamma(s) & \delta(s) \end{pmatrix}$$

The intertwining property of $\mathbf{R}_{\mathbb{C}^2, V}^0(s)$ reads

$$\mathbf{R}_{\mathbb{C}^2, V}^0(s) \cdot \pi_{\mathbb{C}^2(s) \otimes_{\mathbb{D}, 0} V}(x) = (\pi_{V \otimes_{\mathbb{D}, 0} \mathbb{C}^2(s)}(x))^{21} \cdot \mathbf{R}_{\mathbb{C}^2, V}^0(s) \quad (6.5)$$

for any $x \in Y_{\hbar}(\mathfrak{g})$. For $x = \xi_0$,

$$\pi_{\mathbb{C}^2(s) \otimes_{\mathbb{D}, 0} V}(\xi_0) = (\pi_{V \otimes_{\mathbb{D}, 0} \mathbb{C}^2(s)}(\xi_0))^{21} = \xi_0 \otimes \mathbf{1} + \mathbf{1} \otimes \xi_0 = \begin{pmatrix} 1 + \xi_0 & 0 \\ 0 & -1 + \xi_0 \end{pmatrix}$$

the intertwining relation (6.5) yields

$$[\xi_0, \alpha] = 0 \quad [\xi_0, \delta] = 0 \quad [\xi_0, \beta] = -2\beta \quad [\xi_0, \gamma] = 2\gamma$$

For $x = t_1$, $\pi_{V_1 \otimes_{\mathbb{D}, 0} V_2}(t_1) = \pi_{V_1}(t_1) + \pi_{V_2}(t_1)$, so that

$$\pi_{\mathbb{C}^2(s) \otimes_{\mathbb{D}, 0} V}(t_1) = (\pi_{V \otimes_{\mathbb{D}, 0} \mathbb{C}^2(s)}(t_1))^{21} = (t_1 + s\xi_0) \otimes \mathbf{1} + \mathbf{1} \otimes t_1 = (\xi_1 - \frac{\hbar}{2}\xi_0^2 + s\xi_0) \otimes \mathbf{1} + \mathbf{1} \otimes t_1$$

Since ξ_1 acts by 0 on \mathbb{C}^2 , $\pi_{\mathbb{C}^2(s) \otimes_{\mathbb{D}, 0} V}(t_1)$ acts by the matrix

$$\begin{pmatrix} -\hbar/2 + s + t_1 & 0 \\ 0 & \hbar/2 - s + t_1 \end{pmatrix}$$

The relation (6.5) then implies that $[t_1, \alpha] = 0$, $[t_1, \delta] = 0$, $[t_1, \beta] = -2s\beta$ and $[t_1, \gamma] = 2s\gamma$. The last two relations imply in turn that $\beta = \gamma = 0$ since $\text{ad}(t_1) \pm s$ is invertible on $\text{End}(V)$ for all but finitely many values of s .

For $x = x_0^-$, 3.2 implies that the $-+$ coefficient of $\pi_{V \otimes_{\mathbb{D},0} \mathbb{C}^2(s)}(x_0^-)$ is $\mathbf{1}_V$, while that of $\pi_{\mathbb{C}^2(s) \otimes_{\mathbb{D},0} V}(x_0^-)$ is equal to $\oint_{C_1} \frac{\xi(v)}{v-s} = \xi(s)$, where C_1 encloses $\sigma(\mathbb{C}^2(s)) = \{s\}$ and none of the points in $\sigma(V)$. Taking $-+$ coefficients in (6.5) then yields the relation $\alpha(s) = \delta(s) \cdot \xi(s)$. Similarly, taking the $+ -$ coefficients in the intertwining relation (6.5) with $x = x_0^+$ yields $\alpha = \xi \cdot \delta$. Summarising, (6.5) implies that

$$R_{\mathbb{C}^2, V}^0(s) = \begin{pmatrix} \alpha(s) & 0 \\ 0 & \delta(s) \end{pmatrix}$$

where $\alpha, \delta \in \text{End}(V)(s)$ commute with ξ_0, t_1 , and satisfy $\alpha = \delta \cdot \xi = \xi \cdot \delta$.

We now use the cabling relation (6.3)

$$R_{\mathbb{C}_{(1), \mathbb{D}, \hbar}^2 \otimes \mathbb{C}_{(2), V}^2}^0(s) = R_{\mathbb{C}_{(1), V}^2}^0(s + \hbar) \cdot R_{\mathbb{C}_{(2), V}^2}^0(s)$$

in $\text{End}_{\mathbb{C}}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes V)$, where the subscripts are added to emphasize the order of the tensors. The right-hand side applied to $v_+ \otimes v_- \otimes v$ yields $\alpha(s + \hbar) \cdot \delta(s) v$. On the other hand, by naturality,

$$R_{\mathbb{C}_{(1), \mathbb{D}, \hbar}^2 \otimes \mathbb{C}_{(2), V}^2}^0(s) v_+ \otimes v_- \otimes v = R_{\mathbb{C}, V}^0(s) \mathbf{1} \otimes v = v$$

Combining these relations yields $\alpha(s + \hbar) \cdot \alpha(s) = \xi(s)$. Taking now $V = \mathbb{C}^2$, this equation implies that the coefficient $a(s)$ of v_+ in $\alpha(s)v_+$ satisfies the additive difference equation

$$\frac{a(s + 2\hbar)}{a(s)} = \frac{c(s + \hbar)}{c(s)} = \frac{s(s + 2\hbar)}{(s + \hbar)^2}$$

where $c(s) = (s + \hbar)/s$ is the matrix coefficient of $\xi(s)$ corresponding to v_+ . This equation has a unique solution $\varphi(s)$ which is holomorphic and non-zero for $\text{Re}(s/2\hbar) \gg 0$ and is asymptotic to $1 + O(s^{-1})$ in that domain (see, e.g., [11, §4]). Clearly,

$$\varphi(s) = \frac{\Gamma\left(\frac{s}{2\hbar}\right) \cdot \Gamma\left(\frac{s+2\hbar}{2\hbar}\right)}{\Gamma\left(\frac{s+\hbar}{2\hbar}\right)^2}$$

which is not a rational function. \square

6.5. Taylor expansion of $\mathcal{A}_{V_1, V_2}(s)$. Let $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$. We determine in 6.6 the asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ as $s \rightarrow \infty$. As a preliminary step, we compute the Taylor series of the endomorphism $\mathcal{A}_{V_1, V_2}(s)$ introduced in 6.2. Let

$$\mathcal{L}_{V_1, V_2}(s) = - \sum_{\substack{i, j \in \mathbf{1} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} \oint_{C_1} \frac{dt_i(v)}{dv} \otimes t_j \left(v + s + \frac{(\ell + r)\hbar}{2} \right) dv$$

be the logarithm of \mathcal{A}_{V_1, V_2} .

Lemma. *The Taylor series of \mathcal{L}_{V_1, V_2} at $s = \infty$ is given by the action on $V_1 \otimes V_2$ of the element $\mathcal{L}(s) \in s^{-2}(Y_{\hbar}^0(\mathfrak{g}) \otimes Y_{\hbar}^0(\mathfrak{g}))[[s^{-1}]]$ defined by*

$$\mathcal{L}(s) = -\hbar^2 \sum_{\substack{i, j \in \mathbf{1} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{\ell+r}{2}} \sum_{m \geq n \geq 0} (-1)^n (m+1)! s^{-m-2} \frac{t_{i, n}}{n!} \otimes \frac{t_{j, m-n}}{(m-n)!}$$

where $T_m f(s) = f(s + m\hbar)$.

PROOF. Given that

$$\begin{aligned} t_j(v+s) &= \hbar \sum_{m \geq 0} t_{j,m}(v+s)^{-m-1} \\ &= \hbar \sum_{m \geq 0} s^{-m-1} t_{j,m} \sum_{n \geq 0} (-1)^n \binom{m+n}{n} v^n s^{-n} \\ &= \hbar \sum_{m \geq 0} s^{-m-1} \sum_{n=0}^m (-1)^n \binom{m}{n} t_{j,n-m} v^n \end{aligned}$$

and that $t_i(v)' = -\hbar \sum_{a \geq 0} (a+1) t_{i,a} v^{-a-2}$, the expansion of the summand of \mathcal{L}_{V_1, V_2} corresponding to a triple i, j, r is equal to

$$\hbar^2 c_{ij}^{(r)} T_{\frac{\ell+r}{2}} \sum_{m \geq 0} s^{-m-1} \sum_{n=1}^m (-1)^n \binom{m}{n} n t_{i,n-1} \otimes t_{j,n-m}$$

□

Remark. Since $(m+1)!s^{-m-2} = (-1)^m \partial_s^m s^{-2}$, the above reads

$$\mathcal{L}(s) = \sum_{\substack{i,j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{\ell+r}{2}} B_i(\partial_s) \otimes B_j(-\partial_s) (-s^{-2}) \quad (6.6)$$

where $B_k(z) = \hbar \sum_{n \geq 0} \frac{t_{k,n}}{n!} z^n$ is the inverse Borel transform of $t_k(v)$.

6.6. Asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$. Let⁹ $g(x) \in x^{-1} \mathbb{C}[[x^{-1}]]$ be the unique solution of the difference equation $g(x+1) = g(x) - x^{-2}$.

Define $\mathcal{R}^0(s) \in 1 + s^{-1} (Y_{\hbar}^0(\mathfrak{g}) \otimes Y_{\hbar}^0(\mathfrak{g}))[[s^{-1}]]$ by

$$\log(\mathcal{R}^0(s)) = \frac{1}{\ell^2 \hbar^2} \sum_{\substack{i,j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{\ell+r}{2}} B_i(\partial_s) \otimes B_j(-\partial_s) g\left(\frac{s}{\ell \hbar}\right) \quad (6.7)$$

Let $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$. By Theorem 6.2, $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ has an asymptotic expansion as $s \rightarrow \infty$ in any halfplane $\text{Re}(s/\ell \hbar) \geq m$, which is of the form $1 + O(s^{-1})$.

Proposition. *The asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ as $s \rightarrow \infty$ is given by*

$$\mathcal{R}_{V_1, V_2}^0(s) = \pi_{V_1} \otimes \pi_{V_2}(\mathcal{R}^0(s))$$

PROOF. By definition of $\log(\mathcal{R}^0(s))$ and (6.6), we have

$$\begin{aligned} (T_{\ell} - 1) \log(\mathcal{R}^0(s)) &= \frac{1}{\ell^2 \hbar^2} \sum_{\substack{i,j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{\ell+r}{2}} B_i(\partial_s) \otimes B_j(-\partial_s) \left(g\left(\frac{s}{\ell \hbar} + 1\right) - g\left(\frac{s}{\ell \hbar}\right) \right) \\ &= \sum_{\substack{i,j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{\ell+r}{2}} B_i(\partial_s) \otimes B_j(-\partial_s) \left(-\frac{1}{s^2} \right) \\ &= \mathcal{L}(s) \end{aligned}$$

⁹ $g(x)$ is equal to $\sum_{k \geq 0} B_k x^{-k-1}$, and is the asymptotic expansion at $x = \infty$ of the trigamma function $\Psi_1(x) = \frac{d^2}{dx^2} \ln \Gamma(x) = \sum_{n \geq 0} (x+n)^{-2}$.

Thus, $\mathcal{R}_{V_1, V_2}^0(s)$ is the unique formal solution of

$$\mathcal{R}_{V_1, V_2}^0(s + \ell\hbar) = \mathcal{A}_{V_1, V_2}(s) \cdot \mathcal{R}_{V_1, V_2}^0(s)$$

and therefore equals the asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$. \square

6.7. Properties of $\mathcal{R}^0(s)$. The following is the universal analogue of Theorem 6.2.

Theorem. $\mathcal{R}^0(s) \in (Y_{\hbar}^0(\mathfrak{g}) \otimes Y_{\hbar}^0(\mathfrak{g}))[[s^{-1}]]$ has the following properties.

(1) For every $x \in Y_{\hbar}(\mathfrak{g})$, the following holds in $Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}]$

$$\mathcal{R}^0(s) \cdot \Delta_{D, s}(x) = \Delta_{D, -s}^{(21)}(\tau_s(x)) \cdot \mathcal{R}^0(s)$$

(2) The cabling identities

$$\begin{aligned} \Delta_{D, s_1} \otimes \mathbf{1}(\mathcal{R}^0(s_2)) &= \mathcal{R}_{13}^0(s_1 + s_2) \cdot \mathcal{R}_{23}^0(s_2) \\ \mathbf{1} \otimes \Delta_{D, s_2}(\mathcal{R}^0(s_1 + s_2)) &= \mathcal{R}_{13}^0(s_1 + s_2) \cdot \mathcal{R}_{12}^0(s_1) \end{aligned}$$

hold in $Y_{\hbar}^0(\mathfrak{g})^{\otimes 3}[s_1][s_2^{-1}]$ and $Y_{\hbar}^0(\mathfrak{g})^{\otimes 3}[s_2][s_1^{-1}]$ respectively.

(3) $\mathcal{R}^0(s)$ is unitary

$$\mathcal{R}^0(s)^{-1} = \mathcal{R}_{21}^0(-s)$$

(4) For any $a, b \in \mathbb{C}$

$$(\tau_a \otimes \tau_b)(\mathcal{R}^0(s)) = \mathcal{R}^0(s + a - b)$$

(5) $\mathcal{R}^0(s) - 1 \in \mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$, with semiclassical limit given by

$$\overline{\mathcal{R}^0(s) - 1} = \frac{\hbar\Omega_{\mathfrak{h}}}{z + s - w} \in (U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))[[s^{-1}]]$$

Statements (1)–(4) follow from that fact that $\mathcal{R}_{V_1, V_2}^0(s)$ is the asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$ and Theorem 6.2, since finite-dimensional representations separate points in $Y_{\hbar}(\mathfrak{g})$ (Proposition A.1). For completeness, we give a direct proof of Theorem 6.7 below, which does not rely on this fact.

6.8. Direct proof of Theorem 6.7.

Proof of (2). Let $P^0 \subset Y_{\hbar}^0(\mathfrak{g})$ be the \mathbb{C} -linear span of $\{t_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}_{\geq 0}}$, so that $\log(\mathcal{R}^0(s)) \in (P^0 \otimes P^0)[[s^{-1}]]$. The cabling identities follow from the fact that each $t_{i,r}$ is primitive with respect to the Drinfeld coproduct, that is satisfies

$$\Delta_{D, s}(t_{i,r}) = \tau_s(t_{i,r}) \otimes \mathbf{1} + \mathbf{1} \otimes t_{i,r}$$

and the fact that $Y_{\hbar}^0(\mathfrak{g})$ is a commutative subalgebra.

Proof of (3). Write

$$\begin{aligned} \log(\mathcal{R}^0(s)) &= \frac{1}{\ell^2 \hbar^2} \sum_{\substack{i, j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{r}{2}} B_i(\partial_s) \otimes B_j(-\partial_s) g\left(\frac{s}{\ell\hbar} + \frac{1}{2}\right) \\ &= \frac{1}{\ell^2 \hbar^2} \sum_{i, j \in \mathbf{I}} c_{ij} (e^{\frac{\hbar}{2} \partial_s}) B_i(\partial_s) \otimes B_j(-\partial_s) g\left(\frac{s}{\ell\hbar} + \frac{1}{2}\right) \\ &= \frac{1}{\ell^2 \hbar^2} \Omega(\partial_s) g\left(\frac{s}{\ell\hbar} + \frac{1}{2}\right) \end{aligned}$$

where $\Omega(z) = \sum_{i,j \in \mathbf{I}} c_{ij}(z) B_i(z) \otimes B_j(-z)$. The unitary condition follows from

$$\Omega^{21}(z) = \Omega(-z) \quad \text{and} \quad g\left(\frac{1}{2} + x\right) = -g\left(\frac{1}{2} - x\right)$$

The first identity holds because $c_{ij}(q) = c_{ij}(q^{-1}) = c_{ji}(q)$, and the second because $g(x) = -g(1-x)$, since both sides are solutions of the same difference equation.

Proof of (4). Since $\tau_a B_i(z) = e^{az} B_i(z)$,

$$\begin{aligned} \tau_a \otimes \tau_b \log(\mathcal{R}^0(s)) &= \frac{1}{\ell^2 \hbar^2} \sum_{\substack{i,j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{\ell+r}{2}} B_i(\partial_s) \otimes B_j(-\partial_s) e^{(a-b)\partial_s} g\left(\frac{s}{\ell\hbar}\right) \\ &= \frac{1}{\ell^2 \hbar^2} \sum_{\substack{i,j \in \mathbf{I} \\ r \in \mathbb{Z}}} c_{ij}^{(r)} T_{\frac{\ell+r}{2}} B_i(\partial_s) \otimes B_j(-\partial_s) g\left(\frac{s+a-b}{\ell\hbar}\right) \\ &= \log(\mathcal{R}^0(s+a-b)) \end{aligned}$$

Proof of (5). By (2.7), it suffices to prove that $\log(\mathcal{R}^0(s)) \in \mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$, and that

$$\overline{\log(\mathcal{R}^0(s))} = \hbar \frac{\Omega_{\hbar}}{z+s-w}$$

Note first that

$$t_{i,n} \otimes t_{j,m} (-1)^m \partial_s^m g\left(\frac{s}{\ell\hbar}\right) = t_{i,n} \otimes t_{j,m} \left(\frac{\ell\hbar \cdot m!}{s^{m+1}} + O(s^{-m-2}) \right)$$

lies in $\mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$, and has symbol $\ell\hbar \cdot m! d_i h_i z^n \otimes d_j h_j w^m s^{-m-1}$. Since the shifts T_x preserve the filtration $\mathcal{F}_{\bullet}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$ and act as the identity on its associated graded space, it follows that $\log(\mathcal{R}^0(s)) \in \mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$, and that

$$\begin{aligned} \overline{\log(\mathcal{R}^0(s))} &= \frac{\hbar}{\ell} \sum_{i,j \in \mathbf{I}} c_{ij}(1) \sum_{m \geq n \geq 0} (-1)^n \binom{m}{n} d_i h_i z^n \otimes d_j h_j w^{m-n} s^{-m-1} \\ &= \hbar \frac{\Omega_{\hbar}}{z+s-w} \end{aligned}$$

where the last equality follows from the fact that

$$\sum_{j \in \mathbf{I}} c_{ij}(1) d_j h_j = \ell \sum_{j \in \mathbf{I}} (\mathbf{B}^{-1})_{ij} d_j h_j = \ell \varpi_i^{\vee}$$

with $\mathbf{B} = (d_i a_{ij})$, and $\varpi_i^{\vee} \in \mathfrak{h}$ the fundamental coweights.

Proof of (1). The intertwining relation is obvious for $x \in Y_{\hbar}^0(\mathfrak{g})$, since the latter is the commutative algebra generated by the elements $t_{i,r}$, which satisfy

$$\Delta_{\mathbf{D},s}(t_{i,r}) = \tau_s(t_{i,r}) \otimes 1 + 1 \otimes t_{i,r} = \Delta_{\mathbf{D},-s}^{(21)}(\tau_s t_{i,r})$$

Thus, it suffices to prove that, for any $k \in \mathbf{I}$

$$\mathcal{R}^0(s) \Delta_{\mathbf{D},s}(x_{k,0}^{\pm}) = \Delta_{\mathbf{D},-s}^{(21)}(x_{k,0}^{\pm}) \mathcal{R}^0(s)$$

We verify this identity for the $+$ case only. By Proposition 3.3, $\Delta(x_{k,0}^+)$ is equal to $x_{k,0}^+ \otimes 1 + 1 \otimes x_{k,0}^+ + \mathfrak{X}_k(s)$, where

$$\mathfrak{X}_k(s) = \hbar \sum_{N \geq 0} s^{-N-1} \sum_{n=0}^N (-1)^{n+1} \binom{N}{n} \xi_{k,n} \otimes x_{k,N-n}^+$$

We therefore have to prove that

$$\text{Ad}(\mathcal{R}^0(s)) \cdot (x_{k,0}^+ \otimes 1 + 1 \otimes x_{k,0}^+) = x_{k,0}^+ \otimes 1 + 1 \otimes x_{k,0}^+ + \mathfrak{X}_k^{(21)}(-s) - \text{Ad}(\mathcal{R}^0(s)) \cdot \mathfrak{X}_k(s)$$

We claim that $\text{Ad}(\mathcal{R}^0(s)) \cdot (x_{k,0}^+ \otimes 1) = x_{k,0}^+ \otimes 1 + \mathfrak{X}_k^{(21)}(-s)$. Given this, the unitary condition (3) then implies that $\text{Ad}(\mathcal{R}^0(s))^{-1} \cdot (1 \otimes x_{k,0}^+) = 1 \otimes x_{k,0}^+ + \mathfrak{X}_k(s)$ which, combined with the claim yields the required intertwining equation for $x_{k,0}^+$.

To prove the claim, we rely on the following commutation relation, which was obtained in [10, §2.9]

$$[B_i(z), x_{k,n}^+] = \frac{e^{\frac{d_i \alpha_{ik} \hbar}{2} z} - e^{-\frac{d_i \alpha_{ik} \hbar}{2} z}}{z} \cdot \left(\sum_{p \geq 0} x_{k,n+p}^+ \frac{z^p}{p!} \right)$$

Combining with the definition of $\Omega(z)$ given above, we can carry out the following computation, for each $k \in \mathbf{I}$, $n \in \mathbb{Z}_{\geq 0}$, and $y \in Y_{\hbar}^0(\mathfrak{g})$.

$$\begin{aligned} [\Omega(z), x_{k,n}^+ \otimes y] &= \frac{1}{\ell^2 \hbar^2} \sum_{j \in \mathbf{I}} \left(\sum_{i \in \mathbf{I}} c_{ij} \left(e^{\frac{\hbar}{2} z} \right) \left(\frac{e^{\frac{d_i \alpha_{ik} \hbar}{2} z} - e^{-\frac{d_i \alpha_{ik} \hbar}{2} z}}{z} \right) \right) \\ &\quad \cdot \left(\sum_{p \geq 0} x_{k,n+p}^+ \frac{(-z)^p}{p!} \right) \otimes B_j(z) y \\ &= \frac{1}{\ell^2 \hbar^2} \frac{e^{\frac{\ell \hbar}{2} z} - e^{-\frac{\ell \hbar}{2} z}}{z} \cdot \left(\sum_{p \geq 0} x_{k,n+p}^+ \frac{(-z)^p}{p!} \right) \otimes B_k(z) y \end{aligned}$$

Note that we used the equation (6.1) satisfied by $(c_{ij}(q))$ above.

This calculation, combined with

$$\frac{e^{\frac{\ell \hbar}{2} \partial_s} - e^{-\frac{\ell \hbar}{2} \partial_s}}{\partial_s} \cdot g \left(\frac{s}{\ell \hbar} + \frac{1}{2} \right) = \frac{\ell^2 \hbar^2}{s}$$

yields the commutation relation

$$[\log(\mathcal{R}^0(s)), x_{k,n}^+ \otimes y] = \sum_{p \geq 0} (-1)^p x_{k,n+p}^+ \otimes \left(\hbar \sum_{r \geq 0} \binom{r+p}{r} t_{k,r} s^{-r-p-1} \right) y$$

The claim now follows from

$$\text{Ad}(\mathcal{R}^0(s)) = \exp(\text{ad}(\log(\mathcal{R}^0(s))))$$

where both sides are acting on $V_k^+ \otimes Y_{\hbar}^0(\mathfrak{g})$, where V_k^+ is the \mathbb{C} -linear span of $\{x_{k,n}\}_{n \geq 0}$.

7. THE UNIVERSAL AND THE MEROMORPHIC R -MATRICES OF $Y_{\hbar}(\mathfrak{g})$

In this section, we construct the meromorphic and universal R -matrices of $Y_{\hbar}(\mathfrak{g})$.

7.1. The meromorphic R -matrix. Given $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ and $\varepsilon \in \{\uparrow, \downarrow\}$, define $\mathcal{R}_{V_1, V_2}^{\varepsilon} : \mathbb{C} \rightarrow \text{End}(V_1 \otimes V_2)$ by

$$\mathcal{R}_{V_1, V_2}^{\varepsilon}(s) = \mathcal{R}_{V_1, V_2}^{+}(s) \cdot \mathcal{R}_{V_1, V_2}^{0, \varepsilon}(s) \cdot \mathcal{R}_{V_1, V_2}^{-}(s),$$

where $\mathcal{R}_{V_1, V_2}^{+}(s) = (1\ 2) \circ \mathcal{R}_{V_2, V_1}^{-}(-s)^{-1} \circ (1\ 2)$.

Theorem. *The meromorphic function $\mathcal{R}_{V_1, V_2}^{\varepsilon}(s)$ has the following properties.*

(1) *The map*

$$(1\ 2) \circ \mathcal{R}_{V_1, V_2}^{\varepsilon}(s) : V_1(s) \otimes V_2 \rightarrow V_2 \otimes V_1(s)$$

is a morphism of $Y_{\hbar}(\mathfrak{g})$ -modules, which is natural in V_1, V_2 .

(2) *For any $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$,*

$$\begin{aligned} \mathcal{R}_{V_1 \otimes V_2, V_3}^{\varepsilon}(s_2) &= \mathcal{R}_{V_1, V_3}^{\varepsilon}(s_1 + s_2) \cdot \mathcal{R}_{V_2, V_3}^{\varepsilon}(s_2) \\ \mathcal{R}_{V_1, V_2 \otimes V_3}^{\varepsilon}(s_1 + s_2) &= \mathcal{R}_{V_1, V_3}^{\varepsilon}(s_1 + s_2) \cdot \mathcal{R}_{V_1, V_2}^{\varepsilon}(s_1). \end{aligned}$$

In particular, the QYBE holds on $V_1 \otimes V_2 \otimes V_3$:

$$\mathcal{R}_{V_1, V_2}^{\varepsilon}(s_1) \mathcal{R}_{V_1, V_3}^{\varepsilon}(s_1 + s_2) \mathcal{R}_{V_2, V_3}^{\varepsilon}(s_2) = \mathcal{R}_{V_2, V_3}^{\varepsilon}(s_2) \mathcal{R}_{V_1, V_3}^{\varepsilon}(s_1 + s_2) \mathcal{R}_{V_1, V_2}^{\varepsilon}(s_1).$$

(3) *For any $a, b \in \mathbb{C}$,*

$$\mathcal{R}_{V_1(a), V_2(b)}^{\varepsilon}(s) = \mathcal{R}_{V_1, V_2}^{\varepsilon}(s + a - b).$$

(4) $\mathcal{R}_{V_1, V_2}^{\uparrow}(s)$ and $\mathcal{R}_{V_2, V_1}^{\downarrow}(s)$ *are related by the unitarity relation:*

$$(1\ 2) \circ \mathcal{R}_{V_1, V_2}^{\uparrow}(-s) \circ (1\ 2) = \mathcal{R}_{V_2, V_1}^{\downarrow}(s)^{-1}.$$

(5) $\mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s)$ *have the same asymptotic expansion, which is of the form*

$$\mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) \sim 1 + \hbar \Omega_{\mathfrak{g}} s^{-1} + O(s^{-2})$$

as $s \rightarrow \infty$ in any halfplane of the form $\text{Re}(s/\hbar) \geq m$.

PROOF. (1) By definition,

$$(1\ 2) \circ \mathcal{R}_{V_1, V_2}^{\varepsilon}(s) = \mathcal{R}_{V_2, V_1}^{-}(-s)^{-1} \cdot \left((1\ 2) \circ \mathcal{R}_{V_1, V_2}^{0, \varepsilon}(s) \right) \cdot \mathcal{R}_{V_1, V_2}^{-}(s).$$

The result therefore follows from the fact that $\mathcal{R}_{V_1, V_2}^{-}(s)$ is a morphism of $Y_{\hbar}(\mathfrak{g})$ -modules $V_1 \otimes V_2 \rightarrow V_1 \otimes V_2$ (Theorem 4.1), and Theorem 6.2 (1).

(2) We will prove the following equivalent version of the first cabling identity

$$(1\ 2\ 3) \circ \mathcal{R}_{V_1 \otimes V_2, V_3}^{\varepsilon}(s_2) = (1\ 2) \left(\mathcal{R}_{V_1, V_3}^{\varepsilon}(s_1 + s_2) \otimes \mathbf{1} \right) (2\ 3) \left(\mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^{\varepsilon}(s_2) \right) \quad (7.1)$$

By definition, the left-hand side is equal to

$$\begin{aligned}
& \mathcal{R}_{V_3, V_1 \otimes_{s_1} V_2}^-(-s_2)^{-1} \cdot \left((1 \ 2 \ 3) \circ \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^{0, \varepsilon}(s_2) \right) \cdot \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^-(s_2) \\
&= \mathcal{R}_{V_3, V_1 \otimes_{s_1} V_2}^-(-s_2)^{-1} \cdot \left(\mathbf{1} \otimes \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \right) \cdot \left((1 \ 2 \ 3) \circ \mathcal{R}_{V_1 \otimes_{D, s_1} V_2, V_3}^{0, \varepsilon}(s_2) \right) \cdot \\
&\quad \left(\mathcal{R}_{V_1, V_2}^-(s_1) \otimes \mathbf{1} \right) \cdot \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^-(s_2) \\
&= \mathcal{R}_{V_3, V_1 \otimes_{s_1} V_2}^-(-s_2)^{-1} \cdot \left(\mathbf{1} \otimes \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \right) \cdot (1 \ 2) \left(\mathcal{R}_{V_1, V_3}^{0, \varepsilon}(s_1 + s_2) \otimes \mathbf{1} \right) \cdot \\
&\quad (2 \ 3) \left(\mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^{0, \varepsilon}(s_2) \right) \cdot \left(\mathcal{R}_{V_1, V_2}^-(s_1) \otimes \mathbf{1} \right) \cdot \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^-(s_2)
\end{aligned}$$

In the first equality, we used Theorem 4.1 in order to change $V_1 \otimes_{s_1} V_2$ to $V_1 \otimes_{D, s_1} V_2$, while the second equality follows from the cabling identity satisfied by $\mathcal{R}^{0, \varepsilon}(s)$ (Theorem 6.2 (2)).

Note that we have the following identity, which follows from the cocycle equation (4.7) after renaming variables

$$\left(\mathcal{R}_{V_1, V_3}^-(s_1 + s_2) \otimes \mathbf{1} \right) \cdot \mathcal{R}_{V_1 \otimes_{s_1 + s_2} V_3, V_2}^-(-s_2) = \left(\mathbf{1} \otimes \mathcal{R}_{V_3, V_2}^-(-s_2) \right) \cdot \mathcal{R}_{V_1, V_3 \otimes_{-s_2} V_3}^-(s_1)$$

Inserting this operator and its inverse in the last line of the computation above allows us to write the left-hand side of (7.1) as $A(s_1, s_2) \cdot B(s_1, s_2)$, where

$$\begin{aligned}
A(s_1, s_2) &= \mathcal{R}_{V_3, V_1 \otimes_{s_1} V_2}^-(-s_2)^{-1} \cdot \left(\mathbf{1} \otimes \mathcal{R}_{V_1, V_2}^-(s_1)^{-1} \right) \cdot (1 \ 2) \left(\mathcal{R}_{V_1, V_3}^{0, \varepsilon}(s_1 + s_2) \otimes \mathbf{1} \right) \cdot \\
&\quad \left(\mathcal{R}_{V_1, V_3}^-(s_1 + s_2) \otimes \mathbf{1} \right) \cdot \mathcal{R}_{V_1 \otimes_{s_1 + s_2} V_3, V_2}^-(-s_2) \\
B(s_1, s_2) &= \left(\left(\mathbf{1} \otimes \mathcal{R}_{V_3, V_2}^-(-s_2) \right) \cdot \mathcal{R}_{V_1, V_3 \otimes_{-s_2} V_3}^-(s_1) \right)^{-1} \cdot \\
&\quad (2 \ 3) \left(\mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^{0, \varepsilon}(s_2) \right) \cdot \left(\mathcal{R}_{V_1, V_2}^-(s_1) \otimes \mathbf{1} \right) \cdot \mathcal{R}_{V_1 \otimes_{s_1} V_2, V_3}^-(s_2)
\end{aligned}$$

Thus, in order to prove (7.1), it is enough to show that

$$A(s_1, s_2) = (1 \ 2) \circ \mathcal{R}_{V_1, V_3}^\varepsilon(s_1 + s_2) \otimes \mathbf{1} \quad \text{and} \quad B(s_1, s_2) = (2 \ 3) \circ \mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^\varepsilon(s_2)$$

We verify the latter below, the proof of the former, being entirely analogous, is omitted.

Using the cocycle equation (4.7) again, we have:

$$\begin{aligned}
B(s_1, s_2) &= \mathcal{R}_{V_1, V_3 \otimes_{-s_2} V_2}^-(s_1)^{-1} \cdot \left(\mathbf{1} \otimes \mathcal{R}_{V_3, V_2}^-(-s_2)^{-1} \right) \cdot \left((2 \ 3) \circ \mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^{0, \varepsilon}(s_2) \right) \cdot \\
&\quad \left(\mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^-(s_2) \right) \cdot \mathcal{R}_{V_1, V_2 \otimes_{s_2} V_3}^-(s_1 + s_2) \\
&= \mathcal{R}_{V_1, V_3 \otimes_{-s_2} V_2}^-(s_1)^{-1} \cdot \left((2 \ 3) \circ \mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^\varepsilon(s_2) \right) \cdot \mathcal{R}_{V_1, V_2 \otimes_{s_2} V_3}^-(s_1 + s_2) \\
&= \mathcal{R}_{V_1, V_3 \otimes_{-s_2} V_2}^-(s_1)^{-1} \cdot \mathcal{R}_{V_1, V_3 \otimes_{-s_2} V_2}^-(s_1) \cdot \left((2 \ 3) \circ \mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^\varepsilon(s_2) \right) \\
&= (2 \ 3) \circ \mathbf{1} \otimes \mathcal{R}_{V_2, V_3}^\varepsilon(s_2)
\end{aligned}$$

In the second line, we have used the definition of $\mathcal{R}^\varepsilon(s)$, and in the third, Part (1) of this theorem.

(3) follows from Theorem 4.1 (2), and Theorem 6.2 (3).

(4) By definition,

$$\begin{aligned}
(12) \circ \mathcal{R}_{V_1, V_2}^\uparrow(-s) \circ (12) &= \mathcal{R}_{V_2, V_1}^-(s)^{-1} \cdot \left((12) \circ \mathcal{R}_{V_1, V_2}^{0, \uparrow}(-s) \circ (12) \right) \cdot \mathcal{R}_{V_2, V_1}^+(s)^{-1} \\
&= \mathcal{R}_{V_2, V_1}^-(s)^{-1} \cdot \left(\mathcal{R}_{V_2, V_1}^{0, \downarrow}(s)^{-1} \right) \cdot \mathcal{R}_{V_2, V_1}^+(s)^{-1} \\
&= \mathcal{R}_{V_2, V_1}^\downarrow(s)^{-1}
\end{aligned}$$

where the second equality uses Theorem 6.2 (4).

(5) follows from Theorem 6.2 (5) and the Taylor series expansion of $\mathcal{R}_{V_1, V_2}^-(s) = 1 + \hbar r/s + O(s^{-2})$ given in Theorem 4.1 (3). \square

7.2. Existence of a rational intertwiner. The following extends to an arbitrary pair of representations $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ a result due to Drinfeld, which is valid when V_1, V_2 are irreducible [4, Thm. 4], and Maulik–Okounkov, which is valid when \mathfrak{g} is simply-laced, and V_1, V_2 arise from geometry [20].

Theorem. *There is a rational map $R_{V_1, V_2}(s) : \mathbb{C} \rightarrow \text{End}_{\mathbb{C}}(V_1 \otimes V_2)$, which is normalised by $R_{V_1, V_2}(\infty) = \mathbf{1}$ and such that*

$$(12) \circ R_{V_1, V_2}(s) : V_1(s) \otimes V_2 \rightarrow V_2 \otimes V_1(s)$$

intertwines the action of $Y_{\hbar}(\mathfrak{g})$. In particular, $V_1(s) \otimes V_2$ and $V_2 \otimes V_1(s)$ are isomorphic as $Y_{\hbar}(\mathfrak{g})$ -modules for all but finitely many values of s .

PROOF. Let $R_{V_1, V_2}^0(s)$ be the rational operator such that $(12) \circ R_{V_1, V_2}^0(s)$ intertwines the action on $V_1(s) \otimes_{D,0} V_2$ and $V_2 \otimes_{D,0} V_1(s)$, as obtained in Theorem 6.3. Then,

$$R_{V_1, V_2}(s) = \mathcal{R}_{V_1, V_2}^+(s) \cdot R_{V_1, V_2}^0(s) \cdot \mathcal{R}_{V_1, V_2}^-(s)$$

yields the required map. \square

7.3. Non existence of rational commutativity constraints.

Theorem. *There is no function $R_{V_1, V_2} : \mathbb{C} \rightarrow \text{Aut}_{\mathbb{C}}(V_1 \otimes V_2)$ which is rational, defined for any $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, and such that the following holds*

- (1) $(12) \circ R_{V_1, V_2}(s) : V_1(s) \otimes V_2 \rightarrow V_2 \otimes V_1(s)$ intertwines the action of $Y_{\hbar}(\mathfrak{g})$, and is natural in V_1 and V_2 .
- (2) For any $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$,

$$\begin{aligned} R_{V_1 \otimes_{s_1} V_2, V_3}(s_2) &= R_{V_1, V_3}(s_1 + s_2) \cdot R_{V_2, V_3}(s_2) \\ R_{V_1, V_2 \otimes_{s_2} V_3}(s_1 + s_2) &= R_{V_1, V_3}(s_1 + s_2) \cdot R_{V_1, V_2}(s_2) \end{aligned}$$

PROOF. Assume that such a rational $R_{V_1, V_2}(s)$ exists. Set

$$R_{V_1, V_2}^0(s) = \mathcal{R}_{V_1, V_2}^+(s)^{-1} \cdot R_{V_1, V_2}(s) \cdot \mathcal{R}_{V_1, V_2}^-(s)^{-1}$$

Then, the rational operator $R_{V_1, V_2}^0(s)$ contradicts Theorem 6.4, and the claim follows. \square

7.4. The universal R -matrix. We now turn our attention to obtaining a formal, universal analogue of Theorem 7.1. Consider the formal power series

$$\mathcal{R}(s) = \mathcal{R}^+(s) \cdot \mathcal{R}^0(s) \cdot \mathcal{R}^-(s) \in (Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))[[s^{-1}]]$$

where $\mathcal{R}^+(s) = \mathcal{R}_{21}^-(-s)^{-1}$. This series admits an expansion

$$\mathcal{R}(s) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k s^{-k}, \quad \mathcal{R}_k \in \mathcal{F}_{k-1}(Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})).$$

Theorem. *The formal power series $\mathcal{R}(s)$ has the following properties.*

- (1) For every $x \in Y_{\hbar}(\mathfrak{g})$, the following holds in $Y_{\hbar}(\mathfrak{g})^{\otimes 2}[s; s^{-1}]$

$$\tau_s \otimes \mathbf{1} \circ \Delta^{\text{op}}(x) = \mathcal{R}(s) \cdot \tau_s \otimes \mathbf{1} \circ \Delta(x) \cdot \mathcal{R}(s)^{-1}$$

- (2) $\mathcal{R}(s)$ satisfies the cabling identities

$$\begin{aligned} \Delta \otimes \mathbf{1}(\mathcal{R}(s)) &= \mathcal{R}_{13}(s) \mathcal{R}_{23}(s) \\ \mathbf{1} \otimes \Delta(\mathcal{R}(s)) &= \mathcal{R}_{13}(s) \mathcal{R}_{12}(s) \end{aligned}$$

- (3) $\mathcal{R}(s)$ is unitary

$$\mathcal{R}(s)^{-1} = \mathcal{R}_{21}(-s).$$

- (4) For any $a, b \in \mathbb{C}$, we have

$$(\tau_a \otimes \tau_b) \mathcal{R}(s) = \mathcal{R}(s + a - b)$$

- (5) $\mathcal{R}(s) - 1 \in \mathcal{F}_{-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}[[s^{-1}]])$, with semiclassical limit given by

$$\overline{\mathcal{R}(s) - 1} = \frac{\hbar \Omega_{\mathfrak{g}}}{s + z - w} \in (U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]))[[s^{-1}]]$$

In particular, $\mathcal{R}(s) = 1 + \hbar s^{-1} \Omega_{\mathfrak{g}} + O(s^{-2})$.

- (6) For any $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ and $\varepsilon \in \{\uparrow, \downarrow\}$, we have

$$\mathcal{R}_{V_1, V_2}^{\varepsilon}(s) \sim \mathcal{R}_{V_1, V_2}(s)$$

as $s \rightarrow \infty$ in any halfplane of the form $\text{Re}(s/\hbar) \geq m$. Here $\mathcal{R}_{V_1, V_2}(s) = \pi_{V_1} \otimes \pi_{V_2}(\mathcal{R}(s))$. That is, $\mathcal{R}_{V_1, V_2}(s)$ is equal to the asymptotic expansion of $\mathcal{R}_{V_1, V_2}^{\varepsilon}(s)$ from (5) of Theorem 7.1.

PROOF. Parts (1) and (3)–(6) are deduced directly from the definition of $\mathcal{R}(s)$ using the properties of $\mathcal{R}^-(s)$ and $\mathcal{R}^0(s)$ established in Theorems 4.1 and 6.7, respectively.

We note, however, that the cabling identities (2) do not follow directly from Theorems 4.1 and 6.7, as we do not have access to a formal, universal version of the cocycle equation (4.3) of Theorem 4.1. To remedy this, we shall instead make use of the fact that $\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ is sufficiently large to distinguish elements of $Y_{\hbar}(\mathfrak{g})$. More precisely, by Proposition A.1 (with $n = 3$), it is enough to prove that the following identities hold on $V_1 \otimes V_2 \otimes V_3$, for every $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$:

$$\begin{aligned}\mathcal{R}_{V_1 \otimes V_2, V_3}(s) &= \mathcal{R}_{V_1, V_3}(s) \mathcal{R}_{V_2, V_3}(s) \\ \mathcal{R}_{V_1, V_2 \otimes V_3}(s) &= \mathcal{R}_{V_1, V_3}(s) \mathcal{R}_{V_1, V_2}(s)\end{aligned}\tag{7.2}$$

Fix $\varepsilon \in \{\uparrow, \downarrow\}$ and consider the first equality. By (6), the left-hand side (resp. right-hand side) is equal to the uniquely determined asymptotic expansion of $\mathcal{R}_{V_1 \otimes V_2, V_3}^\varepsilon(s)$ (resp. $\mathcal{R}_{V_1, V_3}^\varepsilon(s) \mathcal{R}_{V_2, V_3}^\varepsilon(s)$) as $s \rightarrow \infty$ in any halfplane of the form $\text{Re}(s/\hbar) \geq m$. Moreover, by the first equality in (2) of Theorem 7.1, taken with $s_1 = 0$ and $s_2 = s$, we have

$$\mathcal{R}_{V_1 \otimes V_2, V_3}^\varepsilon(s) = \mathcal{R}_{V_1, V_3}^\varepsilon(s) \mathcal{R}_{V_2, V_3}^\varepsilon(s).$$

Thus, we can conclude that the first cabling identity of (7.2) necessarily holds. An identical argument establishes the second identity. \square

As an immediate consequence of the above theorem and the uniqueness assertion of Theorem 1.1 (see also Appendix B), we obtain the following corollary.

Corollary. $\mathcal{R}(s)$ is equal to Drinfeld’s universal R -matrix.

In particular, Theorem 7.4 provides an independent, and constructive proof of the existence of Drinfeld’s universal R -matrix.

8. MEROMORPHIC TENSOR STRUCTURES

In this section, we reinterpret our results in the language of meromorphic tensor categories. We refer to [22, 23] for a more abstract and general treatment of meromorphic tensor categories. We caution the reader, however, that the framework developed in [22, 23] does not include examples where the tensor product depends non-trivially on a parameter, as is the case for the deformed Drinfeld tensor product. The setup of [22, 23] is also more general than needed for our purposes, in that it is adapted to *pseudo-tensor categories*, where the tensor product need not be defined for all pairs of representations, or be representable.

8.1. Drinfeld tensor product.

Proposition.

- (1) The category $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes_{\mathbb{D}, s})$ is a meromorphic (in fact, rational) tensor category over $(\mathbb{C}, +)$.
- (2) Each of the resummed abelian R -matrices $\mathcal{R}^{0, \uparrow/\downarrow}(s)$ is a meromorphic braiding on $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes_{\mathbb{D}, s})$.
- (3) $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes_{\mathbb{D}, s})$ does not admit a rational braiding.

PROOF. (1) $\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ admits an action of the additive group $(\mathbb{C}, +)$ given by $V \mapsto V(s)$. As recalled in 3.2, for every $V, W \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, there is a rational action of $Y_{\hbar}(\mathfrak{g})$ on $V \otimes W$ given by the deformed Drinfeld tensor product. The properties (1)–(5) of Theorem 3.2 mean exactly that $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes)_{\mathbb{D}, s}$ is a rational tensor category over $(\mathbb{C}, +)$.

(2) is the content of Theorem 6.2 (1)–(3).

(3) is a rephrasing of Theorem 6.4. \square

8.2. Standard tensor product.

Proposition.

- (1) The category $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes)_{\mathbb{D}, s}$ is a meromorphic (in fact, polynomial) tensor category over $(\mathbb{C}, +)$.
- (2) Each of the resummed R -matrices $\mathcal{R}^{\uparrow/\downarrow}(s)$ is a meromorphic braiding on $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes)_{\mathbb{D}, s}$.
- (3) $(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes)_{\mathbb{D}, s}$ does not admit a rational braiding.

PROOF. (1) This is a consequence of the fact that $V_1 \otimes_{\mathbb{D}, s} V_2$ arises from the algebra homomorphism $\Delta_s : Y_{\hbar}(\mathfrak{g}) \rightarrow (Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))[s]$. This tensor product satisfies the properties analogous to (3)–(5) of Theorem 3.2.

(2) is the content of (1)–(3) of Theorem 7.1.

(3) is a rephrasing of Theorem 7.3. \square

8.3. Meromorphic tensor structures.

Proposition. $\mathcal{R}^-(s)$ is a rational braided tensor structure on the identity functor

$$\left(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes, \mathcal{R}^{0, \uparrow/\downarrow}(s) \right) \longrightarrow \left(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes, \mathcal{R}^{\uparrow/\downarrow}(s) \right)$$

PROOF. By definition of a tensor structure on a functor, the statement means that, for every $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, there is a rational $\text{End}(V_1 \otimes_{\mathbb{D}, s} V_2)$ -valued function of s , $\mathcal{R}_{V_1, V_2}^-(s)$, which satisfies (1)–(3) of Theorem 4.1. Namely,

$$\mathcal{R}_{V_1, V_2}^-(s) : V_1 \otimes_{\mathbb{D}, s} V_2 \rightarrow V_1 \otimes_{\mathbb{D}, s} V_2$$

is a $Y_{\hbar}(\mathfrak{g})$ -intertwiner such that $\mathcal{R}_{V_1(a), V_2(b)}^-(s) = \mathcal{R}_{V_1, V_2}^-(s+a-b)$, and the following diagram commutes, for every $V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$

$$\begin{array}{ccc}
 (V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 & \xlongequal{\quad} & V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3) \\
 \downarrow \mathcal{R}_{V_1, V_2}^-(s_1) \otimes \mathbf{1}_{V_3} & & \downarrow \mathbf{1}_{V_1} \otimes \mathcal{R}_{V_2, V_3}^-(s_1) \\
 (V_1 \otimes_{\mathbb{D}, s_1} V_2) \otimes_{s_2} V_3 & & V_1 \otimes_{s_1+s_2} (V_2 \otimes_{\mathbb{D}, s_2} V_3) \\
 \downarrow \mathcal{R}_{V_1 \otimes_{\mathbb{D}, s_1} V_2, V_3}^-(s_2) & & \downarrow \mathcal{R}_{V_1, V_2 \otimes_{\mathbb{D}, s_2} V_3}^-(s_1+s_2) \\
 (V_1 \otimes_{\mathbb{D}, s_1} V_2) \otimes_{\mathbb{D}, s_2} V_3 & \xlongequal{\quad} & V_1 \otimes_{\mathbb{D}, s_1+s_2} (V_2 \otimes_{\mathbb{D}, s_2} V_3)
 \end{array}$$

Lastly, it is claimed in (3) that $\mathcal{R}^-(s)$ is compatible with the braidings on the two categories, that is satisfies

$$\begin{array}{ccc}
V_1(s) \otimes V_2 & \xrightarrow{(12) \circ \mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s)} & V_2 \otimes V_1(s) \\
\mathcal{R}_{V_1, V_2}^-(s) \downarrow & & \downarrow \mathcal{R}_{V_2, V_1}^-(s) \\
V_1(s) \otimes_{D,0} V_2 & \xrightarrow{(12) \circ \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)} & V_2 \otimes_{D,0} V_1(s)
\end{array}$$

The commutativity of the diagram follows from the fact that, by definition

$$\mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) = \mathcal{R}_{V_1, V_2}^+(s) \cdot \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdot \mathcal{R}_{V_1, V_2}^-(s)$$

where $\mathcal{R}_{V_1, V_2}^+(s) = (12) \circ \mathcal{R}_{V_2, V_1}^-(s)^{-1} \circ (12)$. \square

9. RELATION TO THE QUANTUM LOOP ALGEBRA $U_q(L\mathfrak{g})$

In this section, we review the construction of the tensor functor between finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$ and the quantum loop algebra $U_q(L\mathfrak{g})$ obtained in [11, 12]. We then disprove a conjecture stated in [12], and relate the meromorphic R -matrices of $Y_{\hbar}(\mathfrak{g})$ and $U_q(L\mathfrak{g})$.

9.1. The functor Γ [11]. Set $q = \exp(\pi i \hbar)$ and assume that $\hbar \in \mathbb{C} \setminus \mathbb{Q}$, so that q is not a root of unity. Let $U_q(L\mathfrak{g})$ be the quantum loop algebra of \mathfrak{g} . We refer to [2, Ch. 12] and references therein for the definition and basic properties of $U_q(L\mathfrak{g})$.

A finite-dimensional representation $V \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$ is said to be *non-congruent* if, for every $a, b \in \sigma(V)$, $a - b \notin \mathbb{Z}_{\neq 0}$. The full subcategory of such representations is denoted by $\text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g})) \subset \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$.

In [11, §5], an exact, essentially surjective and faithful functor

$$\Gamma : \text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g})) \rightarrow \text{Rep}_{\text{fd}}(U_q(L\mathfrak{g}))$$

is defined. Γ is such that $\Gamma(V) = V$ as vector spaces for any $V \in \text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g}))$, and restricts to an isomorphism of categories on an explicit subcategory of $\text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g}))$ determined by a choice of log.

Given $V \in \text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g}))$, and $i \in \mathbf{I}$, consider the additive difference equation

$$\phi_i(u+1) = \xi_i(u)\phi_i(u)$$

determined by the action of the commuting current $\xi_i(u)$ of $Y_{\hbar}(\mathfrak{g})$ on V . The action of the commuting current $\psi_i(z)$ of $U_q(L\mathfrak{g})$ on V is given by the monodromy of this equation, that is by

$$\psi_i(z) = \lim_{n \rightarrow \infty} \xi_i(u+n) \cdots \xi_i(u-n) \Big|_{z=e^{2\pi i u}}$$

The action of the raising and lowering operators of $U_q(L\mathfrak{g})$ on V requires the non-congruence hypothesis. It is not relevant for our current discussion, and we refer to [11, §5] for details.

9.2. Meromorphic tensor structure on Γ [12]. Let $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g}))$, and consider the abelian qKZ equation determined by $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$, that is the difference equation

$$\Phi(s+1) = \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdot \Phi(s) \quad (9.1)$$

Let $\mathcal{J}_{V_1, V_2}^{\uparrow/\downarrow}(s) : \mathbb{C} \rightarrow \text{End}_{\mathbb{C}}(V_1 \otimes V_2)$ be the left canonical solution of (9.1), which is uniquely determined by the requirement that it be holomorphic and invertible for $\text{Re}(s) \ll 0$, and possess an asymptotic expansion of the form $(-s)^{\hbar\Omega} (1 + O(s^{-1}))$ as $s \rightarrow \infty$ in any halfplane of the form $\text{Re}(s) < m$.

The deformed Drinfeld tensor product $\otimes_{D, \zeta}$ on finite-dimensional representations of $U_q(L\mathfrak{g})$ was introduced by D. Hernandez in [15], and further studied in [12].

Theorem. [12, Thm. 7.3] $\mathcal{J}^{\uparrow/\downarrow}(s)$ is a meromorphic tensor structure on the functor Γ , with respect to the deformed Drinfeld tensor products

$$\left(\Gamma, \mathcal{J}^{\uparrow/\downarrow}(s) \right) : \left(\text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g})), \otimes_{D, s} \right) \longrightarrow \left(\text{Rep}_{\text{fd}}(U_q(L\mathfrak{g})), \otimes_{D, \zeta} \right)$$

where $\zeta = \exp(2\pi i s)$.¹⁰

9.3. Tensor structure with respect to the standard coproducts. The Drinfeld coproduct of $U_q(L\mathfrak{g})$ is known to be conjugated to the standard coproduct by the lower triangular part $\mathcal{R}^-(\zeta)$ of the universal R -matrix of $U_q(L\mathfrak{g})$ (see, for example, [6]). Thus, for any $V_1, V_2 \in \text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g}))$, we have the following isomorphisms of $U_q(L\mathfrak{g})$ -modules

$$\begin{array}{ccc} \Gamma(V_1) \otimes_{\zeta} \Gamma(V_2) & \xrightarrow{J_{V_1, V_2}^{\uparrow/\downarrow}(s)} & \Gamma \left(V_1 \otimes_s V_2 \right) \\ \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^-(\zeta) \downarrow & & \downarrow \mathcal{R}_{V_1, V_2}^-(s) \\ \Gamma(V_1) \otimes_{D, \zeta} \Gamma(V_2) & \xrightarrow{J_{V_1, V_2}^{\uparrow/\downarrow}(s)} & \Gamma \left(V_1 \otimes_{D, s} V_2 \right) \end{array}$$

where $\mathcal{V}_1 \otimes_{\zeta} \mathcal{V}_2 = \tau_{\zeta}^* \mathcal{V}_1 \otimes \mathcal{V}_2$ for any $\mathcal{V}_1, \mathcal{V}_2 \in \text{Rep}_{\text{fd}}(U_q(L\mathfrak{g}))$, and $J_{V_1, V_2}^{\uparrow/\downarrow}(s)$ is defined as the composition

$$J_{V_1, V_2}^{\uparrow/\downarrow}(s) = \mathcal{R}_{V_1, V_2}^-(s)^{-1} \cdot \mathcal{J}_{V_1, V_2}^{\uparrow/\downarrow}(s) \cdot \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^-(\zeta) \quad (9.2)$$

Theorem 9.2 and Proposition 8.3 therefore imply the following

Corollary. $J_{V_1, V_2}^{\uparrow/\downarrow}(s)$ is a meromorphic tensor structure on the functor Γ , with respect to the standard tensor products

$$\left(\Gamma, J^{\uparrow/\downarrow}(s) \right) : \left(\text{Rep}_{\text{fd}}^{\text{NC}}(Y_{\hbar}(\mathfrak{g})), \otimes_s \right) \longrightarrow \left(\text{Rep}_{\text{fd}}(U_q(L\mathfrak{g})), \otimes_{\zeta} \right)$$

¹⁰In [12], the right canonical solution of the equations $\phi(s+1) = \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdot \phi(s)$ is shown to give rise to a tensor structure on Γ . A similar computation shows that the left solution yields a tensor structure on a variant of Γ , which we denote Γ for simplicity.

9.4. Non regularity of $J_{V_1, V_2}^{\uparrow/\downarrow}(s)$. Since the tensor products \otimes_s and \otimes_ζ are polynomial, the first two authors conjectured in [12, §2.13] that $J_{V_1, V_2}^{\uparrow/\downarrow}(s)$ is regular and invertible at $s = 0$. If so, $J_{V_1, V_2}^{\uparrow/\downarrow}(0)$ would give rise to a (non-meromorphic) tensor structure on the functor Γ with respect to the standard (unshifted) tensor products on $\text{Rep}_{\text{fd}}^{\text{nc}}(Y_{\hbar}(\mathfrak{g}))$ and $\text{Rep}_{\text{fd}}(U_q(L\mathfrak{g}))$. The following shows that this is not the case.

Proposition. *The meromorphic tensor structure $J^{\uparrow/\downarrow}(s)$ is either singular or not invertible at $s = 0$.*

PROOF. Since $\Gamma(V(a)) = \Gamma(V)(e^{2\pi i a})$ and each of the factors in the definition (9.2) of $J^{\uparrow/\downarrow}$ is compatible with shifts, we have

$$J_{V_1(a), V_2(b)}^{\uparrow/\downarrow}(s) = J_{V_1, V_2}^{\uparrow/\downarrow}(s + a - b)$$

Thus, if $J_{V_1, V_2}^{\uparrow/\downarrow}(s)$ were regular and invertible at $s = 0$ for every V_1, V_2 , then for a fixed V_1, V_2 it would be holomorphic and invertible for all s . This cannot be true, as the following argument shows.

Assume that V_1, V_2 are irreducible, with highest weight vectors v_1, v_2 of weights $\lambda_1, \lambda_2 \in \mathfrak{h}^*$, and that $(\lambda_1, \lambda_2) \neq 0$. Let $W \subset V_1 \otimes V_2$ be the subspace spanned by $v_1 \otimes v_2$. The triangularity of $\mathcal{R}^-(s)$ and $\mathcal{R}^-(\zeta)$ implies that

$$J_{V_1, V_2}^{\uparrow}(s)|_W = \mathcal{J}_{V_1, V_2}^{\uparrow}(s)|_W$$

If the restriction of $\mathcal{J}_{V_1, V_2}^{\uparrow}(s)$ to W were holomorphic and invertible for every $s \in \mathbb{C}$, the same would be true for $\mathcal{R}_{V_1, V_2}^{0, \uparrow}(s)$, since by 9.1

$$\mathcal{J}_{V_1, V_2}^{\uparrow}(s) = \mathcal{R}_{V_1, V_2}^{0, \uparrow}(s+1) \cdot \mathcal{J}_{V_1, V_2}^{\uparrow}(s+1).$$

In turn, $\mathcal{A}_{V_1, V_2}(s)|_W$ would also be holomorphic and invertible, since (see equation (6.2) above)

$$\mathcal{R}_{V_1, V_2}^{0, \uparrow}(s + \ell\hbar) = \mathcal{A}_{V_1, V_2}(s) \mathcal{R}_{V_1, V_2}^{0, \uparrow}(s).$$

Since $\mathcal{A}_{V_1, V_2}(s)$ is a rational function of s such that $\mathcal{A}_{V_1, V_2}(\infty) = \mathbf{1}$, this implies that $\mathcal{A}_{V_1, V_2}(s)|_W \equiv \mathbf{1}$. The expansion $\mathcal{A}(s) = 1 - \frac{\ell\hbar}{s^2} \Omega_{\mathfrak{h}} + O(s^{-3})$, then yields $\Omega_{\mathfrak{h}}|_W = 0$, which contradicts the fact that $\Omega_{\mathfrak{h}} v_1 \otimes v_2 = (\lambda_1, \lambda_2) v_1 \otimes v_2$. \square

Remark. The above result leaves open the question of whether there exists a tensor functor between the (non meromorphic) tensor categories

$$(\text{Rep}_{\text{fd}}(Y_{\hbar}(\mathfrak{g})), \otimes) \rightarrow (\text{Rep}_{\text{fd}}(U_q(L\mathfrak{g})), \otimes)$$

9.5. The meromorphic abelian R -matrix of $U_q(L\mathfrak{g})$. Assume now that $|q| \neq 1$. We review below the analogue of Theorem 6.2 for $U_q(L\mathfrak{g})$ obtained in [12], based on the results of [3].

Let $\mathcal{V}_1, \mathcal{V}_2$ be two finite-dimensional representations of $U_q(L\mathfrak{g})$. In [12, §8], a rational function $\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta) : \mathbb{P}^1 \rightarrow \text{End}_{\mathbb{C}}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ is defined via a contour integral formula involving the commuting generators of $U_q(L\mathfrak{g})$, which is analogous to the one given in 6.1. $\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta)$ is regular at $\zeta = 0, \infty$, and such that

$$\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(0) = \mathbf{1} = \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\infty)$$

Moreover, $[\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta), \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta')] = 0$ for any ζ, ζ' .

Consider the (regular) difference equation with step $q^{2\ell}$ determined by $\mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta)$

$$\overline{\mathcal{R}}(q^{2\ell}\zeta) = \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta) \cdot \overline{\mathcal{R}}(\zeta) \quad (9.3)$$

This equation admits two meromorphic solutions $\overline{\mathcal{R}}^\uparrow(\zeta), \overline{\mathcal{R}}^\downarrow(\zeta)$, which are uniquely determined by the requirement that $\overline{\mathcal{R}}^{\uparrow/\downarrow}(\zeta)$ be holomorphic near $z = q^{\pm\infty}$ and such that $\overline{\mathcal{R}}^{\uparrow/\downarrow}(q^{\pm\infty}) = 1$. These are explicitly given by

$$\overline{\mathcal{R}}^\uparrow(\zeta) = \prod_{n \geq 0}^{\rightarrow} \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(q^{2\ell n}\zeta)^{-1} \quad \text{and} \quad \overline{\mathcal{R}}^\downarrow(\zeta) = \prod_{n \geq 1}^{\leftarrow} \mathcal{A}_{\mathcal{V}_1, \mathcal{V}_2}(q^{-2\ell n}\zeta)$$

Now set

$$\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{0, \uparrow/\downarrow}(\zeta) = \begin{cases} q^{\mp\Omega_b} \cdot \overline{\mathcal{R}}^{\uparrow/\downarrow}(\zeta) & \text{if } |q| < 1 \\ q^{\pm\Omega_b} \cdot \overline{\mathcal{R}}^{\uparrow/\downarrow}(\zeta) & \text{if } |q| > 1 \end{cases}$$

Theorem. [12, §8] *The category $\left(\text{Rep}_{\text{fd}}(U_q(L\mathfrak{g})), \otimes_{\text{D}, \zeta}, \mathcal{R}^{0, \uparrow/\downarrow}(\zeta)\right)$ is a meromorphic braided tensor category.*

Remark. Let \mathcal{R}^0 be the abelian part of the universal R -matrix of $U_q(L\mathfrak{g})$, and set

$$\mathcal{R}^0(\zeta) = \tau_\zeta \otimes \mathbf{1}(\mathcal{R}^0) \in U_q(L\mathfrak{g})^{\otimes 2}[[\zeta]]$$

It is easy to see that $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^0(\zeta)$ satisfies the difference equation (9.3) [12, §8]. It follows by uniqueness that $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^0(\zeta)$ is the Taylor expansion at $\zeta = 0$ of $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{0, \uparrow}(\zeta)$ if $|q| < 1$, and of $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{0, \downarrow}(\zeta)$ if $|q| > 1$. Similarly, if

$$\mathcal{R}^{0, \vee}(\zeta) = \mathbf{1} \otimes \tau_\zeta(\mathcal{R}^0) \in U_q(L\mathfrak{g})^{\otimes 2}[[\zeta^{-1}]]$$

then $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{0, \vee}(\zeta)$ is the Taylor expansion at $\zeta = \infty$ of $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{0, \downarrow}(\zeta)$ if $|q| < 1$, and of $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{0, \uparrow}(\zeta)$ if $|q| > 1$.

9.6. Abelian q Drinfeld–Kohno theorem. Let now $V_1, V_2 \in \text{Rep}_{\text{fd}}(Y_h(\mathfrak{g}))$, and consider the abelian q KZ equations (9.1) determined by $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$.

Theorem. [12, Thm. 9.3] *The monodromy of the abelian q KZ equations (9.1) is equal to $\mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^{0, \uparrow/\downarrow}(\zeta)$. In other words, the following holds*

$$\mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^{0, \uparrow/\downarrow}(\zeta) = \lim_{n \rightarrow \infty} \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s+n) \cdots \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdots \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s-n) \Big|_{\zeta = \exp(2\pi i s)} \quad (9.4)$$

In [12, §9.6], this assertion is strengthened to include the abelian q KZ equations with values in $V_1 \otimes \cdots \otimes V_n$, where $V_i \in \text{Rep}_{\text{fd}}^{\text{NC}}(Y_h(\mathfrak{g}))$ and $n \geq 3$. Thus, Theorem 9.6 is an analogue of the Drinfeld–Kohno theorem for the abelian q KZ equations determined by $\mathcal{R}^{0, \uparrow/\downarrow}(s)$.

As is the case for the Drinfeld–Kohno theorem, Theorems 9.2 and 9.6 can be understood as defining a meromorphic braided tensor functor

$$\left(\text{Rep}_{\text{fd}}^{\text{NC}}(Y_h(\mathfrak{g})), \otimes_{\text{D}, s}, \mathcal{R}^{0, \uparrow/\downarrow}(s)\right) \longrightarrow \left(\text{Rep}_{\text{fd}}(U_q(L\mathfrak{g})), \otimes_{\text{D}, \zeta}, \mathcal{R}^{0, \uparrow/\downarrow}(\zeta)\right)$$

Unlike its non-meromorphic counterpart, this notion involves an ordered *pair* $(\mathcal{K}, \check{\mathcal{K}})$ of meromorphic tensor structures on the functor Γ , such that the following holds [12, Rem. 9.3]¹¹

$$\mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^{0, \uparrow/\downarrow}(\zeta) = \check{\mathcal{K}}_{V_2, V_1}(-s)_{21}^{-1} \cdot \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdot \mathcal{K}_{V_1, V_2}(s) \quad (9.5)$$

Comparing (9.5) with (9.4), and using the definition of \mathcal{J}_{V_1, V_2} given in 9.1, we see that one can take $\mathcal{K}_{V_1, V_2} = \mathcal{J}_{V_1, V_2}^{\uparrow/\downarrow}$, which is a regularisation of the product

$$\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s-1) \cdot \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s-2) \cdots$$

and $\check{\mathcal{K}}_{V_1, V_2} = \mathcal{J}_{V_1, V_2}^{\downarrow/\uparrow}$, which is a regularisation of

$$\mathcal{R}_{V_2, V_1}^{0, \downarrow/\uparrow}(s-1) \cdot \mathcal{R}_{V_1, V_2}^{0, \downarrow/\uparrow}(s-2) \cdots = \mathcal{R}_{V_2, V_1}^{0, \uparrow/\downarrow}(-s+1)_{21}^{-1} \cdot \mathcal{R}_{V_2, V_1}^{0, \uparrow/\downarrow}(-s+2)_{21}^{-1} \cdots$$

where we used the unitarity relation $\mathcal{R}_{V_1, V_2}^{0, \downarrow/\uparrow}(s) = \mathcal{R}_{V_2, V_1}^{0, \uparrow/\downarrow}(s)_{21}^{-1}$.

9.7. Meromorphic braided tensor equivalence for standard coproducts.

Let \mathcal{R} be the universal R -matrix of $U_q(L\mathfrak{g})$, $\mathcal{R} = \mathcal{R}^+ \cdot \mathcal{R}^0 \cdot \mathcal{R}^-$ its Gauss decomposition, set $\mathcal{R}(\zeta) = \tau_\zeta \otimes \mathbf{1}(\mathcal{R})$, and $\mathcal{R}^\pm(\zeta) = \tau_\zeta \otimes \mathbf{1}(\mathcal{R}^\pm)$. Then, if $\mathcal{V}_1, \mathcal{V}_2 \in \text{Rep}_{\text{fd}}(U_q(L\mathfrak{g}))$, $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^\pm(\zeta)$ are rational functions of ζ such that

$$\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^+(\zeta) = \mathcal{R}_{\mathcal{V}_2, \mathcal{V}_1}^-(\zeta^{-1})_{21}^{-1}$$

Define the meromorphic R -matrix of $U_q(L\mathfrak{g})$ by

$$\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{\uparrow/\downarrow}(\zeta) = \mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^+(\zeta) \cdot \mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{0, \uparrow/\downarrow}(\zeta) \cdot \mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^-(\zeta) \quad (9.6)$$

By Remark 9.5, $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}(\zeta)$ is the Taylor expansion at $\zeta = 0$ of $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{\uparrow/\downarrow}(\zeta)$ if $|q| < 1$, and of $\mathcal{R}_{\mathcal{V}_1, \mathcal{V}_2}^{\downarrow/\uparrow}(\zeta)$ if $|q| > 1$.

Proposition. *The pair $(J_{V_1, V_2}^{\uparrow/\downarrow}(s), J_{V_1, V_2}^{\downarrow/\uparrow}(s))$ is a meromorphic braided tensor structure on the functor Γ with respect to the standard tensor products and meromorphic R -matrices*

$$\left(\text{Rep}_{\text{fd}}^{\text{nc}}(Y_{\hbar}(\mathfrak{g})), \otimes_s, \mathcal{R}^{\uparrow/\downarrow}(s) \right) \longrightarrow \left(\text{Rep}_{\text{fd}}(U_q(L\mathfrak{g})), \otimes, \mathcal{R}^{\uparrow/\downarrow}(\zeta) \right)$$

PROOF. By Corollary 9.3, and 9.6, we only need to check the compatibility of $(J_{V_1, V_2}^{\uparrow/\downarrow}(s), J_{V_1, V_2}^{\downarrow/\uparrow}(s))$ with the meromorphic braidings, that is the relation

$$\mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^{\uparrow/\downarrow}(\zeta) = J_{V_2, V_1}^{\downarrow/\uparrow}(-s)_{21}^{-1} \cdot \mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) \cdot J_{V_1, V_2}^{\uparrow/\downarrow}(s) \quad (9.7)$$

The Gauss decomposition (9.6) yields

$$\begin{aligned} \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^{\uparrow/\downarrow}(\zeta) &= \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^+(\zeta) \cdot \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^{0, \uparrow/\downarrow}(\zeta) \cdot \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^-(\zeta) \\ &= \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^+(\zeta) \cdot \mathcal{J}_{V_2, V_1}^{\downarrow/\uparrow}(-s)_{21}^{-1} \cdot \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdot \mathcal{J}_{V_1, V_2}^{\uparrow/\downarrow}(s) \cdot \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^-(\zeta) \\ &= \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^+(\zeta) \cdot \mathcal{J}_{V_2, V_1}^{\downarrow/\uparrow}(-s)_{21}^{-1} \cdot \mathcal{R}_{V_1, V_2}^+(s)^{-1} \cdot \mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) \\ &\quad \cdot \mathcal{R}_{V_1, V_2}^-(s)^{-1} \cdot \mathcal{J}_{V_1, V_2}^{\uparrow/\downarrow}(s) \cdot \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^-(\zeta) \\ &= J_{V_2, V_1}^{\downarrow/\uparrow}(-s)_{21}^{-1} \cdot \mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) \cdot J_{V_1, V_2}^{\uparrow/\downarrow}(s) \end{aligned}$$

¹¹For the meromorphic braided tensor structures discussed in 8.3, $\check{\mathcal{K}} = \mathcal{K}$.

where the second equality follows from the twist relation (9.5), the third from the definition of $\mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s)$ given in 7.1, and the fourth from the definition (9.2) of $J_{V_1, V_2}^{\uparrow/\downarrow}(s)$, together with the unitarity relations $\mathcal{R}_{V_1, V_2}^+(z) = \mathcal{R}_{V_2, V_1}^-(z^{-1})_{21}^{-1}$ and $\mathcal{R}_{V_1, V_2}^+(s) = \mathcal{R}_{V_2, V_1}^-(s)_{21}^{-1}$. \square

Remark. Much like (9.4), the twist relation (9.7) can be regarded as a monodromy relation. Indeed, both

$$J_{V_1, V_2}^{\uparrow/\downarrow}(s) = \mathcal{R}_{V_1, V_2}^-(s)^{-1} \cdot \mathcal{J}_{V_1, V_2}^{\uparrow/\downarrow}(s) \cdot \mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^-(z)$$

and

$$\begin{aligned} \left(J_{V_2, V_1}^{\downarrow/\uparrow}(-s)_{21}^{-1} \cdot \mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) \right)^{-1} &= \mathcal{R}_{V_1, V_2}^-(s)^{-1} \cdot \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)^{-1} \cdot \mathcal{J}_{V_2, V_1}^{\downarrow/\uparrow}(-s)_{21} \\ &\quad \cdot \mathcal{R}_{\Gamma(V_2), \Gamma(V_1)}^-(z^{-1})_{21} \\ &= \mathcal{R}_{V_1, V_2}^-(s)^{-1} \cdot \mathcal{R}_{V_2, V_1}^{0, \downarrow/\uparrow}(-s)_{21} \cdot \mathcal{J}_{V_2, V_1}^{\downarrow/\uparrow}(-s)_{21} \\ &\quad \cdot \mathcal{R}_{\Gamma(V_2), \Gamma(V_1)}^-(z^{-1})_{21} \end{aligned}$$

are solutions of the difference equation

$$\Phi(s+1) = \left(\mathcal{R}^-(s+1)_{V_1, V_2}^{-1} \cdot \mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s) \cdot \mathcal{R}^-(s)_{V_1, V_2} \right) \cdot \Phi(s)$$

though, due to the presence of the factors $\mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^-(z)$ and $\mathcal{R}_{\Gamma(V_2), \Gamma(V_1)}^-(z^{-1})_{21}$, neither is a canonical left or right solution. Unlike (9.4), however, the twist relation (9.7) should not be considered as a difference version of the (non-abelian) Drinfeld–Kohno Theorem on $n = 2$ points for several reasons.

- (1) As just pointed out, the difference equations underlying (9.7) are not the q KZ equations determined by $\mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s)$, but (a gauge transformation of) the abelian q KZ equations determined by $\mathcal{R}_{V_1, V_2}^{0, \uparrow/\downarrow}(s)$.
- (2) The q KZ equations of Frenkel–Reshetikhin [9] include a *dynamical parameter* $\lambda \in \mathfrak{h}$, and are given by

$$\Phi(s+1) = e^\lambda \otimes \mathbf{1} \cdot \mathcal{R}_{V_1, V_2}^{\uparrow/\downarrow}(s) \cdot \Phi(s)$$

Since the form of the asymptotics of solutions of this equation depends on the regularity of e^λ , its monodromy is a meromorphic function of λ , which is conjectured to be equivalent to $\mathcal{R}_{\Gamma(V_1), \Gamma(V_2)}^\varepsilon(z)$, via a λ -dependent gauge transformation.

APPENDIX A. SEPARATION OF POINTS

A.1. Let \mathcal{V} be a collection of finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$ such that

- (1) $\mathbb{C} \in \mathcal{V}$, and $V_1 \otimes V_2 \in \mathcal{V}$ for all $V_1, V_2 \in \mathcal{V}$,
- (2) $\exists V \in \mathcal{V}$ such that $\text{Ker}(\pi_V|_{\mathfrak{g}}) = \{0\}$.

The goal of this section is to prove that the elements of \mathcal{V} separate points in $Y_{\hbar}(\mathfrak{g})$. More generally, the following holds.

Proposition. *Let $\mathcal{V}_1, \dots, \mathcal{V}_n$ be collections of finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$ satisfying the conditions (1) and (2) above. Then, the ideal $\mathcal{J}_n \subset Y_{\hbar}(\mathfrak{g})^{\otimes n}$ defined by*

$$\mathcal{J}_n = \bigcap_{V_i \in \mathcal{V}_i} \text{Ker}(\pi_{V_1} \otimes \cdots \otimes \pi_{V_n})$$

is the zero ideal.

Remark. The analogous statement for $U(\mathfrak{g}[z])$ fails. Indeed, take $n = 1$ and let $V_{\mathfrak{g}}$ be any faithful, finite-dimensional \mathfrak{g} -module. Set $V = \text{ev}^*(V_{\mathfrak{g}})$, where ev is the evaluation morphism

$$\text{ev} : \mathfrak{g}[z] \rightarrow \mathfrak{g}, \quad f(z) \rightarrow f(0).$$

Then, V is a $U(\mathfrak{g}[z])$ -module satisfying (2), and $\mathcal{V} = \{V^{\otimes n}\}_{n \in \mathbb{Z}_{\geq 0}}$ satisfies (1)–(2). However,

$$z\mathfrak{g}[z] \subset \bigcap_{k \geq 0} \text{Ker}(\pi_{V^{\otimes k}})$$

The proof of the proposition is given in Sections A.2–A.5. In Section A.2, we reduce the task to proving that $\mathcal{J}_1 = \{0\}$. The reduction step is elementary, but is included for the sake of completeness. That the ideal $\mathcal{J}_1 \subset Y_{\hbar}(\mathfrak{g})$ vanishes is an unpublished result of V. Drinfeld’s, whose proof in the \hbar -formal setting has been reproduced in [10, Prop. 8.8]. After recalling relevant background material on co-Poisson Hopf algebras and Lie bialgebras in Sections A.3 and A.4, we explain in Section A.5 how to modify the argument given in [10] to deduce that $\mathcal{J}_1 = \{0\}$.

A.2. Reduction step. Let \mathcal{H}_1 and \mathcal{H}_2 be associative algebras over \mathbb{C} . Assume in addition that, for each $i \in \{1, 2\}$, \mathcal{H}_i is equipped with a family of representations \mathcal{V}_i such that

$$\mathcal{J}_{\mathcal{V}_i} := \bigcap_{V \in \mathcal{V}_i} \text{Ker}(\pi_V) = \{0\}$$

The following general result, coupled with a simple induction on n , implies that Proposition A.1 holds, provided $\mathcal{J}_1 = \{0\}$.

Lemma. *We have*

$$\bigcap_{V_i \in \mathcal{V}_i} \text{Ker}(\pi_{V_1} \otimes \pi_{V_2}) = \{0\}$$

PROOF. The lemma follows easily from the the identity

$$\text{Ker}(\pi_{V_1} \otimes \pi_{V_2}) = (\mathbf{1} \otimes \pi_{V_2})^{-1}(\text{Ker}(\pi_{V_1}) \otimes \text{End}(V_2)),$$

the assumption $\mathcal{J}_{\mathcal{V}_i} = \{0\}$, and the fact that, for any vector spaces M, N and collection of subspaces $\{M_{\lambda}\}_{\lambda \in \Lambda} \subset M$, we have the following equality in $M \otimes N$:

$$\bigcap_{\lambda \in \Lambda} (M_{\lambda} \otimes N) = \left(\bigcap_{\lambda \in \Lambda} M_{\lambda} \right) \otimes N$$

□

A.3. We now pause to collect pertinent facts from the theories of co-Poisson Hopf algebras and Lie bialgebras. Fix a Lie algebra \mathfrak{a} over \mathbb{C} , and recall that a co-Poisson algebra structure on $U(\mathfrak{a})$ is given by the additional data of a Poisson cobracket δ , i.e. a linear map

$$\delta : U(\mathfrak{a}) \rightarrow U(\mathfrak{a}) \wedge U(\mathfrak{a}) \subset U(\mathfrak{a})^{\otimes 2}$$

satisfying the co-Jacobi and co-Leibniz identities:

$$\begin{aligned} (\mathbf{1} + (123) + (132)) \circ \delta \otimes \mathbf{1} \circ \delta &= 0 \\ \mathbf{1} \otimes \Delta \circ \delta &= \delta \otimes \mathbf{1} \circ \Delta + (12) \circ \mathbf{1} \otimes \delta \circ \Delta \end{aligned}$$

If in addition δ satisfies the compatibility condition

$$\delta(xy) = \delta(x)\Delta(y) + \Delta(x)\delta(y)$$

then $(U(\mathfrak{a}), \delta)$ is said to be a co-Poisson Hopf algebra. In this case, the cobracket δ is uniquely determined by its restriction $\delta|_{\mathfrak{a}}$ to \mathfrak{a} , which can be shown to define a Lie bialgebra structure on \mathfrak{a} . Conversely, every Lie bialgebra cocommutator on \mathfrak{a} uniquely extends to a co-Poisson Hopf cobracket on $U(\mathfrak{a})$: see [2, Prop. 6.2.3].

A perhaps less well-known correspondence, which we shall exploit, takes place at the level of ideals. Recall that a co-Poisson bialgebra ideal of $(U(\mathfrak{a}), \delta)$ is a two-sided ideal J of $U(\mathfrak{a})$ satisfying the coideal and co-Poisson conditions

$$\begin{aligned} \Delta(J) &\subset J \otimes U(\mathfrak{a}) + U(\mathfrak{a}) \otimes J, \quad \epsilon(J) = 0 \\ \delta(J) &\subset J \otimes U(\mathfrak{a}) + U(\mathfrak{a}) \otimes J \end{aligned}$$

where ϵ is the counit for $U(\mathfrak{a})$. Similarly, a Lie bialgebra ideal of $(\mathfrak{a}, \delta|_{\mathfrak{a}})$ is a Lie algebra ideal \mathfrak{l} of \mathfrak{a} satisfying

$$\delta(\mathfrak{l}) \subset \mathfrak{l} \otimes \mathfrak{a} + \mathfrak{a} \otimes \mathfrak{l}.$$

Lemma. *Fix a co-Poisson Hopf algebra structure on $U(\mathfrak{a})$. Then the assignment*

$$J \subset U(\mathfrak{a}) \mapsto J \cap \mathfrak{a} \subset \mathfrak{a} \tag{A.1}$$

determines a bijective correspondence between co-Poisson bialgebra ideals J of $U(\mathfrak{a})$ and Lie bialgebra ideals $\mathfrak{l} \subset \mathfrak{a}$, with inverse

$$\mathfrak{l} \subset \mathfrak{a} \mapsto \langle \mathfrak{l} \rangle \subset U(\mathfrak{a}) \tag{A.2}$$

where $\langle \mathfrak{l} \rangle$ is the two-sided ideal generated by \mathfrak{l} .

When the underlying cobracket is taken to be trivial (that is, $\delta \equiv 0$), this reduces to the more familiar assertion (see [21, Prop. 4.8], for example) that (A.1) determines a bijective correspondence between bialgebra ideals of $U(\mathfrak{a})$ and Lie algebra ideals in \mathfrak{a} , with inverse (A.2). The lemma follows readily from this special case by a straightforward verification that (A.1) and (A.2) preserve any additional co-Poisson structure.

A.4. We now narrow our focus to $\mathfrak{a} = \mathfrak{g}[z]$. Let us begin by recalling how one passes from $Y_{\hbar}(\mathfrak{g})$ to the standard Lie bialgebra structure on $\mathfrak{g}[z]$, given by (A.4) below. Since $Y_{\hbar}(\mathfrak{g})$ is a filtered Hopf deformation of the cocommutative Hopf algebra $U(\mathfrak{g}[z])$ (see Section 2.6), the linear map

$$\Delta_{\text{Alt}} := \Delta - \Delta^{\text{op}} : Y_{\hbar}(\mathfrak{g}) \rightarrow Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})$$

is a filtered linear map of degree -1 . That is, it satisfies

$$\Delta_{\text{Alt}}(\mathcal{F}_k(Y_{\hbar}(\mathfrak{g}))) \subset \mathcal{F}_{k-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}) \quad \forall k \in \mathbb{Z}_{\geq 0}$$

where $\mathcal{F}_{-n}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}) = \{0\}$ for $n > 0$. We may therefore view it as a filtered map

$$(Y_{\hbar}(\mathfrak{g}), \mathcal{F}_{\bullet}(Y_{\hbar}(\mathfrak{g}))) \rightarrow (Y_{\hbar}(\mathfrak{g})^{\otimes 2}, \mathcal{F}_{\bullet}(Y_{\hbar}(\mathfrak{g})^{\otimes 2}))$$

where $\mathcal{F}_{\bullet}(Y_{\hbar}(\mathfrak{g})^{\otimes 2})$ is the vector space filtration on $Y_{\hbar}(\mathfrak{g})^{\otimes 2}$ defined by $\mathcal{F}_k(Y_{\hbar}(\mathfrak{g})^{\otimes 2}) := \mathcal{F}_{k-1}(Y_{\hbar}(\mathfrak{g})^{\otimes 2})$ for all $k \in \mathbb{Z}_{\geq 0}$. By definition, the semiclassical limit of Δ is the associated graded map

$$\delta := \text{gr}\left(\frac{\Delta_{\text{Alt}}}{\hbar}\right) : U(\mathfrak{g}[z]) \rightarrow \text{gr}_{\mathbb{F}}(Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})) \cong U(\mathfrak{g}[z]) \otimes U(\mathfrak{g}[w]) \quad (\text{A.3})$$

It is a Poisson cobracket which endows $U(\mathfrak{g}[z])$ with the structure of a co-Poisson Hopf algebra. Moreover, this co-Poisson structure induces the standard Lie bialgebra structure on $\mathfrak{g}[z]$, with cocommutator

$$\delta|_{\mathfrak{g}[z]} : \mathfrak{g}[z] \rightarrow \mathfrak{g}[z] \otimes \mathfrak{g}[w] \cong (\mathfrak{g} \otimes \mathfrak{g})[z, w]$$

given on $f(z) \in \mathfrak{g}[z]$ by the formula

$$\delta(f(z)) = \left[f(z) \otimes 1 + 1 \otimes f(w), \frac{\Omega_{\mathfrak{g}}}{z-w} \right] \in (\mathfrak{g} \otimes \mathfrak{g})[z, w] \quad (\text{A.4})$$

We may summarize the above discussion concisely by saying that $Y_{\hbar}(\mathfrak{g})$ is a *filtered quantization* of the Lie bialgebra $\mathfrak{g}[z]$, equipped with the above cocommutator.

The last ingredient we shall need is the following lemma, which is immediately obtained from Corollary 8.9 of [10] with the help of Lemma A.3.

Lemma. *Let $\epsilon_U : U(\mathfrak{g}[z]) \rightarrow \mathbb{C}$ be the counit.*

- (1) *If $\mathfrak{l} \subset \mathfrak{g}[z]$ is a Lie bialgebra ideal, then $\mathfrak{l} = \{0\}$ or $\mathfrak{l} = \mathfrak{g}[z]$.*
- (2) *If $\mathfrak{J} \subset U(\mathfrak{g}[z])$ is a co-Poisson bialgebra ideal, then $\mathfrak{J} = \{0\}$ or $\mathfrak{J} = \text{Ker}(\epsilon_U)$.*

A.5. Proof that \mathcal{J} vanishes. Consider now the filtration $\mathcal{F}_{\bullet}(\mathcal{J})$ on \mathcal{J} given by $\mathcal{F}_k(\mathcal{J}) = \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})) \cap \mathcal{J}$ for all $k \in \mathbb{Z}_{\geq 0}$, and the associated graded ideal

$$\text{gr}(\mathcal{J}) = \bigoplus_{k \geq 0} \mathcal{F}_k(\mathcal{J})/\mathcal{F}_{k-1}(\mathcal{J}) \subset \text{gr}(Y_{\hbar}(\mathfrak{g})) = U(\mathfrak{g}[z])$$

If $x \in \mathcal{J}$ is nonzero and $k \in \mathbb{Z}_{\geq 0}$ is minimal such that $x \in \mathcal{F}_k(\mathcal{J})$, then the image of x in $\text{gr}_k(\mathcal{J})$ is a nonzero element. Hence, our task is reduced to proving that $\text{gr}(\mathcal{J}) = \{0\}$.

Next, note that the condition (1) imposed on \mathcal{V} guarantees that \mathcal{J} is a bialgebra ideal in $Y_{\hbar}(\mathfrak{g})$. As Δ and ϵ are filtered, it follows that $\text{gr}(\mathcal{J})$ is necessarily a co-Poisson bialgebra ideal of $U(\mathfrak{g}[z])$. Using Lemma A.4, we deduce that $\text{gr}(\mathcal{J}) = \{0\}$ or $\text{gr}(\mathcal{J}) = \text{Ker}(\epsilon_U)$. If $\text{gr}(\mathcal{J}) = \text{Ker}(\epsilon_U)$, then $\mathfrak{g} \subset \text{gr}(\mathcal{J})$, and thus

$$\mathfrak{g} \subset \text{gr}_0(\mathcal{J}) = \mathcal{F}_0(\mathcal{J}) \subset \mathcal{J}$$

This contradicts the assumption (2) on \mathcal{V} , which guarantees that \mathcal{J} intersects \mathfrak{g} trivially. Hence, we may conclude that $\text{gr}(\mathcal{J})$, and thus \mathcal{J} itself, vanishes.

APPENDIX B. UNIQUENESS OF THE UNIVERSAL R -MATRIX

The aim of this section is to give a proof of the uniqueness part of Drinfeld's theorem (Theorem 1.1). Namely, we assume that we are given two formal series

$$\mathcal{R}^{(1)}(s), \mathcal{R}^{(2)}(s) \in 1 + s^{-1}(Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))[[s^{-1}]]$$

satisfying the hypotheses of Theorem 1.1. That is, for $i = 1, 2$,

$$\begin{aligned}\Delta \otimes \mathbf{1}(\mathcal{R}^{(i)}(s)) &= \mathcal{R}_{13}^{(i)}(s)\mathcal{R}_{23}^{(i)}(s) \\ \mathbf{1} \otimes \Delta(\mathcal{R}^{(i)}(s)) &= \mathcal{R}_{13}^{(i)}(s)\mathcal{R}_{12}^{(i)}(s)\end{aligned}$$

and, for every $a \in Y_{\hbar}(\mathfrak{g})$:

$$\tau_s \otimes \mathbf{1} \circ \Delta^{\text{op}}(a) = \mathcal{R}^{(i)}(s) \cdot \tau_s \otimes \mathbf{1} \circ \Delta(a) \cdot \mathcal{R}^{(i)}(s)^{-1}.$$

Our main tool will be the following lemma.

B.1.

Lemma. *The Lie algebra of primitive elements*

$$\text{Prim}^{\Delta}(Y_{\hbar}(\mathfrak{g})) = \{y \in Y_{\hbar}(\mathfrak{g}) : \Delta(y) = y \otimes 1 + 1 \otimes y\}$$

is equal to \mathfrak{g} .

PROOF. We shall again exploit the fact, recalled in Section A.4, that the Yangian $Y_{\hbar}(\mathfrak{g})$ provides a filtered quantization of the Lie bialgebra structure $(\mathfrak{g}[z], \delta)$ on $\mathfrak{g}[z]$ given by (A.4). By (A.3), this means that, for each $x \in \mathfrak{g}$ and $k \geq 0$, we have

$$\hbar \cdot \delta(x.z^k) = \Delta(y) - \Delta^{\text{op}}(y) \pmod{\mathcal{F}_{k-2}(Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g}))} \quad (\text{B.1})$$

for any $y \in \mathcal{F}_k(Y_{\hbar}(\mathfrak{g}))$ whose image $\bar{y} \in \text{gr}_k(Y_{\hbar}(\mathfrak{g})) \subset U(\mathfrak{g}[z])$ coincides with $x.z^k$.

Now let $y \in Y_{\hbar}(\mathfrak{g})$ be an arbitrary nonzero primitive element. Assume that $k \geq 0$ is such that $y \in \mathcal{F}_k(Y_{\hbar}(\mathfrak{g})) \setminus \mathcal{F}_{k-1}(Y_{\hbar}(\mathfrak{g}))$. As Δ is filtered with $\text{gr}(\Delta)$ recovering the standard coproduct on $U(\mathfrak{g}[z])$ (see Section 2.6), we can conclude that the image \bar{y} of y in $\text{gr}_k(Y_{\hbar}(\mathfrak{g})) \subset U(\mathfrak{g}[z])$ is a nonzero, primitive degree k element, and thus of the form $\bar{y} = x.z^k$ for some $x \in \mathfrak{g}$.

Using that $\Delta(y) = \Delta^{\text{op}}(y)$, we deduce from (B.1) that $\delta(x.z^k) = 0$. On the other hand, by definition of δ , we have

$$0 = \delta(x.z^k) = \frac{z^k - w^k}{z - w} [x \otimes 1, \Omega_{\mathfrak{g}}]$$

Hence, $k = 0$ and $y \in \mathfrak{g} \subset Y_{\hbar}(\mathfrak{g})$. \square

B.2. Now let $n \geq 1$ and $X \in Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})$ be such that $\mathcal{R}^{(1)}(s) - \mathcal{R}^{(2)}(s) = s^{-n}X + O(s^{-n-1})$. We will prove that $X = 0$.

Comparing the coefficients of s^{-n} on both sides of the cabling identities, we obtain the following

$$\begin{aligned}\Delta \otimes \mathbf{1}(X) &= X_{13} + X_{23} \\ \mathbf{1} \otimes \Delta(X) &= X_{13} + X_{12}\end{aligned}$$

In other words, $X \in \text{Prim}^{\Delta}(Y_{\hbar}(\mathfrak{g}))^{\otimes 2}$, that is, $X \in \mathfrak{g} \otimes \mathfrak{g} \subset Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})$, by the lemma above.

By the intertwining equation for $a \in \mathfrak{g}$, we conclude that $X \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$. Hence X is a scalar multiple of the Casimir tensor: $X = c\Omega_{\mathfrak{g}}$, for some $c \in \mathbb{C}$.

Let us now consider the intertwining equation for $a = T(h)$

$$\text{ad}(T(h) \otimes 1 + 1 \otimes T(h) + sh \otimes 1) \cdot \mathcal{R}^{(i)}(s) = \hbar r^{21}(h)\mathcal{R}^{(i)}(s) + \hbar \mathcal{R}^{(i)}r(h)$$

Take the difference of the two equations, for $i = 1, 2$, and compare the coefficient of s^{-n+1} , to get $c[h \otimes 1, \Omega_{\mathfrak{g}}] = 0$, for every $h \in \mathfrak{h}$. But that means $c = 0$ and hence $X = 0$, which is what we wanted to show.

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