THE R-MATRIX OF THE AFFINE YANGIAN

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Abstract. Let $\mathfrak{g}$ be an affine Lie algebra with associated Yangian $Y_\hbar(\mathfrak{g})$. We prove the existence of two meromorphic R–matrices associated to any pair of representations of $Y_\hbar(\mathfrak{g})$ in the category $\mathcal{O}$. They are related by a unitary constraint and constructed as products of the form $\mathcal{R}^{\uparrow/\downarrow}(s) = \mathcal{R}^+(s)\cdot\mathcal{R}^{0,\uparrow/\downarrow}(s)\cdot\mathcal{R}^-(s)$, where $\mathcal{R}^+(s) = \mathcal{R}^-_{21}(-s)^{-1}$. The factors $\mathcal{R}^{0,\uparrow/\downarrow}(s)$ are meromorphic, abelian R–matrices, with a WKB–type singularity in $\hbar$, and $\mathcal{R}^-(s)$ is a rational twist. Our proof relies on two novel ingredients. The first is an irregular, abelian, additive difference equation whose difference operator is given in terms of the q–Cartan matrix of $\mathfrak{g}$. The regularisation of this difference equation gives rise to $\mathcal{R}^{0,\uparrow/\downarrow}(s)$ as the exponentials of the two canonical fundamental solutions. The second key ingredient is a higher order analogue of the adjoint action of the affine Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ on $Y_\hbar(\mathfrak{g})$. This action has no classical counterpart, and produces a system of linear equations from which $\mathcal{R}^-(s)$ is recovered as the unique solution. Moreover, we show that both $\mathcal{R}^{\uparrow/\downarrow}(s)$ give rise to the same rational R–matrix on the tensor product of any two highest–weight representations.

Contents

1. Introduction

1.1. Let $\mathfrak{g}$ be an affine Kac-Moody algebra. In contrast to its counterpart for a finite–dimensional, semisimple Lie algebra, the affine Yangian $Y_\hbar(\mathfrak{g})$ is not known to have a universal R–matrix. In particular, it is not known whether an arbitrary representation V of $Y_\hbar(\mathfrak{g})$ gives rise to a solution of the quantum Yang-Baxter equation (QYBE) with additive spectral parameter.

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A notable exception arises when $\mathfrak{g}$ is simply–laced, and V is the equivariant cohomology of a quiver variety for the underlying affine Dynkin diagram. In this setting, a rational solution of the (QYBE) corresponding to V has been constructed by Maulik and Okounkov in [MO19] using stable envelopes.

The main goal of this paper is to construct solutions $\mathcal{R}(s)$ of the (QYBE) for an arbitrary affine Yangian $Y_\hbar(\mathfrak{g})$, with the exception of those of types $\mathsf{A}_1^{(1)}$ and $\mathsf{A}_2^{(2)}$, and a representation V whose restriction to $\mathfrak{g}$ lies in category $\mathcal{O}$. Our solutions are *meromorphic* with respect to the spectral parameter s, natural with respect to the underlying representations, and compatible with the tensor product.

We also show that, on representations generated by a highest–weight vector, they can be normalized so as to be rational in s, and conjecture that they then coincide with those constructed in [MO19] when $\mathfrak{g}$ is simply-laced and V arises from geometry.

1.2. Our approach builds upon the recent works of the last two authors with Valerio Toledano Laredo [GTL17, GTLW21], where meromorphic braidings were constructed on the category of finite–dimensional representations of the Yangian of any semisimple Lie algebra over the complex numbers. It hinged on constructing (1) the negative part of $\mathcal{R}(s)$ as an intertwiner between the standard and Drinfeld coproduct [GTLW21] and (2) the abelian part of $\mathcal{R}(s)$ via methods of resummation [GTL17].

One striking difference that arises while passing from the finite to affine setting is that the resulting R–matrices are no longer regular at $\hbar = 0$, and in fact possess an essential singularity there. Remarkably, this singularity has the same form as a WKB expansion [FF65] (see also [BO99, Ch. 10]). Indeed, we show that, as a function of $\hbar$, $\mathcal{R}(s)$ has the form

$$\mathcal{R}(s) = \exp\left(\frac{1}{\hbar}\mathsf{g}_{\mathrm{sing}} s^{-1}\right)(1 + O(\hbar)), \tag{1.1}$$

where $\mathsf{g}_{\mathrm{sing}}$ is a 2–tensor which we determine explicitly (see Theorem 7.1 (7) and the subsequent remark). This equation strongly hints at yet another approach to solving (QYBE) by varying $\hbar$ in an annular neighbourhood of 0. We believe that this observation is more than a mere analogy and will return to it in a future work.

An *a priori* physical justification for the appearance of the singular term in (1.1) may arise from Costello–Witten–Yamazaki's 4–dimensional gauge theory [CWY18a, CWY18b]. Specifically, and up to central elements, as a renormalization term coming from an affine extension of the theory.

We now provide an overview of the previously known results that motivated the present paper.

1.3. Recall that the quantum Yang–Baxter equation is the following equation for an $\mathrm{End}(V^{\otimes 2})$–valued function (or formal power series) $\mathcal{R}(s)$:

$$\mathcal{R}_{12}(s_1)\mathcal{R}_{13}(s_1+s_2)\mathcal{R}_{23}(s_2) = \mathcal{R}_{23}(s_2)\mathcal{R}_{13}(s_1+s_2)\mathcal{R}_{12}(s_1). \tag{QYBE}$$

This equation is $\mathrm{End}(V^{\otimes 3})$–valued, and the right–hand subscripts indicate on which tensor factors $\mathcal{R}$ acts.

Drinfeld's theory of Yangians [Dri85] provides a uniform method for constructing rational solutions of this equation. The Yangian $Y_\hbar(\mathfrak{a})$ of a finite-dimensional simple Lie algebra $\mathfrak{a}$ was introduced in [Dri85]. It is the canonical Hopf algebra deformation of the current algebra $U(\mathfrak{a}[z])$, and gives rise to rational solutions of (QYBE) on its irreducible, finite–dimensional representations.

To state Drinfeld's results more precisely, let $\Delta : Y_\hbar(\mathfrak{a}) \to Y_\hbar(\mathfrak{a})^{\otimes 2}$ be the coproduct and $\tau_s \in \mathrm{Aut}(Y_\hbar(\mathfrak{a}))$ the one–parameter group of Hopf algebra automorphisms, quantizing the shift automorphisms $z \mapsto z + s$ of $U(\mathfrak{a}[z])$. We abbreviate $\Delta_s = \tau_s \otimes \mathrm{Id} \circ \Delta$ and $\Delta_s^{\mathrm{op}} = \tau_s \otimes \mathrm{Id} \circ \Delta^{\mathrm{op}}$. Then, by [Dri85, Thms. 3], there exists a unique formal series

$$\mathcal{R}^{\mathrm{D}}(s) \in 1^{\otimes 2} + s^{-1} Y_\hbar(\mathfrak{a})^{\otimes 2}[\![s^{-1}]\!]$$

satisfying the intertwining equation

$$\Delta_s^{\mathrm{op}}(x) = \mathcal{R}^{\mathrm{D}}(s)\Delta_s(x)\mathcal{R}^{\mathrm{D}}(s)^{-1} \quad \forall \quad x \in Y_\hbar(\mathfrak{a}),$$

in addition to the cabling identities

$$\begin{aligned} \Delta \otimes \mathrm{Id}(\mathcal{R}^{\mathrm{D}}(s)) &= \mathcal{R}^{\mathrm{D}}_{13}(s)\mathcal{R}^{\mathrm{D}}_{23}(s), \\ \mathrm{Id} \otimes \Delta(\mathcal{R}^{\mathrm{D}}(s)) &= \mathcal{R}^{\mathrm{D}}_{13}(s)\mathcal{R}^{\mathrm{D}}_{12}(s). \end{aligned}$$

These readily imply that $\mathcal{R}^{\mathrm{D}}(s)$ is a solution to (QYBE) on $Y_\hbar(\mathfrak{a})^{\otimes 3}$.

Let V_1, V_2 be two finite–dimensional irreducible representations of $Y_\hbar(\mathfrak{a})$, and $\mathcal{R}^{\mathrm{D}}_{V_1,V_2}(s)$ the evaluation of $\mathcal{R}^{\mathrm{D}}(s)$ on $V_1 \otimes V_2$. Drinfeld also showed that, upon normalizing by its eigenvalue on the tensor product of highest–weight spaces, its dependence on s becomes rational [Dri85, Thm. 4]. Hence, in particular, one obtains rational solutions to (QYBE) valued in any finite–dimensional, irreducible representation V of $Y_\hbar(\mathfrak{a})$.

1.4. In [Dri88], a different presentation of $Y_\hbar(\mathfrak{a})$ is given called Drinfeld's new, or loop, presentation. It takes as an input only the Cartan matrix of $\mathfrak{a}$, and can be used to define the Yangian $Y_\hbar(\mathfrak{g}_{\mathsf{KM}})$ of any symmetrizable Kac–Moody algebra $\mathfrak{g}_{\mathsf{KM}}$, only as an algebra. In light of Drinfeld's foundational results, it is natural to ask whether (QYBE) can be solved for a suitable category of representations of $Y_\hbar(\mathfrak{g}_{\mathsf{KM}})$, in a functorial manner. However, the following two obstacles present themselves almost immediately.

(O1) The universal R–matrix $\mathcal{R}^{\mathrm{D}}(s)$ is only known to exist in finite types. Moreover, our results suggest, *a posteriori*, that there is no such object in the affine setting, without modifying the definition of the affine Yangian, cf. (1.1).

(O2) The coproduct Δ has only been defined in the case where $\mathfrak{g}_{\mathsf{KM}}$ is of finite or affine type [GNW18].[1]

[1]More precisely, the results from [GNW18] require to exclude types $\mathsf{A}_1^{(1)}$ and $\mathsf{A}_2^{(2)}$. The latter case, however, has been considered separately in [Ued20].

1.5. We answer in the affirmative the question raised in the previous paragraph for the Yangian $Y_\hbar(\mathfrak{g})$ associated to any affine Lie algebra[2] which is not of type $\mathsf{A}_1^{(1)}$ or $\mathsf{A}_2^{(2)}$. We consider the category $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, consisting of the representations of $Y_\hbar(\mathfrak{g})$ whose restriction to $\mathfrak{g}$ lies in category $\mathcal{O}$. We prove that there are two meromorphic braidings on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, related by a unitarity relation (see Theorem 2.2 (5) below). These give rise to rational solutions of (QYBE) on tensor products of highest–weight representations.

The construction of the corresponding meromorphic R–matrices relies on a technique which we refer to as the *abelianization method* (see Section 2.4 for a precise description). Roughly, this amounts to building $\mathcal{R}(s)$ from the three component of its Gaussian decomposition:

$$\mathcal{R}(s) = \mathcal{R}^+(s) \cdot \mathcal{R}^0(s) \cdot \mathcal{R}^-(s)\ . \tag{1.2}$$

Note that, in the case of $Y_\hbar(\mathfrak{a})$, such a factorization first appeared, as a conjecture, in the work [KT96] of Khoroshkin and Tolstoy. We refer the reader to the works of the third author [Wen22a, Wen22b] for the precise statement and proof of this conjecture, and its implications for $\mathcal{R}^{\mathrm{D}}(s)$.

Since in (1.2) we further have $\mathcal{R}^+(s) = \mathcal{R}^-_{21}(s)^{-1}$, this equation can be viewed as an abelianization of $\mathcal{R}(s)$ via a (rational) twist.

1.6. We conclude with a brief review of the vast literature on affine Yangians. With the exception of the aforementioned work of Maulik–Okounkov [MO19], there are no results predicting the existence of R–matrices associated to representations of affine Yangians in the generality considered in the present paper.

We first recall that, in the case of $\widehat{\mathfrak{sl}}_2$, the affine Yangian originally appeared in [BL94]. However, the definition provided in *loc. cit.* is now considered incomplete, see [Kod19, Rem. 5.2]. The currently accepted definition is given in [Tsy17a, Tsy17b], see also [BT19]. The available literature on affine Yangians has significantly expanded, in particular along the following two research directions. The first one aims to the construction of new representations in terms of Schur–Weyl type dualities, *e.g.,* [Gua07], vertex operators, *e.g.,* [GRW19, Kod19], or W–algebras, *e.g.,* [KU22, Ued22]. The second one, instead, stems from the theory of cohomological Hall algebras, *e.g.,* [SV17, SV20, YZ20] (see also [VV98]).

The theory of affine Yangians is closely related to quantum toroidal algebras [BT19, GTL13, GTL16, Tsy17a]. The latter have been studied extensively in the $\mathfrak{gl}_n$ case, see *e.g.,* [FJMM16, FJM23] and references therein, and in other types, see *e.g.,* [GN23, Neg23], in relation to shuffle algebras.

The category $\mathcal{O}$ for quantum toroidal algebras, and more generally for quantum affinization, was introduced, and its simple objects classified, by Hernandez [Her05a]. In particular, a formula for the abelian R–matrix $\mathfrak{R}^0(\zeta)$ is given in [Her05a, Sec. 5.4]. This is a purely formal object, analogous to the one obtained by Damiani [Dam98] in finite type, and it is a fundamental ingredient in the construction of q–characters. It would be interesting to understand its functional nature and compare it to the abelian R–matrix $\mathcal{R}^0(s)$, relying on the functor defined in [GTL16], in the same spirit as [GTL17, Thm. 9.6].

[2] We expect our results to readily extend to type $\mathsf{A}_2^{(2)}$.

2. Main results

This section contains the main theorems of this paper. We also explain the technique employed to prove these, *i.e.*, the abelianization method.

2.1. Let $\mathfrak{g}$ be a Kac–Moody algebra of affine type, with the exception of $\mathsf{A}_1^{(1)}$ and $\mathsf{A}_2^{(2)}$, $\hbar \in \mathbb{C}^\times$ and let $Y_\hbar(\mathfrak{g})$ be the Yangian associated to $\mathfrak{g}$ (see Section 3.2 for its definition). In [GNW18], Guay, Nakajima, and the third author define an algebra homomorphism, called the *twisted coproduct* $\Delta^z : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})^{\otimes 2}[z; z^{-1}]\!]$ (see Section 3.9).

Evaluating Δ^z at $z = 1$ leads to terms with infinitely many tensors, which must be carefully interpreted as defining a topological coproduct. Nevertheless, this yields a well–defined action of $Y_\hbar(\mathfrak{g})$ on any tensor product of modules in category $\mathcal{O}$, *i.e.*, $Y_\hbar(\mathfrak{g})$–modules whose restriction to $\mathfrak{g} \subset Y_\hbar(\mathfrak{g})$ lies in category $\mathcal{O}$ (see Section 4.1). We denote this category by $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, and the tensor product of $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ by $V_1 \underset{\mathrm{KM},0}{\otimes} V_2$.

Let $\mathbb{C} \ni s \mapsto \tau_s \in \mathrm{Aut}(Y_\hbar(\mathfrak{g}))$ be the one–parameter group of bialgebra automorphisms (see Section 3.6). For $V \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, set $V(s) = \tau_s^*(V)$, and for $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ define:

$$V_1 \underset{\mathrm{KM},s}{\otimes} V_2 = V_1(s) \underset{\mathrm{KM},0}{\otimes} V_2\,.$$

2.2. **Meromorphic R–matrices.** We are now in position to state the first main theorem of this paper.

Theorem. *Let V_1, V_2 be $Y_\hbar(\mathfrak{g})$–modules in category $\mathcal{O}$. Then, there are two meromorphic functions $\mathcal{R}^\eta_{V_1,V_2} \colon \mathbb{C} \to \mathrm{End}(V_1 \otimes V_2)$, $\eta \in \{\uparrow, \downarrow\}$, with the following properties.*

(1) *$\mathcal{R}^\eta_{V_1,V_2}(s)$ is holomorphic on a half–plane $\varepsilon(\eta)\,\mathrm{Re}(s/\hbar) \gg 0$ and approaches $\mathrm{Id}_{V_1 \otimes V_2}$ as $\varepsilon(\eta)\,\mathrm{Re}(s/\hbar) \to \infty$. Here, $\varepsilon(\uparrow) = +1$ and $\varepsilon(\downarrow) = -1$.*

(2) *The following is a $Y_\hbar(\mathfrak{g})$–intertwiner which is natural in V_1 and V_2:*
$$(1\,2) \circ \mathcal{R}^\eta_{V_1,V_2}(s) : \; V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \to (V_2 \underset{\mathrm{KM},-s}{\otimes} V_1)(s)$$

(3) *For $V_1, V_2, V_3 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, the following cabling identities hold.*
$$\mathcal{R}^\eta_{V_1 \underset{\mathrm{KM},s_1}{\otimes} V_2, V_3}(s_2) = \mathcal{R}^\eta_{V_1,V_3}(s_1 + s_2) \cdot \mathcal{R}^\eta_{V_2,V_3}(s_2)\,,$$
$$\mathcal{R}_{V_1, V_2 \underset{\mathrm{KM},s_2}{\otimes} V_3}(s_1 + s_2) = \mathcal{R}^\eta_{V_1,V_3}(s_1 + s_2) \cdot \mathcal{R}^\eta_{V_1,V_2}(s_1)\,.$$
In particular, the quantum Yang–Baxter equation holds on $V_1 \otimes V_2 \otimes V_3$:
$$\mathcal{R}^\eta_{V_1,V_2}(s_1) \cdot \mathcal{R}^\eta_{V_1,V_3}(s_1 + s_2) \cdot \mathcal{R}^\eta_{V_2,V_3}(s_2) = \\ = \mathcal{R}^\eta_{V_2,V_3}(s_2) \cdot \mathcal{R}^\eta_{V_1,V_3}(s_1 + s_2) \cdot \mathcal{R}^\eta_{V_1,V_2}(s_1)\,.$$

(4) *For any $a, b \in \mathbb{C}$ and $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, we have*
$$\mathcal{R}^\eta_{V_1(a),V_2(b)}(s) = \mathcal{R}^\eta_{V_1,V_2}(s + a - b)\,.$$

(5) *The following unitarity relation holds*

$$\mathcal{R}^{\uparrow}_{V_1,V_2}(s)^{-1} = (1\,2)\circ\mathcal{R}^{\downarrow}_{V_2,V_1}(-s)\circ(1\,2)\,.$$

(6) $\mathcal{R}^{\eta}_{V_1,V_2}(s)$ *have the same asymptotic expansion*[3] *as* $\mathrm{Re}(s/\hbar)\to\varepsilon(\eta)\infty$. *This expansion remains valid in a larger sector* Σ^{η}_{δ}, *for any* $\delta>0$, *where, if* $\theta=\arg(\hbar)$, *then*

$$\Sigma^{\uparrow}_{\delta} = \{re^{\iota\phi} : r\in\mathbb{R}_{>0}\ \text{ and } \phi\in(\theta-\pi+\delta,\theta+\pi-\delta)\} = -\Sigma^{\downarrow}_{\delta}\,.$$

(see Figures A.3 given in Appendix A).

Remarks.

(1) One can state this theorem more compactly as the existence of two meromorphic braidings, related by the unitary relation (5), on the meromorphic (in fact, polynomial) tensor category $\left(\mathcal{O}(Y_\hbar(\mathfrak{g})), \underset{\mathrm{KM},s}{\otimes}\right)$.

(2) The 1–jet of the asymptotic expansion of $\mathcal{R}^{\eta}(s)$ can be obtained by combining Theorems 6.1 (4) and 7.1 (7).

2.3. **Rational R–matrices.** Our results also imply the existence of a *rational R–matrix* on any tensor product of highest weight modules, denoted below by $\mathsf{R}(s)$. Namely, we prove the following:

Theorem. *Let* $V_1,V_2\in\mathcal{O}(Y_\hbar(\mathfrak{g}))$ *be two representations, generated by highest–weight vectors* $\mathsf{v}_1,\mathsf{v}_2$ *respectively. Then,* $\mathcal{R}^{\uparrow/\downarrow}_{V_1,V_2}(s)$ *yield, after normalizing to take value* 1 *on* $\mathsf{v}_1\otimes\mathsf{v}_2$, *the same operator* $\mathsf{R}_{V_1,V_2}(s)$, *whose matrix entries are rational in* s. *Moreover,* $\mathsf{R}_{V_1,V_2}(\infty)=\mathrm{Id}_{V_1\otimes V_2}$.

Remarks.

(1) The factorization of $\mathcal{R}^{\eta}(s)$ as a product of a scalar–valued meromorphic function and a matrix–valued rational function is known to be not natural with respect to morphisms in $\mathcal{O}(Y_\hbar(\mathfrak{g}))$. Indeed, it was shown in [GTLW21, Thms. 6.4, 7.3] that there is no rational braiding on the category of finite–dimensional representations of a finite–type Yangian.

(2) We also want to highlight the fact that the theorem above is more general than [Dri85, Thm. 4] for finite types. The latter assumes that the representations are finite–dimensional and irreducible, and its proof is presumably[4] based on a "generic irreducibility" argument. Our proof of Theorem 2.3 (see Section 7.2) uses explicit calculation of $\mathrm{Ad}(\mathcal{R}^0)(s)$ acting on operators from $Y_\hbar(\mathfrak{g})\otimes Y_\hbar(\mathfrak{g})$. It is valid for highest–weight representations and does not assume integrability or generic cyclicity.

[3] In this paper we use the classical notion of asymptotic expansions à la Poincaré. Namely, for an unbounded set $S\subset\mathbb{C}$, we say $f(z)\sim\sum_{r=0}^{\infty}f_r z^{-r}$, as $z\to\infty$ in S, if for every $n\geq 0$, we have

$$\lim_{\substack{z\to\infty\\ z\in S}} z^n\left(f(z)-\sum_{r=0}^{n}f_r z^{-r}\right)=0.$$

[4] The proof does not appear in print.

2.4. **Abelianization method.** The strategy employed in [GTLW21] is based on an interplay between the standard tensor product and the *Drinfeld tensor product*. The latter was introduced by the second author and Toledano Laredo in [GTL17] and defines a meromorphic tensor structure on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ which depends *rationally* on s. Namely, it gives rise to a family of actions of $Y_\hbar(\mathfrak{g})$ on $V_1 \otimes V_2$, which is a rational function in s, denoted by $V_1 \underset{\mathrm{D},s}{\otimes} V_2$. (see Theorem 4.4). The abelianization method decouples the construction of $\mathcal{R}(s)$ into two separate, independent problems. Assume the following datum is given:

(M1) two meromorphic braidings with respect to the Drinfeld tensor product on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, *i.e.*, a natural system of invertible intertwiners

$$(1\,2) \circ \mathcal{R}^{0,\uparrow/\downarrow}_{V_1,V_2}(s) : V_1 \underset{\mathrm{D},s}{\otimes} V_2 \to (V_2 \underset{\mathrm{D},-s}{\otimes} V_1)(s)\,,$$

for any $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, satisfying the analogue of Theorems 2.2 and 2.3 with respect to the Drinfeld tensor product;

(M2) a meromorphic twist intertwining the standard tensor product and the Drinfeld tensor product on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, *i.e.*, a natural system of invertible intertwiners

$$\mathcal{R}^{-}_{V_1,V_2}(s) : V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \to V_1 \underset{\mathrm{D},s}{\otimes} V_2\,,$$

for any $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, such that the cocycle equation

$$\mathcal{R}^{-}_{V_1 \underset{\mathrm{D},s_1}{\otimes} V_2, V_3}(s_2) \cdot \mathcal{R}^{-}_{V_1,V_2}(s_1) = \mathcal{R}^{-}_{V_1, V_2 \underset{\mathrm{D},s_2}{\otimes} V_3}(s_1+s_2) \cdot \mathcal{R}^{-}_{V_2,V_3}(s_2)$$

holds for any $V_1, V_2, V_3 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$. We require the matrix entries of $\mathcal{R}^-(s)$ to depend rationally on s, and $\mathcal{R}^-(\infty) = \mathrm{Id}$. These conditions ensure that the functional nature of $\mathcal{R}^0(s)$ does not change upon multiplication by $\mathcal{R}^-(s)$ on the right and by $\mathcal{R}^-_{21}(-s)^{-1}$ on the left.

Then, the isomorphism of $Y_\hbar(\mathfrak{g})$–modules

$$(1\,2) \circ \mathcal{R}^{\uparrow/\downarrow}_{V_1,V_2}(s) : V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \to (V_2 \underset{\mathrm{KM},-s}{\otimes} V_1)(s)\,,$$

defined by the commutativity of the following diagram:

$$\begin{tikzcd}
V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \arrow[r, "(1\,2)\circ\mathcal{R}^{\uparrow/\downarrow}_{V_1,V_2}(s)"] \arrow[d, "\mathcal{R}^{-}_{V_1,V_2}(s)"'] & (V_2 \underset{\mathrm{KM},-s}{\otimes} V_1)(s) \arrow[d, "\mathcal{R}^{-}_{V_2,V_1}(-s)"] \\
V_1 \underset{\mathrm{D},s}{\otimes} V_2 \arrow[r, "(1\,2)\circ\mathcal{R}^{0,\uparrow/\downarrow}_{V_1,V_2}(s)"'] & (V_2 \underset{\mathrm{D},-s}{\otimes} V_1)(s)
\end{tikzcd} \tag{2.1}$$

is readily seen to satisfy the properties 1–6 of Theorem 2.2. The details of this calculation can be found in [GTLW21, §7.1]

Finally, note that the intertwiner

$$\mathcal{R}^{+}_{V_1,V_2}(s) = (1\,2) \circ \mathcal{R}^{-}_{V_2,V_1}(-s)^{-1} \circ (1\,2) : V_1 \underset{\mathrm{D},s}{\otimes} V_2 \to V_1 \underset{\mathrm{KM},s}{\otimes} V_2$$

automatically provides a meromorphic twist in the opposite direction. Thus, (2.1) reads

$$\mathcal{R}^{\uparrow/\downarrow}_{V_1,V_2}(s) = \mathcal{R}^{+}_{V_1,V_2}(s) \circ \mathcal{R}^{0,\uparrow/\downarrow}_{V_1,V_2}(s) \circ \mathcal{R}^{-}_{V_1,V_2}(s)\,.$$

Problems (M1) and (M2) are solved in Theorems 7.1 and 6.1, respectively, as we now spell out.

2.5. **Construction of $\mathcal{R}^0(s)$.** The following difference equation is crucial to our construction of $\mathcal{R}^0(s)$:

$$(\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p}))\cdot\Lambda(s) = \mathcal{G}(s), \tag{2.2}$$

where

- p is the shift operator defined by $\mathsf{p}\cdot f(s) = f(s-\hbar/2)$,
- $\mathbf{B}(\mathsf{p})$ is the symmetrized affine p–Cartan matrix (see Section 7.4),
- $\mathcal{G}(s) = \sum_{ij}\mathbf{B}(\mathsf{p})^*_{ji}\cdot\mathcal{T}_{ij}(s)$, where $\mathbf{B}(\mathsf{p})^*$ is the adjoint matrix and $\mathcal{T}_{ij}(s)$ is the element
$$\mathcal{T}_{ij}(s) = \hbar^2\sum_{m\geqslant 1} m!s^{-m-1}\sum_{\substack{a,b\geqslant 0\\ a+b=m-1}}(-1)^a\frac{t_{i,a}}{a!}\otimes\frac{t_{j,b}}{b!}$$
in $\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))\otimes\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))[\![s^{-1}]\!]$. Here, $\{t_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}\subset Y^0_\hbar(\mathfrak{g})$ are generators of the commatative subalgebra of $Y_\hbar(\mathfrak{g})$, defined in Section 3.5 below, and $\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))$ is the linear subspace spanned by them. These elements are primitive with respect to Drinfeld coproduct.

Remarks.

(1) The difference equation (2.2) makes sense for any type. In finite type, it reduces to the one considered in [KT96, GTL17]. Its derivation, carried out in Proposition 7.5 (4) and Corollary 7.6, rests on the intertwining equation for $\mathcal{R}^0(s)$. This is in contrast with the finite–type case, where Khoroshkin–Tolstoy [KT96] obtained the corresponding difference equation in order to compute the canonical tensor of a non–degenerate pairing. Their construction also uses the fact that for finite–type Cartan matrices, the determinant of the q–symmetrized Cartan matrix divides a q–number. The analogue of the non–degenerate pairing for affine Yangians is not known, and the latter statement is false (see the list of determinants given in Appendix C).

(2) We observe that (2.2) is *irregular*, meaning that the difference operator has order of vanishing 3 at $\mathsf{p}=1$ (Lemma 7.4) and the right–hand side $\mathcal{G}(s)$ is $O(s^{-2})$. Thus, there is no solution in $(Y^0_\hbar(\mathfrak{g})\otimes Y^0_\hbar(\mathfrak{g}))[\![s^{-1}]\!]$. However, a direct computation shows that the coefficient of s^{-2} and s^{-3} in $\mathcal{G}(s)$ are central (see Lemma 7.7).

As central elements play no role in interwining equations, the last remark allows us to regularize our difference equation and obtain $\Lambda(s)\in\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))^{\otimes 2}[\![s]\!]$ as its unique formal solution (see Proposition 7.9). We then prove that $\mathcal{R}^0(s)=\exp(\Lambda(s))$ is an abelian R–matrix for $Y_\hbar(\mathfrak{g})$, *i.e.*, it satisfies the same properties from Section 1.1 with $\tau_s\otimes\mathbf{1}\circ\Delta$ replaced by $\underset{\mathrm{D},s}{\Delta}$ (see Theorem 7.1).

To determine the functional nature of this formal object, we evaluate (2.2) on a tensor product of two representations from $\mathcal{O}(Y_\hbar(\mathfrak{g}))$. Focusing on one matrix entry at a time, the problem becomes scalar valued. In Theorem A.1, we prove the existence and uniqueness of two fundamental solutions to the scalar analogue of (2.2). Hence, (2.2) admits two fundamental solutions $\Lambda^{\uparrow/\downarrow}_{V_1,V_2}(s)$ whose exponentials yield the meromorphic braiding $\mathcal{R}^{0,\uparrow/\downarrow}_{V_1V_2}(s)$. Moreover, their asymptotic expansions as $\mathrm{Re}(s/\hbar)\to\pm\infty$ coincide and recover the action of $\Lambda(s)$.

The rationality of (a normalization of) these operators is shown in Theorem 7.2, which combined with the rational nature of $\mathcal{R}^-(s)$ proves Theorem 2.3.

2.6. **Construction of $\mathcal{R}^-(s)$.** We now describe the solution to the problem (M2). The rational twist is obtained from the action of a canonical element in a suitable completion of $(Y_\hbar(\mathfrak{g})\otimes Y_\hbar(\mathfrak{g}))[\![s^{-1}]\!]$, which is uniquely, and explicitly, determined by an intertwining equation.

To state our result precisely, let

$$\sigma_z : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})[z,z^{-1}] \quad \text{and} \quad \underset{\mathrm{D},s}{\Delta} : Y_\hbar(\mathfrak{g}) \to (Y_\hbar(\mathfrak{g})\otimes Y_\hbar(\mathfrak{g}))[s;s^{-1}]\!]$$

be, respectively, the principal grading shift in z (see 3.9) and the *deformed Drinfeld coproduct*. The latter was introduced in [GTLW21, Thm. 3.4] and gives rise to the Drinfeld tensor product (see Section 4.4). Consider the homomorphisms

$$\underset{\mathrm{D},s}{\Delta^z} = (\mathrm{Id}\otimes\sigma_z)\circ\underset{\mathrm{D},s}{\Delta} \quad \text{and} \quad \underset{\mathrm{KM},s}{\Delta^z} = (\tau_s\otimes\mathrm{Id})\circ\Delta^z\,.$$

The sought–after $\mathcal{R}^-(s,z)$ is of the following form

$$\mathcal{R}^-(s,z) = \sum_{\beta\in\mathsf{Q}_+}\mathcal{R}^-(s)_\beta z^{\mathsf{ht}(\beta)} \in (Y_\hbar(\mathfrak{g})\otimes Y_\hbar(\mathfrak{g}))[\![s^{-1}]\!][\![z]\!],$$

where ht is the height function. Here, $Y^\pm_\hbar(\mathfrak{g})$ (resp. $Y^0_\hbar(\mathfrak{g})$) are the unital subalgebras of $Y_\hbar(\mathfrak{g})$ generated by the Drinfeld generators $\{x^\pm_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}$ (resp. by $\mathfrak{h}$ and the commuting Drinfeld generators $\{t_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{>0}}$) and equipped with the standard grading over the root lattice Q (see Sections 3.2, 3.5, and 3.7).

The operator $\mathcal{R}^-(s)$ is assumed to satisfy the following properties.

(P1) **Normalization:** $\mathcal{R}^-(s)_0 = 1\otimes 1$.

(P2) **Triangularity:** for any $\beta\in\mathsf{Q}_+$,
$$\mathcal{R}^-(s)_\beta \in (Y^-_\hbar(\mathfrak{g})_{-\beta}\otimes Y^+_\hbar(\mathfrak{g})_\beta)[\![s^{-1}]\!]\,.$$

(P3) **Intertwining equation:** for any $x\in Y_\hbar(\mathfrak{g})$,
$$\mathcal{R}^-(s,z)\underset{\mathrm{KM},s}{\Delta^z}(x) = \underset{\mathrm{D},s}{\Delta^z}(x)\,\mathcal{R}^-(s,z)\,.$$

The matrix entries of $\mathcal{R}^-(s,z)$ acting on a tensor product of category $\mathcal{O}$ modules become polynomials in z. Thus, by evaluating at $z=1$, one obtains an invertible intertwiner between $\underset{\mathrm{KM},s}{\otimes}$ and $\underset{\mathrm{D},s}{\otimes}$. In finite–type, the crucial observation in [GTLW21, §4] is that the conditions (P1), (P2) and (P3) for $x=t_{i,1}$, where i is an arbitrary node of the Dynkin diagram already have a unique solution. A very general rank 1 reduction argument, valid for this solution, reduces (P3) for arbitrary $x\in Y_\hbar(\mathfrak{g})$, to $x = x_0^{\pm}$ in $Y_\hbar(\mathfrak{sl}_2)$, which has been verified in [GTLW21, §4.8]. By this analogy, we consider the analogue of the *second embedding* of §2.7 in *loc. cit.*:

$$\mathrm{T}' : \mathfrak{h}' \to Y_\hbar^0(\mathfrak{g}) \qquad \text{given by} \qquad \mathrm{T}'(d_i h_i) = t_{i,1}\,,$$

where $\mathsf{D} = (d_i)_{i\in\mathbf{I}}$ is the symmetrising diagonal matrix associated to $\mathfrak{g}$, $\mathfrak{h}' \subset \mathfrak{h}$ is the span of the coroots $\{h_i\}_{i\in\mathbf{I}}$ (see Section 3.1). This is a natural choice, since the formulae of the standard coproduct are explicit and fairly manageable in this case. It is clear that if $h \in \mathfrak{h}'$ and $\beta \in \mathsf{Q}_+$ are such that $\beta(h) \neq 0$, then, the intertwining equation (P3) for $\mathrm{T}'(h)$ produces an explicit expression of the block $\mathcal{R}^-(s)_\beta$ in terms of lower blocks $\mathcal{R}^-(s)_\gamma$, $\gamma < \beta$. For Yangians of finite type, this is enough to determine $\mathcal{R}^-(s,z)$ entirely.

In our case, however, it fails almost completely: since the imaginary root δ vanishes on $\mathfrak{h}'$. The resulting system is unable to determine recursively the blocks $\mathcal{R}^-(s)_{n\delta}$, $n \in \mathbb{Z}_{>0}$, and, consequently, any block $\mathcal{R}^-(s)_\beta$ with $\beta > n\delta$ for some $n \in \mathbb{Z}_{>0}$.

2.7. **Extension of the second embedding.** We overcome this issue by constructing an extension of the second embedding

$$\mathrm{T} : \mathfrak{h} \to Y_\hbar^0(\mathfrak{g})$$

satisfying the following properties (see Theorem 5.1).

(T1) For any $h \in \mathfrak{h}$ and $i \in \mathbf{I}$, one has
$$[\mathrm{T}(h), x_{i,r}^{\pm}] = \pm\alpha_i(h) x_{i,r+1}^{\pm}\,.$$

(T2) For any $h \in \mathfrak{h}$ and $s \in \mathbb{C}$, one has
$$\mathrm{ad}(\tau_s(\mathrm{T}(h))) = \mathrm{ad}(\mathrm{T}(h) + sh)\,.$$

(T3) There is a family of translation–invariant elements $\mathrm{Q}_{n\delta} \in Y_\hbar^-(\mathfrak{g})_{n\delta} \otimes Y_\hbar^+(\mathfrak{g})_{n\delta}$, $n > 0$, such that, for any $h \in \mathfrak{h}$, one has
$$\Delta^z(\mathrm{T}(h)) = \square\,\mathrm{T}(h) + \hbar[h\otimes 1, \Omega_z + \mathrm{Q}_z]\,,$$
where $\mathrm{Q}_z = \displaystyle\sum_{n>0} \mathrm{Q}_{n\delta} z^{n\,\mathsf{ht}(\delta)}$.

Relying on the map T, we reduce the intertwining equation (P3) to a system of linear equations. By choosing an element $\rho^\vee \in \mathfrak{h}$ such that $\alpha_i(\rho^\vee) = 1$ for all $i \in \mathbf{I}$, we obtain the following recursive expression of the blocks of $\mathcal{R}^-(s,z)$:

$$\mathcal{R}^-(s)_\beta = \hbar \sum_{k\geqslant 0} \frac{\mathrm{ad}(\square\,\mathrm{T}(\rho^\vee))^k}{(s\,\mathsf{ht}(\beta))^{k+1}} \left(\sum_{\alpha\in\Phi_+} \mathsf{ht}(\alpha) \mathcal{R}^-(s)_{\beta-\alpha}(\Omega_\alpha + \mathrm{Q}_\alpha) \right).$$

By (P1), the blocks are uniquely determined and satisfy (P2). By a rank one reduction argument, we finally prove that $\mathcal{R}^-(s,z)$ satisfies also the intertwining equation (P3).

Therefore, it yields an intertwiner

$$\mathcal{R}^-_{V_1,V_2}(s) : V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \to V_1 \underset{\mathrm{D},s}{\otimes} V_2$$

which is readily seen to depend rationally on s, equal $\mathrm{Id}_{V_1\otimes V_2}$ at $s=\infty$, and satisfy the cocycle equation (M2)[5].

2.8. **Outline of the paper.** We review the definition of $Y_\hbar(\mathfrak{g})$ and the basic properties of its category $\mathcal{O}$ modules in Sections 3 and 4, respectively. In Section 5, we introduce the transformation $\mathrm{T}:\mathfrak{h}\to Y_\hbar^0(\mathfrak{g})$ and prove its fundamental properties in Theorem 5.1. It is used in Section 6 to construct the operator $\mathcal{R}^-(s)$ as a rational twist relating the standard coproduct and the Drinfeld coproduct on category $\mathcal{O}$ modules (see Theorem 6.2). In Section 7, we construct the meromorphic braidings $\mathcal{R}^{0,\eta}$ in Theorem 7.1 and prove that they both have the same asymptotic expansion given by the formal abelian R–matrix $\mathcal{R}^0(s)$. The proof relies on well–known techniques to solve additive difference equations, which we review in Appendix A for the reader's convenience. Finally, in Appendices B and C, we provide the proofs of Proposition 3.8 and Lemma 7.4, respectively.

3. The affine Yangian $Y_\hbar(\mathfrak{g})$

3.1. **Affine Lie algebras.** Throughout this paper, we fix a symmetrizable, indecomposable Cartan matrix of *affine type* $\mathbf{A}=(a_{ij})_{i,j\in\mathbf{I}}$ and let $(d_i)_{i\in\mathbf{I}}$ be the associated symmetrizing integers, taken to be positive and relatively prime. As in [GNW18], we further assume that $\mathbf{A}$ is not of type $\mathsf{A}_1^{(1)}$ or $\mathsf{A}_2^{(2)}$. Let $\mathfrak{g}$ be the Kac–Moody Lie algebra associated to $\mathbf{A}$. Recall that $\mathfrak{g}$ is generated, as a Lie algebra, by the following set of generators:

- A realization of $\mathbf{A}$ [Kac90, §1.1]. That is, a $|\mathbf{I}|+1$–dimensional $\mathbb{C}$–vector space $\mathfrak{h}$, together with two linearly independent subsets $\{h_i\}_{i\in\mathbf{I}}\subset\mathfrak{h}$ and $\{\alpha_i\}_{i\in\mathbf{I}}\subset\mathfrak{h}^*$ related by $\alpha_j(h_i)=a_{ij}$ for every $i,j\in\mathbf{I}$.
- Raising and lowering operators $\{e_i,f_i\}_{i\in\mathbf{I}}$.

These generators are subject to the usual Chevalley–Serre relations:

- $\mathfrak{h}$ is abelian.
- $[h,e_i]=\alpha_i(h)e_i$ and $[h,f_i]=-\alpha_i(h)f_i$, for every $h\in\mathfrak{h}$ and $i\in\mathbf{I}$.
- $[e_i,f_j]=\delta_{ij}h_i$ for each $i,j\in\mathbf{I}$.
- $\mathrm{ad}(e_i)^{1-a_{ij}}e_j=0=\mathrm{ad}(f_i)^{1-a_{ij}}f_j$ for each $i,j\in\mathbf{I}$ with $i\neq j$.

Let $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) denote the Lie subalgebra of $\mathfrak{g}$ generated by $\{e_i\}_{i\in\mathbf{I}}$ (resp. $\{f_i\}_{i\in\mathbf{I}}$).

We recall some of the important ingredients in the theory of affine Lie algebras. We mostly follow [Kac90, Ch. 6]. For the list of tables of affine Dynkin diagrams, see [Kac90, §4.8].

- By [Kac90, Prop. 4.7], there is a unique $\mathbf{I}$-tuple $(a_i)_{i\in\mathbf{I}}$ of positive, relatively prime integers satisfying $\sum_{i\in\mathbf{I}}a_{ji}a_i=0$ for all $j\in\mathbf{I}$. These numbers are

[5]Note that the cocycle equation only holds as an identity of rational functions valued in $\mathrm{End}(V_1\otimes V_2\otimes V_3)$, and does not seem to possess a natural lift to $Y_\hbar(\mathfrak{g})$, see [GTLW21, Rmk. 4.1].

listed explicitly in [Kac90, Ch. 4, Table Aff]. Following [Kac90, Thm. 5.6], we define $\delta \in \mathfrak{h}^*$ by $\delta = \sum_{i\in\mathbf{I}} a_i\alpha_i \in \sum_i \mathbb{Z}_{\geqslant 0}\alpha_i$. In particular, this element satisfies $\delta(h_j) = 0$ for all $j \in \mathbf{I}$.

- Using $A^T = DAD^{-1}$, we get that $(a_i^\vee = d_ia_i)_{i\in\mathbf{I}}$ are the coefficients of a linear dependence relation among the rows of $\mathbf{A}$. We record the linear relations: $\sum_{i\in\mathbf{I}} a_i^\vee a_{ij} = 0$, $\forall\, j \in \mathbf{I}$. Let $\mathcal{C} = \sum_i a_i^\vee h_i$. Note that $\mathcal{C}$ is central in $\mathfrak{g}$; as in [Kac90, §6.2], we call it the *canonical central element* of $\mathfrak{g}$.
- Let $\mathfrak{h}' \subset \mathfrak{h}$ denote the span of $\{h_i : i \in \mathbf{I}\}$. Let Φ denote the set of roots of $(\mathfrak{g}, \mathfrak{h})$, which comes naturally with the polarization $\Phi = \Phi_+ \sqcup \Phi_-$. Let $\mathsf{Q} = \mathbb{Z}\Phi$ be the root lattice, and $\mathsf{Q}_+ = \sum_{\alpha\in\Phi_+} \mathbb{Z}_{\geqslant 0}\alpha$. Set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

3.2. **The Yangian $Y_\hbar(\mathfrak{g})$ [Dri88].** Let $\hbar \in \mathbb{C}^\times$. The Yangian $Y_\hbar(\mathfrak{g})$ is the unital, associative $\mathbb{C}$–algebra generated by elements $\mathfrak{h} \cup \{x_{i,r}^\pm, \xi_{i,r}\}_{i\in\mathbf{I}, r\in\mathbb{Z}_{\geqslant 0}}$, subject to the following relations:

(Y1) $\xi_{i,0} = d_ih_i \in \mathfrak{h}$ for every $i \in \mathbf{I}$. $\mathfrak{h}$ is abelian, and $[h, \xi_{i,r}] = 0 = [\xi_{i,r}, \xi_{j,s}]$ for every $h \in \mathfrak{h}$, $i, j \in \mathbf{I}$ and $r, s \in \mathbb{Z}_{\geqslant 0}$.

(Y2) For $i, j \in \mathbf{I}$ and $s \in \mathbb{Z}_{\geqslant 0}$: $[\xi_{i,0}, x_{j,s}^\pm] = \pm d_ia_{ij}x_{j,s}^\pm$.

(Y3) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{Z}_{\geqslant 0}$:
$$[\xi_{i,r+1}, x_{j,s}^\pm] - [\xi_{i,r}, x_{j,s+1}^\pm] = \pm\hbar\frac{d_ia_{ij}}{2}\left(\xi_{i,r}x_{j,s}^\pm + x_{j,s}^\pm\xi_{i,r}\right) .$$

(Y4) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{Z}_{\geqslant 0}$:
$$[x_{i,r+1}^\pm, x_{j,s}^\pm] - [x_{i,r}^\pm, x_{j,s+1}^\pm] = \pm\hbar\frac{d_ia_{ij}}{2}\left(x_{i,r}^\pm x_{j,s}^\pm + x_{j,s}^\pm x_{i,r}^\pm\right) .$$

(Y5) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{Z}_{\geqslant 0}$: $[x_{i,r}^+, x_{j,s}^-] = \delta_{ij}\xi_{i,r+s}$.

(Y6) Let $i \neq j \in \mathbf{I}$ and set $m = 1 - a_{ij}$. For any $r_1, \cdots, r_m, s \in \mathbb{Z}_{\geqslant 0}$:
$$\sum_{\pi\in\mathfrak{S}_m} \left[x_{i,r_{\pi(1)}}^\pm, \left[x_{i,r_{\pi(2)}}^\pm, \left[\cdots, \left[x_{i,r_{\pi(m)}}^\pm, x_{j,s}^\pm\right]\cdots\right]\right]\right] = 0.$$

Note that it follows from this definition that the assignment
$$\sqrt{d_i}e_i \mapsto x_{i,0}^+, \quad \sqrt{d_i}f_i \mapsto x_{i,0}^-, \quad d_ih_i \mapsto \xi_{i,0}, \quad h \mapsto h,$$
for all $i \in \mathbf{I}$ and $h \in \mathfrak{h}$, extends to an algebra homomorphism $U(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})$. We shall frequently work through this homomorphism without further comment.

We denote by $Y_\hbar^0(\mathfrak{g})$ and $Y_\hbar^\pm(\mathfrak{g})$ the unital subalgebras of $Y_\hbar(\mathfrak{g})$ generated by $\mathfrak{h}\cup\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}$ and $\{x_{i,r}^\pm\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}$, respectively. Let $Y_\hbar^\geqslant(\mathfrak{g})$ (resp. $Y_\hbar^\leqslant(\mathfrak{g})$) denote the subalgebras of $Y_\hbar(\mathfrak{g})$ generated by $Y_\hbar^0(\mathfrak{g})$ and $Y_\hbar^+(\mathfrak{g})$ (resp. $Y_\hbar^0(\mathfrak{g})$ and $Y_\hbar^-(\mathfrak{g})$).

3.3. **Filtration.** The Yangian $Y_\hbar(\mathfrak{g})$ is a filtered algebra once given the *loop filtration*. That is, let $\deg(y_r) = r$, where y is one of ξ_i or $x_i^\pm$. For each $k \geqslant 0$, set
$$\mathbf{F}_k(Y_\hbar(\mathfrak{g})) = \text{Linear subspace spanned by monomials of degree } \leqslant k\ .$$

The induced filtration on $Y_\hbar(\mathfrak{g})^{\otimes 2}$ will again be denoted by $\mathbf{F}_\bullet(Y_\hbar(\mathfrak{g})^{\otimes 2})$.

3.4. **Formal currents.** For each $i \in \mathbf{I}$, we define $\xi_i(u), x_i^\pm(u) \in Y_\hbar(\mathfrak{g})[\![u^{-1}]\!]$ by

$$\xi_i(u) = 1 + \hbar \sum_{r \in \mathbb{Z}_{\geqslant 0}} \xi_{i,r} u^{-r-1} \quad \text{and} \quad x_i^\pm(u) = \hbar \sum_{r \in \mathbb{Z}_{\geqslant 0}} x_{i,r}^\pm u^{-r-1}.$$

The relations (Y1)–(Y6) can be written in terms of these formal series, see [GTL16, Prop. 3.3]. Namely, it is proven that these relations are equivalent to the following identities:

($\mathcal{Y}$1) For any $i \in \mathbf{I}$, $\xi_{i,0} = d_i h_i \in \mathfrak{h}$. For any $i, j \in \mathbf{I}$ and $h, h' \in \mathfrak{h}$,

$$[\xi_i(u), \xi_j(v)] = 0 \qquad [\xi_i(u), h] = 0 \qquad [h, h'] = 0.$$

($\mathcal{Y}$2) For any $i \in \mathbf{I}$, and $h \in \mathfrak{h}$, $[h, x_i^\pm(u)] = \pm\alpha_i(h) x_i^\pm(u)$.

($\mathcal{Y}$3) For any $i, j \in \mathbf{I}$, and $a = \hbar d_i a_{ij}/2$

$$(u - v \mp a)\xi_i(u) x_j^\pm(v) = (u - v \pm a) x_j^\pm(v) \xi_i(u) \mp 2a x_j^\pm(u \mp a) \xi_i(u).$$

($\mathcal{Y}$4) For any $i, j \in \mathbf{I}$, and $a = \hbar d_i a_{ij}/2$

$$(u - v \mp a) x_i^\pm(u) x_j^\pm(v) = (u - v \pm a) x_j^\pm(v) x_i^\pm(u) + \hbar \left([x_{i,0}^\pm, x_j^\pm(v)] - [x_i^\pm(u), x_{j,0}^\pm] \right).$$

($\mathcal{Y}$5) For any $i, j \in \mathbf{I}$

$$(u - v)[x_i^+(u), x_j^-(v)] = -\delta_{ij} \hbar \left(\xi_i(u) - \xi_i(v) \right).$$

($\mathcal{Y}$6) For any $i \neq j \in \mathbf{I}$, $m = 1 - a_{ij}$, $r_1, \cdots, r_m \in \mathbb{Z}_{\geqslant 0}$, and $s \in \mathbb{Z}_{\geqslant 0}$

$$\sum_{\pi \in \mathfrak{S}_m} \left[x_i^\pm(u_{\pi_1}), \left[x_i^\pm(u_{\pi(2)}), \left[\cdots, \left[x_i^\pm(u_{\pi(m)}), x_j^\pm(v) \right] \cdots \right] \right] \right] = 0.$$

Here the relations ($\mathcal{Y}$1)–($\mathcal{Y}$5) are identities in $Y_\hbar(\mathfrak{g})[u, v; u^{-1}, v^{-1}]\!]$, while the Serre relation ($\mathcal{Y}$6) (for a fixed pair $i, j \in \mathbf{I}$ with $i \neq j$) is an equality in the formal series space $Y_\hbar(\mathfrak{g})[\![u_1^{-1}, \ldots, u_m^{-1}, v^{-1}]\!]$, where $m = 1 - a_{ij}$.

3.5. **Alternate set of generators for $Y_\hbar^0(\mathfrak{g})$.** For each $i \in \mathbf{I}$, let $t_i(u)$ and $B_i(z)$ be the formal series defined by

$$t_i(u) = \hbar \sum_{r \geqslant 0} t_{i,r} u^{-r-1} = \log(\xi_i(u)) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \left(\hbar \sum_{r \in \mathbb{Z}_{\geqslant 0}} \xi_{i,r} u^{-r-1} \right)^n,$$

$$B_i(z) = \hbar \sum_{r \geqslant 0} t_{i,r} \frac{z^r}{r!}.$$

That is, $t_i(u)$ is the formal series logarithm of $\xi_i(u)$ and $B_i(z)$ is the formal Borel transform of $t_i(u)$. The elements $\{t_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}_{\geqslant 0}} \subset Y_\hbar^0(\mathfrak{g})$ are polynomials in $\{\xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}}$, and together with $\mathfrak{h}$, generated $Y_\hbar^0(\mathfrak{g})$. We record the formulae for the first few terms, for future use:

$$t_{i,0} = \xi_{i,0}, \tag{3.1}$$

$$t_{i,1} = \xi_{i,1} - \frac{\hbar}{2} \xi_{i,0}^2, \tag{3.2}$$

$$t_{i,2} = \xi_{i,2} - \hbar\xi_{i,1}\xi_{i,0} + \frac{\hbar^2}{3}\xi_{i,0}^3, \tag{3.3}$$

$$t_{i,3} = \xi_{i,3} - \hbar\xi_{i,2}\xi_{i,0} - \frac{\hbar}{2}\xi_{i,1}^2 + \hbar^2\xi_{i,1}\xi_{i,0}^2 - \frac{\hbar^3}{4}\xi_{i,0}^4 \ . \tag{3.4}$$

The following commutation relation was obtained in [GTL13, §2.9]:

$$[B_i(z), x_{k,n}^\pm] = \pm\frac{e^{\frac{d_ia_{ik}\hbar}{2}z} - e^{-\frac{d_ia_{ik}\hbar}{2}z}}{z}\left(\sum_{r\geqslant 0} x_{k,n+r}^\pm \frac{z^r}{r!}\right). \tag{3.5}$$

Comparing coefficients of z, one obtains (see [GTL13, Remark 2.9]):

$$[t_{i,m}, x_{k,n}^\pm] = \pm d_ia_{ik}\sum_{\ell=0}^{\lfloor\frac{m}{2}\rfloor}\begin{pmatrix} m \\ 2\ell \end{pmatrix}\frac{(\hbar d_ia_{ik}/2)^{2\ell}}{2\ell+1}x_{k,m+n-2\ell}^\pm \ .$$

A few special cases of this relation which will be particularly relevant to us are

$$[t_{i,1}, x_{j,n}^\pm] = \pm d_ia_{ij}x_{j,n+1}^\pm \ , \tag{3.6}$$

$$[t_{i,2}, x_{j,n}^\pm] = \pm d_ia_{ij}x_{j,n+2}^\pm \pm \frac{\hbar^2}{12}(d_ia_{ij})^3x_{j,n}^\pm, \ , \tag{3.7}$$

$$[t_{i,3}, x_{j,n}^\pm] = \pm d_ia_{ij}x_{j,n+3}^\pm \pm \frac{\hbar^2}{4}(d_ia_{ij})^3x_{j,n+1}^\pm \ . \tag{3.8}$$

3.6. **Shift automorphism.** The group of translations of the complex plane acts on $Y_\hbar(\mathfrak{g})$ as follows. For $s \in \mathbb{C}$, $\tau_s(h) = h$, $\forall\ h \in \mathfrak{h}$, and $\tau_s(y(u)) = y(u-s)$, where y is one of $\xi_i, x_i^\pm$. In terms of modes, we have

$$\tau_s(y_r) = \sum_{i=0}^{r}\begin{pmatrix} r \\ i \end{pmatrix} s^{r-i}y_i \ .$$

If s is instead viewed as a formal variable, then these same formulae define an algebra homomorphism $Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})[s]$, still denoted τ_s. This is a filtered algebra homomorphism, provided $Y_\hbar(\mathfrak{g})[s] \cong Y_\hbar(\mathfrak{g}) \otimes \mathbb{C}[s]$ is equipped with the standard tensor product filtration in which $\deg s = 1$.

3.7. Q–**grading.** Viewed as a module over $\mathfrak{h}$, we have $Y_\hbar(\mathfrak{g}) = \bigoplus_{\beta\in\mathsf{Q}} Y_\hbar(\mathfrak{g})_\beta$, where

$$Y_\hbar(\mathfrak{g})_\beta = \{y \in Y_\hbar(\mathfrak{g}) : [h, y] = \beta(h)y, \ \forall\ h \in \mathfrak{h}\}.$$

This gives rise to a Q-graded algebra structure on $Y_\hbar(\mathfrak{g})$ for which $Y_\hbar^\pm(\mathfrak{g})$, $Y_\hbar^\geqslant(\mathfrak{g})$ and $Y_\hbar^\leqslant(\mathfrak{g})$ are all Q-graded subalgebras. In particular, we have

$$Y_\hbar^\geqslant(\mathfrak{g}) = \bigoplus_{\beta\in\mathsf{Q}_+} Y_\hbar^\geqslant(\mathfrak{g})_\beta \quad \text{and} \quad Y_\hbar^\leqslant(\mathfrak{g}) = \bigoplus_{\beta\in\mathsf{Q}_+} Y_\hbar^\leqslant(\mathfrak{g})_{-\beta}.$$

3.8. **The elements** $\mathcal{C}_r$**.** Recall from Section 3.1 that $\mathcal{C} = \sum_{i\in\mathbf{I}} a_id_ih_i \in \mathfrak{h}$ is the canonical central element of $\mathfrak{g}$. We define higher order analogues of $\mathcal{C}$ in $Y_\hbar(\mathfrak{g})$ by setting

$$\mathcal{C}_r = \sum_{i\in\mathbf{I}} a_it_{i,r} \quad \forall \quad r \geqslant 0.$$

In particular, $\mathcal{C}_0$ is the image of $\mathcal{C}$ in $Y_\hbar(\mathfrak{g})$. In general, these elements do not belong to the center of $Y_\hbar(\mathfrak{g})$. However, we do have the following corollary of the relations (3.6)–(3.8).

Corollary. *The elements* $\mathcal{C}_0, \mathcal{C}_1$ *are central, while* $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *satisfy*

$$[\mathcal{C}_2, x_{j,s}^\pm] = \pm\frac{\hbar^2}{12}\mu_j x_{j,s}^\pm \quad \textit{and} \quad [\mathcal{C}_3, x_{j,s}^\pm] = \pm\frac{\hbar^2}{4}\mu_j x_{j,s+1}^\pm,$$

for all $j \in \mathbf{I}$ *and* $s \geqslant 0$. *Here* μ_j *is defined by*

$$\mu_j = \sum_{i\in\mathbf{I}} a_i (d_i a_{ij})^3 \quad \forall \quad j \in \mathbf{I}.$$

That $\mathrm{ad}(\mathcal{C}_j)$ is a non-trivial derivation for $j = 2, 3$ will play a crucial role in the constructions of this article. As will the following proposition, which is proven in detail in Appendix B.

Proposition. *Let* $\mathbf{B} = (d_i a_{ij})_{i,j\in\mathbf{I}}$ *be the symmetrized Cartan matrix of* $\mathfrak{g}$*, and let* $\underline{\mu} = (\mu_j)_{i\in\mathbf{I}} \in \mathbb{Z}^{\mathbf{I}}$*, where* μ_j *is defined above. Then the augmented matrix* $(\mathbf{B}\,|\,\underline{\mu})$ *has rank* $|\mathbf{I}|$.

3.9. **The coproduct** Δ^z**.** Finally, we recall the definition of the twisted standard coproduct Δ^z on $Y_\hbar(\mathfrak{g})$, which was introduced in [GNW18, §6.1]. By [GNW18, §6.1], the assignment

$$\sigma_z(x_{i,r}^\pm) = z^{\pm 1} x_{i,r}^\pm, \quad \sigma_z(\xi_{i,r}) = \xi_{i,r} \quad \text{and} \quad \sigma_z(h) = h,$$

for all $i \in \mathbf{I}$, $r \geqslant 0$ and $h \in \mathfrak{h}$, extends to an injective algebra homomorphism

$$\sigma_z : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})[z^{\pm 1}].$$

Note that if $\rho^\vee \in \mathfrak{h}$ is chosen so that $\alpha_i(\rho^\vee) = 1$, for every $i \in \mathbf{I}$, then $\sigma_z = \mathrm{Ad}(z^{\rho^\vee})$. Next, for each $\beta \in \Phi_+ \cup \{0\}$, let $\Omega_\beta \in \mathfrak{g}_{-\beta} \otimes \mathfrak{g}_\beta$ be the canonical element defined by the restriction of the standard invariant form on $\mathfrak{g}$ to $\mathfrak{g}_{-\beta} \times \mathfrak{g}_\beta$. We then define Ω_z and Ω_z^- in $(\mathfrak{g}\otimes\mathfrak{g})[\![z]\!]$ by

$$\Omega_z^- = \sum_{\beta\in\Phi_+} \Omega_\beta z^{\mathsf{ht}(\beta)} \quad \text{and} \quad \Omega_z = \Omega_0 + \Omega_z^-,$$

where $\mathsf{ht} : \mathsf{Q}_+ \to \mathbb{Z}_{\geqslant 0}$ is the additive *height* function, defined on $\beta = \sum_j n_j \alpha_j$ by $\mathsf{ht}(\beta) = \sum_j n_j$. Note in particular that $\Omega_\beta z^{\mathsf{ht}(\beta)} = (\mathrm{Id}\otimes\sigma_z)(\Omega_\beta)$ for each $\beta \in \mathsf{Q}_+$.

By Theorem 6.2 of [GNW18], there is an algebra homomorphism

$$\Delta^z : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})^{\otimes 2}[z^{-1}; z]\!]$$

uniquely determined by the formulae

$$\begin{aligned} \Delta^z(y) &= y \otimes 1 + 1 \otimes \sigma_z(y), \\ \Delta^z(t_{i,1}) &= t_{i,1} \otimes 1 + 1 \otimes t_{i,1} + \hbar[\xi_{i,0} \otimes 1, \Omega_z], \end{aligned} \tag{3.9}$$

for each $y \in \mathfrak{h} \cup \{x_{j,0}^\pm\}_{j\in\mathbf{I}}$ and $i \in \mathbf{I}$. We will call Δ^z the (twisted) *standard coproduct* on $Y_\hbar(\mathfrak{g})$. It is not coassociative or counital, but satisfies the twisted coalgebra relations

$$\begin{aligned} (\Delta^z \otimes \mathrm{Id}) \circ \Delta^{zw} &= (\mathrm{Id}\otimes\Delta^w) \circ \Delta^z, \\ (\varepsilon\otimes\mathrm{Id})\Delta^z(x) = 1 \otimes \sigma_z(x) \quad &\text{and} \quad (\mathrm{Id}\otimes\varepsilon)\Delta^z(x) = x \otimes 1 \quad \forall\, x \in Y_\hbar(\mathfrak{g}), \end{aligned} \tag{3.10}$$

where $\varepsilon : Y_\hbar(\mathfrak{g}) \to \mathbb{C}$ is the counit, defined by $\varepsilon(y) = 0$ for all $y \in \mathfrak{h}\cup\{\xi_{i,r}, x_{i,r}^\pm\}_{i\in\mathbf{I},r\geqslant 0}$.

It also follows easily from the formulae given above that Δ^z preserves the loop filtration on $Y_\hbar(\mathfrak{g})$ introduced in Section 3.3 above. That is, for every $k \in \mathbb{Z}_{\geqslant 0}$ we have

$$\Delta^z(y) \in \mathbf{F}_k(Y_\hbar(\mathfrak{g})^{\otimes 2})[\![z]\!],\ \forall\, y \in \mathbf{F}_k(Y_\hbar(\mathfrak{g}))\ . \tag{3.11}$$

We shall make use of the linear map $\square : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})$ defined by

$$\square(y) = y \otimes 1 + 1 \otimes y \quad \forall\, y \in Y_\hbar(\mathfrak{g}).$$

Though it is not an algebra homomorphism, it satisfies $\square([x,y]) = [\square(x), \square(y)]$ for all $x, y \in Y_\hbar(\mathfrak{g})$. In particular, by (4.13) of [GNW18], we have

$$\begin{aligned} \Delta^z(x_{i,1}^+) &= \square^z(x_{i,1}^+) - z\hbar[1 \otimes x_{i,0}^+, \Omega_z], \\ \Delta^z(x_{i,1}^-) &= \square^z(x_{i,1}^-) + \hbar[x_{i,0}^- \otimes 1, \Omega_z], \end{aligned} \tag{3.12}$$

for each $i \in \mathbf{I}$, where $\square^z = (\mathrm{Id} \otimes \sigma_z) \circ \square$. Using the fact that $\xi_{i,r} - t_{i,r} \in \mathbf{F}_{r-1}(Y_\hbar(\mathfrak{g}))$, it follows from the proof of [GW23, Prop. 2.6] that

$$(\Delta^z - \square^z)(t_{i,r}) \in \mathbf{F}_{r-1}(Y_\hbar(\mathfrak{g})^{\otimes 2})[\![z]\!],\ \forall\, i \in \mathbf{I}, r \in \mathbb{Z}_{\geqslant 0}. \tag{3.13}$$

4. Representations of affine Yangians

In this section we give the definitions of the relevant categories of representations of $Y_\hbar(\mathfrak{g})$, namely $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ and $\mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g})) \subset \mathcal{O}(Y_\hbar(\mathfrak{g}))$. We remind the reader of the standard tensor product $\underset{\text{KM},s}{\otimes}$ and the Drinfeld tensor product $\underset{\text{D},s}{\otimes}$, two different s–dependent tensor structures on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$. The dependence is polynomial in s for $\underset{\text{KM},s}{\otimes}$ and rational in s for $\underset{\text{D},s}{\otimes}$.

4.1. Categories $\mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g})) \subset \mathcal{O}(Y_\hbar(\mathfrak{g}))$.

A representation V of $Y_\hbar(\mathfrak{g})$ is said to be in category $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ if its restriction to $U(\mathfrak{g})$ is in category $\mathcal{O}$. That is,

(1) V is a direct sum of finite–dimensional weight spaces.
$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu],\ \dim(V[\mu]) < \infty,$$
where $V[\mu] = \{v \in V : h \cdot v = \mu(h)v,\ \forall h \in \mathfrak{h}\}$.

(2) Let $P(V) = \{\mu : V[\mu] \neq \{0\}\}$ be the set of weight of V. Then, there exist $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ such that
$$P(V) \subset \bigcup_{j=1}^{r} \lambda_j - \mathsf{Q}_+.$$

Let $\mathcal{O}_{\text{int}}(Y_\hbar(\mathfrak{g}))$ be the full subcategory of $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ consisting of those $V \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ which satisfy the following *integrability condition*: for every $\mu \in P(V)$ and $i \in \mathbf{I}$, there exists $N \gg 0$ such that $V[\mu - j\alpha_i] = 0,\ \forall\, j \geqslant N$.

Given $V \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ and $s \in \mathbb{C}$, let $V(s)$ denote the pull–back representation $\tau_s^*(V)$.

Remark. These are the Yangian analogues of the corresponding categories for quantum affinizations studied by Hernandez in [Her05b] (see also [GTL16, §3]).

The analogue of the following *rationality property* for quantum affine algebras was obtained in [BK96, §6] and [Her07, Prop. 3.8]. The Yangian version, stated below, can be found in [GTL16, Prop. 3.6].

Proposition. *Let V be a representation of $Y_\hbar(\mathfrak{g})$ which is $\mathfrak{h}$–diagonalizable, with finite–dimensional weight spaces. Then, for every weight $\mu \in \mathfrak{h}^*$ of V, the generating series*

$$\xi_i(u) \in \mathrm{End}(V[\mu])[\![u^{-1}]\!], \qquad x_i^\pm(u) \in \mathrm{Hom}(V[\mu], V[\mu \pm \alpha_i])[\![u^{-1}]\!],$$

defined in 3.4 above, are the Taylor series expansions at ∞ of rational functions of u.

4.2. **Matrix logarithms.** Let $i \in \mathbf{I}$, $V \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ and $\mu \in P(V)$. By the previous proposition, the operator $\xi_i(u)$ acting on the finite–dimensional weight space $V[\mu]$ becomes a rational, abelian, function of $u \in \mathbb{C}$, taking value 1 at $u = \infty$. Let $A \subset \mathbb{C}^\times$ be the set of poles of $\xi_i(u)^{\pm 1}$. As shown in [GTL17, Prop. 5.4], we can view $t_i(u)$ as Taylor series near ∞ of a single–valued function defined on the cut plane:

$$t_i(u) = \log(\xi_i(u)) : \mathbb{C} \setminus \bigcup_{a \in A} [0, a] \to \mathrm{End}(V[\mu]).$$

4.3. **Standard tensor product.** Let Δ^z be the twisted standard coproduct introduced in Section 3.9. Given $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, with action homomorphisms $\pi_\ell : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V_\ell)$, the composition

$$(\pi_1 \otimes \pi_2) \circ \Delta^z : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V_1 \otimes V_2)[\![z]\!]$$

can be evaluated at $z = 1$ to yield an action of $Y_\hbar(\mathfrak{g})$ on $V_1 \otimes V_2$ (see [GNW18, Cor. 6.9]). The resulting representation of $Y_\hbar(\mathfrak{g})$ is denoted $V_1 \underset{\mathrm{KM},0}{\otimes} V_2$. More generally, we set

$$V_1 \underset{\mathrm{KM},s}{\otimes} V_2 = V_1(s) \underset{\mathrm{KM},0}{\otimes} V_2 \quad \forall\, s \in \mathbb{C}.$$

The properties of this tensor product are summarized in the following theorem, which is a consequence of the results of [GNW18]; see also [GTLW21, Prop. 8.2].

Theorem. *The category $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ together with the tensor product $\underset{\mathrm{KM},s}{\otimes}$ is a (polynomial) tensor category. In more detail, we have the following properties:*

(1) *For $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, $V_1 \underset{\mathrm{KM},s}{\otimes} V_2$ depends polynomially in s.*

(2) *The tensor product is compatible with the shift automorphism. That is, for every $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, we have:*

$$(V_1 \underset{\mathrm{KM},s_1}{\otimes} V_2)(s_2) = V_1(s_2) \underset{\mathrm{KM},s_1}{\otimes} V_2(s_2), \qquad V_1(s_1) \underset{\mathrm{KM},s_2}{\otimes} V_2 = V_1 \underset{\mathrm{KM},s_1+s_2}{\otimes} V_2.$$

(3) *Let $\mathbf{1}$ denote the 1–dimensional, trivial representation of $Y_\hbar(\mathfrak{g})$. Then the following natural identifications of vector spaces are $Y_\hbar(\mathfrak{g})$–intertwiners.*

$$\mathbf{1} \underset{\mathrm{KM},s}{\otimes} V_2 \cong V_2, \qquad V_1 \underset{\mathrm{KM},s}{\otimes} \mathbf{1} \cong V_1(s).$$

(4) *The tensor product is asssociative in the following sense. For any $V_1, V_2, V_3 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, the natural identification of vector spaces is a $Y_\hbar(\mathfrak{g})$–intertwiner:*

$$(V_1 \underset{\mathrm{KM},s_1}{\otimes} V_2) \underset{\mathrm{KM},s_2}{\otimes} V_3 \cong V_1 \underset{\mathrm{KM},s_1+s_2}{\otimes} (V_2 \underset{\mathrm{KM},s_2}{\otimes} V_3).$$

4.4. **Drinfeld tensor product.** The following version of the Drinfeld tensor product on representation of Yangians was introduced in [GTL17, §4] (see also [GTLW21, §3.2–3.4, and Prop. 8.1]).

Let $\underset{\mathrm{D},s}{\Delta}$ be the assignment on the generating set $\{h, \xi_{i,r}, x^{\pm}_{i,r}\}_{h\in\mathfrak{h}, i\in\mathbf{I}, r\geq 0}$ of $Y_\hbar(\mathfrak{g})$, with values in $Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})[s; s^{-1}]\!]$, defined as follows:

- For any $h \in \mathfrak{h}$, $\underset{\mathrm{D},s}{\Delta}(h) = \square(h) = h \otimes 1 + 1 \otimes h$.
- For any $i \in \mathbf{I}$, $\underset{\mathrm{D},s}{\Delta}(\xi_i(u)) = \xi_i(u-s) \otimes \xi_i(u)$. Thus,
$$\underset{\mathrm{D},s}{\Delta}(\xi_{i,r}) = \tau_s(\xi_{i,r}) \otimes 1 + 1 \otimes \xi_{i,r} + \hbar \sum_{p=0}^{r-1} \tau_s(\xi_{i,p}) \otimes \xi_{i,r-1-p}\ .$$
Note that the elements $\{t_{i,r}\}$ introduced in Section 3.5 are primitive with respect to the Drinfeld coproduct. That is,
$$\underset{\mathrm{D},s}{\Delta}(t_{i,r}) = \tau_s(t_{i,r}) \otimes 1 + 1 \otimes t_{i,r}\ .$$
- For each $i \in \mathbf{I}$, we have:
$$\begin{aligned}\underset{\mathrm{D},s}{\Delta}(x^+_{i,r}) &= \tau_s(x^+_{i,r}) \otimes 1 + 1 \otimes x^+_{i,r} \\ &\quad + \hbar \sum_{N\geqslant 0} s^{-N-1} \left(\sum_{n=0}^{N} (-1)^{n+1} \begin{pmatrix} N \\ n \end{pmatrix} \xi_{i,n} \otimes x^+_{i,r+N-n} \right), \\ \underset{\mathrm{D},s}{\Delta}(x^-_{i,r}) &= \tau_s(x^-_{i,r}) \otimes 1 + 1 \otimes x^-_{i,r} \\ &\quad + \hbar \sum_{N\geqslant 0} s^{-N-1} \left(\sum_{n=0}^{N} (-1)^{n+1} \begin{pmatrix} N \\ n \end{pmatrix} x^-_{i,r+n} \otimes \xi^+_{i,N-n} \right).\end{aligned}$$

It was shown in [GTLW21, Thm. 3.4] that the analogue of this assignment for the Yangian $Y_\hbar(\mathfrak{a})$ of a finite-dimensional simple Lie algebra $\mathfrak{a}$ defines an algebra homomorphism $Y_\hbar(\mathfrak{a}) \to Y_\hbar(\mathfrak{a}) \otimes Y_\hbar(\mathfrak{a})[s; s^{-1}]\!]$. This result extends naturally to any Kac–Moody algebra satisfying the condition that each rank 2 diagram subalgebra is of finite type, since the defining relations of the Yangian are inherently of rank 2 type. In particular, this applies to the affine Kac–Moody algebra $\mathfrak{g}$: the above assignment extends to an algebra homomorphism

$$\underset{\mathrm{D},s}{\Delta} : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})[s; s^{-1}]\!].$$

In contrast with the infinite sums encountered in the definition of Δ from Section 3.9, the infinite sums written above do not truncate. It is a consequence of the rationality property provided by Proposition 4.1, that these formal Laurent series in s^{-1} become rational functions of s, once evaluated on $V_1 \otimes V_2$. More precisely, given $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, the composition

$$(\pi_1 \otimes \pi_2) \circ \underset{\mathrm{D},s}{\Delta} : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V_1 \otimes V_2)[s; s^{-1}]\!]$$

takes values in $\mathrm{End}(V_1 \otimes V_2)(s)$ (rational functions of s). We let $V_1 \underset{\mathrm{D},s}{\otimes} V_2$ denote the resulting representation of $Y_\hbar(\mathfrak{g})$. The following theorem summarizes the results of [GTLW21, §3.2–3.4].

Theorem. *The category $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ together with the tensor product $\underset{\mathrm{D},s}{\otimes}$ is a (rational) tensor category. In more detail, we have the following properties:*

(1) *For $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, $V_1 \underset{\mathrm{D},s}{\otimes} V_2$ depends rationally on s.*

(2) *The tensor product is compatible with the shift automorphism. That is, for every $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, we have:*

$$(V_1 \underset{\mathrm{D},s_1}{\otimes} V_2)(s_2) = V_1(s_2) \underset{\mathrm{D},s_1}{\otimes} V_2(s_2), \qquad V_1(s_1) \underset{\mathrm{D},s_2}{\otimes} V_2 = V_1 \underset{\mathrm{D},s_1+s_2}{\otimes} V_2.$$

(3) *Let $\mathbb{1}$ denote the 1–dimensional, trivial representation of $Y_\hbar(\mathfrak{g})$. Then the following natural identifications of vector spaces are $Y_\hbar(\mathfrak{g})$–intertwiners.*

$$\mathbb{1} \underset{\mathrm{D},s}{\otimes} V_2 \cong V_2, \qquad V_1 \underset{\mathrm{D},s}{\otimes} \mathbb{1} \cong V_1(s).$$

(4) *The tensor product is asssociative in the following sense. For any $V_1, V_2, V_3 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, the natural identification of vector spaces is a $Y_\hbar(\mathfrak{g})$–intertwiner:*

$$(V_1 \underset{\mathrm{D},s_1}{\otimes} V_2) \underset{\mathrm{D},s_2}{\otimes} V_3 \cong V_1 \underset{\mathrm{D},s_1+s_2}{\otimes} (V_2 \underset{\mathrm{D},s_2}{\otimes} V_3).$$

Remark. We wish to stress the point that $\underset{\mathrm{D},s}{\Delta}$ is *not* coassociative. The associativity of $\underset{\mathrm{D},s}{\otimes}$ rests on an identity among rational functions of two variables, which does not seem to have a lift at the level of algebra (see [GTLW21, Remark 3.1]).

5. The transformation T

In this section, we describe a linear map $\mathrm{T} : \mathfrak{h} \to Y_\hbar^0(\mathfrak{g})$, which plays a crucial role in our construction of $\mathcal{R}^-(s)$ in Section 6 below.

5.1. **The transformation** T. Let $\circ \in \mathbf{I}$ denote the *extending vertex*, as in [Kac90, §6.1] and let $d \in \mathfrak{h}$ denote the *scaling element*, chosen so as to have $\alpha_i(d) = \delta_{i,\circ}$ (see [Kac90, §6.2]).

By Proposition 3.8, the augmented matrix $(\mathbf{B} \,|\, \underline{\mu})$ has full rank and thus the unit vector $(\delta_{j,\circ})_{j\in\mathbf{I}} \in \mathbb{Q}^{\mathbf{I}}$ lies in its range. In particular, there exists a tuple $(\zeta_i)_{i\in\mathbf{I}} \in \mathbb{Q}^{\mathbf{I}}$ and $\zeta \in \mathbb{Q}^\times$ such that

$$\frac{1}{4}\zeta\mu_j + \sum_{i\in\mathbf{I}} d_j a_{ji} \zeta_i = \delta_{j,\circ} \quad \forall\, j \in \mathbf{I}.$$

Since $\{d\} \cup \{d_i h_i\}_{i\in\mathbf{I}}$ is a basis of $\mathfrak{h}$, the assignment $\mathrm{T} : \{d\} \cup \{d_i h_i\}_{i\in\mathbf{I}} \to Y_\hbar^0(\mathfrak{g})$ defined by

$$\mathrm{T}(d) = \sum_{i\in\mathbf{I}} \zeta_i t_{i,1} + \frac{1}{\hbar^2}\zeta \mathcal{C}_3 \quad \text{and} \quad \mathrm{T}(d_i h_i) = t_{i,1} \quad \forall\, i \in \mathbf{I}$$

uniquely extends to a linear map $\mathrm{T} : \mathfrak{h} \to Y_\hbar^0(\mathfrak{g})$. The following theorem provides the main result of this section.

Theorem. *The linear map* T *has the following properties:*

(1) *For each* $h \in \mathfrak{h}$, $j \in \mathbf{I}$ *and* $r \geqslant 0$, *one has*
$$[\mathrm{T}(h), x_{j,r}^{\pm}] = \pm\alpha_j(h) x_{j,r+1}^{\pm}.$$

(2) *For each* $h \in \mathfrak{h}$ *and* $s \in \mathbb{C}$, *one has*
$$\mathrm{ad}(\tau_s(\mathrm{T}(h))) = \mathrm{ad}(\mathrm{T}(h) + sh).$$

(3) *For each* $h \in \mathfrak{h}$, $\Delta^z \circ \mathrm{T}$ *satisfies*
$$\Delta^z(\mathrm{T}(h)) = \mathrm{T}(h) \otimes 1 + 1 \otimes \mathrm{T}(h) + \hbar[h \otimes 1, \Omega_z + \mathrm{Q}_z]$$
where Q_z *is a formal series in* z *with the following properties:*

i) $\mathrm{Q}_z = \sum_{n>0} \mathrm{Q}_{n\delta} z^{n\mathsf{ht}(\delta)}$, *where each coefficient* $\mathrm{Q}_{n\delta}$ *satisfies*
$$\mathrm{Q}_{n\delta} \in Y_\hbar^-(\mathfrak{g})_{-n\delta} \otimes Y_\hbar^+(\mathfrak{g})_{n\delta}.$$

ii) *For each* $a, b \in \mathbb{C}$, *one has*
$$(\tau_a \otimes \tau_b)(\mathrm{Q}_z) = \mathrm{Q}_z.$$

iii) *The tensor factors of* Q_z *are primitive:*
$$(\Delta^w \otimes \mathrm{Id})(\mathrm{Q}_z) = \mathrm{Q}_z^{13} + \mathrm{Q}_{z/w}^{23}$$
$$(\mathrm{Id} \otimes \Delta^w)(\mathrm{Q}_z) = \mathrm{Q}_z^{12} + \mathrm{Q}_{zw}^{13}.$$

iv) Q_z *belongs to the centralizer of* $U(\mathfrak{g}')^{\otimes 2}$ *in* $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$.

Proof. Let us begin with some preliminary observations that will be useful throughout the proof. Since the constant ζ is nonzero, we can set $\mathsf{h} = \frac{\hbar^2}{\zeta}(d - \sum_{i\in\mathbf{I}} \zeta_i d_i h_i) \in \mathfrak{h}$. Then $\{\mathsf{h}\} \cup \{d_i h_i\}_{i\in\mathbf{I}}$ is a basis of $\mathfrak{h}$, and we have

$$(5.1)\qquad \begin{aligned} \mathrm{T}(\mathsf{h}) &= \frac{\hbar^2}{\zeta}\left(\mathrm{T}(d) - \sum_{i\in\mathbf{I}} \zeta_i t_{i,1}\right) = \mathcal{C}_3, \\ [\mathsf{h}, x_{j,r}^{\pm}] &= \pm\frac{\hbar^2}{\zeta}\left(\delta_{j,\circ} - \sum_{i\in\mathbf{I}} \zeta_i d_i a_{ij}\right) x_{j,r}^{\pm} = \pm\frac{\hbar^2}{4}\mu_j x_{j,r}^{\pm}, \end{aligned}$$

for all $j \in \mathbf{I}$ and $r \geqslant 0$. It thus follows by Corollary 3.8 that we have the equality of operators $\mathrm{ad}(\mathsf{h}) = \mathrm{ad}(3\mathcal{C}_2)$ on $Y_\hbar(\mathfrak{g})$.

Consider now Parts (1) and (2) of the theorem. They clearly hold when $h \in \{d_i h_i\}_{i\in\mathbf{I}}$, so it is sufficient to verify them for $h = \mathsf{h}$. For Part (1), this is immediate from (5.1) and Corollary 3.8. As for Part (2), since $\mathrm{T}(\mathsf{h}) = \mathcal{C}_3$, we have

$$\begin{aligned} \mathrm{ad}(\tau_s(\mathrm{T}(\mathsf{h}))) &= \mathrm{ad}(\mathcal{C}_3 + 3s\mathcal{C}_2 + 3s^2\mathcal{C}_1 + s^3\mathcal{C}_0) \\ &= \mathrm{ad}(\mathcal{C}_3 + 3s\mathcal{C}_2) \\ &= \mathrm{ad}(\mathrm{T}(\mathsf{h}) + s\mathsf{h}), \end{aligned}$$

where in the second equality we have used that $\mathcal{C}_0$ and $\mathcal{C}_1$ are central, and in the last equality we have applied (5.1).

Let us now turn to Part (3). We will construct the unique series Q_z satisfying

$$\Delta^z(\mathrm{T}(h)) = \mathrm{T}(h)\otimes 1 + 1\otimes \mathrm{T}(h) + \hbar[h\otimes 1, \Omega_z + \mathrm{Q}_z]$$

for all $h \in \mathfrak{h}$, in addition to the conditions *i)*–*iv)*, in several steps which will be carried out in Sections 5.2–5.5. In order to explain these steps, we introduce $\Theta_z \in Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$ via the following equation:

$$\Delta^z(\mathcal{C}_3) = \mathcal{C}_3\otimes 1 + 1\otimes \mathcal{C}_3 + \hbar\Theta_z. \tag{5.2}$$

It follows from the definitions of $\mathcal{C}_3$ and Δ^z that $\Theta_z = \sum_{\beta>0}\Theta_\beta z^{\mathsf{ht}(\beta)}$, where the summation is taken over all nonzero $\beta \in \mathsf{Q}_+$ and Θ_β satisfies

$$\Theta_\beta \in Y_\hbar^{\leqslant}(\mathfrak{g})_{-\beta}\otimes Y_\hbar^{\geqslant}(\mathfrak{g})_\beta.$$

In more detail, that Θ_z takes this form follows from Proposition 2.9 of [GW23] (or, more precisely, its proof). Next, let $\mathcal{K}_\beta = \Theta_\beta + \beta(\mathsf{h})\Omega_\beta$ for each nonzero $\beta\in\Phi_+$, so that

$$\mathcal{K}_z = \sum_{\beta>0}\mathcal{K}_\beta z^{\mathsf{ht}(\beta)} = \Theta_z - [\mathsf{h}\otimes 1, \Omega_z]. \tag{5.3}$$

These elements are of interest to us because of the equation

$$\Delta^z(\mathcal{C}_3) = \mathcal{C}_3\otimes 1 + 1\otimes\mathcal{C}_3 + \hbar\left(\mathcal{K}_z + [\mathsf{h}\otimes 1, \Omega_z]\right).$$

Thus, for the equation in (3) to hold, we must find Q_z so that $[\mathsf{h}\otimes 1, \mathrm{Q}_z] = \mathcal{K}_z$ and establish its stated properties. We proceed as follows:

1) In Section 5.2, we will prove that $\mathcal{K}_\beta = 0$ for all $\beta\in\mathsf{Q}_+\setminus\mathbb{Z}_{\geqslant 0}\delta$, and that $\mathcal{K}_z$ belongs to the centralizer of $U(\mathfrak{g}')^{\otimes 2}$ in $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$. This will be achieved in Proposition 5.2 with the help of Lemma 5.2, which computes the commutation relations between Θ_z and $\Delta^z(y)$, for all $y\in\{x_{i,0}^\pm, t_{i,1}\}_{i\in\mathbf{I}}$.

2) By the previous step, $\mathcal{K}_z$ takes the form $\mathcal{K}_z = \sum_{n>0}\mathcal{K}_{n\delta}z^{n\mathsf{ht}(\delta)}$. In Sections 5.3 and 5.4, we will explicitly compute each coefficient $\mathcal{K}_{n\delta}$ and study some of their properties; see Lemma 5.3 and Proposition 5.4.

3) In Section 5.5, we finally define $\mathrm{Q}_z = \sum_{n>0}\mathrm{Q}_{n\delta}z^{n\mathsf{ht}(\delta)}$ by setting $\mathrm{Q}_{n\delta} = -\frac{1}{n\delta(\mathsf{h})}\mathcal{K}_{n\delta}$ for each positive integer n, so that $[\mathsf{h}\otimes 1, \mathrm{Q}_z] = \mathcal{K}_z$. In Proposition 5.5, we show that Q_z has all the properties stated in Part (3) of the theorem. □

5.2. **Proof that $\mathcal{K}_\beta = 0$ for $\beta\in\mathsf{Q}_+\setminus\mathbb{Z}_{\geqslant 0}\delta$.** Our goal in this section is to show that the series $\mathcal{K}_z\in Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$ defined in (5.3) belongs to the centralizer of $U(\mathfrak{g}')^{\otimes 2}$ in $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$ and, consequently, that its component $\mathcal{K}_\beta$ is zero for $\beta\in\mathsf{Q}_+\setminus\mathbb{Z}_{\geqslant 0}\delta$. We begin with the following lemma, which spells out some of the commutation relations satisfied by Θ_z.

Lemma. *For each $i\in\mathbf{I}$, one has*

$$[\Theta_z, \Delta^z(t_{i,1})] = -[\xi_{i,0}\otimes 1, [\square(\mathcal{C}_3), \Omega_z]],$$

$$[\Theta_z, \Delta^z(x_{i,0}^+)] = \frac{\hbar^2}{4}z\mu_i[\Omega_z, 1\otimes x_{i,0}^+] \quad \textit{and} \quad [\Theta_z, \Delta^z(x_{i,0}^-)] = \frac{\hbar^2}{4}\mu_i[\Omega_z, x_{i,0}^-\otimes 1].$$

PROOF. Since $\mathcal{C}_3$ and $t_{i,1}$ commute and Δ^z is an algebra homomorphism, we have

$$[\hbar\Theta_z, \Delta^z(t_{i,1})] = [\Delta^z(\mathcal{C}_3), \Delta^z(t_{i,1})] - [\square(\mathcal{C}_3), \Delta^z(t_{i,1})] = -[\square(\mathcal{C}_3), \Delta^z(t_{i,1})].$$

Since $\square(\mathcal{C}_3)$ and $\square(t_{i,1})$ commute, we obtain

$$[\hbar\Theta_z, \Delta^z(t_{i,1})] = -\hbar[\square(\mathcal{C}_3), [\xi_{i,0} \otimes 1, \Omega_z]] = -\hbar[\xi_{i,0} \otimes 1, [\square(\mathcal{C}_3), \Omega_z]],$$

which yields the first identity of the lemma. Similarly, we have

$$[\hbar\Theta_z, \Delta^z(x^\pm_{i,0})] = \Delta^z([\mathcal{C}_3, x^\pm_{i,0}]) - \square^z[\mathcal{C}_3, x^\pm_{i,0}] = \pm\frac{\hbar^2}{4}\mu_i(\Delta^z - \square^z)(x^\pm_{i,1}),$$

where we recall that $\square^z = (1 \otimes \sigma_z) \circ \square$. The second and third relations of the lemma now follow from (3.12). $\square$

Proposition. *The element $\mathcal{K}_z$ has the following properties:*

(1) *For each $x \in \mathfrak{g}'$, one has*

$$[\mathcal{K}_z, x \otimes 1] = 0 = [1 \otimes x, \mathcal{K}_z].$$

(2) $\mathcal{K}_\beta = 0$ *for all* $\beta \in \mathsf{Q}_+ \setminus \mathbb{Z}_{\geqslant 0}\delta$.

PROOF. If Part (1) holds, then by taking $x = \xi_{i,0}$ in the relation $[\mathcal{K}_z, x \otimes 1] = 0$ we find that

$$0 = [\mathcal{K}_z, \xi_{i,0} \otimes 1] = \sum_{\beta>0} (\alpha_i, \beta)\mathcal{K}_\beta z^{\mathsf{ht}(\beta)} \quad \forall\, i \in \mathbf{I}.$$

Projecting onto the $Y_\hbar(\mathfrak{g})_{-\beta} \otimes Y_\hbar(\mathfrak{g})_\beta$-component, we obtain $(\alpha_i, \beta)\mathcal{K}_\beta = 0$ for all $i \in \mathbf{I}$, and thus $\mathcal{K}_\beta = 0$ provided $(\beta, \alpha_i) \neq 0$ for some $i \in \mathbf{I}$. This shows that Part (2) follows from Part (1).

Let us now turn to proving Part (1). For each nonzero $\beta \in \mathsf{Q}_+$, set

$$\Theta_\beta(s) = (\tau_s \otimes \mathrm{Id})(\Theta_\beta) - \Theta_\beta \in s(Y^\leqslant_\hbar(\mathfrak{g})_{-\beta} \otimes Y^\geqslant_\hbar(\mathfrak{g})_\beta)[s].$$

Note that each of these is a polynomial of degree at most 2. Indeed, by (3.13), $(\Delta_z - \square)(t_{i,3})$ belongs to the subspace $\mathbf{F}_2(Y_\hbar(\mathfrak{g})^{\otimes 2})[\![z]\!]$ of $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$. So, the same is true for $(\Delta_z - \square)(\mathcal{C}_3)$ and thus Θ_z. As $\tau_s \otimes \mathrm{Id}$ sends any element of $\mathbf{F}_k(Y_\hbar(\mathfrak{g})^{\otimes 2})$ to a polynomial in s of degree at most k (see Section 3.6 above), the assertion follows. This allows us to write

$$\Theta_z(s) = \sum_{\beta>0} \Theta_\beta(s) z^{\mathsf{ht}(\beta)} = \Theta_z^{(1)} s + \Theta_z^{(2)} s^2,$$

where $\Theta_z^{(a)} \in Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$ for each $a = 1, 2$.

Furthermore, $\Theta_z(s)$ is $\mathfrak{g}$-invariant, in the sense that it satisfies

$$[\Delta^z(x), \Theta_z(s)] = 0 \quad \forall\, x \in \mathfrak{g}. \tag{5.4}$$

For $x \in \mathfrak{h}$ this is clear from the form of $\Theta_z(s)$, and for $x = x_i^\pm$ this follows by applying $\tau_s \otimes \mathrm{Id}$ to the second line of identities from Lemma 5.2.

Applying $\tau_s \otimes \mathrm{Id}$ instead to the first equation of Lemma 5.2 while using that $\mathcal{C}_0$ and $\mathcal{C}_1$ are central, we obtain

$$[\Theta_z + \Theta_z(s), \Delta^z(t_{i,1}) + s\xi_{i,0} \otimes 1] = -[\xi_{i,0} \otimes 1, [\square(\mathcal{C}_3) + 3s\mathcal{C}_2 \otimes 1, \Omega_z]].$$

Expanding both sides and subtracting the first relation of Lemma 5.2 yields

$$s[\Theta_z, \xi_{i,0} \otimes 1] + s[\Theta_z(s), \xi_{i,0} \otimes 1] + [\Theta_z(s), \Delta^z(t_{i,1})] = -3s[\xi_{i,0} \otimes 1, [\mathcal{C}_2 \otimes 1, \Omega_z]] \tag{5.5}$$

Comparing powers of s, we immediately deduce that

$$[\Theta_z^{(2)}, \xi_{i,0}\otimes 1] = 0 \quad \text{and} \quad [\Theta_z^{(1)}, \xi_{i,0}\otimes 1] = -[\Theta_z^{(2)}, \Delta^z(t_{i,1})]. \tag{5.6}$$

It follows from the first identity that the β-component $\Theta_{z,\beta}^{(2)}$ of $\Theta_z^{(2)}$ is zero unless $(\beta,\alpha_i)=0$ for all $i\in\mathbf{I}$. Furthermore, by using this identity, and applying $\mathrm{ad}(\xi_{i,0}\otimes 1)$ to (5.4) with $x = x_{i,0}^{\pm}$, we deduce that

$$[\Theta_z^{(2)}, x\otimes 1] = 0 = [\Theta_z^{(2)}, 1\otimes x] \quad \forall\, x\in\mathfrak{g}'. \tag{5.7}$$

Consequently, the bracket $[\Theta_z^{(2)}, \Delta^z(t_{i,1})]$ coincides with $[\Theta_z^{(2)}, \square(t_{i,1})]$, and hence the second equation of (5.6) yields

$$[\Theta_{z,\beta}^{(1)}, \xi_{i,0}\otimes 1] = -[\Theta_{z,\beta}^{(2)}, \square(t_{i,1})] \quad \forall\, \beta\in\mathsf{Q}_+\setminus\{0\},\ i\in\mathbf{I}.$$

Note that the left–hand side is zero if $\beta\in\alpha_i^{\perp}$, and the right–hand side is zero if $\beta\notin\alpha_i^{\perp}$. Hence both sides are identically zero for all nonzero $\beta\in\mathsf{Q}_+$. Using this observation together with (5.4), (5.5), and (5.7), and the fact that $\mathrm{ad}(\mathsf{h}) = 3\,\mathrm{ad}(\mathcal{C}_2)$, we conclude that

$$\begin{gathered}[\Theta_z(s), x\otimes 1] = 0 = [\Theta_z(s), 1\otimes x]\\ s[\mathcal{K}_z, \xi_{i,0}\otimes 1] = -[\Theta_z(s), \Delta^z(t_{i,1})] = -[\Theta_z(s), \square(t_{i,1})]\end{gathered} \tag{5.8}$$

for all $x\in\mathfrak{g}'$ and $i\in\mathbf{I}$. From the first equality with $x\in\mathfrak{h}'$, we find that the component $\Theta_\beta(s)$ is zero unless $\beta\in\cap_{i\in\mathbf{I}}\alpha_i^{\perp}$. As $s[\mathcal{K}_z,\xi_{i,0}\otimes 1]$ has no such component and $\mathrm{ad}(\square(t_{i,1}))$ is weight zero, we deduce from the second line that

$$[\mathcal{K}_z, \xi_{i,0}\otimes 1] = 0 \quad \forall\, i\in\mathbf{I}.$$

To prove that $[\mathcal{K}_z, x\otimes 1] = 0 = [\mathcal{K}_z, 1\otimes x]$ for general $x\in\mathfrak{g}'$, it now suffices to show that $\mathcal{K}_z$ commutes with $\Delta^z(x_{i,0}^{\pm})$ for each $i\in\mathbf{I}$. We have

$$3[[\mathcal{C}_2\otimes 1, \Omega_z], \Delta^z(x_{i,0}^{\pm})] = \pm\frac{\hbar^2}{4}\mu_i[x_{i,0}^{\pm}\otimes 1, \Omega_z] + 3[\mathcal{C}_2\otimes 1, [\Omega_z, \square^z(x_{i,0}^{\pm})]].$$

By Lemma 4.2 and §6.3 of [GNW18], we have $[\Omega_z, \square^z(x_{i,0}^{+})] = x_{i,0}^{+}\otimes\xi_{i,0}$ and $[\Omega_z, \square^z(x_{i,0}^{-})] = -z^{-1}\xi_{i,0}\otimes x_{i,0}^{-}$. Hence, we obtain

$$3[[\mathcal{C}_2\otimes 1, \Omega_z], \Delta^z(x_{i,0}^{-})] = -\frac{\hbar^2}{4}\mu_i[x_{i,0}^{-}\otimes 1, \Omega_z],$$

$$3[[\mathcal{C}_2\otimes 1, \Omega_z], \Delta^z(x_{i,0}^{+})] = \frac{\hbar^2}{4}\mu_i[x_{i,0}^{+}\otimes 1, \Omega_z] + \frac{\hbar^2}{4}\mu_i x_{i,0}^{+}\otimes\xi_{i,0} = -\frac{\hbar^2}{4}z\mu_i[1\otimes x_{i,0}^{+}, \Omega_z].$$

Combining these calculations with Lemma 5.2 yields $[\mathcal{K}_z, \Delta^z(x_{i,0}^{\pm})] = 0$ for all $i\in\mathbf{I}$, as desired. This completes the proof of Part (1). □

5.3. **Computing $\mathcal{K}_{n\delta}$ explicitly, I.** By the results of the previous section, $\mathcal{K}_z$ is a series of the form $\sum_{n>0}\mathcal{K}_{n\delta}z^{n\mathsf{ht}(\delta)}$ with each coefficient $\mathcal{K}_{n\delta}$ belonging to the centralizer of $U(\mathfrak{g}')^{\otimes 2}$. The goal of this section and Section 5.4 is to compute these coefficients explicitly. Our starting point is the following lemma, which will also play a crucial role in Section 5.5.

Lemma. *Let n be a positive integer and set $\beta = n\delta$. Then, for each $i\in\mathbf{I}$, one has*

$$[\square(t_{i,1}), \Theta_\beta] = d_i\hbar\sum_{\alpha+\gamma=\beta}\alpha(\mathsf{h})\gamma(h_i)\,[\Omega_\alpha, \Omega_\gamma]\,,$$

where the summation runs over all positive roots $\alpha,\gamma\in\Phi_+$ such that $\alpha+\gamma=\beta$. Moreover, the element $\Gamma_{i,\beta}=[\square(t_{i,1}),\mathcal{K}_\beta]$ has the following properties:

(1) *For each $a,b\in\mathbb{C}$, one has $(\tau_a\otimes\tau_b)(\Gamma_{i,\beta})=\Gamma_{i,\beta}$.*

(2) *One has*

$$(\Delta^z\otimes\mathrm{Id})(\Gamma_{i,\beta})=\Gamma^{13}_{i,\beta}+\Gamma^{23}_{i,\beta}z^{-\mathsf{ht}(\beta)}+\Gamma^-_{i,\beta}(z),$$
$$(\mathrm{Id}\otimes\Delta^z)(\Gamma_{i,\beta})=\Gamma^{12}_{i,\beta}+\Gamma^{13}_{i,\beta}z^{\mathsf{ht}(\beta)}+\Gamma^+_{i,\beta}(z),$$

where $\Gamma^-_{i,\beta}(z)$ and $\Gamma^+_{i,\beta}(z)$ are given explicitly by

$$\Gamma^-_{i,\beta}(z)=d_i\hbar\sum_{\alpha+\gamma=\beta}(\alpha(\mathsf{h})\gamma(h_i)-\gamma(\mathsf{h})\alpha(h_i))[\Omega^{13}_\alpha,\Omega^{23}_\gamma]z^{-\mathsf{ht}(\gamma)}$$
$$+\hbar\beta(\mathsf{h})\mathrm{ad}(\xi^{(1)}_{i,0})\left([\Omega^{12}_z,\Omega^{13}_\beta+\Omega^{23}_\beta z^{-\mathsf{ht}(\beta)}]\right),$$

$$\Gamma^+_{i,\beta}(z)=d_i\hbar\sum_{\alpha+\gamma=\beta}(\alpha(\mathsf{h})\gamma(h_i)-\gamma(\mathsf{h})\alpha(h_i))\,[\Omega^{12}_\alpha,\Omega^{13}_\gamma]z^{\mathsf{ht}(\gamma)}$$
$$+\hbar\beta(\mathsf{h})\mathrm{ad}(\xi^{(2)}_{i,0})\left([\Omega^{23}_z,\Omega^{12}_\beta+\Omega^{13}_\beta z^{\mathsf{ht}(\beta)}]\right),$$

with both summations taken over all $\alpha,\gamma\in\Phi_+$ such that $\alpha+\gamma=\beta$.

(3) *The series $\Gamma^+_{i,\beta}(z)$ and $\Gamma^-_{i,\beta}(z)$ satisfy*

$$\Gamma^\pm_{i,\beta}(z)\in(\mathfrak{n}^-\otimes\mathfrak{n}^\pm\otimes\mathfrak{n}^+)[z^{\pm1}]$$

Proof. From the first identity of Lemma 5.2, we obtain

$$[\Theta_z,\square(t_{i,1})]+\hbar[\Theta_z,[\xi_{i,0}\otimes 1,\Omega_z]]=-[\xi_{i,0}\otimes 1,[\square(\mathcal{C}_3),\Omega_z]].$$

As $\mathrm{ad}(\mathcal{C}_3)$ is weight zero, the right-hand side has no $Y_\hbar(\mathfrak{g})_{-\beta}\otimes Y_\hbar(\mathfrak{g})_\beta$ component. Since $\mathrm{ad}(t_{i,1})$ is weight zero, this implies that

$$[\square(t_{i,1}),\Theta_\beta]=\hbar\sum_{\alpha+\gamma=\beta}[\Theta_\alpha,[\xi_{i,0}\otimes 1,\Omega_\gamma]]=-d_i\hbar\sum_{\alpha+\gamma=\beta}\gamma(h_i)[\Theta_\alpha,\Omega_\gamma],\tag{5.9}$$

where the summations are taken over all $\gamma\in\Phi_+$ and $\alpha\in\mathsf{Q}_+\setminus\{0\}$ such that $\alpha+\gamma=\beta$. However, for the term $\gamma(h_i)[\Theta_\alpha,\Omega_\gamma]$ to provide a nonzero contribution, we must have $\gamma\notin\alpha_i^\perp$. As $\alpha+\gamma=\beta$ and $\beta\in\alpha_i^\perp$, this implies that $\alpha\notin\alpha_i^\perp$ as well. By Part (2) of Proposition 5.2, for any such α we have

$$\mathcal{K}_\alpha=\Theta_\alpha+\alpha(\mathsf{h})\Omega_\alpha=0.$$

The formula for $[\square(t_{i,1}),\Theta_\beta]$ stated in the lemma now follows from (5.9) after replacing Θ_α by $-\alpha(\mathsf{h})\Omega_\alpha$.

We now turn to establishing Parts (1)–(3) of the lemma.

Proof of Part (1). From what has been proven above, we have

$$\Gamma_{i,\beta}=d_i\hbar\sum_{\alpha+\gamma=\beta}\alpha(\mathsf{h})\gamma(h_i)\,[\Omega_\alpha,\Omega_\gamma]+\beta(\mathsf{h})[\square(t_{i,1}),\Omega_\beta].$$

It is therefore sufficient to prove that $[\square(t_{i1}),\Omega_\beta]$ is fixed by $\tau_a\otimes\tau_b$ for any $a,b\in\mathbb{C}$. We have

$$(\tau_a\otimes\tau_b)([\square(t_{i,1}),\Omega_\beta])=[\square(t_{i,1})+a\xi_{i,0}\otimes 1+1\otimes b\xi_{i,0},\Omega_\beta]$$
$$=[\square(t_{i,1}),\Omega_\beta]+(\beta,\alpha_i)(b-a)\Omega_\beta.$$

Since $(\beta, \alpha_i) = n(\delta, \alpha_i) = 0$, this coincides with $[\square(t_{i,1}), \Omega_\beta]$.

Proof of Part (2). By the first assertion of the lemma, we have

$$\begin{aligned}
&(\Delta^z \otimes \mathrm{Id})([\square(t_{i,1}), \Theta_\beta]) \\
&= d_i\hbar \sum_{\alpha+\gamma=\beta} \alpha(\mathsf{h})\gamma(h_i) \left[\Omega_\alpha^{13} + \Omega_\alpha^{23} z^{-\mathsf{ht}(\alpha)}, \Omega_\gamma^{13} + \Omega_\gamma^{23} z^{-\mathsf{ht}(\gamma)}\right] \\
&= (\square^z \otimes \mathrm{Id})[\square(t_{i,1}), \Theta_\beta] \\
&\qquad + d_i\hbar \sum_{\alpha+\gamma=\beta} \alpha(\mathsf{h})\gamma(h_i)([\Omega_\alpha^{13}, \Omega_\gamma^{23}] z^{-\mathsf{ht}(\gamma)} + [\Omega_\alpha^{23}, \Omega_\gamma^{13}] z^{-\mathsf{ht}(\alpha)}) \\
&= (\square^z \otimes \mathrm{Id})[\square(t_{i,1}), \Theta_\beta] + d_i\hbar \sum_{\alpha+\gamma=\beta} (\alpha(\mathsf{h})\gamma(h_i) - \gamma(\mathsf{h})\alpha(h_i))[\Omega_\alpha^{13}, \Omega_\gamma^{23}] z^{-\mathsf{ht}(\gamma)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&(\Delta^z \otimes \mathrm{Id})[\square(t_{i,1}), \Omega_\beta] \\
&= [t_{i,1}^{(3)} + \square(t_{i,1}) \otimes 1 + \hbar[\xi_{i,0} \otimes 1, \Omega_z] \otimes 1, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}] \\
&= (\square^z \otimes \mathrm{Id})[\square(t_{i,1}), \Omega_\beta] + \hbar[\mathrm{ad}(\xi_{i,0}^{(1)})(\Omega_z^{12}), \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}] \\
&= (\square^z \otimes \mathrm{Id})[\square(t_{i,1}), \Omega_\beta] + \hbar \mathrm{ad}(\xi_{i,0}^{(1)})([\Omega_z^{12}, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}]),
\end{aligned}$$

where in the last line we have used that $\mathrm{ad}(\xi_{i,0}^{(1)})$ commutes with Ω_β^{13} and Ω_β^{23} as $\beta = n\delta$. As $\Gamma_{i,\beta} = [\square(t_{i,1}), \Theta_\beta + \beta(\mathsf{h})\Omega_\beta]$, this implies that

$$(\Delta^z \otimes \mathrm{Id})(\Gamma_{i,\beta}) = \Gamma_{i,\beta}^{13} + \Gamma_{i,\beta}^{23} z^{-\mathsf{ht}(\beta)} + \Gamma_{i,\beta}^-(z)$$

with $\Gamma_{i,\beta}^-(z)$ as in the statement of the lemma. The computation of $(\mathrm{Id} \otimes \Delta^z)(\Gamma_{i,\beta})$ is nearly identical, and hence omitted.

Proof of Part (3). As the proofs for $\Gamma_{i,\beta}^-(z)$ and $\Gamma_{i,\beta}^+(z)$ are the same, we will focus on the former. By the formulas of Part (2), it is sufficient to prove that

$$\mathrm{ad}(\xi_{i,0}^{(1)}) \left([\Omega_z^{12}, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}]\right) \in (\mathfrak{n}^- \otimes \mathfrak{n}^- \otimes \mathfrak{n}^+)[z^{-1}].$$

It is clear that $\mathrm{ad}(\xi_{i,0}^{(1)}) \left([\Omega_z^{12}, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}]\right)$ is a Laurent series in z with coefficients in $\mathfrak{n}^- \otimes \mathfrak{g}' \otimes \mathfrak{n}^+$. Let $\widetilde{\Omega}_z = \Omega_z + (\Omega_{z^{-1}}^-)^{21} \in \mathfrak{g}^{\otimes 2}[\![z^{\pm 1}]\!]$. Note that this is just $(\mathrm{Id} \otimes \sigma_z)(\widetilde{\Omega})$ where $\widetilde{\Omega}$ is the *full* Casimir tensor of $\mathfrak{g}$. It satisfies

$$[\widetilde{\Omega}_z, \square^z(x)] = 0 \quad \forall\, x \in \mathfrak{g}.$$

Hence, we have

$$[\Omega_z^{12}, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}] = -[(\Omega_{z^{-1}}^-)^{21}, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}],$$

which is a formal series in z^{-1} with coefficients in $\mathfrak{g}' \otimes \mathfrak{n}^- \otimes \mathfrak{n}^+$. Hence, we have

$$\mathrm{ad}(\xi_{i,0}^{(1)}) \left([\Omega_z^{12}, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}]\right) \in (\mathfrak{n}^- \otimes \mathfrak{g}' \otimes \mathfrak{n}^+)[z^{-1}; z]\!] \cap (\mathfrak{g}' \otimes \mathfrak{n}^- \otimes \mathfrak{n}^+)[\![z^{-1}]\!]$$

This shows that $\mathrm{ad}(\xi_{i,0}^{(1)}) \left([\Omega_z^{12}, \Omega_\beta^{13} + \Omega_\beta^{23} z^{-\mathsf{ht}(\beta)}]\right)$ is a polynomial in z^{-1} with coefficients in $\mathfrak{n}^- \otimes \mathfrak{n}^- \otimes \mathfrak{n}^+$, and thus that $\Gamma_{i,\beta}^-(z) \in (\mathfrak{n}^- \otimes \mathfrak{n}^- \otimes \mathfrak{n}^+)[z^{-1}]$. □

5.4. **Computing $\mathcal{K}_{n\delta}$ explicitly, II.** We now turn towards computing each coefficient $\mathcal{K}_{n\delta}$ of the series $\mathcal{K}_z$. For the sake of brevity, let us define a linear operator $\mathrm{ad}_{x,y}$ on $Y_\hbar(\mathfrak{g})$, for each $x, y \in Y_\hbar(\mathfrak{g})$, by setting

$$\mathrm{ad}_{x,y} = \mathrm{ad}(x) \circ \mathrm{ad}(y) : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g}).$$

For each $n > 0$ and $\ell \in \{1, 2\}$, we then define

$$\mathcal{K}_{i,n\delta;\ell} = (\mathrm{ad}^{(\ell)}_{x^-_{i,1},x^+_{i,0}} + \mathrm{ad}^{(\ell)}_{x^+_{i,1},x^-_{i,0}})(\Gamma_{i,n\delta}) - \hbar\mathrm{ad}^{(\ell)}_{x^-_{i,0},x^+_{i,0}}(\Gamma_{i,n\delta}) \cdot \xi^{(\ell)}_{i,0},$$

where the superscript in $\mathrm{ad}^{(\ell)}_{x,y}$ indicates that it operates in the ℓ-th tensor factor.

Proposition. *Let n be a positive integer and fix $\ell \in \{1, 2\}$. Then $\mathcal{K}_{n\delta}$ is given explicitly by*

$$\mathcal{K}_{n\delta} = (-1)^\ell \frac{3}{2n\delta(\mathsf{h})} \sum_{i\in\mathbf{I}} \frac{a_i}{di} \mathcal{K}_{i,n\delta;\ell}.$$

PROOF. We shall establish the claimed formula for $\mathcal{K}_{n\delta}$ in the case where $\ell = 1$. The proof in the $\ell = 2$ case follows by a simple modification of the same argument. To begin, note that since $[3\mathcal{C}_2 \otimes 1, \mathcal{K}_{n\delta}] = -n\delta(\mathsf{h})\mathcal{K}_{n\delta}$ (and $\delta(\mathsf{h}) \neq 0$), it is sufficient to establish that

$$[\mathcal{C}_2 \otimes 1, \mathcal{K}_{n\delta}] = \sum_{i\in\mathbf{I}} \frac{a_i}{2di} \mathcal{K}_{i,n\delta;1}. \tag{5.10}$$

By Part (1) of Proposition 5.2, we have

$$[x^\pm_{i,0} \otimes 1, \Gamma_{i,n\delta}] = [x^\pm_{i,0} \otimes 1, [\square(t_{i,1}), \mathcal{K}_{n\delta}]] = \mp 2d_i[x^\pm_{i,1} \otimes 1, \mathcal{K}_{n\delta}],$$

and therefore

$$[x^\pm_{i,1} \otimes 1, \mathcal{K}_{n\delta}] = \mp\frac{1}{2d_i}[x^\pm_{i,0} \otimes 1, \Gamma_{i,n\delta}] \quad \forall\, i \in \mathbf{I}.$$

It follows that, for each $i \in \mathbf{I}$, we have

$$\begin{aligned}[\xi_{i,1} \otimes 1, \mathcal{K}_{n\delta}] &= [[x^+_{i,1} \otimes 1, x^-_{i,0} \otimes 1], \mathcal{K}_{n\delta}] \\ &= -\frac{1}{2d_i}[[x^+_{i,0} \otimes 1, \Gamma_{i,n\delta}], x^-_{i,0} \otimes 1] = \frac{1}{2d_i}\mathrm{ad}^{(1)}_{x^-_{i,0},x^+_{i,0}}(\Gamma_{i,n\delta})\end{aligned}$$

where in the second equality we have used that, by Proposition 5.2, $[x^-_{i,0} \otimes 1, \mathcal{K}_{n\delta}] = 0$. Similarly, we obtain

$$\begin{aligned}[\xi_{i,2} \otimes 1, \mathcal{K}_{n\delta}] &= [[x^+_{i,1} \otimes 1, x^-_{i,1} \otimes 1], \mathcal{K}_{n\delta}] \\ &= -[x^-_{i,1} \otimes 1, [x^+_{i,1} \otimes 1, \mathcal{K}_{n\delta}]] + [x^+_{i,1} \otimes 1, [x^-_{i,1} \otimes 1, \mathcal{K}_{n\delta}]] \\ &= \frac{1}{2d_i}(\mathrm{ad}^{(1)}_{x^-_{i,1},x^+_{i,0}} + \mathrm{ad}^{(1)}_{x^+_{i,1},x^-_{i,0}})(\Gamma_{i,n\delta}).\end{aligned}$$

Since $t_{i,2} = \xi_{i,2} - \hbar\xi_{i,1}\xi_{i,0} + \frac{\hbar^2}{3}\xi^3_{i,0}$ (see (3.3)) and $[\xi_{i,0} \otimes 1, \mathcal{K}_{n\delta}] = 0$, the above computations yield

$$\begin{aligned}2d_i[t_{i,2} \otimes 1, \mathcal{K}_{n\delta}] &= (\mathrm{ad}^{(1)}_{x^-_{i,1},x^+_{i,0}} + \mathrm{ad}^{(1)}_{x^+_{i,1},x^-_{i,0}})(\Gamma_{i,n\delta}) - \hbar\mathrm{ad}^{(1)}_{x^-_{i,0},x^+_{i,0}}(\Gamma_{i,n\delta}) \cdot \xi^{(1)}_{i,0} \\ &= \mathcal{K}_{i,n\delta;1}.\end{aligned}$$

As $\mathcal{C}_2 = \sum_{i\in\mathbf{I}} a_i t_{i,2}$, this implies the formula (5.10). $\square$

5.5. **The series Q_z.** In this section, we use the results of the previous subsections to construct a series Q_z satisfying all the properties listed in Part (3) of Theorem 5.1. To begin, recall from Proposition 5.4 that for each positive integer n, index $i \in \mathbf{I}$, and number $\ell \in \{1,2\}$, we defined

$$\mathcal{K}_{i,n\delta;\ell} = (\mathrm{ad}^{(\ell)}_{x^-_{i,1},x^+_{i,0}} + \mathrm{ad}^{(\ell)}_{x^+_{i,1},x^-_{i,0}})(\Gamma_{i,n\delta}) - \hbar \mathrm{ad}^{(\ell)}_{x^-_{i,0},x^+_{i,0}}(\Gamma_{i,n\delta}) \cdot \xi^{(\ell)}_{i,0},$$

where $\xi^{(\ell)}_{i,0} = 1^{\otimes(\ell-1)} \otimes \xi_{i,0} \otimes 1^{\otimes(2-\ell)}$ and $\Gamma_{i,n\delta}$ and $\mathrm{ad}^{(\ell)}_{x,y}$ are given as follows:

i) $\Gamma_{i,n\delta} = [\square(t_{i,1}), \mathcal{K}_{n\delta}]$. By Lemma 5.3, this may be written equivalently as

$$\Gamma_{i,n\delta} = d_i\hbar \sum_{\alpha+\gamma=n\delta} \alpha(\mathsf{h})\gamma(h_i)\,[\Omega_\alpha, \Omega_\gamma] + n\delta(\mathsf{h})[\square(t_{i,1}), \Omega_{n\delta}],$$

with $\mathsf{h} = \frac{\hbar^2}{\zeta}(d - \sum_{i\in\mathbf{I}} \zeta_i d_i h_i) \in \mathfrak{h}$, as in (5.1).

ii) $\mathrm{ad}_{x,y} = \mathrm{ad}(x) \circ \mathrm{ad}(y)$ for all $x, y \in Y_\hbar(\mathfrak{g})$, while

$$\mathrm{ad}^{(1)}_{x,y} = \mathrm{ad}_{x,y} \otimes \mathrm{Id} \quad \text{and} \quad \mathrm{ad}^{(2)}_{x,y} = \mathrm{Id} \otimes \mathrm{ad}_{x,y}.$$

We now introduce the distinguished series $\mathrm{Q}_z \in Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$ by setting

$$\mathrm{Q}_z = -\sum_{n>0} \frac{1}{n\delta(\mathsf{h})} \mathcal{K}_{n\delta} z^{n\mathsf{ht}(\delta)}.$$

By Proposition 5.4, the coefficient $\mathrm{Q}_{n\delta} = \frac{1}{n\delta(\mathsf{h})}\mathcal{K}_{n\delta}$ is given by

$$\mathrm{Q}_{n\delta} = \frac{3}{2n^2\delta(\mathsf{h})^2} \sum_{i\in\mathbf{I}} \frac{a_i}{di} \mathcal{K}_{i,n\delta;1} = -\frac{3}{2n^2\delta(\mathsf{h})^2} \sum_{i\in\mathbf{I}} \frac{a_i}{di} \mathcal{K}_{i,n\delta;2}. \tag{5.11}$$

The following proposition completes the proof of Theorem 5.1 by showing that Q_z satisfies all the conditions spelled out in Part (3) therein.

Proposition. *The series Q_z has the following properties:*

(1) *For each $h \in \mathfrak{h}$, one has*

$$\Delta^z(\mathrm{T}(h)) = \mathrm{T}(h) \otimes 1 + 1 \otimes \mathrm{T}(h) + \hbar[h \otimes 1, \Omega_z + \mathrm{Q}_z].$$

(2) *For each $n > 0$, one has*

$$\mathrm{Q}_{n\delta} \in Y^-_\hbar(\mathfrak{g})_{-n\delta} \otimes Y^+_\hbar(\mathfrak{g})_{n\delta}.$$

(3) *For each $a, b \in \mathbb{C}$, one has*

$$(\tau_a \otimes \tau_b)(\mathrm{Q}_z) = \mathrm{Q}_z.$$

(4) *The tensor factors of Q_z are primitive:*

$$(\Delta^w \otimes \mathrm{Id})(\mathrm{Q}_z) = \mathrm{Q}^{13}_z + \mathrm{Q}^{23}_{z/w} \quad \textit{and} \quad (\mathrm{Id} \otimes \Delta^w)(\mathrm{Q}_z) = \mathrm{Q}^{12}_z + \mathrm{Q}^{13}_{zw}.$$

(5) *Q_z belongs to the centralizer of $U(\mathfrak{g}')^{\otimes 2}$ in $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!]$.*

Proof. Note that Part (5) follows immediately from the definition of Q_z and Part (1) of Proposition 5.2. We shall prove the remaining four statements in order.

Proof of (1). If $h \in \mathfrak{h}'$, then $[h \otimes 1, \mathrm{Q}_z] = 0$ and so the claimed formula for $\Delta^z(\mathrm{T}(h))$ holds by definition of Δ^z (see (3.9)). It thus suffices to verify the formula in the case where $h = \mathsf{h}$. Since $\mathrm{T}(\mathsf{h}) = \mathcal{C}_3$, this amounts to checking

$$\Delta^z(\mathcal{C}_3) = \mathcal{C}_3 \otimes 1 + 1 \otimes \mathcal{C}_3 + \hbar[\mathsf{h} \otimes 1, \Omega_z + \mathrm{Q}_z].$$

This is a consequence of the definition of Q_z. Indeed, we have

$$[\mathsf{h} \otimes 1, \mathrm{Q}_z] = -\sum_{n>0} n\delta(\mathsf{h})\mathrm{Q}_{n\delta} z^{n\mathsf{ht}(\delta)} = \mathcal{K}_z = \Theta_z - [\mathsf{h} \otimes 1, \Omega_z].$$

Proof of (2). By Lemma 5.3, we have

$$\Gamma_{i,n\delta} = d_i\hbar \sum_{\alpha+\gamma=n\delta} \alpha(\mathsf{h})\gamma(h_i)\,[\Omega_\alpha, \Omega_\gamma] + n\delta(\mathsf{h})[\square(t_{i,1}), \Omega_{n\delta}] \in Y_\hbar^-(\mathfrak{g})_{-n\delta} \otimes Y_\hbar^+(\mathfrak{g})_{n\delta}.$$

It follows by definition of $\mathcal{K}_{i,n\delta;1}$ and $\mathcal{K}_{i,n\delta;2}$ that

$$\mathcal{K}_{i,n\delta;1} \in Y_\hbar(\mathfrak{g})_{-n\delta} \otimes Y_\hbar^+(\mathfrak{g})_{n\delta} \quad \text{and} \quad \mathcal{K}_{i,n\delta;2} \in Y_\hbar^-(\mathfrak{g})_{-n\delta} \otimes Y_\hbar(\mathfrak{g})_{n\delta}.$$

Thus, we conclude from (5.11) that $\mathrm{Q}_{n\delta}$ belongs to $Y_\hbar^-(\mathfrak{g})_{-n\delta} \otimes Y_\hbar^+(\mathfrak{g})_{n\delta}$.

Proof of (3). Fix a positive integer n and complex numbers $a, b \in \mathbb{C}$. Then, by definition of $\mathcal{K}_{i,n\delta;1}$ and $\mathcal{K}_{i,n\delta;2}$, we have

$$\begin{aligned}
&(\mathrm{Id} \otimes \tau_b)(\mathcal{K}_{i,n\delta;1})\\
&= (\mathrm{ad}^{(1)}_{x_{i,1}^-, x_{i,0}^+} + \mathrm{ad}^{(1)}_{x_{i,1}^+, x_{i,0}^-})((\mathrm{Id} \otimes \tau_b)\Gamma_{i,n\delta}) - \hbar \mathrm{ad}^{(1)}_{x_{i,0}^-, x_{i,0}^+}((\mathrm{Id} \otimes \tau_b)\Gamma_{i,n\delta}) \cdot \xi_{i,0}^{(1)},\\
&(\tau_a \otimes \mathrm{Id})(\mathcal{K}_{i,n\delta;2})\\
&= (\mathrm{ad}^{(2)}_{x_{i,1}^-, x_{i,0}^+} + \mathrm{ad}^{(2)}_{x_{i,1}^+, x_{i,0}^-})((\tau_a \otimes \mathrm{Id})\Gamma_{i,n\delta}) - \hbar \mathrm{ad}^{(2)}_{x_{i,0}^-, x_{i,0}^+}((\tau_a \otimes \mathrm{Id})\Gamma_{i,n\delta}) \cdot \xi_{i,0}^{(2)}.
\end{aligned}$$

Since $(\tau_a \otimes \tau_b)(\Gamma_{i,n\delta}) = \Gamma_{i,n\delta}$ by Part (1) of Lemma 5.3, we get that $(\mathrm{Id} \otimes \tau_b)(\mathcal{K}_{i,n\delta;1}) = \mathcal{K}_{i,n\delta;1}$ and $(\tau_a \otimes \mathrm{Id})(\mathcal{K}_{i,n\delta;2}) = \mathcal{K}_{i,n\delta;2}$. By (5.11), this implies that

$$(\tau_a \otimes \tau_b)(\mathrm{Q}_z) = \mathrm{Q}_z \quad \forall\, a, b \in \mathbb{C}.$$

Proof of (4). By Part (2) of Lemma 5.3, we have

$$\begin{aligned}
&((\Delta^z - \square^z) \otimes \mathrm{Id})(\mathcal{K}_{i,n\delta;2})\\
&= (\mathrm{ad}^{(3)}_{x_{i,1}^-, x_{i,0}^+} + \mathrm{ad}^{(3)}_{x_{i,1}^+, x_{i,0}^-})(\Gamma_{i,n\delta}^-(z)) - \hbar \mathrm{ad}^{(3)}_{x_{i,0}^-, x_{i,0}^+}(\Gamma_{i,n\delta}^-(z)) \cdot \xi_{i,0}^{(3)}.
\end{aligned}$$

By Part (3) of Lemma 5.3 and Proposition 5.4, this yields that

$$\mathcal{K}_{n\delta}^-(z) = ((\Delta^z - \square^z) \otimes \mathrm{Id})(\mathcal{K}_{n\delta}) \in (\mathfrak{n}^- \otimes \mathfrak{n}^- \otimes Y_\hbar(\mathfrak{g}))[z^{-1}] \tag{5.12}$$

Similarly, Lemma 5.3 and Proposition 5.4 imply that

$$\mathcal{K}_{n\delta}^+(z) = (\mathrm{Id} \otimes (\Delta^z - \square^z))(\mathcal{K}_{n\delta}) \in (Y_\hbar(\mathfrak{g}) \otimes \mathfrak{n}^+ \otimes \mathfrak{n}^+)[z]. \tag{5.13}$$

Note that, by definition of Q_z, the statement of Part (4) is equivalent to the assertion that $\mathcal{K}_{n\delta}^+(z) = 0 = \mathcal{K}_{n\delta}^-(z)$. This is established in the following claim.

Claim. $\mathcal{K}_{n\delta}^+(z)$ and $\mathcal{K}_{n\delta}^+(z)$ are both equal to 0.

Proof of Claim. Define $\Theta^+_{z,w}$ and $\Theta^-_{z,w}$ by the equations

$$(\Delta^z \otimes \mathrm{Id})(\Theta_{zw}) = \Theta^{13}_{zw} + \Theta^{23}_{w} + \Theta^-_{z,w} \quad \text{and} \quad (\mathrm{Id} \otimes \Delta^w)(\Theta_z) = \Theta^{12}_z + \Theta^{13}_{zw} + \Theta^+_{z,w}.$$

By the twisted coassociativity of Δ^z (see (3.10)), we have

$$\begin{aligned} 0 =& \hbar^{-1}(\Delta^z \otimes \mathrm{Id} \circ \Delta^{zw} - \mathrm{Id} \otimes \Delta^w \circ \Delta^z)(\mathcal{C}_3) \\ =& \Theta^{12}_z - \Theta^{23}_w + (\Delta^z \otimes \mathrm{Id})(\Theta_{zw}) - (\mathrm{Id} \otimes \Delta^w)(\Theta_z) = \Theta^-_{z,w} - \Theta^+_{z,w}. \end{aligned}$$

Hence, we have $\Theta^+_{z,w} = \Theta^-_{z,w}$. Therefore, we drop the superscript and write $\Theta_{z,w}$ for $\Theta^{\pm}_{z,w}$. Next, observe that since the coefficients of $[\mathsf{h} \otimes 1, \Omega_z]$ belong to $\mathfrak{g} \otimes \mathfrak{g}$, the element $\mathcal{K}_z = \Theta_z - [\mathsf{h} \otimes 1, \Omega_z]$ satisfies

$$(\Delta^z \otimes \mathrm{Id})(\mathcal{K}_{zw}) = \mathcal{K}^{13}_{zw} + \mathcal{K}^{23}_{w} + \Theta_{z,w} \quad \text{and} \quad (\mathrm{Id} \otimes \Delta^w)(\mathcal{K}_z) = \mathcal{K}^{12}_z + \mathcal{K}^{13}_{zw} + \Theta_{z,w}.$$

In particular, we have

$$\sum_{n>0} \mathcal{K}^-_{n\delta}(z)(zw)^{n\mathsf{ht}(\delta)} = \Theta_{z,w} = \sum_{n>0} \mathcal{K}^+_{n\delta}(w) z^{n\mathsf{ht}(\delta)}.$$

However, by (5.12) and (5.13), this is only possible if both the left-hand side and right-hand side are identically zero. Therefore, $\mathcal{K}^-_{n\delta}(z) = 0 = \mathcal{K}^+_{n\delta}(w)$ for each positive integer n, as desired. This completes the proof of Part (4), and thus the proof of the proposition. □

6. Construction of $\mathcal{R}^-(s)$

In this section, we show that the standard and Drinfeld tensor products on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ are conjugate to each other by a triangular element $\mathcal{R}^-(s)$. In other words, $\mathcal{R}^-(s)$ is a rational tensor structure on the identity functor

$$(\mathrm{Id}, \mathcal{R}^-(s)) : \left(\mathcal{O}(Y_\hbar(\mathfrak{g})), \underset{\mathrm{D},s}{\otimes}\right) \to \left(\mathcal{O}(Y_\hbar(\mathfrak{g})), \underset{\mathrm{KM},s}{\otimes}\right).$$

We refer the reader to Theorem 6.2 below for the precise meaning of this statement. Our proof is almost verbatim to the one for finite type [GTLW21, Thm. 4.1], with some subtle differences highlighted in this section. Additionally, the argument in *loc. cit.* rests on the linear map $\mathrm{T} : \mathfrak{h} \to Y^0_\hbar(\mathfrak{g})$ and the formulae for $\Delta_s(\mathrm{T}(h))$. For affine Yangians, the map only exists, *a priori*, from $\mathfrak{h}' \to Y^0_\hbar(\mathfrak{g})$. That is the reason we had to extend it, using $\mathcal{C}_3$ as in Theorem 5.1.

6.1. The formal series $\mathcal{R}^-(s,z)$.

Recall from Section 5.1 that there is a series $\mathrm{Q}_z = \sum_{n>0} \mathrm{Q}_{n\delta} z^{n\mathsf{ht}(\delta)}$, with $\mathrm{Q}_{n\delta}$ defined explicitly in (5.11), which satisfies

$$\Delta^z(\mathrm{T}(h)) = \mathrm{T}(h) \otimes 1 + 1 \otimes \mathrm{T}(h) + \hbar[h \otimes 1, \Omega_z + \mathrm{Q}_z] \quad \forall\, h \in \mathfrak{h}$$

in addition to the conditions of Part (3) in Theorem 5.1. Here the transformation T itself is defined in Theorem 5.1 above, while $\Omega_z \in (\mathfrak{g} \otimes \mathfrak{g})[\![z]\!]$ is as in Section 3.9. Furthermore, we introduce the algebra homomorphisms

$$\underset{\mathrm{D},s}{\Delta}{}^z = (\mathrm{Id} \otimes \sigma_z) \circ \underset{\mathrm{D},s}{\Delta} \quad \text{and} \quad \Delta^z_s = (\tau_s \otimes \mathrm{Id}) \circ \Delta^z$$

in addition to the function $\nu : \mathsf{Q}_+ \to \mathbb{Z}_{\geqslant 0}$ defined by

$$\nu(\beta) = \min\{k \in \mathbb{Z}_{\geqslant 0} : \beta = \alpha^{(1)} + \cdots + \alpha^{(k)} \text{ for } \alpha^{(1)}, \ldots, \alpha^{(k)} \in \Phi_+\}.$$

The following theorem provides the formal version of the main result of this section.

Theorem. *There exists a unique family of elements* $\{\mathcal{R}^-(s)_\beta\}_{\beta\in\mathsf{Q}_+}$, *with* $\mathcal{R}^-(s)_\beta \in (Y_\hbar(\mathfrak{g})_{-\beta}\otimes Y_\hbar(\mathfrak{g})_\beta)[\![s^{-1}]\!]$ *for each* $\beta\in\mathsf{Q}_+$, *satisfying* $\mathcal{R}^-(s)_0 = 1\otimes 1$ *in addition to the intertwiner equation*

$$\mathcal{R}^-(s,z)\Delta_s^z(\mathrm{T}(h)) = \underset{\mathrm{D},s}{\Delta^z}(\mathrm{T}(h))\mathcal{R}^-(s,z) \quad \forall\, h\in\mathfrak{h} \tag{6.1}$$

in $Y_\hbar(\mathfrak{g})^{\otimes 2}[s;s^{-1}]\!][\![z]\!]$, *where* $\mathcal{R}^-(s,z)=\sum_{\beta\in\mathsf{Q}_+}\mathcal{R}^-(s)_\beta z^{\mathsf{ht}(\beta)}$. *Moreover:*

(1) *For each* $x\in Y_\hbar(\mathfrak{g})$, *one has*
$$\mathcal{R}^-(s,z)\Delta_s^z(x) = \underset{\mathrm{D},s}{\Delta^z}(x)\mathcal{R}^-(s,z).$$

(2) *For each* $\beta\in\mathsf{Q}_+$, $\mathcal{R}^-(s)_\beta$ *satisfies*
$$\mathcal{R}^-(s)_\beta \in s^{-\nu(\beta)}\left(Y_\hbar^-(\mathfrak{g})\otimes Y_\hbar^+(\mathfrak{g})\right)[\![s^{-1}]\!].$$

(3) *For each* $\beta\in\mathsf{Q}_+$ *and* $a,b\in\mathbb{C}$, $\mathcal{R}^-(s)_\beta$ *satisfies*
$$\tau_a\otimes\tau_b(\mathcal{R}^-(s)_\beta) = \mathcal{R}^-(s+a-b)_\beta.$$

(4) *As an element of* $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![z]\!][\![s^{-1}]\!]$, $\mathcal{R}^-(s,z)$ *is of the form*
$$\mathcal{R}^-(s,z) = 1+\hbar s^{-1}(\Omega_z^- + \mathrm{Q}_z) + O(s^{-2}).$$

Proof. With Theorem 5.1 at our disposal, the proof of the finite-type counterpart to this result, given in [GTLW21, §4], carries over to the present setting with only minor adjustments. In what follows we summarize the key steps while making transparent the role played by z (which does not appear in [GTLW21]) and Theorem 5.1.

Proof of Existence and Uniqueness. By Theorem 5.1, $\mathrm{ad}(\tau_s(\mathrm{T}(h))) = \mathrm{ad}(\mathrm{T}(h)+sh)$ for all $h\in\mathfrak{h}$ and $(\tau_s\otimes\mathrm{Id})(\mathrm{Q}_z)=\mathrm{Q}_z$. It follows that the intertwiner equation (6.1) is equivalent to

$$(\mathcal{T}(h)+\mathrm{ad}(sh\otimes\mathrm{Id}))\cdot\mathcal{R}^-(s,z) = \hbar\mathcal{R}^-(s,z)[h\otimes 1,\Omega_z+\mathrm{Q}_z]$$

for all $h\in\mathfrak{h}$, where we have set $\mathcal{T}(h)=\mathrm{ad}(\square(\mathrm{T}(h)))$. Taking the $Y_\hbar(\mathfrak{g})_{-\beta}\otimes Y_\hbar(\mathfrak{g})_\beta$ component of this equation, for any fixed $\beta\in\mathsf{Q}_+$, and dividing both sides of the resulting identity by $z^{\mathsf{ht}(\beta)}$ yields

$$(\mathcal{T}(h)-s\beta(h))\cdot\mathcal{R}^-(s)_\beta = -\hbar\sum_{\alpha\in\Phi_+}\alpha(h)\mathcal{R}^-(s)_{\beta-\alpha}(\Omega_\alpha+\mathrm{Q}_\alpha), \tag{6.2}$$

where it is understood that $\mathcal{R}^-(s)_\gamma = 0 = \mathrm{Q}_\alpha$ for $\gamma\notin\mathsf{Q}_+$ and $\alpha\notin\mathbb{Z}\delta$. In particular, the right–hand side of the equation above is a finite sum. The proof that there is at most one family $\{\mathcal{R}^-(s)_\beta\}_{\beta\in\mathsf{Q}_+}$ satisfying the conditions of the theorem now proceeds identically to the proof of the analogous assertion for the Yangian of a finite–dimensional simple Lie algebra given in [GTLW21, §4.2]. Indeed, if $h\in\mathfrak{h}$ is chosen so that $\beta(h)\neq 0$, then the operator on the left–hand side of the above equation can be inverted to yield

$$\begin{aligned}\mathcal{R}^-(s)_\beta &= \frac{\hbar}{s\beta(h)}\left(1-\frac{\mathcal{T}(h)}{s\beta(h)}\right)^{-1}\sum_{\alpha\in\Phi_+}\alpha(h)\mathcal{R}^-(s)_{\beta-\alpha}(\Omega_\alpha+\mathrm{Q}_\alpha)\\ &= \hbar\sum_{k\geqslant 0}\frac{\mathcal{T}(h)^k}{(s\beta(h))^{k+1}}\sum_{\alpha\in\Phi_+}\alpha(h)\mathcal{R}^-(s)_{\beta-\alpha}(\Omega_\alpha+\mathrm{Q}_\alpha).\end{aligned} \tag{6.3}$$

This equation determines $\mathcal{R}^-(s)_\beta$ recursively in terms of $\mathcal{R}^-(s)_\gamma$ with $\mathsf{ht}(\gamma) < \mathsf{ht}(\beta)$, and thus establishes the uniqueness of $\{\mathcal{R}^-(s)_\beta\}_{\beta\in\mathsf{Q}_+}$ satisfying (6.1) and the initial condition $\mathcal{R}^-(s)_0 = 1\otimes 1$.

To prove existence, we proceed as in [GTLW21, §4.2]. Choose $h \in \mathfrak{h}$ such that $\beta(h)\neq 0$ for every nonzero $\beta\in\mathsf{Q}_+$. Corresponding to this choice of h, we may define a family of elements $\{\mathcal{R}^-(s)_\beta\}_{\beta\in\mathsf{Q}_+}$ recursively on the height of β using (6.3) and the condition $\mathcal{R}^-(s)_0 = 1$. This produces elements $\mathcal{R}^-(s)_\beta \in (Y_\hbar(\mathfrak{g})_{-\beta}\otimes Y_\hbar(\mathfrak{g})_\beta)[\![s^{-1}]\!]$ which by definition satisfy

$$\mathcal{R}^-(s,z)\Delta_s^z(\mathrm{T}(h)) = \underset{\mathrm{D},s}{\Delta^z}(\mathrm{T}(h))\mathcal{R}^-(s,z) \tag{6.4}$$

for our fixed choice of h, where $\mathcal{R}^-(s,z) = \sum_{\beta\in\mathsf{Q}_+}\mathcal{R}^-(s)_\beta z^{\mathsf{ht}(\beta)}$. To see that $\mathcal{R}^-(s,z)$ in fact satisfies (6.1) in general, let $h'\in\mathfrak{h}$ and note that, since Δ_s^z and $\underset{\mathrm{D},s}{\Delta^z}$ are algebra homomorphisms, the two series

$$\mathcal{R}^-(s,z)\Delta_s^z(\mathrm{T}(h')) \quad \text{and} \quad \underset{\mathrm{D},s}{\Delta^z}(\mathrm{T}(h'))\mathcal{R}^-(s,z)$$

both solve (6.4), are of the form $\sum_{\beta\in\mathsf{Q}_+}\mathcal{R}(s)_\beta z^{\mathsf{ht}(\beta)}$ with $\mathcal{R}(s)_\beta \in (Y_\hbar(\mathfrak{g})_{-\beta}\otimes Y_\hbar(\mathfrak{g})_\beta)[s;s^{-1}]\!]$ for each $\beta\in\mathsf{Q}_+$, and have $Y_\hbar(\mathfrak{g})_0\otimes Y_\hbar(\mathfrak{g})_0$ components equal to $(\tau_s\otimes\mathrm{Id})\square(\mathrm{T}(h'))$. They therefore coincide by the uniqueness argument given above (see (6.3)).

Proof of Parts (2)–(4). By Part (3) of Theorem 5.1, $\Omega_\alpha+\mathrm{Q}_\alpha$ lies in $Y_\hbar^-(\mathfrak{g})_{-\alpha}\otimes Y_\hbar^+(\mathfrak{g})_\alpha$ for each $\alpha\in\Phi_+$. Therefore, the recursion (6.3) implies that

$$\mathcal{R}^-(s)_\beta \in s^{-\nu(\beta)}\left(Y_\hbar^-(\mathfrak{g})_{-\beta}\otimes Y_\hbar^+(\mathfrak{g})_\beta\right)[\![s^{-1}]\!]$$

for all $\beta\in\mathsf{Q}_+$, with $\mathcal{R}^-(s)_\alpha = (\Omega_\alpha+\mathrm{Q}_\alpha)s^{-1}+O(s^{-2})$ for all $\alpha\in\Phi_+$. This proves Parts (2) and (4) of the theorem. Similarly, given $a,b\in\mathbb{C}$, Parts (2) and (3) of Theorem 5.1 imply that the elements $(\tau_a\otimes\tau_b)(\mathcal{R}^-(s)_\beta)$ and $\mathcal{R}^-(s+a-b)_\beta$ both satisfy (6.2) with s replaced by $s+a-b$, and thus coincide by uniqueness (see also [GTLW21, §4.3]). Note that the argument given in *loc. cit.* doesn't work as stated since $\mathrm{T}(\mathfrak{h})\oplus\mathfrak{h}$ is not stable under τ_b. Still ad of this subspace is, which is all that is needed here.

Proof of Part (1). Since the desired identity is satisfied for $x\in\mathfrak{h}\oplus\mathrm{T}(\mathfrak{h})$, it suffices to prove that

$$\mathcal{R}^-(s,z)\Delta_s^z(x_{i,0}^\pm) = \underset{\mathrm{D},s}{\Delta^z}(x_{i,0}^\pm)\mathcal{R}^-(s,z)$$

for each $i\in\mathbf{I}$. This is proven using a rank 1 reduction argument, exactly as in [GTLW21, §4.7], with the $\mathfrak{sl}_2$ case having been verified in [GTLW21, §4.8]. □

Remark. We emphasize that the use of Theorem 5.1 is essential in the above proof, and in particular in the existence and uniqueness argument for $\{\mathcal{R}^-(s)_\beta\}_{\beta\in\mathsf{Q}_+}$. Indeed, unlike in the setting of [GTLW21, Thm 4.1], it is not sufficient to replace $\{\mathrm{T}(h)\}_{h\in\mathfrak{h}}$ by the family $\{t_{i,1}\}_{i,\in\mathbf{I}}$ in (6.1), as it is not true that for any nonzero $\beta\in\mathsf{Q}_+$ there is $h\in\mathfrak{h}'$ such that $\beta(h)\neq 0$. The existence of such an h is necessary to pass from (6.2) to (6.3).

In addition, unlike in the finite case considered in [GTLW21], the infinite sum $\sum_\beta \mathcal{R}^-(s)_\beta$ does not converge to an element of $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$. This is the reason behind the introduction of the auxiliary parameter z in the statement of the theorem.

6.2. **The operators $\mathcal{R}^-_{V_1,V_2}(s)$.** Let $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, with $\pi_\ell : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V_\ell)$ being the action homomorphisms. By the category $\mathcal{O}$ condition, the matrix entries of $\pi_1 \otimes \pi_2(\Omega_z)$ and $\pi_1 \otimes \pi_2(\mathrm{Q}_z)$ are polynomials in z and therefore can be evaluated at $z = 1$. Let

$$\Omega_{V_1,V_2} = \pi_1 \otimes \pi_2(\Omega_z)\big|_{z=1} \quad \text{and} \quad \mathrm{Q}_{V_1,V_2} = \pi_1 \otimes \pi_2(\mathrm{Q}_z)\big|_{z=1}.$$

We have the following representation–theoretic version of Theorem 6.1.

Theorem. *Let $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$. Then, there exist a unique* $\mathrm{End}(V_1 \otimes V_2)$*–valued, rational function of s, denoted by $\mathcal{R}^-_{V_1,V_2}(s)$, satisfying the following properties.*

(1) *(Normalization)* $\mathcal{R}^-_{V_1,V_2}(\infty) = \mathrm{Id}_{V_1 \otimes V_2}$.

(2) *(Triangularity)* $\mathcal{R}^-_{V_1,V_2}(s) = \sum_{\beta \in \mathsf{Q}_+} \mathcal{R}^-_{V_1,V_2}(s)_\beta$*, where, for any* $\mu_1, \mu_2 \in \mathfrak{h}^*$

$$\mathcal{R}^-_{V_1,V_2}(s)_\beta : V_1[\mu_1] \otimes V_2[\mu_2] \to V_1[\mu_1 - \beta] \otimes V_2[\mu_2 + \beta].$$

(3) *(Intertwiner) For every $h \in \mathfrak{h}$, the following intertwining relation holds:*

$$\left[\square(\mathrm{T}(h)) + sh \otimes 1, \mathcal{R}^-_{V_1,V_2}(s)\right] = \hbar \mathcal{R}^-_{V_1,V_2}(s)[h \otimes 1, \Omega_{V_1,V_2} + \mathrm{Q}_{V_1,V_2}].$$

Moreover, this operator has the following properties:

(4) *If $\mathcal{R}^-(s,z)$ is as in Theorem 6.1, then*

$$\mathcal{R}^-_{V_1,V_2}(s) = \pi_1 \otimes \pi_2(\mathcal{R}^-(s,z))\big|_{z=1}.$$

(5) $\mathcal{R}^-_{V_1,V_2}(s) : V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \to V_1 \underset{\mathrm{D},s}{\otimes} V_2$ *is a $Y_\hbar(\mathfrak{g})$–intertwiner, which is natural in V_1, V_2 and compatible with the shift automorphism:*

$$\mathcal{R}^-_{V_1(a),V_2(b)}(s) = \mathcal{R}^-_{V_1,V_2}(s + a - b).$$

(6) *(Cocycle equation) The following diagram commutes, for every $V_1, V_2, V_3 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$:*

$$\begin{tikzcd}
(V_1 \underset{\mathrm{KM},s_1}{\otimes} V_2) \underset{\mathrm{KM},s_2}{\otimes} V_3 \arrow[r, equal] \arrow[d, "\mathcal{R}^-_{V_1,V_2}(s_1)\otimes \mathrm{Id}_{V_3}"'] & V_1 \underset{\mathrm{KM},s_1+s_2}{\otimes} (V_2 \underset{\mathrm{KM},s_2}{\otimes} \otimes V_3) \arrow[d, "\mathrm{Id}_{V_1} \otimes \mathcal{R}^-_{V_2,V_3}(s_2)"] \\
(V_1 \underset{\mathrm{D},s_1}{\otimes} V_2) \underset{\mathrm{KM},s_2}{\otimes} V_3 \arrow[d, "\mathcal{R}^-_{V_1 \underset{\mathrm{D},s_1}{\otimes} V_2, V_3}(s_2)"'] & V_1 \underset{\mathrm{KM},s_1+s_2}{\otimes} (V_2 \underset{\mathrm{D},s_2}{\otimes} \otimes V_3) \arrow[d, "\mathcal{R}^-_{V_1, V_2 \underset{\mathrm{D},s_2}{\otimes} V_3}(s_1+s_2)"] \\
(V_1 \underset{\mathrm{D},s_1}{\otimes} V_2) \underset{\mathrm{D},s_2}{\otimes} V_3 \arrow[r, equal] & V_1 \underset{\mathrm{D},s_1+s_2}{\otimes} (V_2 \underset{\mathrm{D},s_2}{\otimes} \otimes V_3)
\end{tikzcd}$$

Remark. We wish to highlight that the cocycle equation only holds for the rational intertwiners, not for the formal $\mathcal{R}^-(s,z)$; see [GTLW21, Remark 4.1].

Proof. For notational simplicity, we drop the subscript V_1, V_2. Let us fix a $\mu \in \mathfrak{h}^*$ and consider the intertwining equation valued in the finite–dimensional vector space $\mathcal{E} = \mathrm{End}((V_1 \otimes V_2)[\mu])$. For $h \in \mathfrak{h}$, let $\mathsf{r}(h) = [h \otimes 1, \Omega + \mathrm{Q}]$, viewed as a strictly lower triangular operator on $\mathcal{E}$. Now, we have:

$$A(h,s) = \mathrm{ad}(\square(\mathrm{T}(h)) + sh \otimes 1) - \rho(\hbar \mathsf{r}(h)) \in \mathrm{End}(\mathcal{E})[s],$$

where $\rho(X)$ denotes the operator of right multiplication by X. This operator is triangular in the weight grading on $\mathcal{E}$, with diagonal blocks being linear in s, for generic h. The same argument given in the proof of Theorem 6.1, carried out in $\mathrm{End}(\mathcal{E})$, shows that there is a unique unipotent solution to (6.1). Moreover, each $\mathcal{R}^-(s)_\beta$ is a rational function of s, vanishing at $s = \infty$ for $\beta > 0$. This proves the first part of the theorem.

The uniqueness of the solution, together with Theorem 6.1, imply that the Taylor series expansion of this rational function at $s = \infty$ is the same as the evaluation of $\mathcal{R}^-(s,z)$ on $V_1 \otimes V_2$, specialized at $z = 1$. This proves (4). Part (5) is now a consequence of Theorem 6.1.

The cocycle equation also rests on uniqueness, and coassociativity of Δ_s and $\underset{\mathrm{D},s}{\Delta}$ on representations (see [GTLW21, §4.7]). Namely, consider the two sides of the cocycle equation:

$$L(s_1,s_2) = \mathcal{R}^-_{V_1 \underset{\mathrm{D},s_1}{\otimes} V_2, V_3}(s_2) \circ \left(\mathcal{R}^-_{V_1,V_2}(s_1) \otimes \mathrm{Id}_{V_3} \right),$$

$$R(s_1,s_2) = \mathcal{R}^-_{V_1, V_2 \underset{\mathrm{D},s_2}{\otimes} V_3}(s_1+s_2) \circ \left(\mathrm{Id}_{V_1} \otimes \mathcal{R}^-_{V_2,V_3}(s_2) \right).$$

It is straightforward to see that both of these operators intertwine the two action of $\mathrm{T}(h)$. That is, using the formulae for Δ_s and Theorem 5.1 (3), these operators solve the following equation:

$$\operatorname{ad}\left(\sum_{a=1}^{3} \mathrm{T}(h)^{(a)} + (s_1+s_2)h^{(1)} + s_2 h^{(2)}\right) \cdot X(s_1,s_2) =$$
$$\hbar X(s_1,s_2)(\mathsf{r}_{12}(h) + \mathsf{r}_{23}(h) + \mathsf{r}_{13}(h)).$$

As in [GTLW21, §4.6], there is at most one solution to this equation, which is triangular in the sense that $X = \sum_{\beta,\gamma\in\mathsf{Q}_+} X_{\beta,\gamma}$, where, for $\mu_1,\mu_2,\mu_3 \in \mathfrak{h}^*$,

$$X_{\beta,\gamma} : V_1[\mu_1]\otimes V_2[\mu_2]\otimes V_3[\mu_3] \to V_1[\mu_1-\beta]\otimes V_2[\mu_2+\beta-\gamma]\otimes V_3[\mu_3+\gamma].$$

Hence, $L(s_1,s_2) = R(s_1,s_2)$. □

7. Construction of $\mathcal{R}^0(s)$

In this section, we introduce an additive, regular difference equation, whose coefficients come from $\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))^{\otimes 2}$, where $\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))$ is the linear span of $\{t_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}$. We show that the exponential of the two fundamental solutions of this difference equation give rise to two meromorphic braidings on $(\mathcal{O}(Y_\hbar(\mathfrak{g})), \underset{\mathrm{D},s}{\otimes})$, related by a unitarity condition. They have the same asymptotic expansion, which is the unique formal solution of our difference equation, *i.e.*, a formal abelian R–matrix.

7.1. Main construction and result.

We begin by stating the main result of this section.

Theorem. *Given $V_1,V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, there exist two meromorphic* $\mathrm{End}(V_1\otimes V_2)$*–valued functions, $\mathcal{R}^{0,\eta}_{V_1,V_2}(s)$, $\eta\in\{\uparrow,\downarrow\}$, which are natural in V_1 and V_2 and have the following properties:*

(1) *Let $\mathbb{H}^\uparrow = \{s\in\mathbb{C} : \mathrm{Re}(s/\hbar) \gg 0\} = -\mathbb{H}^\downarrow$. Then, for every $V_1,V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, $\mathcal{R}^{0,\eta}_{V_1,V_2}(s)$ is holomorphic and invertible in $\mathbb{H}^\eta$ and approaches $\mathrm{Id}_{V_1\otimes V_2}$ as $|s|\to\infty$.*

(2) *For every V_1,V_2, the following is a $Y_\hbar(\mathfrak{g})$–intertwiner:*
$$(1\ 2)\circ\mathcal{R}^{0,\eta}_{V_1,V_2}(s) : V_1 \underset{\mathrm{D},s}{\otimes} V_2 \to (V_2 \underset{\mathrm{D},-s}{\otimes} V_1)(s)\ .$$

(3) *The following holds, for every $V_1,V_2\in\mathcal{O}(Y_\hbar(\mathfrak{g}))$ and $a,b\in\mathbb{C}$:*
$$\mathcal{R}^{0,\eta}_{V_1(a),V_2(b)}(s) = \mathcal{R}^{0,\eta}_{V_1,V_2}(s+a-b)\ .$$

(4) *For every $V_1,V_2,V_3\in\mathcal{O}(Y_\hbar(\mathfrak{g}))$, we have:*
$$\mathcal{R}^{0,\eta}_{V_1\underset{\mathrm{D},s_1}{\otimes}V_2,V_3}(s_2) = \mathcal{R}^{0,\eta}_{V_1,V_3}(s_1+s_2)\mathcal{R}^{0,\eta}_{V_2,V_3}(s_2)\ ,$$
$$\mathcal{R}^{0,\eta}_{V_1,V_2\underset{\mathrm{D},s_2}{\otimes}V_3}(s_1+s_2) = \mathcal{R}^{0,\eta}_{V_1,V_3}(s_1+s_2)\mathcal{R}^{0,\eta}_{V_1,V_2}(s_1)\ .$$

(5) *For every $V_1,V_2\in\mathcal{O}(Y_\hbar(\mathfrak{g}))$, we have:*
$$\mathcal{R}^{0,\uparrow}_{V_1,V_2}(s)^{-1} = (1\ 2)\circ\mathcal{R}^{0,\downarrow}_{V_2,V_1}(-s)\circ(1\ 2)\ .$$

(6) *There exists a unique formal series* $\mathcal{R}^0(s) \in Y_\hbar^0(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$ *such that, for every* $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$*, with* $\pi_\ell : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(V_\ell)$ *the corresponding action homomorphism, we have:*
$$\mathcal{R}^{0,\eta}_{V_1,V_2}(s) \sim \pi_1 \otimes \pi_2(\mathcal{R}^0(s)), \ \text{as} \ \pm \mathrm{Re}(s/\hbar) \to \infty.$$
This asymptotic expansion remains valid in a larger sector Σ^η_δ*, for any* $\delta > 0$*, where, if* $\theta = \arg(\hbar)$ *then:*
$$\Sigma^\uparrow_\delta = \{re^{\iota\phi} : r \in \mathbb{R}_{>0}, \phi \in (\theta - \pi + \delta, \theta + \pi - \delta)\} = -\Sigma^\downarrow_\delta.$$
(see Figure A.3 in Appendix A where $\chi = \hbar/2$*).*

(7) *The first order term of* $\mathcal{R}^0(s)$ *is given by:*
$$\mathcal{R}^0(s) = \exp\left(s^{-1}\left(\frac{1}{\hbar}\mathsf{g}_{\text{sing}} + \frac{\hbar}{4\mathsf{d}^{(0)}}\mathsf{g}_0\right) + O(s^{-2})\right).$$
Here, the constant $\ell^{(0)}$ *and the tensor* g_0 *are given in Lemma 7.7,* $\mathsf{d}^{(0)}$ *in Lemma 7.4 below, and*
$$\mathsf{g}_{\text{sing}} = \frac{\ell^{(0)}}{\mathsf{d}^{(0)}}\left(\frac{\mathcal{C}_2 \otimes \mathcal{C}_0}{2} - \mathcal{C}_1 \otimes \mathcal{C}_1 + \frac{\mathcal{C}_0 \otimes \mathcal{C}_2}{2}\right).$$

Remark. Note that by Corollary 3.8 the tensor g_{sing} commutes with zero–weight elements of $Y_\hbar(\mathfrak{g})^{\otimes 2}$, and hence with $\mathcal{R}^\pm(s)$. Thus, it can be taken to the left, as stated in (1.1).

The proof of this theorem is outlined in Section 7.3 and worked out in the following sections. Before going into the proof, we will prove the rationality property of $\mathcal{R}^0(s)$.

7.2. **Rationality.** Assume that $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ are two highest–weight representations. Let $\lambda_\ell \in \mathfrak{h}^*$ be their highest weights, and fix $\mathsf{v}_\ell \in V_\ell[\lambda_\ell]$ highest–weight vectors ($\ell = 1, 2$). For $\eta \in \{\uparrow, \downarrow\}$, define $\mathfrak{f}^\eta(s)$ as the eigenvalue of $\mathcal{R}^{0,\eta}_{V_1,V_2}(s)$ on $V_1[\lambda_1] \otimes V_2[\lambda_2]$:
$$\mathcal{R}^{0,\eta}(s) \cdot \mathsf{v}_1 \otimes \mathsf{v}_2 = \mathfrak{f}^\eta(s)\mathsf{v}_1 \otimes \mathsf{v}_2 \ .$$

Theorem. *With the notational set up as above, the normalized operator*
$$\mathsf{R}^0_{V_1,V_2}(s) = \mathfrak{f}^\eta(s)^{-1}\mathcal{R}^{0,\eta}_{V_1,V_2}(s)$$
is independent of η*, and is rational in* s*.*

Proof. Let $\mu_\ell \in P(V_\ell)$ be two weights ($\ell = 1, 2$). We argue by induction on $\mathrm{ht}(\lambda_1 - \mu_1) + \mathrm{ht}(\lambda_2 - \mu_2)$. For notational convenience we will drop V_1, V_2 from the subscripts of our operators. The base case being clear, we focus on the induction step. Note that, by the highest–weight property, we have
$$V_1[\mu_1] \otimes V_2[\mu_2] = \sum_{\substack{k_1,k_2 \in \mathbf{I} \\ r_1,r_2 \in \mathbb{Z}_{\geqslant 0}}} x^-_{k_1,r_1}(V_1[\mu_1 + \alpha_{k_1}]) \otimes x^-_{k_2,r_2}(V_2[\mu_2 + \alpha_{k_2}]).$$

Moreover, by the commutation relation $[t_{k,1}, x^-_{k,r}] = -2d_k x^-_{k,r+1}$, for any $V \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ and $\mu \in P(V)$, we have
$$\sum_{r \in \mathbb{Z}_{\geqslant 0}} x^-_{k,r}(V[\mu]) = Y^0_\hbar(\mathfrak{g}) \cdot x^-_{k,0}(V[\mu]).$$

Recall that $\mathcal{R}^{0,\eta}(s)$ commutes with the action of $Y_\hbar^0(\mathfrak{g}) \otimes Y_\hbar^0(\mathfrak{g})$ on $V_1 \otimes V_2$. Thus, it suffices to show that the action of $\mathsf{R}^0(s)$ on the image of $x_{k,0}^- \otimes 1$ (and $1 \otimes x_{k,0}^-$) is rational, and independent of η. We focus of the former, as the latter will follow either with a similar proof, or using unitarity. We will use the following commutation relation between $\mathcal{R}^{0,\eta}(s)$ and $x_{k,0}^- \otimes 1$, which will be established in the proof of Part (2) of Theorem 7.1, outlined in Section 7.3:

$$\mathrm{Ad}(\mathcal{R}^{0,\eta}(s)^{-1}) \cdot (x_{k,0}^- \otimes 1) = x_{k,0}^- \otimes 1 + \mathfrak{Y}_k(s), \tag{7.1}$$

where

$$\mathfrak{Y}_k(s) = \hbar \sum_{N \geqslant 0} s^{-N-1} \left(\sum_{n=0}^{N} (-1)^{n+1} \left(\begin{array}{c} N \\ n \end{array} \right) x_{k,n}^- \otimes \xi_{k,N-n} \right).$$

The right–hand side of (7.1) has the following contour integral representation, which immediately shows that it is rational in s (see [GTL17, §4.5]):

$$x_{k,0}^- \otimes 1 + \mathfrak{Y}_k(s) = \frac{1}{\hbar} \oint_{C_1} x_k^-(u) \otimes \xi_k(u+s)\, du\ .$$

Here, C_1 is a contour enclosing all the poles of $x_k^-(u)$ acting on $V_1[\mu_1 + \alpha_k]$, and s is so large that $\xi_k(u+s)$ acting on $V_2[\mu_2]$ is holomorphic on and within C_1.

Now, let $w_1 \in V_1[\mu_1 + \alpha_k]$ and $w_2 \in V_2[\mu_2]$. Then, since $\mathrm{Ad}(\mathcal{R}^{0,\eta}(s)) = \mathrm{Ad}(\mathsf{R}^0(s))$, we have

$$\begin{aligned}
&\mathsf{R}^0(s)^{-1} \circ (x_{k,0}^- \otimes 1) \cdot (w_1 \otimes w_2) \\
&\qquad = \left(\mathrm{Ad}(\mathcal{R}^{0,\eta}(s)^{-1}) \cdot (x_{k,0}^- \otimes 1) \right) \left(\mathsf{R}^0(s)^{-1} \cdot (w_1 \otimes w_2) \right) \\
&\qquad = \left(x_{k,0}^- \otimes 1 + \mathfrak{Y}_k(s) \right) \cdot \left(\mathsf{R}^0(s)^{-1} \cdot (w_1 \otimes w_2) \right).
\end{aligned}$$

The last line gives a vector in $V_1[\mu_1] \otimes V_2[\mu_2]$ depending rationally on s, by the induction hypothesis combined with the rationality of $x_{k,0}^- \otimes 1 + \mathfrak{Y}_k(s)$. The theorem is proved. □

7.3. **Proof of Theorem 7.1.** Our proof is based on an explicit construction of $\mathcal{R}^0(s)$, both as a formal series, and as an $\mathrm{End}(V_1 \otimes V_2)$-valued meromorphic function of s. This $\mathcal{R}^0(s)$ has the form

$$\mathcal{R}^0(s) = \exp(\Lambda(s)), \quad \text{where } \Lambda(s) \in s^{-1} \mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))^{\otimes 2} [\![s^{-1}]\!].$$

The $\Lambda(s)$, in turn, is obtained in a few steps. We outline the steps here, which are carried out in Sections 7.4–7.9. To begin, consider the difference equation

$$(\mathsf{p} - \mathsf{p}^{-1}) \det(\mathbf{B}(\mathsf{p})) \cdot \Lambda(s) = \mathcal{G}(s), \tag{7.2}$$

where p, $\mathbf{B}(\mathsf{p})$ and $\mathcal{G}(s)$ are defined as follows:

- p is the shift operators: $\mathsf{p} \cdot f(s) = f(s - \hbar/2)$.
- $\mathbf{B}(\mathsf{p})$ is the symmetrized, affine p–Cartan matrix (see Section 7.4 below).
- $\mathcal{G}(s) = \sum_{ij} \mathbf{B}(\mathsf{p})^*_{ji} \cdot \mathcal{T}_{ij}(s)$, where $\mathbf{B}(\mathsf{p})^*$ is the adjoint matrix to $\mathbf{B}(\mathsf{p})$ and

$$\mathcal{T}_{ij}(s) = \hbar^2 \sum_{m \geqslant 1} m! s^{-m-1} \sum_{\substack{a,b \geqslant 0 \\ a+b=m-1}} (-1)^a \frac{t_{i,a}}{a!} \otimes \frac{t_{j,b}}{b!}$$

In Section 7.5, we prove important properties of the operators $\mathcal{T}_{ij}(s)$. These are used to establish nearly identical properties for the operator $\mathcal{G}(s)$ in Section 7.6. We compute the first few terms of $\mathcal{G}(s)$ in Section 7.7 and show that the coefficients of s^{-2} and s^{-3} are central. This fact is used in Section 7.8 to regularize the difference equation (7.2) to

$$(\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p}))\cdot\Lambda(s)=\mathcal{G}_{\mathrm{reg}}(s), \tag{7.3}$$

where $\mathcal{G}_{\mathrm{reg}}(s)$ is obtained by removing the aforementioned central terms from $\mathcal{G}(s)$, and is defined explicitly in (7.11). We show in Corollary 7.8 that this equation has a unique formal solution. The properties of $\Lambda(s)$, analogous to the assertions in this theorem, are obtained in Proposition 7.9.

Now, given $V_1,V_2\in\mathcal{O}(Y_\hbar(\mathfrak{g}))$, the evaluation of the difference equation (7.3) on $V_1\otimes V_2$ has coefficients from the following subalgebra of $\mathrm{End}(V_1\otimes V_2)$:

$$\bigoplus_{\mu_1\in P(V_1),\mu_2\in P(V_2)}\mathrm{End}_{Y_\hbar^0(\mathfrak{g})}(V_1[\mu_1])\otimes\mathrm{End}_{Y_\hbar^0(\mathfrak{g})}(V_2[\mu_2]).$$

Thus, fixing μ_1,μ_2, we can view (7.3) as an equation for a finite size matrix–valued function of s. In Appendix A (see Theorem A.1), we establish the existence and uniqueness of the solutions of such equations with the prescribed asymptotics, for any right–hand side of $O(s^{-4})$ (this 4 is 1 plus the order of vanishing of the polynomial on the left–hand side at $\mathsf{p}=1$; see Lemma 7.4 below). We verify that our difference equation satisfies the hypotheses of Theorem A.1 in Section 7.10 (see Corollary 7.10).

Given these preparatory results, let us prove the theorem. Note that (1) and (6) follow from definitions. Parts (3) and (5) of the theorem are a direct consequence of Parts (4) and (3) of Proposition 7.9, respectively. Part (4) follows from the fact that the coefficients of $\Lambda(s)$ are tensors of primitive elements with respect to the Drinfeld coproduct.

Let us prove (2). It is clear that $\mathcal{R}^{0,\eta}_{V_1,V_2}(s)$ commutes with operators from $Y_\hbar^0(\mathfrak{g})$. Therefore, it is enough to show that $(1\ 2)\circ\mathcal{R}^{0,\eta}_{V_1,V_2}(s)$ commutes with $\underset{\mathrm{D},s}{\Delta}(x^{\pm}_{k,0})$, for all $k\in\mathbf{I}$. We focus on the $+$ case, the other one being similar. Recall that

$$\underset{\mathrm{D},s}{\Delta}(x^+_{k,0})=\square(x^+_{k,0})+\hbar\sum_{N\geqslant 0}s^{-N-1}\left(\sum_{a+b=n}(-1)^{a+1}\left(\begin{array}{c}N\\a\end{array}\right)\xi_{k,a}\otimes x^+_{k,b}\right).$$

Let $\mathfrak{X}_k(s)$ denote the second term on the right–hand side. The desired commutation relation decouples to the following two:

$$\mathrm{Ad}(\mathcal{R}^{0,\eta}_{V_1,V_2}(s))\cdot x^+_{k,0}\otimes 1=x^+_{k,0}\otimes 1+\mathfrak{X}_k^{\mathrm{op}}(-s),$$
$$\mathrm{Ad}(\mathcal{R}^{0,\eta}_{V_1,V_2}(s)^{-1})\cdot 1\otimes x^+_{k,0}=1\otimes x^+_{k,0}+\mathfrak{X}_k(s).$$

Note that the second follows from the first, given Part (5) of the theorem (unitarity). For the first, we write

$$x^+_{k,0}\otimes 1+\mathfrak{X}_k^{\mathrm{op}}(-s)=\sum_{a=0}^{\infty}x^+_{k,a}\otimes\partial_s^{(a)}(\xi_k(s)).$$

The desired relation is then a consequence of the following claim.

Claim: For every $k \in \mathbf{I}, n \in \mathbb{Z}_{\geqslant 0}$ and $y \in Y_\hbar^0(\mathfrak{g})$, we have

$$\mathrm{ad}(\Lambda^\eta_{V_1,V_2}(s)) \cdot (x^+_{k,n} \otimes y) = \sum_{a=0}^{\infty} x^+_{k,n+a} \otimes \partial_s^{(a)}(t_k(s))y.$$

We show that this identity is true for the formal $\Lambda(s)$ in Proposition 7.9 (5). To deduce it for the meromorphic functions $\Lambda^\eta_{V_1,V_2}(s)$, we apply the difference operator $\mathcal{D}(\mathsf{p}) = (\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p}))$ to both sides to conclude that they are solutions to the same difference equation, with the same asymptotic expansion as $s \to \infty$. Hence, by the uniqueness result of Theorem A.1, they are equal.

A word on set up and proofs. Our construction of $\mathcal{R}^0(s)$ is weight preserving. So, let $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ and $\mu_\ell \in P(V_\ell)$ $(\ell = 1,2)$ be two fixed weights. All the operators considered in Sections 7.5–7.7 can be viewed as meromorphic $\mathrm{End}(V_1[\mu_1]\otimes V_2[\mu_2])$–valued functions of a complex variable s, which are regular near $s=\infty$. We will be interested in both their functional nature, and naturality with regards to V_1, V_2. Therefore, their Taylor series expansions will be shown to be the evaluation on $V_1 \otimes V_2$ of an element of $Y_\hbar^0(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$.

An identity among such operators can be shown either functionally, or formally. The functional proofs, by which we mean proofs involving the usual "contour deformation" trick, can be found in [GTL17] (we will make precise citation when needed). Here, we also give the formal proofs — both for completeness and for their aesthetic beauty.

7.4. p–**Cartan matrix.** Let p be an indeterminate and let $\mathbf{B}(\mathsf{p}) = ([d_i a_{ij}]_{\mathsf{p}}) \in \mathrm{M}_{\mathbf{I}\times\mathbf{I}}(\mathbb{Z}[\mathsf{p},\mathsf{p}^{-1}])$. Here, we use the standard notation of Gaussian numbers:

$$[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}.$$

Let $\mathbf{B}(\mathsf{p})^*$ be the adjoint matrix of $\mathbf{B}(\mathsf{p})$, so that

$$\mathbf{B}(\mathsf{p})^*\mathbf{B}(\mathsf{p}) = \mathbf{B}(\mathsf{p})\mathbf{B}(\mathsf{p})^* = \det(\mathbf{B}(\mathsf{p}))\,\mathrm{Id}\,.$$

The following lemma is crucial, and is proved by direct inspection (see Appendix C for the table of determinants of symmetrized, affine p–Cartan matrices).

Lemma. *The order of vanishing of* $\det(\mathbf{B}(\mathsf{p}))$ *at* $\mathsf{p}=1$ *is* 2*. That is,*

$$\mathsf{d}^{(0)} = \left.\frac{\det(\mathbf{B}(\mathsf{p}))}{(\mathsf{p}-\mathsf{p}^{-1})^2}\right|_{\mathsf{p}=1} \text{ exists and } \neq 0\ .$$

Moreover, all the zeroes of $\det(\mathbf{B}(\mathsf{p}))$ *have modulus* 1.

Below we will view p as an operator on either rational, matrix–valued functions of s, regular near $s=\infty$; or formal series in s^{-1}, via:

$$\mathsf{p}\cdot f(s) = f(s-\hbar/2)\ . \tag{7.4}$$

7.5. **The $\mathcal{T}$ operators.** Recall that for $i,j \in \mathbf{I}$, we defined

$$\mathcal{T}_{ij}(s) = \hbar^2 \sum_{m\geqslant 1} m! s^{-m-1} \sum_{\substack{a,b\geqslant 0\\ a+b=m-1}} (-1)^a \frac{t_{i,a}}{a!} \otimes \frac{t_{j,b}}{b!}\ . \tag{7.5}$$

In Lemma 6.5 of [GTLW21], it was shown that, when evaluated on $V_1[\mu_1]\otimes V_2[\mu_2]$, $\mathcal{T}_{ij}(s)$ is the Taylor series expansion near $s=\infty$ of the contour integral

$$\mathcal{T}_{ij}(s) = \oint_{C_1} t_i'(u)\otimes t_j(u+s)\,du, \tag{7.6}$$

where C_1 is a contour enclosing zeroes and poles of $\xi_i(u)$ acting on $V_1[\mu_1]$, and s is large enough so that $t_j(u+s)$ acting on $V_2[\mu_2]$ is holomorphic on and within C_1. As usual, we suppress $2\pi\iota$ factor in our notations $\oint = \frac{1}{2\pi\iota}\int$. Here, $t_j(w)$ is viewed as a single–valued function on a cut plane as in Section 4.2. This expression is used to show that $\exp(\mathcal{T}_{ij}(s))$ becomes a rational function of s on $V_1[\mu_1]\otimes V_2[\mu_2]$. (see the proof of Proposition 7.5 (1) below). We can also rewrite (7.5) above using $B_i(z)$ (see Section 3.5 above), and the fact that $m!s^{-m-1} = (-\partial_s)^m\cdot s^{-1}$, as

$$\mathcal{T}_{ij}(s) = zB_i(-z)\otimes B_j(z)\big|_{z=-\partial_s}\cdot s^{-1} = B_i(-z)\otimes B_j(z)\big|_{z=-\partial_s}\cdot s^{-2}. \tag{7.7}$$

Proposition. *The elements $\{\mathcal{T}_{ij}(s)\}_{i,j\in\mathbf{I}}$ have the following properties:*

(1) *As an operator on $V_1[\mu_1]\otimes V_2[\mu_2]$, $\exp(\mathcal{T}_{ij}(s))$ is a rational function of s, regular at $s=\infty$ with value $\mathrm{Id}_{V_1[\mu_1]\otimes V_2[\mu_2]}$ at $s=\infty$.*

(2) *$\mathcal{T}_{ij}^{\mathrm{op}}(s) = \mathcal{T}_{ji}(-s)$.*

(3) *$\tau_a\otimes\tau_b(\mathcal{T}_{ij}(s)) = \mathcal{T}_{ij}(s+a-b)$ for each $a,b\in\mathbb{C}$.*

(4) *Let $k\in\mathbf{I}$, $n\in\mathbb{Z}_{\geqslant 0}$ and let $y\in Y_\hbar^0(\mathfrak{g})$. Then, the following commutation relations hold, where p is the shift operator (7.4):*

$$\begin{aligned}
[\mathcal{T}_{ij}(s), x_{k,n}^\pm\otimes y] &= \pm(\mathsf{p}^{d_ia_{ik}} - \mathsf{p}^{-d_ia_{ik}})\cdot\left(\sum_{a=0}^\infty x_{k,n+a}^\pm\otimes\partial_s^{(a)}(t_j(s))y\right)\\
&= \pm(\mathsf{p}^{d_ia_{ik}} - \mathsf{p}^{-d_ia_{ik}})\cdot\\
&\qquad\hbar\left(\sum_{N\geqslant 0} N!s^{-N-1}\sum_{a+b=N}(-1)^a\frac{x_{k,n+a}^\pm}{a!}\otimes\frac{t_{j,b}y}{b!}\right),\\
[\mathcal{T}_{ij}(s), y\otimes x_{k,n}^\pm] &= \mp(\mathsf{p}^{d_ja_{jk}} - \mathsf{p}^{-d_ja_{jk}})\cdot\left(\sum_{b=0}^\infty y(-\partial_s)^{(b)}(t_i(-s))\otimes x_{k,n+b}^\pm\right)\\
&= \pm(\mathsf{p}^{d_ja_{jk}} - \mathsf{p}^{-d_ja_{jk}})\cdot\\
&\qquad\hbar\left(\sum_{N\geqslant 0} N!s^{-N-1}\sum_{a+b=N}(-1)^a\frac{t_{i,a}y}{a!}\otimes\frac{x_{k,n+b}^\pm}{b!}\right).
\end{aligned}$$

Proof. We remind the reader of the following general fact, proved in Claims 1 and 2 of the proof of [GTL17, Thm. 5.5]:

Let V,W be two finite–dimensional $\mathbb{C}$–vector spaces, $A:\mathbb{C}\to\mathrm{End}(V)$ and $B:\mathbb{C}\to\mathrm{End}(W)$ two rational functions, taking value Id at ∞, such that $[A(s),A(s')] = 0 = [B(s),B(s')]$ for all $s,s'\in\mathbb{C}$. Let $\sigma(A)$ and $\sigma(B)$ be the set of poles of $A(s)^{\pm1}$ and $B(s)^{\pm1}$ respectively. Then the following is a rational $\mathrm{End}(V\otimes W)$–valued function of s, taking value Id at $s=\infty$:

$$X(s) = \exp\left(\oint_{C_1} A(u)^{-1}A'(u)\otimes\log(B(u+s))\,du\right).$$

Here C_1 is a countour enclosing $\sigma(A)$, and s is large enough so that $\log(B(u+s))$ is analytic within and on C_1. Moreover, we have

$$X(s) = \exp\left(\oint_{C_2} \log(A(u-s)) \otimes B(u)^{-1}B'(u)\,du\right).$$

This proves (1) and (2). Note that (2) can also be easily deduced from the formal expansion (7.5).

To prove (3), note that $\tau_a B_i(z) = e^{az}B_i(z)$. Therefore, using (7.7), we have:

$$\begin{aligned}(\tau_a \otimes \tau_b)(\mathcal{T}_{ij}(s) &= \tau_a(B_i(-z)) \otimes \tau_b(B_j(z))|_{z=-\partial_s} \cdot s^{-2} \\ &= e^{-(a-b)z}B_i(-z) \otimes B_j(z)\Big|_{z=-\partial_s} \cdot s^{-2} \\ &= e^{(a-b)\partial_s} \cdot \mathcal{T}_{ij}(s) = \mathcal{T}_{ij}(s+a-b).\end{aligned}$$

In the last line, we used Taylor's theorem $e^{c\partial_s} \cdot f(s) = f(s+c)$.

We remark that a "contour deformation" style proof of (4) is given in [GTL17, Prop. 5.10]. Here, we give a different "formal" proof, using the expression (7.7) of $\mathcal{T}_{ij}(s)$ in terms of $B_i(z)$ and the commutation relation (3.5). Let us focus on the first relation (the second one can be deduced easily from the first, using the unitarity relation (2)). Setting $c_{ik} = d_i a_{ik}\hbar/2$, we have

$$\begin{aligned}[\mathcal{T}_{ij}(s), x^{\pm}_{k,n} \otimes y] &= z[B_i(-z), x^{\pm}_{k,n}] \otimes B_j(z)y\Big|_{z=-\partial_s} \cdot s^{-1} \\ &= \pm(e^{c_{ik}z} - e^{-c_{ik}z})\left(\hbar \sum_{a,b\geqslant 0} (-1)^a \frac{x^{\pm}_{k,n+a}}{a!} \otimes \frac{t_{j,b}y}{b!} z^{a+b}\right)\Bigg|_{z=-\partial_s} \cdot s^{-1} \\ &= \pm(e^{-c_{ik}\partial_s} - e^{c_{ik}\partial_s}) \cdot X^{\pm}_{k,j;n}(s)(1 \otimes y)\ ,\end{aligned}$$

where

$$\begin{aligned}X^{\pm}_{k,j;n}(s) &= \hbar \sum_{N\geqslant 0}\left(\sum_{a+b=N} (-1)^a \frac{x^{\pm}_{k,n+a}}{a!} \otimes \frac{t_{j,b}}{b!}\right)(-\partial_s)^N \cdot s^{-1} \\ &= \hbar \sum_{N\geqslant 0} N!s^{-N-1}\left(\sum_{a+b=N} (-1)^a \frac{x^{\pm}_{k,n+a}}{a!} \otimes \frac{t_{j,b}}{b!}\right) \\ &= \sum_{a=0}^{\infty} x^{\pm}_{k,n+a} \otimes \partial_s^{(a)}(t_j(s)).\end{aligned} \tag{7.8}$$

Since $\mathsf{p} = e^{-(\hbar/2)\partial_s}$, the first relation of (4) follows immediately. □

7.6. The operator $\mathcal{G}(s)$.

We now define

$$\mathcal{G}(s) = \sum_{i,j\in\mathbf{I}} \mathbf{B}(\mathsf{p})^*_{ji} \cdot \mathcal{T}_{ij}(s). \tag{7.9}$$

The following is a direct corollary of Proposition 7.5 and the symmetry of $\mathbf{B}(\mathsf{p})^*$.

Corollary. *The element $\mathcal{G}(s)$ has the following properties:*

(1) $\exp(\mathcal{G}(s))$ *is a rational function of s, taking value 1 at $s = \infty$.*

(2) $\mathcal{G}^{\mathrm{op}}(s) = \mathcal{G}(-s)$.

(3) $\tau_a \otimes \tau_b(\mathcal{G}(s)) = \mathcal{G}(s+a-b)$ *for each* $a, b \in \mathbb{C}$.

(4) *Let* $k \in \mathbf{I}, n \in \mathbb{Z}_{\geqslant 0}$ *and let* $y \in Y_\hbar^0(\mathfrak{g})$. *Then, we have the following commutation relations:*

$$[\mathcal{G}(s), x_{k,n}^\pm \otimes y] = \pm(\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p})) \cdot \sum_{a=0}^\infty x_{k,n+a}^\pm \otimes y\partial_s^{(a)}(t_k(s)),$$

$$[\mathcal{G}(s), y \otimes x_{k,n}^\pm] = \mp(\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p})) \cdot \sum_{b=0}^\infty y(-\partial_s)^{(b)}(t_k(s)) \otimes x_{k,n+b}^\pm.$$

7.7. **Expansion and regularization of $\mathcal{G}$.** The following lemma computes the expansion of $\mathcal{G}(s)$ in s^{-1}, and is crucial in carrying out a "regularization" argument later.

Lemma. $\mathcal{G}(s)$ *admits the expansion*

$$\hbar^{-2}\mathcal{G}(s) = \ell^{(0)}\mathcal{C}_0 \otimes \mathcal{C}_0 s^{-2} + 2\ell^{(0)}\left(-\mathcal{C}_1 \otimes \mathcal{C}_0 + \mathcal{C}_0 \otimes \mathcal{C}_1\right)s^{-3}$$
$$+ 6\left(\ell^{(0)}\left(\frac{\mathcal{C}_2 \otimes \mathcal{C}_0}{2} - \mathcal{C}_1 \otimes \mathcal{C}_1 + \frac{\mathcal{C}_0 \otimes \mathcal{C}_2}{2}\right) + \frac{\hbar^2}{4}\mathsf{g}_0\right)s^{-4} + O(s^{-5}),$$

where $\ell^{(0)}$ *and* g_0 *are given as follows:*

- $\ell^{(0)} \in \mathbb{Z}_{>0}$ *is a constant depending on* $\mathfrak{g}$ *defined by* $\ell^{(0)}a_i a_j = \mathbf{B}(1)^*_{i,j}$ *for every* $i, j \in \mathbf{I}$ *(see Table 1 below).*
- $\mathsf{g}_0 = \sum_{i,j}\mathbf{B}^{*,(2)}_{ij} t_{i,0} \otimes t_{j,0} \in \mathfrak{h} \otimes \mathfrak{h}$, *where* $\mathbf{B}^{*,(2)}$ *is the coefficient of* t^2 *in the Taylor series expansion of* $\mathbf{B}(e^t)^*$ *near* $t = 0$*:*
$$\mathbf{B}(e^t)^* = \mathbf{B}^* + t^2\mathbf{B}^{*,(2)} + O(t^4).$$

Proof. Using the definition of $\mathcal{G}(s)$, and the formula (7.7) for $\mathcal{T}_{ij}(s)$, we obtain

$$\mathcal{G}(s) = \sum_{i,j}\mathbf{B}(\mathsf{p})^*_{ij} \cdot \mathcal{T}_{ij}(s) = \sum_{i,j}\mathbf{B}(e^{\frac{\hbar}{2}z})^* B_i(-z) \otimes B_j(z)\Big|_{z=-\partial_s} \cdot s^{-2}.$$

For notational simplicity, let us write

$$\mathcal{T}_{ij}^{(n)} = \sum_{a=0}^n (-1)^a \frac{t_{i,a}}{a!} \otimes \frac{t_{j,n-a}}{(n-a)!},$$

so that $B_i(-z) \otimes B_j(z) = \hbar^2 \sum_{n\geqslant 0}\mathcal{T}_{ij}^{(n)} z^n$. With this notation at hand, we can write the expansion of $\mathcal{G}(s)$ as

$$\mathcal{G}(s) = \hbar^2 \sum_{N\geqslant 0}(N+1)! s^{-N-2}\left(\sum_{n=0}^{\lfloor \frac{N}{2}\rfloor}\frac{\hbar^{2n}}{2^{2n}}\sum_{i,j}\mathbf{B}_{ij}^{*,(2n)}\mathcal{T}_{ij}^{(N-2n)}\right), \tag{7.10}$$

where $\mathbf{B}^{*,(2n)}$ is the coefficient of t^{2n} in the Taylor expansion of $\mathbf{B}(e^t)^*$ at $t = 0$:

$$\mathbf{B}(e^t)^* = \sum_{n=0}^\infty \mathbf{B}^{*,(2n)} t^{2n}.$$

Thus, the coefficient of s^{-2} in $\hbar^{-2}\mathcal{G}(s)$ is given by $\sum_{ij}\mathbf{B}^{*,(0)} t_{i,0} \otimes t_{j,0}$. Note that $\mathbf{B}^{*,(0)} = \mathbf{B}^*$ is the adjoint of the corank 1, symmetric matrix $\mathbf{B} = D\mathbf{A}$, and hence is of rank 1. Thus, its rows (and columns, as it is symmetric) are scalar multiples of $\mathcal{C}_0$. In other words, there is a constant $\ell^{(0)}$ such that

$$\mathbf{B}^*_{ij} = \ell^{(0)} a_i a_j \text{ for every } i,j \in \mathbf{I}\,.$$

This shows that the coefficient of s^{-2} in $\hbar^{-2}\mathcal{G}(s)$ is $\sum_{ij} \mathbf{B}^*_{ij} t_{i,0} \otimes t_{j,0} = \ell^{(0)}\mathcal{C}_0 \otimes \mathcal{C}_0$, as claimed in the statement of the lemma.

Similarly, we obtain from (7.10) that the coefficients of s^{-3} and s^{-4} in $\hbar^{-2}\mathcal{G}(s)$ are given by

$$2\sum_{ij} \mathbf{B}^*_{ij}\mathcal{T}^{(1)}_{ij} \quad \text{and} \quad 6\sum_{ij} \mathbf{B}^*_{ij}\mathcal{T}^{(2)}_{ij} + \frac{6\hbar^2}{4}\mathsf{g}_0,$$

respectively. Since $\mathbf{B}^*_{ij} = \ell^{(0)} a_i a_j$ for each $i,j \in \mathbf{I}$, we have

$$2\sum_{ij} \mathbf{B}^*_{ij}\mathcal{T}^{(1)}_{ij} = 2\ell^{(0)}(\mathcal{C}_0 \otimes \mathcal{C}_1 - \mathcal{C}_1 \otimes \mathcal{C}_0),$$

$$6\sum_{ij} \mathbf{B}^*_{ij}\mathcal{T}^{(2)}_{ij} = 6\ell^{(0)}\left(\frac{\mathcal{C}_0 \otimes \mathcal{C}_2}{2} - \mathcal{C}_1 \otimes \mathcal{C}_1 + \frac{\mathcal{C}_2 \otimes \mathcal{C}_0}{2}\right).$$

Combining these facts, we can conclude that the s^{-3} and s^{-4} coefficients of $\hbar^{-2}\mathcal{G}(s)$ are as stated in the lemma.

We are left to compute the explicit values of the constant $\ell^{(0)}$. To this end, observe that for any fixed $i \in \mathbf{I}$, we have

$$\ell^{(0)} = \frac{\det(\mathbf{B}(1)^{(i)})}{a_i^2},$$

where $\mathbf{B}(1)^{(i)}$ is the submatrix obtained by removing the i–th row and i–th column from $\mathbf{B}$. By choosing i so that $a_i = 1$ and $\mathbf{B}(1)^{(i)}$ is a (connected) finite–type Dynkin diagram, whose determinants are known (see, for instance, [Kac90, §4.8, Table Fin]), we obtain the explicit values of $\ell^{(0)}$ listed in Table 1 below:

Type of $\mathfrak{g}$	$\mathsf{A}_n^{(1)}$	$\mathsf{B}_n^{(1)}$	$\mathsf{C}_n^{(1)}$	$\mathsf{D}_n^{(1)}$	$\mathsf{E}_6^{(1)}$	$\mathsf{E}_7^{(1)}$	$\mathsf{E}_8^{(1)}$	$\mathsf{F}_4^{(1)}$	$\mathsf{G}_2^{(1)}$
$\ell^{(0)}$	$n+1$	2^n	4	4	3	2	1	4	3

Type of $\mathfrak{g}$	$\mathsf{A}_{2n}^{(2)}$	$\mathsf{A}_{2n-1}^{(2)}$	$\mathsf{D}_{n+1}^{(2)}$	$\mathsf{E}_6^{(2)}$	$\mathsf{D}_4^{(3)}$
$\ell^{(0)}$	2^n	4	2^n	4	3

TABLE 1. Table of values of $\ell^{(0)}$

This type-by-type computation completes the proof of the lemma. □

Next, we define $\mathcal{G}_{\mathrm{reg}}(s)$ by setting

$$\mathcal{G}_{\mathrm{reg}}(s) = \mathcal{G}(s) - \hbar^2\left(\ell^{(0)}\mathcal{C}_0 \otimes \mathcal{C}_0 s^{-2} + 2\ell^{(0)}\left(-\mathcal{C}_1 \otimes \mathcal{C}_0 + \mathcal{C}_0 \otimes \mathcal{C}_1\right)s^{-3}\right). \tag{7.11}$$

Since the terms removed from $\mathcal{G}(s)$ are central (see Corollary 3.8 above), we have $\mathrm{ad}(\mathcal{G}(s)) = \mathrm{ad}(\mathcal{G}_{\mathrm{reg}}(s))$.

Corollary. *Properties (2)–(4) of Corollary 7.6 hold for $\mathcal{G}_{\mathrm{reg}}(s)$. In addition,*

$$\mathcal{G}_{\mathrm{reg}}(s) = 6\hbar^2\left(\ell^{(0)}\left(\frac{\mathcal{C}_2 \otimes \mathcal{C}_0}{2} - \mathcal{C}_1 \otimes \mathcal{C}_1 + \frac{\mathcal{C}_0 \otimes \mathcal{C}_2}{2}\right) + \frac{\hbar^2}{4}\mathsf{g}_0\right)s^{-4} + O(s^{-5}),$$

where $\ell^{(0)}$ *and* g_0 *are as in Lemma 7.7.*

7.8. **The difference equation and its formal solution.** Recall the difference equation (7.2):

$$(\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p}))\cdot\Lambda(s)=\mathcal{G}(s)\ .$$

This is an irregular, additive difference equation. Meaning, the difference operator has order of vanishing 3, while the right–hand side is $O(s^{-2})$. In more elementary terms, this equation has no solution in $Y^0_\hbar(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$, since the difference operator applied to any such formal power series results in a series starting with s^{-4}. The following simple lemma makes this more precise.

Lemma. *Let $\mathcal{A}$ be an arbitrary vector space over $\mathbb{C}$, and let $F(s)=\sum_{n\geqslant 0}F_n s^{-n-1}\in\mathcal{A}[\![s^{-1}]\!]$. Let $\hbar\in\mathbb{C}^\times$ and $\mathsf{p}\cdot F(s)=F(s-\hbar/2)$, as above. Then, for any polynomial $D(\mathsf{p})\in\mathbb{C}[\mathsf{p}^{\pm 1}]$, we have*

$$D(\mathsf{p})\cdot F(s)=\sum_{N\geqslant 0}s^{-N-1}\left(\sum_{\ell=0}^{N}\left(\begin{array}{c}N\\ \ell\end{array}\right)F_{N-\ell}\frac{\hbar^\ell}{2^\ell}\left((\mathsf{p}\partial_{\mathsf{p}})^\ell\cdot D(\mathsf{p})\right)\Big|_{\mathsf{p}=1}\right)\ .$$

Proof. The proof of this lemma is a direct verification. Namely, write $D(\mathsf{p})=\sum_{r\in\mathbb{Z}}d_r\mathsf{p}^r$. Then, we have:

$$\begin{aligned}D(\mathsf{p})\cdot F(s)&=\sum_{n\geqslant 0}F_n s^{-n-1}\left(\sum_r d_r\left(1-s^{-1}r\frac{\hbar}{2}\right)^{-n-1}\right)\\&=\sum_{n\geqslant 0}F_n s^{-n-1}\left(\sum_r d_r\sum_{\ell\geqslant 0}\left(\begin{array}{c}n+\ell\\ \ell\end{array}\right)r^\ell\frac{\hbar^\ell}{2^\ell}s^{-\ell}\right)\\&=\sum_{N\geqslant 0}s^{-N-1}\left(\sum_{\ell=0}^{N}F_{N-\ell}\left(\begin{array}{c}N\\ \ell\end{array}\right)\frac{\hbar^\ell}{2^\ell}\left(\sum_r d_r r^\ell\right)\right)\ .\end{aligned}$$

The lemma now follows from the observation that

$$\sum_r d_r r^\ell=(\mathsf{p}\partial_{\mathsf{p}})^\ell\cdot D(\mathsf{p})\big|_{\mathsf{p}=1}\,. \qquad \square$$

Remark. The lemma above implies that if $D(\mathsf{p})$ has order of vanishing $k\in\mathbb{Z}_{\geqslant 0}$ at $\mathsf{p}=1$, then $D(\mathsf{p})\cdot F(s)\in s^{-k-1}\mathcal{A}[\![s^{-1}]\!]$, for every $F(s)\in s^{-1}\mathcal{A}[\![s^{-1}]\!]$. In fact, the operation is invertible, thus yielding an isomorphism of vector spaces

$$D(\mathsf{p}):s^{-1}\mathcal{A}[\![s^{-1}]\!]\xrightarrow{\sim}s^{-k-1}\mathcal{A}[\![s^{-1}]\!]\ .$$

Corollary. *There exists a unique $\Lambda(s)\in s^{-1}Y^0_\hbar(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$ such that*

$$(\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p}))\cdot\Lambda(s)=\mathcal{G}_{\mathrm{reg}}(s)\ .$$

7.9. **Properties of $\Lambda(s)$.** We now turn to establishing the key properties satisfied by the element $\Lambda(s)$ from the previous corollary. Recall that $\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))$ is the linear span of $\{t_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}$.

Proposition. $\Lambda(s)$ *has the following properties:*

(1) $\Lambda(s)\in\mathrm{Prim}^{\mathrm{D}}(Y_\hbar(\mathfrak{g}))^{\otimes 2}[\![s^{-1}]\!]$.

(2) *The leading term of* $\Lambda(s)$ *is given by*

$$\Lambda(s) = s^{-1}\left(\frac{\ell^{(0)}}{\mathsf{d}^{(0)}\hbar}\left(\frac{\mathcal{C}_2\otimes\mathcal{C}_0}{2} - \mathcal{C}_1\otimes\mathcal{C}_1 + \frac{\mathcal{C}_0\otimes\mathcal{C}_2}{2}\right) + \frac{\hbar}{4\mathsf{d}^{(0)}}\mathsf{g}_0\right) + O(s^{-2}),$$

where $\ell^{(0)}, \mathsf{g}_0$ *are as given in Lemma 7.7 and* $\mathsf{d}^{(0)}$ *is given in Lemma 7.4.*

(3) $\Lambda^{\mathrm{op}}(s) = -\Lambda(-s)$.

(4) $(\tau_a\otimes\tau_b)(\Lambda(s)) = \Lambda(s+a-b)$ *for each* $a, b\in\mathbb{C}$.

(5) *For each* $k\in\mathbf{I}$, $n\in\mathbb{Z}_{\geqslant 0}$ *and* $y\in Y_\hbar^0(\mathfrak{g})$, *we have the following commutation relations:*

$$[\Lambda(s), x_{k,n}^{\pm}\otimes y] = \pm\sum_{a=0}^{\infty} x_{k,n+a}^{\pm}\otimes\partial_s^{(a)}(t_k(s))y,$$

$$[\Lambda(s), y\otimes x_{k,n}^{\pm}] = \mp\sum_{b=0}^{\infty} y(-\partial_s)^{(b)}(t_k(-s))\otimes x_{k,n+b}^{\pm}.$$

Proof. Note that (1) is obvious from the construction and (4) follows directly from Part (3) of Proposition 7.5.

Let us prove (2). Let Λ_0 denote the coefficient of s^{-1} in $\Lambda(s)$. In our case, the difference operator is

$$\mathcal{D}(\mathsf{p}) = (\mathsf{p}-\mathsf{p}^{-1})\det(\mathbf{B}(\mathsf{p})) = (\mathsf{p}-\mathsf{p}^{-1})^3\frac{\det(\mathbf{B}(\mathsf{p}))}{(\mathsf{p}-\mathsf{p}^{-1})^2}.$$

Let $C(\mathsf{p}) = \dfrac{\det(\mathbf{B}(\mathsf{p}))}{(\mathsf{p}-\mathsf{p}^{-1})^2}$. The following is an easy computation:

$$(\mathsf{p}\partial_{\mathsf{p}})^{\ell}\cdot\mathcal{D}(\mathsf{p})\big|_{\mathsf{p}=1} = \sum_{j=0}^{\ell}\begin{pmatrix}\ell\\ j\end{pmatrix}3(3^{j-1}-1)(1+(-1)^{j-1})\left((\mathsf{p}\partial_{\mathsf{p}})^{\ell-j}\cdot C(\mathsf{p})\right)\big|_{\mathsf{p}=1}.$$

The relevant coefficient is the $\ell=j=3$ term, where we get $48\mathsf{d}^{(0)}$. Using Lemma 7.8, we compare the coefficients of s^{-4} in $\mathcal{D}(\mathsf{p})\cdot\Lambda(s)$ and $\mathcal{G}_{\mathrm{reg}}(s)$ from Lemma 7.7 to obtain

$$48\frac{\hbar^3}{8}\mathsf{d}^{(0)}\Lambda_0 = 6\hbar^2\left(\ell^{(0)}\left(\frac{\mathcal{C}_2\otimes\mathcal{C}_0}{2} - \mathcal{C}_1\otimes\mathcal{C}_1 + \frac{\mathcal{C}_0\otimes\mathcal{C}_2}{2}\right) + \frac{\hbar^2}{4}\mathsf{g}_0\right),$$

which is precisely (2).

For (3), we flip the tensor factors in the difference equation (7.3) to get

$$\mathcal{D}(\mathsf{p})\cdot\Lambda^{\mathrm{op}}(s) = \mathcal{G}_{\mathrm{reg}}(-s).$$

On the other hand, $\mathsf{p}\cdot f(-s) = (\mathsf{p}^{-1}\cdot f(s))|_{s\mapsto -s}$, and $\mathcal{D}(\mathsf{p}^{-1}) = -\mathcal{D}(\mathsf{p})$. This gives:

$$\mathcal{D}(\mathsf{p})\cdot\Lambda(-s) = -\mathcal{G}_{\mathrm{reg}}(-s).$$

Thus, by uniqueness, we have $\Lambda^{\mathrm{op}}(s) = -\Lambda(-s)$.

The proof of (5) is also based on a uniqueness argument. Namely, apply $\mathcal{D}(\mathsf{p})$ to both sides of the commutation relation. The resulting equation holds by Corollaries 7.6 and 7.7. Thus both sides solve the same difference equation and hence must be equal. □

7.10. **Growth properties of the Borel transform of $\mathcal{G}_{\mathrm{reg}}(s)$.** Recall from Corollary 7.6 that $\exp(\mathcal{G}(s))$ is a rational function of s, taking value 1 at $s=\infty$.

Claim. The matrix entries of $\mathcal{G}(s)$, and hence of $\mathcal{G}_{\mathrm{reg}}(s)$, are of the form

$$\log\left(\prod_j \frac{s-a_j}{s-b_j}\right) + r_0(s),$$

where $\{a_j, b_j\}_j \subset \mathbb{C}$ is a finite set of complex numbers and $r_0(s)$ is a rational function vanishing at $s=\infty$.

Proof of Claim. Write $X(s) = \exp(\mathcal{G}(s))$ in its mulitplicative Jordan decomposition, whose entries are again rational functions of s (see [GTL16, Lemma 4.12]): $X(s) = X_d(s)(1+X_n(s))$. Note that entries of $X_d(s)$ are rational functions taking value 1 at $s=\infty$, while those of $X_n(s)$ vanish at $s=\infty$. Hence,

$$\mathcal{G}(s) = \log(X(s)) = \log(X_d(s)) + \sum_{r\geqslant 0}(-1)^r \frac{X_n(s)^{r+1}}{r+1}.$$

Here, the second sum on the last line is finite, since $X_n(s)$ is nilpotent. Our claim follows. □

Corollary. *Let $g(s)$ be a matrix entry of $\mathcal{G}_{\mathrm{reg}}(s)$. Write $g(s) = \sum_{n=0}^{\infty} g_n s^{-n-1}$ and let $\mathcal{B}(g)(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}$ be its Borel transform. Then, $\mathcal{B}(g)(t)$ is an entire function of t, and there exist constants C_1, C_2, R such that:*

$$|\mathcal{B}(g)(t)| \leqslant C_1 e^{C_2|t|}, \text{ for every } t \text{ with } |t| > R\ .$$

Proof. The proof is an easy computation of the Borel transform of (a) the logarithm of rational function, and (b) a rational function taking value 1 at $s=\infty$. Namely, it follows from the following computations:

$$g(s) = \log\left(\frac{s-a}{s-b}\right) \;\Rightarrow\; \mathcal{B}(g)(t) = \frac{e^{bt}-e^{at}}{t}\ .$$

$$g(s) = \frac{1}{(s-a)^{\ell+1}} \;\Rightarrow\; \mathcal{B}(g)(t) = \frac{t^\ell}{\ell!}e^{at}.$$ □

Appendix A. Laplace transform and regular difference equations

We collect some of the well–known techniques to solve linear, additive, regular difference equations. The material of this section is fairly standard, and can be found in any advanced text on complex analysis, for instance [AF21, Cos09, WW27].

A.1. **Set up and statement of the main theorem.** Let $D(\mathsf{p}) \in \mathbb{C}[\mathsf{p}^{\pm 1}]$ and let $k \in \mathbb{Z}_{\geqslant 0}$ be its order of vanishing at $\mathsf{p}=1$. Let $g(s)$ be a meromorphic, $\mathbb{C}$–valued function of $s \in \mathbb{C}$, which is holomorphic near $s=\infty$, and has order of vanishing $k+1$ there. Thus, $g(s) = \sum_{n=k}^{\infty} g_n s^{-n-1}$. Let $\mathcal{B}(g)$ denote the Borel transform of $g(s)$:

$$\mathcal{B}(g)(t) = \sum_{n=k}^{\infty} g_n \frac{t^n}{n!}\ .$$

It is an easy exercise to show that the series written above has infinite radius of convergence (assuming $g(s)$ has non–zero radius of convergence near ∞), hence defines an entire function of $t \in \mathbb{C}$.

Let us fix $\chi \in \mathbb{C}^\times$ a non–zero step and let p act as shift $\mathsf{p} \cdot F(s) = F(s-\chi)$. Let $\theta = \arg(\chi) \in (-\pi, \pi]$.

Below, we use the following notation for rays and half–planes. Let $\psi \in \mathbb{R}$ and $c \in \mathbb{R}_{>0}$. Let $\ell_\psi = \mathbb{R}_{\geqslant 0} e^{\iota\psi}$ be the ray at phase ψ. And,

$$\mathbb{H}_{\psi,c} = \{z \in \mathbb{C} : \mathrm{Re}(ze^{-\iota\psi}) > c\} \ ,$$

is the half–plane orthogonal to ℓ_ψ, located to the right of the line perpendicular to ℓ_ψ, passing through $ce^{\iota\psi}$.

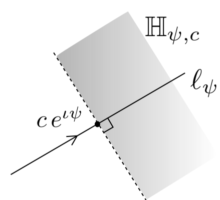


Figure A.1. The ray ℓ_ψ and corresponding half–plane $\mathbb{H}_{\psi,c}$

According to Lemma 7.8, we have a unique formal power series $f(s) \in s^{-1}\mathbb{C}[\![s^{-1}]\!]$, such that $D(\mathsf{p}) \cdot f(s) = \displaystyle\sum_{n=k}^{\infty} g_n s^{-n-1}$.

Theorem. *Assume that the following two conditions hold:*

- *The roots of $D(\mathsf{p}) = 0$ lie on the unit circle.*
- *$\mathcal{B}(g)(t)$ has at most exponential growth as $t \to \infty$. Meaning, there exist constants $R, C_1, C_2 \in \mathbb{R}_{>0}$, such that*
$$|\mathcal{B}(g)(t)| < C_1 e^{C_2|t|}, \text{ for every } t \text{ such that } |t| > R.$$

Then, there exist two meromorphic functions $f^\eta(s)$, $\eta \in \{\uparrow, \downarrow\}$, uniquely determined by the following conditions.

1. *$D(\mathsf{p}) \cdot f^\eta(s) = g(s)$.*
2. *$f^\eta(s)$ is holomorphic in $\mathbb{H}^\eta$, where $\mathbb{H}^\uparrow = \{s : \mathrm{Re}(s/\chi) \gg 0\} = -\mathbb{H}^\downarrow$.*
3. *$f^\eta(s) \sim f(s)$ as $\pm\mathrm{Re}(s/\chi) \to \infty$.*

Moreover, $f^\eta(s)$ is holomorphic in a larger domain $\mathcal{P}^\eta_\delta$, for any $\delta > 0$, where,

$$\mathcal{P}^\uparrow_\delta = \bigcup_{\psi \in \left(\theta - \frac{\pi}{2} + \delta, \theta + \frac{\pi}{2} - \delta\right)} \mathbb{H}_{\psi, C_2} \ , \qquad \mathcal{P}^\downarrow_\delta = -\mathcal{P}^\uparrow_\delta \ .$$

Here C_2 is the constant from our assumption on $\mathcal{B}(g)(t)$ above.

The asymptotic expansion $f^\eta(s) \sim f(s)$ is valid in a larger sector Σ^η_δ, for any $\delta > 0$,

$$\Sigma^\uparrow_\delta = \{re^{\iota\psi} : r \in \mathbb{R}_{>0}, \psi \in (\theta - \pi + \delta, \theta + \pi - \delta)\}, \qquad \Sigma^\downarrow_\delta = -\Sigma^\uparrow_\delta \ .$$

(see Figures A.2 and A.3 below).

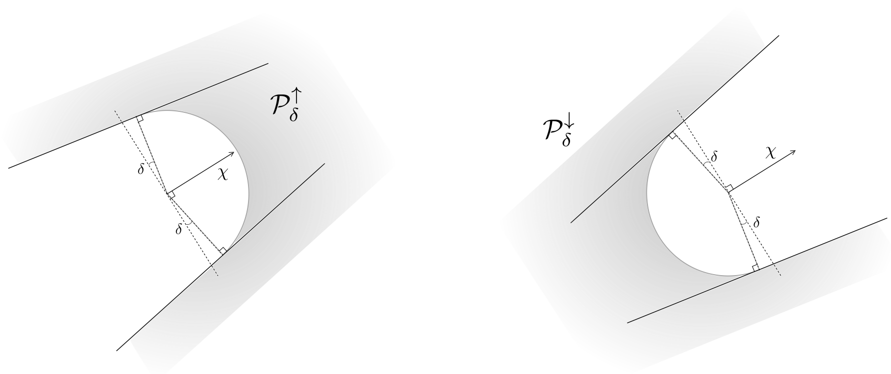


FIGURE A.2. Domains $\mathcal{P}^\eta_\delta$ of holomorphy

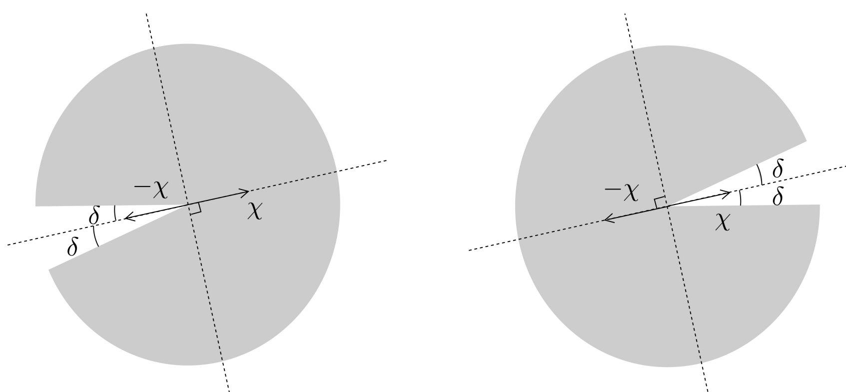


FIGURE A.3. Sectors Σ^η_δ of asymptotic expansion

PROOF. For notational convenience, we set $\chi = 1$. The reader can easily verify that the statement of the theorem can be obtained from its $\chi = 1$, $\theta = 0$ counterpart, by a counterclockwise rotation by θ.

The proof is given in the rest of this section. We show uniqueness of f^η in Section A.2. The existence is based on a general technique of Laplace transforms and their asymptotic expansions obtained via Watson's lemma. We review these results in Section A.3, and use them to show the existence of f^η in Section A.4. We show that the domain of holomorphy and sector of validity of asymptotic expansion can be enlarged, as stated in the theorem, in Section A.5. □

A.2. **Uniqueness.** The uniqueness of $f^\eta(s)$ follows from the following general lemma. For this, we drop the hypothesis that the roots of $D(\mathsf{p})=0$ lie on the unit circle, as it appears naturally from the conclusion of the lemma. We will assume, without loss of generality, that 0 is not a root of $D(\mathsf{p})$.

Let $\overline{\rho}(D)$ (resp. $\underline{\rho}(D)$) be the modulus of the longest (resp. shortest) root of $D(\mathsf{p})=0$. Note that $0<\underline{\rho}(D)\leqslant\overline{\rho}(D)$.

Lemma. *Let $\Sigma\subset\mathbb{C}$ be an unbounded open set satisfying:*

$$\text{For every } z\in\mathbb{C}, \text{ there exists } N\gg 0, \text{ such that}$$
$$z+n\in\Sigma, \quad (\text{resp. } z-n\in\Sigma) \quad ,\forall\ n\geqslant N\ .$$

Assume that $\phi(s)$ is a meromorphic function of s, holomorphic on Σ such that:

(1) $\phi(s)$ is asymptotically zero in Σ. That is,
$$\text{For every } n\in\mathbb{Z}_{\geqslant 0},\ \lim_{\substack{s\to\infty\\ s\in\Sigma}} s^n\phi(s)=0\ .$$
(2) $D(\mathsf{p})\cdot\phi(s)=0$.

If $\overline{\rho}(D)\leqslant 1$ (resp. $\underline{\rho}(D)\geqslant 1$), then $\phi\equiv 0$.

Proof. For the purposes of the proof, we write $D(\mathsf{p})=\mathsf{p}^N-\sum_{r=0}^{N-1}d_r\mathsf{p}^r$, and $d_0\neq 0$ (up to an overall shift, this is the general case). We work with the right fundamental domain. Thus, for every $s\in\mathbb{C}$,
$$\phi(s)=\sum_{n=0}^{N-1}d_n\phi(s+N-n)$$
In vector notation $\vec{\Phi}(s)=(\phi(s)\ \phi(s+1)\ \cdots\ \phi(s+N-1))^{\mathsf{p}}$, we get: $\vec{\Phi}(s)=\mathcal{D}\cdot\vec{\Phi}(s+1)$, where,
$$\mathcal{D}=\begin{bmatrix} d_{N-1} & d_{N-2} & \cdots & \cdots & d_0\\ 1 & 0 & \cdots & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1 & 0\end{bmatrix}$$
is the companion matrix of $D(\mathsf{p})$. The following identity is known as Gelfand's formula. Its proof can be found in any standard functional analysis book, for instance [RSN90, §149]. Here $|\cdot|$ is an arbitrary, fixed norm on the space of $N\times N$ matrices:
$$\lim_{n\to\infty}|\mathcal{D}^n|^{\frac{1}{n}}=\overline{\rho}(D)\ .$$

We will assume that $\overline{\rho}(D)\leqslant 1$. For the left domain, we have to use the fact that $\mathcal{D}$ is invertible ($d_0\neq 0$), thus, $|\mathcal{D}^{-n}|$ grows as $\underline{\rho}(D)^{-n}$.

The remainder of the argument is standard. Let $s_0\in\mathbb{C}$ be fixed, and assume that we are given $\varepsilon>0$. We will show that $|\vec{\Phi}(s_0)|<\varepsilon$, hence showing that it has to be zero.

Choose an $\ell \geqslant 1$ and $\alpha > 0$. Using the asymptotically zero condition, we get $R > 0$ such that $|\vec{\Phi}(s)| < \alpha |s|^{-\ell}$ for every $s \in \Sigma$ with $|s| > R$. Using the hypothesis on Σ, we choose an $N_0 > 0$ so that $s_0 + n \in \Sigma$ and $\mathrm{Re}(s_0 + n) > R$, for every $n \geqslant N_0$. The following inequality follows for every $n \geqslant N_0$:

$$\left|\vec{\Phi}(s_0)\right| = \left|\mathcal{D}^n \cdot \vec{\Phi}(s_0 + n)\right| \leqslant |\mathcal{D}|^n \alpha |s_0 + n|^{-\ell} \ .$$

As $|\mathcal{D}|^n$ grows proportionally to $\overline{\rho}(D)^n$, it is enough to observe that

$$\lim_{n\to\infty} \rho^n |s_0 + n|^{-\ell} = \begin{cases} 0 & \text{if } \rho \leqslant 1 \ , \\ \infty & \text{otherwise.} \end{cases}$$

□

Remark. It is worth mentioning that the uniqueness fails in general. For instance, consider the equation $f(t+1) - 2f(t) = 0$. It has infinitly many solutions $\lambda 2^t$ ($\lambda \in \mathbb{C}$) which are all asymptotically zero as $\mathrm{Re}(t) \to -\infty$.

A.3. **Laplace's theorem and Watson's lemma.** The following theorem summarizes the fundamental results of Laplace transforms and their asymptotic expansions. For a proof, see [AF21, §6.2.2] or [Cos09, §3.3, 3.4].

Theorem. *Assume that $\psi \in \mathbb{R}$ and $\ell_-\psi = \mathbb{R}_{>0} e^{-\iota\psi}$ is the ray at phase $-\psi$. Assume given a continuous function $F(t)$, $t \in \ell_{-\psi}$ such that*

- *$|F(t)|$ grows slower than exponential as $t \to \infty$. That is, there are constants $R, C_1, C_2 \in \mathbb{R}_{>0}$ such that*
$$|F(t)| < C_1 e^{C_2|t|}, \ \textit{for } |t| > R, \ t \in \ell_{-\psi} \ .$$
- *$F(t)$ has at worst logarithmic singularity as $t \to 0$. That is, there are constants $r, c_1, c_2 \in \mathbb{R}_{>0}$ such that*
$$|F(t)| < c_1 t^{c_2 - 1}, \ \textit{for } |t| < r, \ t \in \ell_{-\psi} \ .$$

Then, the following formula defines a function of z, holomorphic on the half plane $\mathbb{H}_{\psi, C_2}$:

$$\mathcal{L}(F)_\psi(z) = \int_{\ell_{-\psi}} F(t) e^{-tz} dt \ .$$

Moreover, if $F(t) \sim \sum F_n \frac{t^n}{n!}$ as $t \to 0$ along $\ell_{-\psi}$, then we have the following asymptotic expansion as $\mathrm{Re}(ze^{-\iota\psi}) \to \infty$:

$$\mathcal{L}(F)_\psi(z) \sim \sum_{n=0}^{\infty} F_n z^{-n-1} \ .$$

This expansion remains valid as $|z| \to \infty$ in

$$z \in \Sigma_\psi = \{re^{\iota t} : r \in \mathbb{R}_{>0}, t \in (\psi - \pi/2 + \delta, \psi + \pi/2 - \delta)\} \ ,$$

for any $\delta > 0$.

Remark. Note that the last part of the theorem is a triviality, since

$$|s| \to \infty, \ s \in \Sigma_\psi^\delta \text{ if, and only if } \mathrm{Re}(se^{-\iota\psi}) \to \infty \ .$$

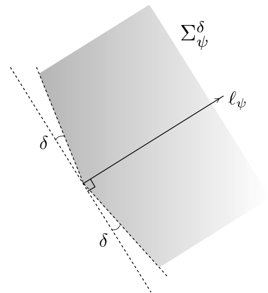


FIGURE A.4. Sector Σ_ψ^δ

A.4. **Existence.** Let $\psi \in \mathbb{R}$, and define:

$$f_\psi(s) = \int_{\ell_{-\psi}} \frac{\mathcal{B}\,(g)\,(t)}{D(e^t)} e^{-ts}\,dt\ .$$

We have to verify that the integrand satisfies the conditions of Theorem A.3 above, as long as $\ell_{-\psi}$ does not contain any roots of $D(e^t) = 0$. As roots of $D(\mathsf{p}) = 0$ are on the unit circle, this excludes $\psi = \pm\frac{\pi}{2}$ rays.

As $t \to \infty$ along $\ell_{-\psi}$, either $\mathrm{Re}(t) \to \infty$ or $-\mathrm{Re}(t) \to \infty$. It is not difficult to see that $|D(e^t)|$ approaches a finite, positive number, or ∞, as $\pm\mathrm{Re}(t) \to \infty$. In either case, we can choose $c > 0$ and $R > 0$, such that $|D(t)| > c$ for every $t \in \ell_{-\psi}$, with $|t| > R$. Combined with our hypothesis on $\mathcal{B}\,(g)\,(t)$, the sub–exponential growth condition is met for our kernel. We note that, by our hypothesis on the order of vanishing, $\frac{\mathcal{B}(g)(t)}{D(e^t)}$ is holomorphic near $t = 0$.

Thus for $\psi \neq \pm\frac{\pi}{2}$ modulo 2π, we get a holomorphic function on $\mathbb{H}_{\psi,C_2}$. Moreover,

$$D(\mathsf{p}) \cdot f_\psi(s) = \int_{\ell_{-\psi}} \frac{\mathcal{B}\,(g)\,(t)}{D(e^t)} D(e^t) e^{-ts}\,dt = \int_{\ell_{-\psi}} \mathcal{B}\,(g)\,(t) e^{-ts}\,dt = g(s)\ .$$

This functional equation allows us to extend $f_\psi(s)$ as a meromorphic function of $s \in \mathbb{C}$.

Moreover, as $|s| \to \infty$, $s \in \Sigma_\psi^\delta$ (see Figure A.4), we get the following asymptotic expansion of f_ψ: let $\dfrac{\mathcal{B}\,(g)\,(t)}{D(e^t)} = \displaystyle\sum_{n=0}^{\infty} \beta_n \frac{t^n}{n!}$ be its Taylor series expansion. Then,

$$f_\psi(s) \sim \sum_{n=0}^{\infty} \beta_n s^{-n-1}\ .$$

By uniqueness of the formal solution to $D(\mathsf{p}) \cdot f(s) = g(s)$, the series on the right–hand side is $f(s)$ for all ψ.

Hence, we obtain $f^{\uparrow}$ for $\psi = 0$ and $f^{\downarrow}$ for $\psi = \pi$. This prove that existence of the claimed meromorphic solutions.

A.5. **Extension of domains.** We will now show that $f_{\psi_1}(s) = f_{\psi_2}(s)$ for any $\psi_1, \psi_2 \in (-\pi/2, \pi/2)$. Similarly, for $\psi_1, \psi_2 \in (\pi/2, 3\pi/2)$. Therefore, $f^{\uparrow} = f_{\psi}$ for any $\psi \in (-\pi/2, \pi/2)$, and $f^{\downarrow} = f_{\psi}$ for any $\psi \in (\pi/2, 3\pi/2)$. This shows that $f^{\uparrow}$ is holomorphic in $\mathcal{P}^{\uparrow}_{\delta}$, and $f^{\uparrow} \sim f$ is valid in $\Sigma^{\uparrow}_{\delta}$ for any $\delta > 0$.

Now, let $\psi_1, \psi_2 \in (-\pi/2, \pi/2)$, $\psi_1 < \psi_2$. As $f_{\psi_1}(s) = f_{\psi_2}(s)$ is an identity between two meromorphic functions solving the same difference equation, it is enough to verify it for $s \in \mathbb{H}_{\psi_1, C_2} \cap \mathbb{H}_{\psi_2, C_2}$. Note that this set satisfies the hypotheses of Lemma A.2.

Fix a constant $C_3 > C_2$, and assume that $s \in \mathbb{H}_{\psi_1, C_2} \cap \mathbb{H}_{\psi_2, C_2}$ is such that $\mathrm{Re}(se^{-\iota\theta}) \geqslant C_3$ for every $\theta \in [\psi_1, \psi_2]$.

Then, we get:

$$f_{\psi_1}(s) - f_{\psi_2}(s) = \int_{\ell_{-\psi_1}} - \int_{\ell_{-\psi_2}} \left(\frac{\mathcal{B}(g)(t)}{D(e^t)} e^{-ts} \right) dt \ .$$

Let $R > 0$ and let $\mathcal{C}_R$ be the closed contour consisting of three smooth components: (a) straight (directed) line segment $L_{1,R}$ from 0 to $Re^{-\iota\psi_1}$, (b) circular arc γ_R from $Re^{-\iota\psi_1}$ to $Re^{-\iota\psi_2}$, and (c) directed segment from $Re^{-\iota\psi_2}$ to 0, denoted by $-L_{2,R}$.

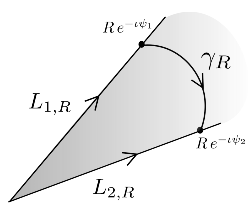


FIGURE A.5. The contour $\mathcal{C}_R$

As the integrand is holomorphic on the right half–plane, by Cauchy's theorem:

$$\int_{\mathcal{C}_R} \frac{\mathcal{B}(g)(t)}{D(e^t)} e^{-ts}\, dt = 0$$

Therefore, we get:

$$\begin{aligned} f_{\psi_1}(s) - f_{\psi_2}(s) &= \lim_{R\to\infty} \int_{L_{1,R}} - \int_{L_{2,R}} \left(\frac{\mathcal{B}(g)(t)}{D(e^t)} e^{-ts}\, dt \right) \\ &= \lim_{R\to\infty} \int_{\mathcal{C}_R} - \int_{\gamma_R} \left(\frac{\mathcal{B}(g)(t)}{D(e^t)} e^{-ts}\, dt \right) \\ &= -\lim_{R\to\infty} \int_{\gamma_R} \frac{\mathcal{B}(g)(t)}{D(e^t)} e^{-ts}\, dt \end{aligned}$$

Thus, it remains to show that the last limit is zero. Recall that we have the following bound on our kernel: there exist constants C_1, C_2 and R_1 such that

$$\left|\frac{\mathcal{B}\left(g\right)\left(t\right)}{D(e^t)}\right| < C_1 e^{C_2|t|}, \text{ for every } t \text{ with } |t| > R_1\ .$$

And, $s \in \mathbb{C}$ is chosen so that $|e^{-st}| = e^{-\operatorname{Re}(st)} \leqslant e^{-C_3 R}$ ($C_3 > C_2$ was a fixed constant). Hence, for $R \gg 0$, we obtain:

$$\left|\int_{\gamma_R} \frac{\mathcal{B}\left(g\right)\left(t\right)}{D(e^t)} e^{-ts}\, dt\right| \leqslant C_1 e^{(C_2 - C_3)R}(\psi_2 - \psi_1)R.$$

The last quantity clearly approaches 0, as $R \to \infty$, and we are done.

Appendix B. The augmented symmetrized Cartan matrix

In this appendix, we give a full proof of Proposition 3.8, which states that the augmented matrix $(\mathbf{B}\,|\,\underline{\mu})$ has full rank, where $\underline{\mu}$ is the column vector with j-th component $\mu_j = \sum_{i\in\mathbf{I}} a_i(d_i a_{ij})^3$ (see Corollary 3.8) and $\mathbf{B} = (d_i a_{ij})_{i,j\in\mathbf{I}}$ is the symmetrized Cartan matrix of $\mathfrak{g}$.

B.1. A characterization of μ_j.

We begin with the following simple lemma.

Lemma. *For each $i, j \in \mathbf{I}$, set $m_{ji} = a_{ji}(1 - a_{ji}^2)$. Then*

$$\mu_j / d_j^3 = 6a_j - \sum_{\substack{i\in\mathbf{I}\\ a_{ji}<-1}} m_{ji} a_i \quad \forall\, j \in \mathbf{I}.$$

In particular, $\mu_j = 6d_j^3 a_j$ unless there is $i_j \in \mathbf{I}$ such that $a_{j,i_j} < -1$. If such an index i_j exists then, provided $\mathfrak{g}$ is not of type $\mathsf{C}_2^{(1)}$, it is unique and one has

$$\mu_j = \begin{cases} 6d_j^3(a_j - a_{i_j}) & \text{if } \ a_{j,i_j} = -2,\\ 6d_j^3(a_j - 4a_{i_j}) & \text{if } \ a_{j,i_j} = -3.\end{cases}$$

Proof. Fix $j \in \mathbf{I}$. Then, since $\mu_j = \sum_{i\in\mathbf{I}} a_i(d_i a_{ji})^3$, we have

$$\mu_j/d_j^3 = \sum_{i\in\mathbf{I}} a_i a_{ji}^3 = \sum_{i\in\mathbf{I}} a_i a_{ji} + \sum_{i\in\mathbf{I}} a_i(a_{ji}^3 - a_{ji}) = \sum_{i\in\mathbf{I}} a_i(a_{ji}^3 - a_{ji}),$$

where we have used that $\mathbf{A}\underline{\delta} = 0$, with $\underline{\delta} = (a_j)_{j\in\mathbf{I}} \in \mathbb{Z}^{\mathbf{I}}$. Since $a_{jj}^3 - a_{jj} = 6$ and $m_{ji} = a_{ji} - a_{ji}^3$ is zero when $a_{ji} = -1$, this proves the first part of the lemma.

As for the second assertion, if $i \in \mathbf{I}$ is such that $a_{ji} < -1$, then the vertices i and j of the Dynkin diagram of $\mathfrak{g}$ are connected by multiple edges, with an arrow going from j to i. By [Kac90, §4.8, Tables Aff 1–3], any such index $i = i_j$ is unique except in type $\mathsf{C}_2^{(1)}$, where there is a single short simple root connected to two long simple roots. The claimed formulas now follow from the fact that, for any such type we have

$$\sum_{\substack{i\in\mathbf{I}\\ a_{ji}<-1}} m_{ji} a_i = \begin{cases} 6a_i & \text{if } a_{ji} = -2\\ 24a_i & \text{if } a_{ji} = -3.\end{cases} \qquad \square$$

B.2. **The rank of $(\mathbf{B}\,|\,\underline{\mu})$.** We now come to the following result, which is a restatement of Proposition 3.8.

Proposition. *The augmented matrix $(\mathbf{B}\,|\,\underline{\mu})$ has rank $|\mathbf{I}|$.*

Proof. Let $\mathbf{B}_j$ denote the j-th column of $\mathbf{B}$. We first note that there does not exist a vector $\underline{\gamma} \in \mathbb{Q}^{\mathbf{I}}$ such that $\mathbf{B}\underline{\gamma} \in (\mathbb{Q}_{>0})^{\mathbf{I}}$. Indeed, if there did, then $\underline{\delta} = (a_j)_{j\in\mathbf{I}} \in (\mathbb{Z}_{>0})^{\mathbf{I}}$ would satisfy $\underline{\delta}^t(\mathbf{B}\underline{\gamma}) > 0$, but this contradicts that $\underline{\delta}^t(\mathbf{B}\underline{\gamma}) = (\underline{\delta}^t\mathbf{B})\underline{\gamma} = 0$.

Hence, to prove the proposition it is sufficient to show that there is a vector $\underline{\gamma} \in \mathbb{Z}^{\mathbf{I}}$ such that

$$\underline{\gamma} \in (\mathbb{Z}\underline{\mu} + \mathrm{range}(\mathbf{B})) \cap (\mathbb{Z}_{>0})^{\mathbf{I}}. \tag{B.1}$$

If the Dynkin diagram associated to $\mathbf{A}$ is simply laced, then Lemma B.1 yields $\mu_j = 6a_j > 0$ for all $j \in \mathbf{I}$, so we may take $\underline{\gamma} = \underline{\mu}$.

Suppose instead that the Dynkin diagram of $\mathbf{A}$ belongs to the following list:

$$\mathsf{C}_\ell^{(1)}\ (\ell > 2),\ \mathsf{F}_4^{(1)},\ \mathsf{A}_{2\ell-1}^{(2)},\ \mathsf{E}_6^{(2)}.$$

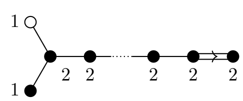
Then it follows from Lemma B.1 and [Kac90, Tables Aff 1–2] that $\mu_j = 6d_j^3 a_j$ for all $j \in \mathbf{I}$ such that $a_{ji} \geqslant -1$ for all $i \in \mathbf{I}$, and $\mu_j = 6d_j^3(a_j - a_i) = 6d_j^3$ if $j \in \mathbf{I}$ is such $a_{ji} = -2$ for some $i \in \mathbf{I}$. Thus, for all these types we may again take $\underline{\gamma} = \underline{\mu}$.

To complete the proof, it remains to show that there is $\underline{\gamma}$ as in (B.1) associated to each of the following affine types:

$$\mathsf{C}_2^{(1)},\ \mathsf{G}_2^{(1)},\ \mathsf{D}_4^{(3)},\ \mathsf{B}_\ell^{(1)},\ \mathsf{A}_{2\ell}^{(2)},\ \mathsf{D}_{\ell+1}^{(2)}.$$

We do this case by case, using the classification provided by [Kac90, Thm. 4.8] and appealing to the notation from Tables Fin and Aff of [Kac90, §4.8] as needed.

The $\mathsf{B}_\ell^{(1)}$ case. In this case, the Dynkin diagram of $\mathfrak{g}$ is



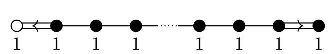
and we take $\mathbf{I} = \{0, l, \ldots, \ell\}$ with $\{1, \ldots, \ell\}$ labelling the type B_ℓ subdiagram with vertices given by the blackened nodes (labelled from left to right in increasing order), and $i = 0$ labels the extending node. In the above diagram, the node $j \in \mathbf{I}$ is marked by the number a_j. By Lemma B.1 we have $\mu_j = 6d_j^3 a_j = 48a_j$ for all $0 \leqslant j < \ell$ and $\mu_\ell = 0$. Equivalently,

$$\underline{\mu}^t = 48 \cdot (1, 1, 2, \ldots, 2, 0).$$

On the other hand, we have $\mathbf{B}_\ell^t = (0, \ldots, -2, 2)$, so we may take

$$\underline{\gamma} = \underline{\mu} + \mathbf{B}_\ell \in (\mathbb{Z}_{>0})^{\mathbf{I}}.$$

The $\mathsf{D}_{\ell+1}^{(2)}$ case. We again take $\mathbf{I} = \{0, 1, \ldots, \ell\}$, and label the nodes of the Dynkin diagram, which is



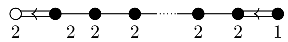
from left to right in increasing order. Since $a_j = 1$ for all $j \in \mathbf{I}$, Lemma B.1 yields

$$\underline{\mu}^t = (0, 6d_1^3, \ldots, 6d_{\ell-1}^3, 0) = 48 \cdot (0, 1, \ldots, 1, 0).$$

As $\mathbf{B}_0^t = (2, -2, 0, \dots, 0)$ and $\mathbf{B}_\ell^t = (0, \dots, 0, -2, 2)$, we deduce that

$$\underline{\gamma} = \underline{\mu} + \mathbf{B}_0 + \mathbf{B}_\ell \in (\mathbb{Z}_{>0})^{\mathbf{I}}.$$

The $\mathsf{A}_{2\ell}^{(2)}$ *case.* In this case, the Dynkin diagram is

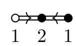
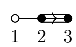
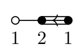


and we choose the same labeling convention as in the $\mathsf{D}_{\ell+1}^{(2)}$ case. We have $d_0 = 1$, $d_j = 2$ for $0 \leqslant j < \ell$, and $d_\ell = 4$. Lemma B.1 thus yields

$$\underline{\mu}^t = (0, 12d_1^3, \dots, 12d_{\ell-2}^3, 6d_{\ell-1}^3(a_{\ell-1} - a_\ell), 6d_\ell^3) = 48 \cdot (0, 2, \dots, 2, 1, 8).$$

Since $\mathbf{B}_0^t = (2, -2, 0, \dots, 0)$, the vector $\underline{\gamma} = \underline{\mu} + \mathbf{B}_0$ lies in $(\mathbb{Z}_{>0})^{\mathbf{I}}$.

The cases $\mathsf{C}_2^{(1)}$, $\mathsf{G}_2^{(1)}$ *and* $\mathsf{D}_4^{(3)}$. In these cases, all the relevant data, together with a choice of $\underline{\gamma}$ satisfying (B.1), is given in the following table:

Type	Diagram	$\mathbf{B}$	$\underline{\mu}$	$\underline{\gamma}$
$\mathsf{C}_2^{(1)}$		$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix}$	$\begin{bmatrix} 48 \\ 0 \\ 48 \end{bmatrix}$	$\underline{\mu} + \mathbf{B}_2$
$\mathsf{G}_2^{(1)}$		$\begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 2 \end{pmatrix}$	$\begin{bmatrix} 162 \\ 324 \\ -30 \end{bmatrix}$	$\underline{\mu} + 16\mathbf{B}_3$
$\mathsf{D}_4^{(3)}$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 6 \end{pmatrix}$	$\begin{bmatrix} 6 \\ -12 \\ 162 \end{bmatrix}$	$\underline{\mu} - 5\mathbf{B}_3$

Therefore, we may conclude that $(\mathbf{B} \,|\, \underline{\mu})$ has rank $|\mathbf{I}|$ for all affine types, where we continue to exclude types $\mathsf{A}_1^{(1)}$ and $\mathsf{A}_2^{(2)}$. $\square$

Appendix C. Determinant of affine quantum Cartan matrices

In this short appendix, we prove Lemma 7.4. We provide explicit formulae for both the determinant of the symmetrized affine quantum Cartan matrix $\mathbf{B}(\mathsf{p})$ and

$$\mathsf{d}^{(0)} = \left. \frac{\det(\mathbf{B}(\mathsf{p}))}{(\mathsf{p} - \mathsf{p}^{-1})^2} \right|_{\mathsf{p}=1}.$$

We exclude type $\mathsf{A}_1^{(1)}$. For the exceptional types, the formulae are obtained by a direct computation while, for the other types, it is enough to proceed by a corank two Lagrange expansion. The resulting formulae are given in Table 2 for untwisted simply laced types; in Table 3 for untwisted non simply laced types; in Table 4 for twisted types.

By direct inspection, this shows that $\det(\mathbf{B}(\mathsf{p}))$ has order of vanishing 2 at $\mathsf{p} = 1$ and all its zeros lie in $U(1)$.

Remark. For simply laced types , the explicit formulae for $\det(\mathbf{B}(\mathsf{p}))$ first appeared in [Sut07, Sec. 4]. Note however that in untwisted type BCFG our formulae differ from those given in *loc. cit.*, due to a different definition of the quantum Cartan matrix in these types.

Type of $\mathfrak{g}$	$\det(\mathbf{B}(\mathsf{p}))$	$\mathsf{d}^{(0)}$
$\mathsf{A}_n^{(1)}$	$\mathsf{p}^{-n-1}(\mathsf{p}^{n+1}-1)^2$	$\frac{(n+1)^2}{4}$
$\mathsf{D}_n^{(1)}$	$\mathsf{p}^{-n-1}(\mathsf{p}^2+1)(\mathsf{p}^{2(n-2)}-1)(\mathsf{p}^4-1)$	$4(n-2)$
$\mathsf{E}_6^{(1)}$	$\mathsf{p}^{-7}(\mathsf{p}^2+1)(\mathsf{p}^6-1)^2$	18
$\mathsf{E}_7^{(1)}$	$\mathsf{p}^{-8}(\mathsf{p}^2+1)(\mathsf{p}^8-1)(\mathsf{p}^6-1)$	24
$\mathsf{E}_8^{(1)}$	$\mathsf{p}^{-9}(\mathsf{p}^2+1)(\mathsf{p}^{10}-1)(\mathsf{p}^6-1)$	30

Table 2. Untwisted type ADE

Type of $\mathfrak{g}$	$\det(\mathbf{B}(\mathsf{p}))$	$\mathsf{d}^{(0)}$
$\mathsf{B}_n^{(1)}$	$\mathsf{p}^{-3n-1}(\mathsf{p}^2+1)^n(\mathsf{p}^{2(2n-3)}-1)(\mathsf{p}^8-1)$	$2^{n+2}(2n-3)$
$\mathsf{C}_n^{(1)}$	$\mathsf{p}^{-3n-11}(\mathsf{p}^2+1)^{n+1}(\mathsf{p}^4+1)(\mathsf{p}^{4(n+2)}-1)(\mathsf{p}^8-1)$	$2^{n+5}(n+2)$
$\mathsf{F}_4^{(1)}$	$\mathsf{p}^{-11}(\mathsf{p}^2+1)^3(\mathsf{p}^{10}-1)(\mathsf{p}^6-1)$	120
$\mathsf{G}_2^{(1)}$	$[3]_{\mathsf{p}}\mathsf{p}^{-9}(\mathsf{p}^2+1)(\mathsf{p}^{10}-1)(\mathsf{p}^6-1)$	90

Table 3. Untwisted type BCFG

Type of $\mathfrak{g}$	$\det(\mathbf{B}(\mathsf{p}))$	$\mathsf{d}^{(0)}$
$\mathsf{A}_{2n}^{(2)}$	$\mathsf{p}^{-6n-5}(\mathsf{p}^2+1)^{2n}(\mathsf{p}^{2(4n+1)}-1)(\mathsf{p}^8-1)$	$4^{n+1}(4n+1)$
$\mathsf{A}_{2n-1}^{(2)}$	$\mathsf{p}^{-2(n+1)}(\mathsf{p}^2+1)(\mathsf{p}^{2(n-2)}-1)(\mathsf{p}^4-1)$	$4(n-2)$
$\mathsf{D}_{n+1}^{(2)}$	$\mathsf{p}^{-3n-2}(\mathsf{p}^2+1)^n(\mathsf{p}^{4n}-1)(\mathsf{p}^4-1)$	$2^{n+2}n$
$\mathsf{E}_6^{(2)}$	$\mathsf{p}^{-9}(\mathsf{p}^2+1)^2(\mathsf{p}^8-1)(\mathsf{p}^6-1)$	48
$\mathsf{D}_4^{(3)}$	$\mathsf{p}^{-7}(\mathsf{p}^2+1)(\mathsf{p}^6-1)^2$	18

Table 4. Twisted type

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