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The Problem of Extrapolation of Quantum States

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Abstract

To what extent is it possible to extrapolate the state of a composite quantum system from the state of its subsystem? We address this general question by examining two particular models of quantum system dynamics. Our main observation is that quantum extrapolation is possible within the frame of nonlinear nonlocal models. We also find that this contrasts with the situation encountered in the linear Jaynes-Cummings model.

Keywords: quantum information, nonlocal quantum dynamics, quantum extrapolation

1 Introduction

Signal and information processing typically involve the task of extrapolation of an unknown larger data set from a known smaller sample. It is therefore not surprising that extrapolation has been studied quite extensively in the context of classical signal processing. Good extrapolation algorithms invariably rely upon an *a priori* model of the signal, or its source. Of course, the model may remain largely implicit, and its role may be quite subtle, e.g. in some situations it is appropriate and sufficient to have an *a priori* view about the signal regularity—an assumption as this enables tasks such as signal denoising, or *upsampling* (extrapolation of denser signal discretization from sparser one).

Much less is known at present about the problem of extrapolation in the context of quantum information. It is, however, interesting and natural to consider. Indeed, a quantum device of some utility for information processing, say, a *bona fide* quantum computer, will necessarily be a composite quantum system comprising a number of components. It may well be expected that in operating a quantum computer one will have access to the state of a subsystem only, rather than the state of the entire system. At the same time, the composite state of the entire system may be relevant to computation, either by design or by some other necessity. In such a case one will naturally encounter the problem of extrapolating the composite quantum state from the state of one of its subsystems. Recall that a subsystem cannot be ascribed a state vector, but only a statistical state in the form of a density matrix. Therefore, the problem of extrapolation is to estimate the unknown state vector of the entire system from the reduced density matrix corresponding to its subsystem.

2 The context and framework of quantum state extrapolation

The broad context in which we encounter composite system dynamics is that of the quantum dynamical semigroups, see [1], [2]. One of the central themes of this area is that of describing the dynamics of an accessible part of a composite system coupled to an unaccessible part with possibly unknown dynamics. It has resulted in remarkable results, such as the theory of the Markovian master equations, [3]. This theme, developed already in the 1970s, gains in importance in light of recent progress in nanotechnologies, e.g. [4], [5], and in quantum computation, [6].

The Markovian models of subsystem dynamics, useful as they are, seem to provide little insight into the state of the unaccessible subsystem, or the sate of the composite system. At the same time, it seems that nonlocal correlations present in composite quantum systems, such as those discussed in the context of Bell inequalities, open a small window of opportunity for estimation of the unaccessible subsystem state. In this article we discuss the essential nature of this opportunity, and propose a basic schema for extrapolating the state of an unaccessible component of an entangled composite system. To be sure, a purely kinematic treatment of correlations, while sufficient for a discussion of the Bell inequalities, is not enough to tackle the problem of extrapolation. Indeed, any type of extrapolation requires an *a priori* restriction on the set of possible quantum states of a system, and this can only be formulated on grounds of dynamics. Dynamics introduces a distinction between stationary and non-stationary states. As it turns out, in some nonlinear types of dynamics this by itself enables extrapolation.

In this article, we focus our attention on two manifestly nonlocal models of dynamics. The first type is the well-known Jaynes-Cummings model of the interaction of matter with an electromagnetic field in a cavity, [7], [8]. We find, however, that the linear Jaynes-Cummings dynamics excludes the possibility of extrapolation. The other type of nonlocal dynamics considered here is nonlinear. It turns out that extrapolation is possible within this framework. The latter type of dynamics has been proposed and investigated by this author in a number of publications, e.g. more recently [9], [10]. The main relevant results, evoked here in Section 3.1, come from reference [11]. For the reader's benefit, we carry out a direct comparison of this type of dynamics with the Jaynes-Cummings model in Section 3.2. The highlight of this comparison is formula (12). We hope that it will help the readers who are already familiar with the Jaynes-Cummings model to quickly grasp the essence of the nonlinear nonlocal dynamics. Both these models are described only to a minimal extent necessary to discuss the idea of extrapolation. Neither do we consider the full ramifications of the extrapolation problem. Our intention is only to clarify the foundational concept.

3 Nonlocal models of composite quantum systems

3.1 Nonlinear nonlocal dynamics

Suppose we are given two isolated quantum systems with their respective Hilbert spaces of state-vectors, denoted $\mathbf{H}$ and $\widehat{\mathbf{H}}$. Their dynamics is determined with Hamiltonians, say, H and $\hat{H}$. Let us say

$$H = \sum_{n\in\mathbb{N}} h_n |e_n\rangle\langle e_n|, \tag{1}$$

where $h_0 \leq h_1 \leq \ldots$, and $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathbf{H}$, while

$$\hat{H} = \sum_{n\in\mathbb{N}} \hat{h}_n |f_n\rangle\langle f_n|, \tag{2}$$

where $\hat{h}_0 \leq \hat{h}_1 \leq \ldots$, and $(f_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\widehat{\mathbf{H}}$.

Suppose the two systems are now allowed to interact, and form a composite. The state vectors of the composite live in $\widehat{\mathbf{H}} \otimes \mathbf{H}$. It is indispensable to us to identify composite states with operators

$$\widehat{\mathbf{H}} \bigotimes \mathbf{H} \ni |\Psi\rangle = \sum_{m,n} k^*_{mn} |f_m\rangle \otimes |e_n\rangle \longleftrightarrow K = \sum k_{mn} |f_m\rangle\langle e_n| : \mathbf{H} \to \widehat{\mathbf{H}}. \tag{3}$$

Recall that once the composite is formed, the subsystems are described by mixed states, say, ρ and $\hat{\rho}$. On easily verifies that $\rho := \mathrm{Tr}_{\widehat{\mathbf{H}}} |\Psi\rangle\langle\Psi| = K^*K$, and similarly $\hat{\rho} = KK^*$. (Our convention is such that $\hat{\rho}$ is the transpose matrix of $\hat{\rho}^T := \mathrm{Tr}_{\mathbf{H}} |\Psi\rangle\langle\Psi|$.)

Next, we prescribe the dynamics of the composite system. We assume that f is an analytic function, well-defined on the entire positive semi-axis $(0, \infty)$, where it assumes real values. In addition, let F denote the antiderivative of f, $F' = f$.

Our preoccupation is with composite-system Hamiltonians of the type

$$\Xi(K) = \mathrm{Tr}\left[KHK^*\right] + \mathrm{Tr}\left[K^*\hat{H}K\right] + \frac{1}{s}\,\mathrm{Tr}\left[F\left(K^*K\right)\right], \quad (s = \text{ real parameter}). \tag{4}$$

Here F is the antiderivative of f, and f is an analytic function assuming real values on $(0,\infty)$. We assume for simplicity that f is $1-1$. The function F is one of the constituents of the model. It is a parameter to be selected when attempting to apply the Ξ based model to a composite quantum system of interest. The criteria for selecting an appropriate F will not be discussed here. Instead we just briefly mention that the criteria may e.g. be based on the spectral characteristic of the system, which allows the matching of an appropriate F, see [12]. Alternatively, F may be found via fundamental quantum-teleportation type experiments with the underlying system, see [9].

From the point of view of our discussion, the main point is that in this model the *stationary states* of the composite system satisfy, [11]:

$$KH + \hat{H}K + \frac{1}{s}Kf(K^*K) = \nu K, \tag{5}$$

and may be represented in the form

$$K = K_{J,\sigma}(s) := \sum_{n\in J} r_n^\sigma(s)e^{i\theta_n}\,|f_{\sigma(n)}\rangle\langle e_n|, \tag{6}$$

where $J \subseteq \mathbb{N}$, $\sigma : J \xrightarrow{1-1} \mathbb{N}$ is an arbitrary immersion of J into $\mathbb{N}$. Moreover, the phases θ_n are arbitrary, while r_n is constraint as follows:

$$r_n^\sigma(s)^2 = f^{-1}\left[s\left(\nu - h_n - \hat{h}_{\sigma(n)}\right)\right] \text{ for all } n \in J. \tag{7}$$

Observe that when the composite system is in the state $K_{J,\sigma}$, the two subsystem mixed states are:

$$\rho_{J,\sigma}(s) = \sum_{n\in J} \rho^n_{J,\sigma}(s)|e_n\rangle\langle e_n|, \quad \rho^n_{J,\sigma}(s) := r_n^\sigma(s)^2, \tag{8}$$

and

$$\hat{\rho}_{J,\sigma}(s) = \sum_{n\in J} \rho^n_{J,\sigma}(s)|f_{\sigma(n)}\rangle\langle f_{\sigma(n)}| \quad \text{(same eigenvalues as } \rho_{J,\sigma}(s)). \tag{9}$$

Since the physical solutions must satisfy $\mathrm{Tr}\ \rho_{J,\sigma}(s) = \mathrm{Tr}\left[K_{J,\sigma}(s)^*K_{J,\sigma}(s)\right] = \sum_{n\in J}\rho^n_{J,\sigma}(s) = 1$ the actual value of ν in formula (7) is found *a posteriori* from this condition.

The purpose of this commentary is to discuss how the stationary states (6) may be used to extrapolate the unknown state of one part of a composite system. Before we embark on this discussion, we will attempt in the next few paragraphs to place this nonlocal model of quantum dynamics in the context of foundational quantum theory.

3.2 Comparison with the Jaynes-Cummings model

Perhaps the best known nonlocal model of a composite system dynamics is the Jaynes-Cummings model, [7]. We will compare this model to the one described in the previous section. The Jaynes-Cummings model describes an interaction of light with matter in a cavity. It is a nonlocal-type model describing the dynamics of a composite quantum system. In particular, it does not come from a quantization of a classical system. The model is based on a Hamiltonian which assumes the following form, [8]:

$$H = I \otimes H_F + H_M \otimes I + H_I : \quad \mathbf{H}_M \otimes \mathbf{H}_F \longrightarrow \mathbf{H}_M \otimes \mathbf{H}_F.$$

We now describe the Hilbert spaces $\mathbf{H}_M$ and $\mathbf{H}_F$, as well as the three terms of the Hamiltonian. H_F describes free EM field in a cavity. It results from a quantization of the EM field with periodic boundary conditions. It turns out that H_F is a harmonic oscillator:

$$H_F = \hbar\omega \hat{a}^* \hat{a} : \quad \mathbf{H}_F \longrightarrow \mathbf{H}_F.$$

The eigenbasis of H_F consists of vectors denoted $|n\rangle$. We have $\mathbf{H}_F = \mathrm{span}\{|n\rangle : n = 0, 1, 2, \ldots\}$. One has

$$\hat{a}\,|n\rangle = \sqrt{n}\;|n-1\rangle \quad \text{and} \quad \hat{a}^*\,|n\rangle = \sqrt{n+1}\;|n+1\rangle,$$

so that

$$\hat{a}^* \hat{a}\,|n\rangle = n\;|n\rangle.$$

For simplicity, H_M, which describes noninteracting matter, is taken to be a two-level Hamiltonian:

$$H_M = \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) : \quad \mathbf{H}_M \longrightarrow \mathbf{H}_M,$$

where $\mathbf{H}_M = \mathrm{span}\{|g\rangle, |e\rangle\}$. Note that the energy gap between the ground state $|g\rangle$ and the excited state $|e\rangle$ is $\hbar\omega_0$.

Finally, the interaction part H_I is defined as

$$H_I = \hbar\lambda\,(\sigma_+ \otimes \hat{a} + \sigma_- \otimes \hat{a}^*) : \quad \mathbf{H}_M \otimes \mathbf{H}_F \longrightarrow \mathbf{H}_M \otimes \mathbf{H}_F$$

where

$$\sigma_+ = |e\rangle\langle g| \quad \text{and} \quad \sigma_- = |g\rangle\langle e|.$$

A direct check shows that:

$$(I \otimes H_F + H_M \otimes I + H_I)\,|g\rangle|n\rangle = (\hbar\omega n - \frac{\hbar\omega_0}{2})\ |g\rangle|n\rangle + \hbar\lambda\sqrt{n}\ |e\rangle|n-1\rangle, \tag{10}$$

as well as

$$(I \otimes H_F + H_M \otimes I + H_I)\,|e\rangle|n-1\rangle = (\hbar\omega(n-1) + \frac{\hbar\omega_0}{2})\ |e\rangle|n-1\rangle + \hbar\lambda\sqrt{n}\ |g\rangle|n\rangle, \tag{11}$$

Let us represent the composite Hilbert space in the form:

$$\mathbf{H}_M \otimes \mathbf{H}_F = S^0 \oplus \bigoplus_{n=1}^{\infty} S^n,$$

where $S^0 = \text{span}\{|g\rangle|0\rangle\}$, and $S^n = \text{span}\{|g\rangle|n\rangle, |e\rangle|n-1\rangle\}$ for $n = 1, 2, 3, \ldots$. Formulas (10), (11) show that H filters through this decomposition, i.e.

$$H\ |_{S^n} \colon S^n \longrightarrow S^n, \quad n = 0, 1, 2, \ldots$$

This fundamental property of the Hamiltonian H allows us to understand its properties via a representation of each component as a 2×2 matrix. Subsequently, one finds the eigenvalues $E_\pm(n)$ of each component $H\ |_{S^n}$ separately. It turns out that

$$E_\pm(n) = (n - \frac{1}{2})\hbar\omega \pm \frac{\hbar}{2}\Omega_n \quad \text{where } \Omega_n = \left[(\omega - \omega_0)^2 + 4\lambda^2 n\right]^{1/2}.$$

The correction terms Ω_n are known as the Rabi frequencies. The spectrum of the composite system is different than that of either one of its components. This effect is the main prediction of the Jaynes-Cummings model.

We will now reinterpret the model in the variable K. Let us select a (component) composite state to be, say,

$$|\Psi\rangle = z_1\,|g\rangle|n\rangle + z_2\,|e\rangle|n-1\rangle.$$

Identifying the state $|\Psi\rangle$ with an operator K as in (3) we obtain

$$K = \bar{z}_1\,|g\rangle\langle n| + \bar{z}_2\,|e\rangle\langle n-1|.$$

Note that K acts between the two subsystem spaces as follows

$$K : \quad \mathbf{H}_F \longrightarrow \mathbf{H}_M.$$

It is easily seen that the energy of the system may be expressed as

$$\langle\Psi|H\Psi\rangle = \mathrm{Tr}\left[KH_FK^*\right] + \mathrm{Tr}\left[K^*H_MK\right] + \hbar\lambda\,\mathrm{Tr}\left[K^*\sigma_+K\hat{a}^*\right] + \hbar\lambda\,\mathrm{Tr}\left[K\hat{a}K^*\sigma_-\right]. \tag{12}$$

It is interesting to compare this formula with the expression for $\Xi(K)$ in (4). While the first two terms are of the same type in both, the interaction terms in the Jaynes-Cummings model remains sesquilinear, while the interaction term in (4) may include any function F (satisfying some rather loose constraints). Note that since the set of nonzero eigenvalues of KK^* is equal to the set of nonzero eigenvalues of K^*K, one has $\mathrm{Tr}\left[F\left(K^*K\right)\right] = \mathrm{Tr}\left[F\left(KK^*\right)\right]$. Using the operator K instead of a composite state vector is not necessary in linear models such as the Jaynes-Cummings model. It becomes necessary, however, when the model incorporates nonlinear terms, such as the entropy. Since transition to operators enables one to use the underlying structure of an operator algebra, it becomes possible to form nonlinear functions of the state without any aid from the local geometry. In this way, one obtains rich models with interesting spectral characteristic. In particular, the spectra sometimes have fractal structure, even when the components (corresponding to H and $\hat{H}$) have very simple spectra, [12].

Reliance on an operator representation of the state vector is sometimes beneficial even when the underlying dynamics is linear, or has an entirely different type of nonlinearity than the one considered here, e.g. in article [13] authors exploit a certain type of linear an nonlinear models for two-strand systems, such as the DNA, which also require an operator interpretation of the state vector. The reader may wish to consult the Appendix of [11] for further comments on nonlinear operator equations encountered in the physics and mathematics literature.

4 Extrapolating quantum states

4.1 In the Jaynes-Cummings model extrapolation is impossible

It is interesting to consider if any type of extrapolation of a composite state from the subsystem state is possible in the Jaynes-Cummings model. Assume that the composite system described by this model is in a stationary state. It is easily seen that the stationary states corresponding to $E_+(n)$ or $E_+(n)$ are (respectively).

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle|n\rangle + |e\rangle|n-1\rangle\right)$$

and

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}} \left(|g\rangle|n\rangle - |e\rangle|n-1\rangle \right).$$

If the composite system is in one of the eigenstates, the subsystem will be in the state

$$\rho_M = \frac{1}{2} \left(|g\rangle\langle g| + |e\rangle\langle e| \right).$$

In other words, the mixed state does not depend on the state of the composite. In consequence, it does not contain any information about the composite state. In particular it is impossible to say anything about the number of quanta n of the electromagnetic field. An experimenter with access to the subsystem $\mathbf{H}_M$, will always find it in the mixed state ρ_M, and will not be able to infer anything about the state of the other subsystem ($\mathbf{H}_F$), nor of the composite state.

4.2 Extrapolation is possible in the nonlinear nonlocal dynamics

From now on, we consider a composite quantum system for which a model of type (4) has been found and established. This means we can assume full knowledge of the constituents of the model, including the eigenvalues and eigenvectors of the two Hamiltonians (1), (2), the parameter s, and the function F. Furthermore we assume that the composite system at hand is in a stationary state. In particular, its pure state is as described in (6), (7), and the mixed states of the subsystems are as in (8), (9). We wish to extrapolate the composite state $|\Psi\rangle$, *a fortiori* $\hat{\rho}$, while having access only to subsystem $\mathbf{H}$. Equation (7) is key in this task. Let us consider for simplicity a particular example with $f(x) = x$, and $s = 1$. Furthermore, let $\dim \mathbf{H} = 2 = \dim \widehat{\mathbf{H}}$. Finally, let (in suitable units)

$$H = \frac{1}{6}|e_1\rangle\langle e_1| + \frac{2}{6}|e_2\rangle\langle e_2|, \qquad \hat{H} = \frac{1}{6}|f_1\rangle\langle f_1| + \frac{2}{6}|f_2\rangle\langle f_2|,$$

so that here

$$h_1 = \hat{h}_1 = \frac{1}{6} \quad h_2 = \hat{h}_2 = \frac{2}{6}.$$

We assume that the composite is in a stationary state. Equation (6) implies that for $J = \{1, 2\}$, the composite state is either of the form

$$|\Psi^I\rangle = \sqrt{\rho_1^I}\, e^{i\theta_1^I}\, |e_1\rangle|f_1\rangle + \sqrt{\rho_2^I}\, e^{i\theta_2^I}\, |e_2\rangle|f_2\rangle,$$

or of the form

$$|\Psi^{II}\rangle = \sqrt{\rho_1^{II}}\, e^{i\theta_1^{II}}\, |e_1\rangle|f_2\rangle + \sqrt{\rho_2^{II}}\, e^{i\theta_2^{II}}\, |e_2\rangle|f_1\rangle,$$

where we have applied the usual identification of operators K with composite states $|\Psi\rangle$. It follows that the corresponding mixed states of the subsystem $\mathbf{H}$ are

$$\rho^{I,II} = \rho_1^{I,II}\, |e_1\rangle\langle e_1| + \rho_2^{I,II}\, |e_2\rangle\langle e_2|.$$

Furthermore, (7) implies that

$$\rho_1^I = \nu - h_1 - \hat{h}_1 = \nu - \frac{1}{3}, \quad \rho_2^I = \nu - h_2 - \hat{h}_2 = \nu - \frac{2}{3}$$

but

$$\rho_1^{II} = \nu - h_1 - \hat{h}_2 = \nu - \frac{1}{2}, \quad \rho_2^{II} = \nu - h_2 - \hat{h}_1 = \nu - \frac{1}{2}$$

In both cases the condition $\rho_1^{I,II} + \rho_2^{I,II} = 1$ implies $\nu = 1$, so that in fact

$$\rho^I = \frac{2}{3}\, |e_1\rangle\langle e_1| + \frac{1}{3}\, |e_2\rangle\langle e_2|,$$

and

$$\rho^{II} = \frac{1}{2}\, |e_1\rangle\langle e_1| + \frac{1}{2}\, |e_2\rangle\langle e_2|.$$

With this understood let us make the following observations: An experimenter estimating ρ will either find that it is closer to ρ^I or to ρ^{II}. The knowledge of the mixed state implies the corresponding composite state. If for example $\rho \approx \rho^I$, we infer $|\Psi\rangle = |\Psi^I\rangle$. Of course, the phase factors $e^{i\theta_1^I}$, and $e^{i\theta_2^I}$ remain arbitrary. Moreover once we know that the subsystem $\mathbf{H}$ is in the state ρ^I, we know the mixed state of $\widehat{\mathbf{H}}$ to be

$$\hat{\rho}^I = \frac{2}{3}\, |f_1\rangle\langle f_1| + \frac{1}{3}\, |f_2\rangle\langle f_2|.$$

This in turn allows us to determine the energy of the second subsystem $\langle \hat{H} \rangle = \mathrm{Tr}[\hat{H}\hat{\rho}]$, as well as the total energy Ξ of the composite system, which is given by (4). Thus, we possess a nearly complete information about the system. The only undetermined parameters are the phases of the composite system state, which are not constrained by the model.

Remarks.

- For the sake of completeness we note that there are additional eigenstates corresponding to $J = \{1\}$ or $J = \{2\}$. In these instances we either have $|\Psi\rangle = |e_1\rangle|f_1\rangle$ or $|\Psi\rangle = |e_1\rangle|f_2\rangle$, both implying $\rho = |e_1\rangle\langle e_1|$, or (symmetrically) $|\Psi\rangle = |e_2\rangle|f_1\rangle$ or $|\Psi\rangle = |e_2\rangle|f_2\rangle$ both resulting in $\rho = |e_2\rangle\langle e_2|$. Of course, in the degenerate case extrapolation is not unique. However, if an experimenter estimating ρ will find that it is close to $\rho = |e_1\rangle\langle e_1|$, the system can be either in a state $|\Psi\rangle = |e_1\rangle|f_1\rangle$ or $|\Psi\rangle = |e_1\rangle|f_2\rangle$. This is still a modicum of information about the composite system state. In particular, the possibility of it being in the state $|\Psi^I\rangle$ or $|\Psi^{II}\rangle$ can be eliminated.

- Furthermore, let us recall that the mutual information corresponding to the composite system, [6], is defined as

$$S(\widehat{\mathbf{H}} : \mathbf{H}) = S(\rho) + S(\hat{\rho}) - S(\rho_{\text{composite}}),$$

where $S(\rho) = -\mathrm{Tr}\,\rho \log \rho$ is the entropy of the system characterized by a mixed state ρ. When the composite system is in a pure state, we have $S(\rho_{\text{composite}}) = S(|\Psi\rangle\langle\Psi|) = 0$. Also, since ρ and $\hat{\rho}$ share all the nonzero eigenvalues, we have $S(\rho) = S(\hat{\rho})$. Therefore, in regard to the stationary state considered here we in fact have

$$S(\widehat{\mathbf{H}} : \mathbf{H}) = 2S(\rho).$$

Note that in the degenerate case the mutual information $S(\widehat{\mathbf{H}} : \mathbf{H}) = 0$, whereas it is nonzero for the nondegenerate case considered above. In other words, the example suggests that extrapolation, as outlined here, is only possible in those cases when the mutual information is nonzero.

- The extrapolation process described here starts with an estimation of the density operator of a subsystem. In regard to this problem, the reader may wish to consult [14].

References

[1] R. Alicki and M. Fannes, Quantum Dynamical Systems, (Oxford University Press 2001)

[2] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems, (Oxford University Press 2002)

[3] G. Lindblad, "On the Generators of Quantum Dynamical Semigroups", Comm. Math. Phys., Vol. **48** (1976), 119-130

[4] S. Datta, "Nanoscale device modeling: the Green's function method", Superlattices and Microstructures, Vol. **28**, 2000, 253-278

[5] M. Paulsson, F. Zahid, S. Datta, "Resistance of a molecule", in: W.A. Goddart III et al. (editors), Handbook of Nanoscience, Engineering, and Technology, (CRC Press 2003)

[6] S. Stenholm, K.-A. Suominen, Quantum approach to informatics, (Wiley 2005)

[7] E. T. Jaynes and F. W. Cummings, "Comparison of Quantum and Semiclassical Radiation Theories with Application to the Beam Maser", Proc. IEEE **51** (1963), 89-109

[8] C.C. Gerry and P. L. Knight, Introductory Quantum Optics, (Cambridge University Press 2005)

[9] A. Sowa, "Quantum entanglement in composite systems", Theor. Math. Phys., Vol. **159**, No. 2 (2009), 655-667

[10] A. Sowa, "A model for nonlocal bonding in bipartite systems", J. Mod. Opt., Vol. **56**, Issue 12 (2009), 1363-1368

[11] A. Sowa, "Stationary states in nonlocal type dynamics of composite systems", J. Geom. Phys., **59** (2009), 1604-1612

[12] A. Sowa, "Spectra of nonlocally bound quantum systems", submitted

[13] D. Aerts and M. Czachor, "Two-State Dynamics for Replicating Two-Strand Systems", Open Sys. & Information Dyn. (2007) 14:397-410

[14] D.F.V. James; P.G. Kwiat, W.J. Munro, A.G. White, "Measurement of qubits", Phys. Rev. A **64** (2001) 052312-115

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