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Sensitivity to Lattice Structure in the Mesoscopic-Loop Models of Planar Systems

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Abstract

Some planar systems display universal magnetic behavior at low temperatures, e.g. either superconductivity or quantization of the Hall conductance. These are the phenomena that traditionally attracted the most interest among researchers. At the same time, nanotechnology inspires us to examine complementary, albeit still strongly quantum, regimes of operation, in which electronic properties depend on the specifics of the material. In particular, one is compelled to address the problem of efficient extraction of material-specific information from essentially quantum systems.

As demonstrated in my recent article, mesoscopic feedback', a certain extension of the Schrödinger equation—namely, an extension via the so-called *mesoscopic loop*—leads to a PDE model of systems operating in the quantum Hall regime. In the present article, we construct lattice-type solutions of the mesoscopic loop model with periodic potential. The method of construction is geometric, and leads to continuous strong solutions. At the same time, the result brings to light a constraint that the quantum Hall phenomenology imposes on the parameters of the model. This suggests that the model has the capacity to transcribe material-specific information inherent in the spatially distributed Hall conductance data outside the FQHE regime.

Keywords: Quantum Hall Effects, nanofabrication process control

1 Motivation

At present nanotechnology is mostly defined through the plethora of novel concepts it brings to the playground of high-tech, and appreciated through the hope it generates as an enabling paradigm for societally beneficial applications. We are already witnessing an inspiring influx of prototype nano-materials and nano-devices. However, as it comes to mass-manufacture and commercialization of quality products based on such inventions there often exist essential technical barriers. Namely, it is a uniquely challenging task to enable relevant process control and quality assurance systems as necessary for automatization of nanofabrication processes. The latter necessarily involves collection of data, and extraction of informational contents in relevance to the physical nano-system and process. In this context, however, the classical principles of signal-processing need to be dramatically updated. How to read out the informational contents from a signal generated by a system that bears an essentially quantum character? How to define and filter out noise in this case?

Signal processing procedures such as feature extraction and denoising can only be successful if they are specialized to the specific type of signal. In particular, construction of a suitable model of a physical process is prerequisite to the task of analyzing signals generated by this process. The model may remain implicit, but will always be present as the backbone of the method of analysis. To illustrate this with an example, which by itself belongs in the domain of classical mechanics, let us briefly envision a chaotic physical system generating signals in a particular way. Suppose further that the type of dynamics governing that system would be known in general, say, in the form of a class nonlinear ordinary differential equations (assuming some freedom in the choice of coefficients). In such a case, the parameters that best describe the specific system would be the same parameters that characterize the relevant specific differential equation. It would then suit us to design the signal processing apparatus in such a way as to estimate those coefficients based on the stream of data collected from the actual process. That would enable us to perform analysis of the source, and answer various reasonable questions. For example, we might be able to tell if two intercepted signals come from one source or two different sources. Similarly, we should be able to define the notion of quality (of a system), if that were our aim. We would then be able to determine if a particular system satisfies our expectations as to quality, hopefully solely based on the signal it generates.

Thus, in principle, one may hope to use an *a priori* assumed parametric model in order to extract from a signal information about its physical source.

An important additional observation is that tasks such as differentiating between different systems and classifying a system as high or low quality do not necessarily require absolute accuracy of the model.

Analysis of quantum mechanical systems typically presents a greater challenge in that their dynamics is captured by partial differential equations, or coupled systems of such equations. The parameter space is also a larger one, e.g. if the dynamics is sufficiently captured by a single-particle Schrödinger equation, parameters consist of the lattice and impurity potentials characteristic of the ambient material. In particular, the task of characterizing such systems, via analysis of data from electronic measurements, is a more daunting one. In particular, it appears crucial that one should undertake it with such models in hand whose complexity would not be an additional impediment. In fact, the general practice of signal processing suggests that information may be effectively reflected in simplified models. In particular, a model that is optimal from the signal processing viewpoint is not necessarily optimal from the ab-initio modeling point of view, and vice versa. An optimal model for information extraction needs to be constructive and computationally effective. Perfect accuracy in all aspects of the physical system, on the other hand, is not essential, if it would be desirable.

This is the point of view we will assume when examining a certain class of models relevant to the low-temperature two dimensional systems in the magnetic field. This area is attractive to us for several reasons:

- We already have good models of 2D systems. Namely, I have in mind the mesoscopic-loop (MeM) model and its various extensions. Since the MeM is essentially a PDE-based model, it is computationally very tractable. The constituents of the PDE provide definition of parameters relevant to the (quantum) Hall platform signal-processing task.

- Rich and exciting phenomena are observed in this category: think of the quantum wells, superconducting films, 2D graphene sheets (i.e. the famous relativistic material, cf. [6], [7]), and the so-called diluted magnetic semiconductors, cf. [8], [2].

- The canonical ab initio type approach to the spin-dependent charge transport is via the current density functional theory (CDFT), cf. [4]. However, the CDFT models are far from achieving a status similar to that of the classical DFT theories. These newer models are burdened with significant computational complexity. Also, it is known that the

time-dependent version (TDCDFT) is burdened with non-uniqueness phenomena, cf. [1].

- In view of complex and emergent nature of the planar-system phenomena, the theoretical foundations for computational modeling may fundamentally need to rely on precepts that break free of the analogy with bulk systems. Consider for example the surprising emergence (within the framework of relativistic QFT in $2+1$ dimensions) of the so-called chiral condensate in the presence of magnetic field, cf. [3].

Apart from all that, it is very much of interest to us, of course, to see to what extent we can build a faithful MeM based model of certain specific 2D systems, but that is a different issue. The MeM amounts to a novel approach to the magnetic transport phenomena, such as the Quantum Hall Effects. Its cornerstone is a nonlinear operator equation dubbed the mesoscopic Schrödinger equation. We emphasize that the model based on equations (1)-(2) corresponds to a special case solution of this universal theory. The overall scope of the MeM is significantly broader. While the MeM and the mesoscopic Schrödinger equation are new, and have been postulated on grounds of essentially geometric considerations relevant to the physical conceptual framework at hand, it is worthwhile to view them in the broader context of emergent physical laws. Some other examples of scale-emergent laws obtained on grounds of semi-classical methods (and *not* related to the QHE) can be found in [5].

In this article, we consider the MeM model from a particular point of view. Namely, we wish to examine, quite roughly, the constraint in the parameter space (of external potentials) introduced by the quantum Hall effect, as it is captured by the MeM model.

2 The basic characteristic of the MeM model

Throughout this article, we will set the familiar constants e, $\hbar$ and m^* to 1. The geometric arena for the following discussion is the Euclidean plane with fixed orientation. In particular, recall that the Hodge star $*$ in two dimensions is a linear operation on differential forms determined by the following relations as $*dx = dy, *dy = -dx, *1 = dx \wedge dy, *(dx \wedge dy) = 1$. In general, the Hodge star defines a conformal structure on a surface. Also, the co-derivative δ is defined on the p-forms via the exterior derivative, i.e. $\delta = - * d *$ (formula specified to a two-dimensional manifold).

In a recent article [11] an analytic model has been introduced, which captures the characteristic of the quantum Hall regime. The model for a system characterized by Hall resistance R_H is based on the following coupling of the vector potential A with the wavefunction Ψ:

$$*dA = R_H|\Psi|^2 \tag{1}$$

$$\nabla_A^*\nabla_A\Psi + V\Psi = E\Psi. \tag{2}$$

Naturally, the second equation is the familiar magnetic Schrödinger equation, corresponding to the variable magnetic field $B(x, y) = *dA$ at energy level E. Recall that

$$\nabla_A = (d - iA)/2, \qquad \nabla_A^* = (\delta + iA)/2.$$

Equation (1), on the other hand, describes a specific for this theory charge-to-flux feedback loop. The constant b is determined by the value of the filling factor, say, $\nu = N/M$. It has been shown in [11] that this model with $V = 0$ predicts vanishing longitudinal resistance and fractionally quantized Hall resistance. In fact (in the physical units of choice)

$$R_H = 2\pi\frac{M}{N}. \tag{3}$$

The model has been derived from the postulates of the mesoscopic mechanics (MeM), proposed in [9]. The MeM's central notion is a nonlinear operator equation dubbed the mesoscopic Schrödinger equation. The MeM postulates an emergent law, based on the mesoscopic equation, which prescribes a particular form of the collective response of electrons in 2D to the external magnetic field. The collective response is mediated via the locally variable magnetic flux density, while being consistent with the quantum-mechanical particle dynamics, cf. [10]. In particular, the distribution of magnetic flux is everywhere proportional to the distribution of the current-carrying charge. Of course the question of exactly how variable the magnetic flux distribution is cannot be settled *a priori*. Will it be more or less flat, or periodic, or quasi-random? Can it be effectively flat at larger scales? These remain open questions. In particular, the answers may depend on the geometric setting, boundary conditions, assumptions about the lattice, whether or not the Coulomb interaction is accounted for in the model, etc. In this article we will show that the model (1)-(2) with periodic potential (of a certain particular type) admits a class of strong lattice-periodic solutions that guarantee quantization of Hall resistance (just as in the $V \equiv 0$ case). By strong solutions we understand solutions that satisfy the equations almost everywhere. In addition, the physically meaningful

quantities resulting from these solutions, such as charge and flux distributions, are continuous. What is equally important (as having new types of solutions) is the fact that the method of construction will bring to light a constraint on the external potential imposed by the requirement of quantization of Hall resistance.

3 Geometric aspects of QHE phenomenology

We will now probe the model in the presence of an external potential V. One might ask: Does the model imply quantization of the Hall resistance independently of the type of V? As it turns out, the answer to this question is negative. We will arrive at this conclusion by way of finding a constraint on the external (periodic) potential which is necessary for the existence of solutions exhibiting quantization of the Hall resistance. This will be formalized via two postulates, which we have dubbed *Hall Conditions* (**HC 1** and **HC 2**). Their role is to reflect the fundamental phenomenological properties of a quantum Hall system into the structure of the single-particle magnetic Schrödinger equation. While the Hall Conditions impose additional constraints on the structure at hand, they will also guide us toward new solutions of (1)-(2). Overall, as the reader will soon note, our methods resemble classical, local differential geometry.

Now, let j be a 1-form, which we will refer to as the current density.

HC 1 *We say that a model (of an electronic system) defined by the current density j satisfies Hall Condition 1, if there exist:*

- *a function u_H, referred to as the Hall potential, and*
- *a constant R_H, referred to as the Hall resistance,*

so that

$$du_H = R_H * j. \tag{4}$$

Suppose that γ is a finite-length curve in the plane, and let $\overrightarrow{n}$ denotes the normal vector orientated to the right from the curve (which has an orientation prescribed by its parametrization). Condition (4) trivially implies

$$R_H \int_\gamma j(\overrightarrow{n}) = u_H(\gamma(1)) - u_H(\gamma(0)). \tag{5}$$

This is interpretted in the following way: The ratio of the Hall voltage between the end-points of the curve γ to the total current through γ is constant, and

equal to the Hall resistance R_H. In particular, the ratio does not depend on the actual shape of the curve, but only on its end-points. This simple condition is interesting because it appears to capture and geometrize the most fundamental property of the FQHE - independence of the Hall resistance on the particular realization of the measurement, including the choice of the points where we attach the electrodes to measure the transversal potential.

with a very strong theoretical paradigm. Next, we will interpret both the current density and the Hall potential within the framework of single-particle Quantum Mechanics. To fix the notation, let us represent the vector potential A in local coordinates as $A = A_1 dx + A_2 dy$, so that in particular $\delta A = -(A_{1,x} + A_{2,y})$. Thus, the magnetic flux density is given via

$$B(x, y) = *dA = A_{2,x} - A_{1,y}.$$

Recall that the electronic state wavefunction Ψ determines the current density 1-form as follows

$$j = -Re\{\Psi^*(i \cdot d\Psi + \Psi A)\}. \tag{6}$$

For our purposes it is beneficial to separate the wavefunction's modulus from its phase

$$\Psi = R \exp(i\theta). \tag{7}$$

Throughout the article we are interested in solutions such that $R^2 > 0$, at least almost everywhere. We will subsequently tacitly pass over all technical points that might arise when $R = 0$. Substituting this into the formula above, we obtain

$$j = R^2(d\theta - A). \tag{8}$$

A straightforward calculation shows that

$$\delta j = -2R\langle dR, d\theta - A\rangle - R^2(\Delta\theta + \delta A). \tag{9}$$

Moreover, representing the Schrödinger equation (2) in the local coordinates as

$$-(\partial_x^2 + \partial_y^2)\Psi + 2i(A_1\Psi_{,x} + A_2\Psi_{,y}) + (A_1^2 + A_2^2)\Psi - i(\delta A)\Psi = 2(E - V)\Psi, \tag{10}$$

and substituting (7), we readily find that it is equivalent to a union of two separate real-variable equations. On the one hand, we obtain

$$-2\langle dR, d\theta - A\rangle - R(\Delta\theta + \delta A) = 0, \tag{11}$$

which in view of (9) is equivalent to

$$\delta j = 0. \tag{12}$$

On the other hand, we also have

$$-\Delta R + |d\theta - A|^2 R = 2(E - V)R. \tag{13}$$

Thus, the Schrödinger equation is equivalent to the system consisting of equations (12) and (13).

Let us also note that condition (4) implies $\delta(R_H j) = 0$, which is consistent with (12). Thus, a necessary condition for the fulfilment of Hall Condition 1 is always satisfied in the Schrödinger dynamics. However, we have yet to identify a candidate for the function u_H. In fact, the only possible choice for the Hall potential is

$$u_H = \frac{1}{2}|d\theta - A|^2 \tag{14}$$

Indeed, this quantity enjoys the same status in equation (13) as the potential V. This allows us to specialize Hall Condition 1 as follows:

HC 2 *We say that a model (of an electronic system) defined by the magnetic Schrödinger equation satisfies Hall Condition 2, if*

$$\frac{1}{2}d|d\theta - A|^2 = R_H * R^2(d\theta - A). \tag{15}$$

Note that (15) descends from (4) by substituting for j its representation (8), and for u_H the quantity selected in (14). We emphasize that relation (15) is external and cannot be deduced from the Schrödinger equation alone. equation (12). When supplemented with equation (13), it will imply the Schrödinger equation in whole. Let us introduce an auxiliary variable

$$a = d\theta - A, \tag{16}$$

and consider the following system of equations

$$\begin{aligned} -*da &= R_H R^2 \\ -\Delta R + |a|^2 R &= 2(E - V)R \\ d|a|^2 &= 2R_H R^2 * a. \end{aligned} \tag{17}$$

Observe that the first equation is a restatement of equation (1). Observe also that (17) automatically implies Hall Condition 2, and *a fortiori* (12). Thus, (17) implies the Schrödinger equation (2). Let us point out that the Schrödinger equation displays the well-known gauge invariance property. Namely, if the vector potential is represented in a new gauge as $A + d\varphi$, then the pair $R, \theta + \varphi$ characterize the new wavefunction (so that only the phase of the solution depends on the choice of gauge). Introduction of the variable a eliminates this degree of freedom from the equations.

4 Implications of the QHE phenomenology

We will now focus on (17). As explained above, solutions of this system of partial differential equations provide solutions of (1)-(2). Moreover, such solutions are guarantied to satisfy **HC 2** (and *a fortiori* **HC 1**).

We will work in the local coordinates. Let a be given as

$$a = \alpha dx + \beta dy, \tag{18}$$

and let us introduce

$$\lambda = R_H R^2.$$

The first equation of (17) now reads

$$\lambda = - * da = \alpha_y - \beta_x, \tag{19}$$

while the third equation assumes the form

$$\begin{bmatrix} \alpha_y - \lambda & \beta_y \\ \alpha_x & \beta_x + \lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0. \tag{20}$$

Equations (19) and (20) jointly lead to

$$\begin{bmatrix} \beta_x & \beta_y \\ \alpha_x & \alpha_y \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0. \tag{21}$$

This equation implies that

$$\frac{\alpha_y}{\alpha_x} = \frac{\beta_y}{\beta_x} = -\frac{\alpha}{\beta} = \mu, \tag{22}$$

where the last equality defines μ. (Naturally, these ratio identities need to be reversed if the expressions are not well defined. The conclusions stemming from reversed equations are equivalent up to renaming the coordinates. Also, in what follows we assume $\mu \neq 0$. The conclusion we will reach is automatic in the case $\mu = 0$.) In particular, we obtain

$$\alpha_x = -\mu_x \beta - \mu \beta_x,$$

$$\alpha_y = -\mu_y \beta - \mu \beta_y.$$

Therefore

$$\mu = \frac{\beta_y}{\beta_x} = \frac{\alpha_y}{\alpha_x} = \frac{-\mu_y \beta - \mu \beta_y}{-\mu_x \beta - \mu \beta_x} = \frac{(\ln|\mu|)_y \beta + \beta_y}{(\ln|\mu|)_x \beta + \beta_x}.$$

This in turn implies

$$\mu = (\ln|\mu|)_y/(\ln|\mu|)_x.$$

Let $\mu = \pm\exp\eta$. Then

$$\pm e^{\eta}\eta_x - \eta_y = 0. \tag{23}$$

This is a first order PDE. The equation of characteristics assumes the form

$$\begin{aligned} \dot{x} &= \pm e^{\eta} \\ \dot{y} &= -1 \end{aligned} \tag{24}$$

The function $\eta = \eta(x, y)$ is constant along the characteristics. Let us pick a function $\eta_0 = \eta_0(y)$, and prescribe the boundary condition

$$\eta(0, y) = \eta_0(y).$$

A characteristic passing through $(0, y)$ will be a straight line with the slope $\pm e^{-\eta_0(y)}$. Thus, any two characteristics will necessarily intersect, unless they have the same slope. Therefore, a globally defined solution $\eta = \ln|\mu|$ must be constant, i.e. $\mu = const$. Furthermore, stepping back to equations (22), we now conclude that α and β are constant along the lines whose slope is determined by μ. In fact,

$$\alpha = \alpha(x + \mu y) \text{ and } \beta = -(1/\mu)\alpha.$$

and (18) implies

$$a = \alpha(x + \mu y)(dx - (1/\mu)dy).$$

Applying an orthogonal change of coordinates, and rescaling α, we can express a in a particularly simple form

$$a = -\alpha(x)dy. \tag{25}$$

Via (17), this leads to

$$R^2 = (1/R_H)\alpha'(x). \tag{26}$$

In other words, charge concentration is constant along linear stipes. In addition, substituting $R = \sqrt{\alpha'(x)/R_H}$ and $a = d\theta - A = -\alpha'(x)dx$ into the middle equation in (17), we obtain an ordinary differential equation for α. Namely,

$$\frac{\alpha'''}{\alpha'} = \frac{1}{2}\left(\frac{\alpha''}{\alpha'}\right)^2 + 2\alpha^2 - 4(E - V). \tag{27}$$

This ODE is quite fundamental to our approach, as it encapsulates the most rudimentary model for the FQHE. (This is the same equation as Eq. (20) in

[11].) We note that for (27) to admit solutions it is necessary that $V = V(x)$. In real crystals this condition could only be satisfied approximately: either for certain types of crystal lattice and at a suitable length-scale , or in situations when V can be suppressed altogether. We obtain the following conclusion

Theorem 1 *For a system defined by the mesoscopic-loop model (1)-(2) to satisfy the Hall Conditions with perfect accuracy globally in the plane it is necessary that the potential display shift invariance, say, $V = V(x)$. Conversely, if (27) with shift-invariant potential is satisfied by α, and $\alpha' > 0$, then a and R as in (25) and (26), provide a solution of the mesoscopic-loop model, which satisfies the Hall Conditions.*

Proof. Direct calculation shows that the system (17) is satisfied.

As a matter of fact, it is quite easy to convince oneself that (27) possesses nontrivial solutions for many choices of the potential. Naturally, they will not always be defined globally in the whole plane. It is not our goal to discuss this ODE in more detail here. self-consistent potential as the latter In fact, in the next section we will assume a somewhat more refined point of view at the system (17).

5 New solutions of the MeM model, and constraints on the lattice potential

We retain the notation of the previous section, but will drop the requirement that all physical variables be defined globally in the plane. In particular, let us observe that μ need not be constant if the characteristics are considered only locally. In fact, we will now focus on systems which are defined only locally, and subsequently systems which consists of locally defined pieces. This will in fact give us insight into new types of solutions of the MeM model, which is the main fallout of this work. Consider a rectangle in the Cartesian plane $P = [0, 1] \times I$, where I is an interval in the y-axis. We will refer to P as tile. Let the tile be equipped with a foliation by straight line intervals. Let us pick an orientation-preserving diffeomorphism of the interval I, say, $h : I \longrightarrow I$, and let the foliation be prescribed by straight line segments stretching between each pair: $(0, y)$ and $(1, h(y))$. This determines the slope function $m : P \longrightarrow R$, where $m(x, y)$ is the slope of the line passing through (x, y). In reference to the notation above

$$m = -1/\mu.$$

We will refer to the leaves of the foliation as characteristics. Note that m is constant along the characteristics, so that

$$m_x + mm_y = 0. \tag{28}$$

As we have shown in the previous section, also

$$\alpha_x + m\alpha_y = 0, \tag{29}$$

so that α is also constant along the characteristics. In particular, both m and α are determied by their trace value on the left edge of the tile P. Subsequently, let a be given in the form

$$a = \alpha(dx + mdy). \tag{30}$$

Furthermore, let a determine the physical variables u_H and R^2 via formulas

$$u_H = \frac{1}{2}|a|^2, \tag{31}$$

and

$$R^2 = -\frac{1}{R_H} * da = \frac{1}{R_H}\left[\alpha_y - (m\alpha)_x\right]. \tag{32}$$

In particular the choice of α is restricted by the requirement that the right-hand side be positive. (In fact zeros in isolated points are also acceptable.) We note that in view of (17) the choice of α is in fact tied to the effective potential V. Namely, we have the following

Theorem 2 *When a and R are as in (30) and (32), equations (17) are satisfied on the tile P with the potential defined by*

$$V = V(E) = E + \frac{1}{2}\left(\frac{\Delta R}{R} - |a|^2\right) \tag{33}$$

*for arbitrary E. In particular **HC 2** is then satisfied.*

Proof. A direct application of (28), (29), (31), and (32) shows that the first and the third equations of (17) are satisfied. The middle equation holds for a suitably defined V as given above.

(33) is viewed as a local constraint that the potential needs to satisfy in order to ensure that **HC 2** will hold (with perfect accuracy). Next, we will discuss how a global lattice-type potential can be constructed from "pieces" prescribed on local tiles that retains this property.

Suppose now that we are given two adjacent tiles, i.e. in addition to P, consider $\tilde{P} = [1, 2] \times I$. Let the latter tile be endowed with its own foliation

determined by a diffeomorphism $\tilde{h} : I \longrightarrow I$, and let the corresponding slope function be denoted $\tilde{m}$. Furthermore, let the relation

$$\tilde{a} = \tilde{\alpha}(dx + \tilde{m}dy)$$

introduce notation for the familiar constituents of the system on the second tile. As we have already demonstrated, **HC 2** is satisfied separately in P and $\tilde{P}$. However, the Hall potential and the charge density functions are defined separately in the two tiles. *A priori* these functions may have jump discontinuities on the contact between the two tiles. In the interest of obtaining physically meaningful solutions stretching across tiles one asks: What are the necessary conditions on the foliations in the two tiles for these functions to remain continuous? We will now find the answer to this question. In order to start, we use (31), (32), (30), and analogous formulas associated to $\tilde{P}$, and observe that this requirement is equivalent to the following two conditions:

$$(1 + m(1, y)^2)\alpha(1, y)^2 = (1 + \tilde{m}(1, y)^2)\tilde{\alpha}(1, y)^2 \text{ for all } y \in I, \tag{34}$$

and

$$\alpha_y(1, y) - (m\alpha)_x(1, y) = \tilde{\alpha}_y(1, y) - (\tilde{m}\tilde{\alpha})_x(1, y) \text{ for all } y \in I. \tag{35}$$

Taking advantage of (28) and (29), we readily reduce (35) to

$$(1 + m^2)\alpha_y - m_x\alpha = \tilde{\alpha}_y - (\tilde{m}\tilde{\alpha})_x \tag{36}$$

(We apply shorthand notation. This and subsequent relations hold only on the interval $\{1\} \times I$, where the functions whose domain is either one tile are all simultaneously defined.) Furthermore, condition (34) implies

$$1 + m^2 = \frac{(1 + \tilde{m}^2)\tilde{\alpha}^2}{\alpha^2}.$$

Differentiating this equation with respect to y and taking advantage of (28), we also obtain

$$m_x = -\frac{1}{2}\partial_y \left(\frac{(1 + \tilde{m}^2)\tilde{\alpha}^2}{\alpha^2} \right).$$

Applying the last two identities to (36), differentiating and observing cancelations, we find that

$$\alpha = \frac{1}{2} \frac{[(1 + \tilde{m}^2)\tilde{\alpha}^2]_y}{\tilde{\alpha}_y - (\tilde{m}\tilde{\alpha})_x}.$$

Now, since $\tilde{\alpha}$ and $\tilde{m}$ satisfy the tilde-ed equations (29) and (28), we have

$$\frac{1}{2} \frac{[(1 + \tilde{m}^2)\tilde{\alpha}^2]_y}{\tilde{\alpha}_y - (\tilde{m}\tilde{\alpha})_x} = \frac{(1 + \tilde{m}^2)\tilde{\alpha}\tilde{\alpha}_y + \tilde{\alpha}^2\tilde{m}\tilde{m}_y}{\tilde{\alpha}_y + \tilde{m}^2\tilde{\alpha}_y + \tilde{m}\tilde{m}_y\tilde{\alpha}} = \tilde{\alpha}.$$

In conclusion,

$$\alpha(1, y) = \tilde{\alpha}(1, y) \text{ for all } y \in I, \tag{37}$$

and at the same time, in view of (34),

$$m(1, y) = \pm\tilde{m}(1, y) \text{ for all } y \in I. \tag{38}$$

That is, the continuity conditions restrict the freedom of building a global continuous solution from small local tiles. However, there is still a possibility of obtaining nontrivial almost everywhere solutions by intertwining a given tile with its mirror reflection, cf. Fig. 1.

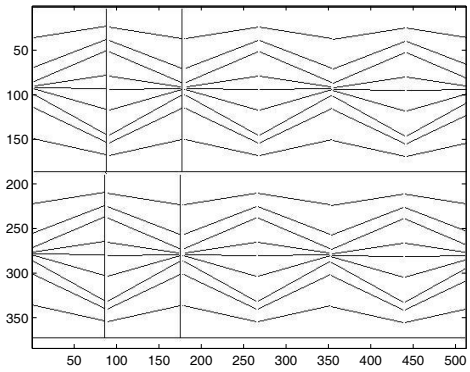

Fig. 1. The lattice of characteristic foliations. The original foliated tile, say, one in the upper left corner, and its mirror reflection tessellate the plane.

The moduli space of solutions of this particular type is parameterized by pairs (m, α). Let us in addition assume that $\alpha > 0$, so as to guarantee that the current direction will be from left to right according to the orientation of the x-axis. Note that the requirement $R^2 > 0$ introduces a constraint via equation (32). Eliminating all derivatives with respect to x in the usual manner, we easily find that the condition may be expressed in the following way:

For $R^2 > 0$ to be satisfied, it is necessary and sufficient that the function $\alpha\sqrt{1+m^2}$ be strictly increasing on the left edge of the tile. (Of course, this

guaranties that the function will be increasing on any vertical section of the tile.)

This concludes our description of strong solutions with lattice-periodicity.

6 Closing remarks

In this article we have explored the mesoscopic loop model from a particular point of view. However, there are many more questions and many more different approaches to this rather pliable mathematical and conceptual structure. To mention just a few themes: It is of interest to study the model with lattice and impurity potentials characterized *a priori*. It is also of interest to extend the model and reformulate it in a manifestly many-body setting, so as to account for the effects of electron-electron interaction. Furthermore, it may be beneficial to reduce the role of microscopic dynamics in the analysis of predictions of the model, e.g. by developing a suitable statistical approach.

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