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NOVEL TYPES OF NONDIFFUSIVE FLOWS WITH APPLICATIONS TO IMAGE ENHANCEMENT

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Abstract: Motivated by the various goals of image enhancement, we introduce and study nonlinear high-dimensional Lipschitz-continuous dynamical systems of a particular type.

The most distinctive feature of this approach is that an initial (digital) function, typically representing luminance of an image, evolves toward a model that is generally less smooth in any classical sense while simultaneously being regularized, in a certain well understood sense, at a coarser scale.

We limit our discussion of applicability of this type of evolution processes to a single rather elemental example of simultaneous image interpolation and deblurring which is essentially out of reach for any linear methods.

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1. The Applicable Goal

Let us alert the reader that the nonlinear flows introduced in this paper, cf. (5), differ significantly from all of the broad family of flows constructed with the aid of the surface curvature. At the level of an infinitesimal step, our flows typically use the median filter in combination with a convolution with an oscillatory kernel. The fact that the kernel we consider is *oscillatory* is one of those features that set our approach apart also from the technique of iteration of morphological operators. Roughly speaking, the high-pass filter facilitates deblurring while the median filter regularizes the evolving image at a coarser scale. Thus, both of the goals of smoothing/denoising and deblurring of images, apparently contradictory when viewed through the prism of linear methods, are here achieved in parallel. The process we introduce may be viewed, at least in the discrete setting, that we are in fact focused on, as a system of (very many) nonlinear ordinary differential equations. We will later show that solutions of the initial value problem exist and are unique. We will also shed some light on the problem of what behavior is to be expected of the solutions by considering simplified models in more detail.

Our initial target problem of image enhancement was to reconstruct a digital image, whose every second row had been lost, cf. Figure 2 at the end of this article. We can think of essentially only three approaches to this problem:

1. Interpolating the missing row. The shortcoming of this approach is that interpolation is designed to work for smooth functions, and will result in rough error whenever a discontinuity (typically presented in an image luminance function) is encountered. In addition, even a small error resulting from interpolation would be amplified to quite a monstrous artifact if a subsequent linear sharpening (deconvolution) was required in order to reduce blurring. On reflection, this seems to be an inherent limitation of all linear methods.

   On the other hand, all nonlinear flows of geometric type, including mean curvature based flows, regularized total variation minimizing flows, as well as iteration of morphological operators, cf. [6], may be seen as iterated interpolation. Indeed, in all those cases the driving force (infinitesimal step) locally depends on Taylor coefficients of order two. In particular, it is now known that all these flows are asymptotically equivalent, cf. [7]. Naturally, their short and medium time-range effects may be giving subtle and desirable effects. However, it seems that these techniques essentially fail for the application at hand.

2. Disocclusion techniques [9]. They are a possibility that might be tested in this context but they typically require more costly computation and involve "messy" programming. Naturally, these are typically based on interpolation in one form or another and may be burdened with the same type of side-effects as the techniques mentioned above.

3. Regularization of the input in some sense while leaving the part of information that is known to be accurate, like e.g. the low frequency contents, intact. This is the route we take in the current approach. It is *a priori* far from obvious how to chose the right regularizing operators in this context. For the application at hand we have settled for the so-called median filter. Median filters have been exploited in the area of image enhancement for a long time, e.g. [8] and [10].

The technique we present provides a good quality solution to the problem above. Moreover, at the expense of some modifications it may be extended to target other applicable problems. As mentioned above, since we allow the possibility that the kernel of the convolutional operator that enters the definition at the infinitesimal level may be oscillatory, this approach yields results that are qualitatively different from those obtained with the technique of iterating morphological operators, or the many geometric nonlinear flows. It also differs from the broad set of techniques resulting from variational and multiresolution type approaches to signal analysis (a review of many nonlinear algorithms for image enhancement may be found in [1] and other articles in that issue of the IEEE Transactions on Image Processing; it may be helpful to the reader in forming his/her opinion on the history of this area of research). Our idea is perhaps closest to that of Frieden, cf. [4], who studied iterative applications of intertwined median filters and low-pass Fourier filters to digital images.

Nonlinear flow (5) is also attractive from the purely mathematical point of view. The natural broader setting for (5) is that of the graph theory, which incorporates all the required geometric and analytic structures, including the crucial notion of scale. On the other hand, considering the flow on manifolds would involve preselecting the characteristic scale as the infinitesimal step is not given by a differential operator but rather a localized global operation (selection of the median). We do not consider these issues at present. Naturally, the particular characteristic of the solutions of (5) depends also on the choice of filter. This feature is exploited when extending the system to other applications, such as simultaneous deblocking and sharpening, etc. We have made an effort to rigorously prove some basic properties of (5) that allow one to make sense of what it is really doing. In particular, existence for all times $t > 0$ and uniqueness of solutions, as well as some basic estimates on their growth are shown in Theorem 1. We have also included remarks that shed some light at the morphology of solutions in some simple cases.

2. The Flow

As stated above, we will only look at the discrete theory. Our functions are in fact vectors in R^n or matrices in R^N, $N = n^2$ that we will often refer to as 1D signals or 2D images. A pixel is one of the components, e.g. $x(i) = x_i$ or $u(i,j) = u_{i,j}$. To simplify notation we will assume throughout our discussion periodic boundary condition, so that in particular $x(n+k) = x(k)$ or $u(n+k, n+l) = u(k,l)$. Similarly, all the standard functional operators will be interpreted as periodic, e.g. convolution with a fixed (periodic) kernel. We will refer to the usual Discrete Fourier Transform as the Fourier Transform denoted FT or $FT(u) = \hat{u}$. We will sometimes consider a broader setting, when the Fourier Transform consists of a transcription of the data into a sequence of coefficients in some basis, e.g. given by the eigenfunctions of an elliptic operator, etc. Now, in order to define the flow we need two geometric operators that, somewhat remarkably, come from two different realms of analysis.

1) **The median filter operator.** Depending on a particular application one may select another suitable operator to play this part. At the most general level, description of the properties of the resulting

flow should be essentially independent of this particular choice (however, in order to guarantee uniqueness of solutions one has to assume that the operator of choice, say T, satisfies the Lipschitz condition $\|Tu - Tv\| \leq C\|u - v\|$). For our purpose at hand we want to focus attention on the median filter M. Led by experimental results, we specifically define it as the median of the 3×3 neighborhood of each pixel with its center punched out, i.e.

$$Mu(k,l) = \text{Median}\left\{ \begin{array}{l} u(k-1,l-1), u(k-1,l), u(k-1,l+1), \\ u(k,l-1), u(k,l+1), \\ u(k+1,l-1), u(k+1,l), u(k+1,l+1) \end{array} \right\}, \quad (1)$$

where the median is defined as the average of the fourth and fifth largest elements. M is subtly nonlinear in 2D, but it is worth keeping in mind that its 1D analog is a linear operator. Namely,

$$Mx(k) = .5\left(x(k-1) + x(k+1)\right). \quad (2)$$

This will enable us to completely understand the properties of a 1D flow, which gives some guidance as to the basic properties of its 2D counterpart.

2) **The filter.** Let H denote a fixed kernel to be thought of as a high pass filter to be specified later, so that $u \to H \star u$ suppresses the low frequency component of the signal. Depending on our knowledge of the structure of data at hand, H can be designed to fulfill any additional specifications. Having tried several such possibilities on our images, we have obtained the most satisfying results with a rotationally symmetric "sin-squared" (Hamming) high-pass filter. More strictly, let (ξ_1, ξ_2) denote the Fourier domain variables normalized so that $-1 < \xi_i \leq 1$ and let $r = (\xi_1^2 + \xi_2^2)^{1/2}$. Let $0 < Lo < Hi < 1$ and $s = \frac{\pi}{2(Hi-Lo)}$, and let I_A denote the characteristic function of the set A. We first define

$$\hat{H}(\xi_1, \xi_2) = \left(I_{\{Lo \leq r \leq Hi\}} \sin\left(s(r - Lo)\right) + I_{\{Hi \leq r\}}\right)^2, \quad (3)$$

and subsequently

$$H \star u = FT^{-1}\left(\hat{H}\hat{u}\right). \quad (4)$$

We are now ready to define the flow that is our main interest in this paper. Let $u = u(t,k,l)$ be a (continuously) evolving image with the initial condition $u(0,k,l) = u_0(k,l)$. We define an evolution process by

$$\frac{\partial}{\partial t}u = H \star Mu\,, \tag{5}$$

for operators H and M as specified above. For the sake of this article let us adopt for it the name Filtered Median Flow or FMF. Since we want to consider only the discrete version of the flow in this article, it will benefit us later if we introduce a more detailed notation for this case. Namely, let $f_1, f_2, \ldots, f_n$ denote the orthonormal Fourier basis – for the sake of simplicity we ignore the more natural in the case of 2D images double indexing (in a more general setting one would understand it broadly as a basis that consists of eigenfunctions of the Laplacian on a given graph). As noted above, the filter H acts via multiplication of the Fourier coefficients by fixed numbers – let us say the i-th coefficient gets multiplied by h_i. For the type of filters we are practically interested in, it is both possible and convenient to fix the indices in such a way that $h_i = 0$ for $i = 1, \ldots, k$ and $0 \leq h_i \leq 1$ for all $i = k+1, \ldots, n$. In this convention, flow (5) can be written as

$$\frac{d}{dt}u = \sum_{i=1}^{n} h_n \langle Mu, f_n \rangle f_n, \tag{6}$$

where $\langle .,. \rangle$ denotes the scalar product corresponding to the Euclidean norm $\|.\|$. Here the partial time derivative has been replaced with the ordinary derivative as we reinterpret the flow in the functional space. Since in addition $u(t) = [u_{i,j}(t)]$ is a discrete matrix we may assume the point of view that (6) is a nonlinear system of ordinary differential equations. It is sometimes convenient to switch from one point of view to the other, but all results of the next section will be presented from the ODE perspective.

It is helpful to realize that evolution of u as prescribed by the flow as above has the following property.

Fact 1. *If $h_0 = 0$ then a solution of (6) has the property that the total sum of its components remains fixed throughout the evolution, i.e.*

$$\int u(t) := \sum_{k,l} u_{k,l}(t) = const. \tag{7}$$

Proof. This observation follows directly from the fact that $\int f_n = 0$ for all $n \neq 0$. □

At this point we would like to mention the heuristics behind the equation. On one hand, we want to emphasize the analogy with the classical heat flow. Indeed, if M in (1) were replaced by the standard average and H were reduced to the subtraction of a multiplicity of u so as to guarantee that the right-hand side has mean zero, we would have obtained a version of the standard discretized heat flow. The advantage of having the median in place of the average is precisely the fact that the former preserves edges. On the other hand, the high-pass filter H enforces that the right-hand side maintains average zero and provides some extra control. In addition, we would like to point out that image enhancement tasks often pose apparently contradicting demands, e.g. the image has to be simultaneously deblurred and denoised. Given an image u_0, it would be good to find its "enhanced model". This stands in contradistinction with the picture typical in nonlinear differential equations approach to image enhancement. There, an image is expected to evolve towards the unique limit, say the constant function, regardless of the initial condition while here an image is expected to evolve until its median is in some sense close to the null space of H, when the rate of evolution should slow down.

Let us also note that if the filter H were a convolution with a positive kernel, the operator $u \to H \star Mu$ would be morphological. In that case the whole process would be *asymptotically* equivalent to an iterative application of a morphological operator. That in turn is known to be equivalent [7] (cf. also [5]) to the viscosity solution of the mean curvature flow (or the regularized Rudin-Osher flow [11]) – the nowadays standard technique for reduction of the total variation of an image. Naturally, one is often interested in the deformation of an image resulting from a short time evolution in which case results are different from the long time asymptotics and interesting in their own right. However, we emphasize that since the application at hand requires we pick an oscillatory kernel, one should expect effects other than those displayed by any regularizing techniques.

Let us illustrate some of these observations using the much easier to analyze 1D flow. In this case the median operator (2) is linear – a convolution with the filter $G = [101]$, i.e. $Mu = G \star u$. Thus, (5) is best described in the frequency domain, where

$$\frac{\partial}{\partial t}\hat{u} = \hat{H} \cdot \hat{G} \cdot \hat{u},$$

which has a unique solution

$$\hat{u}(t,\xi) = \exp\left(t\hat{H}(\xi)\hat{G}(\xi)\right)\hat{u_0}(\xi).$$

Now, $\hat{G} = \cos(\pi\xi)$ so that choosing $\hat{H}$ supported in $\{\xi : |\xi| > .5\}$ will guarantee convergence. As a result of this evolution the high frequency content of the initial signal is completely suppressed while the low frequency content remains intact. On the other hand, a choice of H with the Fourier Transform nonvanishing in the low-pass band $\{\xi : |\xi| < .5\}$ would cause gradual (divergent to infinity) amplification of the low frequency content of the signal. Although this convergence property appears to be in principle correct in 2D as well, its analysis would require much more subtle arguments specifically designed so as to handle the nonlinearity of M as in (1). For illustration we display Figure 1 below. It shows that a long run "model" of an image will typically have a lot of high frequency energy while its median-filtered version is completely contained in the low pass band. Thus, the image has been regularized in the coarser scale, as seen via the nonlinear median filter, while actually loosing regularity in the finer scale.

3. Existence and Uniqueness of Solutions

We now undertake the task of demonstrating that the flow makes good sense in higher dimensions, especially dimension 2, and/or possibly more general settings. In particular, M is unavoidably a *nonlinear* operator now. We begin with a general fact about the median we will later use in order to prove that solutions of FMF exist and are unique. We emphasize that inasmuch as existence for short time may seem obvious, especially when viewed via the prism of numerical simulations, the fact that solutions do not blow up to infinity in finite time as well as uniqueness of solutions are rather unobvious (one should bear in mind the standard examples like $\dot{x} = x^{2/3}$ with the initial condition $x(0) = 0$, when both $x(t) = 0$ and $x(t) = (t/3)^3$ satisfy the ODE with the given initial condition). Let us denote vectors in R^n by

$$X = (x_1, x_2, \ldots, x_n), \quad Y = (y_1, y_2, \ldots, y_n), \text{ etc.},$$

and let $|X|$ denote, say, the standard Euclidean norm (we think of vectors X, Y as low-dimensional; we will later use them in local calculations; we also want to reserve $\|.\|$ to denote the Euclidean norm in the *discrete* functional space that contains high-dimensional u, v, etc.). In order to prove uniqueness we will first show that

$$\mathcal{M} : R^n \to R, \quad \mathcal{M}X = \text{Median}\,\{x_1, x_2, \ldots, x_n\} \tag{8}$$

is a Lipschitz function. We know of no reference to this simple yet nontrivial fact. As a midway fact we need to establish that sorting, viewed as a map $S : R^n \to R^n$, defined as

$$SX = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}), \text{where } x_{\sigma(1)} \le x_{\sigma(2)}, \ldots \le x_{\sigma(n)}\,, \tag{9}$$

is also Lipschitz[1]. Although it does not enter discussion in an essential way, we emphasize that in order to make definition (9) unambiguous one needs to introduce an extra rule for deciding how to order elements when there is a tie. Let us adopt the convention that in case $x_i = x_k$ one puts the number with lower index first. We are now ready to establish the following basic facts.

Proposition 1. *Both the sorting mapping S and the median function $\mathcal{M}$ are Lipschitz, i.e.*

$$|SX - SY| \le |X - Y| \tag{10}$$

and

$$|\mathcal{M}X - \mathcal{M}Y| \le C(n)|X - Y|, \tag{11}$$

where the optimal constants are $C(n) = 1$ for n odd and $C(n) = \sqrt{2}/2$ for n even. In fact, the above inequality remains valid for any median-type function defined by the selection of, say, the k-th smallest element out of an n-tuple.

[1] It was pointed out to the author in private communication by Dr. John Elton that this fact is known in the theory of Banach lattices. I do not know a reference, though. Anyhow, for the reader convenience we provide a simple argument that proves this point. It also prepares ground for the second part of the proof.

Proof. *i)* We prove (10) first and later use it in the proof of the second inequality above. We begin by considering the two-dimensional case. Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ and let $TY = (y_2, y_1)$. We have

$$\begin{aligned} &|X - TY|^2 - |X - Y|^2 \\ &\quad = (x_1 - y_2)^2 + (x_2 - y_1)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 \\ &\qquad = 2(x_1 - x_2)(y_1 - y_2). \end{aligned}$$

The right-hand side is positive if vectors X and Y have their coordinates ordered in the same way, e.g. increasingly in which case $SX = X$, $SY = Y$. Otherwise, switching the order decreases the distance. This shows (10) is true for $n = 2$. Let us now consider the general case. It is a simple fact that sorting can be accomplished by successive transpositions of pairs of elements according to some pattern. The exact declaration of this pattern can be skipped here as it is of no consequence in further considerations. It takes a moment to realize that the same algorithm may be used to sort two vectors X and Y simultaneously by successively applying transpositions of corresponding pairs in the following sense. If (x_k, x_l) and (y_k, y_l) are both in the reversed (decreasing) order, one transposes both; otherwise, only the pair that is in the reversed order gets reversed. According to our initial observation the norm of the difference will decrease or stay at the same level as we are applying this procedure. This proves the first inequality (10).

ii) We may assume without loss of generality that $\mathcal{M}X \le \mathcal{M}Y$. We introduce two index-sets

$$I_x = \{i : x_i \le \mathcal{M}X\}\ , \quad I_y = \{i : y_i \ge \mathcal{M}Y\}\,. \tag{12}$$

If n is odd then both $\#I_x$ and $\#I_y$ exceed $(n+1)/2$ so that in particular $I_x \cap I_y \neq \emptyset$. Let us chose $i \in I_x \cap I_y$. It follows that $|Y - X| \ge y_i - x_i \ge \mathcal{M}Y - \mathcal{M}X$ which proves that (11) holds in this case. In order to see that the constant 1 cannot be improved, consider an example when all $x_i = y_i$ for $i = 1, \ldots, n$, $i \neq (n+1)/2$, except $x_{(n+1)/2} = y_{(n+1)/2} + 1$ and both X and Y are sorted in the increasing order.

Let us now consider the case when n is even. Since $\mathcal{M}SX = \mathcal{M}X$, inequality (10) shows that in order to prove (11) we may assume without loss of generality that X and Y are sorted in the increasing order. With this understood,

$$\mathcal{M}X = \frac{1}{2}(x_{n/2} + x_{n/2+1}), \quad \mathcal{M}Y = \frac{1}{2}(y_{n/2} + y_{n/2+1}),$$

so that in particular

$$\begin{aligned} |\mathcal{M}X - \mathcal{M}Y| &= \frac{1}{2}|x_{n/2} + x_{n/2+1} - y_{n/2} + y_{n/2+1}| \\ &\leq \frac{1}{2}|x_{n/2} - y_{n/2}| + \frac{1}{2}|x_{n/2+1} - y_{n/2+1}| \\ &\leq \left(\frac{1}{2}|x_{n/2} - y_{n/2}|^2 + \frac{1}{2}|x_{n/2+1} - y_{n/2+1}|^2\right)^{1/2} \\ &\leq \frac{\sqrt{2}}{2}|X - Y|. \end{aligned}$$

This shows (11) for n even. One checks that the constant $\frac{\sqrt{2}}{2}$ is achieved for a sorted pair X and Y such that all $x_i = y_i$ for $i = 1, \dots n/2 - 1, n/2+2, \dots, n$, while $x_{n/2} = x_{n/2+1} < y_{n/2} = y_{n/2+1}$. This completes the proof of proposition. □

Corollary 1. *The median filter operator M defined in (1) has the Lipschitz property. In fact for any two digital functions u and v we have*

$$\|Mu - Mv\| \leq 2\|u - v\|. \tag{13}$$

Moreover, the composition of M with a Fourier multiplier as above is also Lipshitz continuous with the same constant, i.e.

$$\|H \star Mu - H \star Mv\| \leq 2\|u - v\|. \tag{14}$$

Proof. It follows from (1) and (11) that for any pair (k, l)

$$|Mu(k,l) - Mv(k,l)|^2 \leq \frac{1}{2} \sum_{(i,j)\in J_{k,l}} |u(i,j) - v(i,j)|^2,$$

where each

$$J_{k,l} = \Big\{(k-1, l-1), (k-1, l), (k-1, l+1), (k, l-1), \dots$$
$$(k, l+1), (k+1, l-1), (k+1, l), (k+1, l+1)\Big\}$$

is the perforated neighborhood of (k,l) containing eight elements (assuming bi-periodicity as we are). Taking the sum over (k,l) of both sides of the inequality leads to (13) as the norm $\|u-v\|^2$ is overestimated by the right-hand side of the resulting inequality exactly four-fold.

Since convolution is linear, in order to prove inequality (14) we only need to recall

$$\begin{aligned}\|H\star v\|^2 &= \|\hat{H}\cdot\hat{v}\|^2 = \sum h_n^2\hat{v}_n^2\\ &\leq \sum \hat{v}_n^2 = \|\hat{v}\|^2 = \|v\|^2,\end{aligned}$$

where we have used the Parseval identity. □

This allows us to prove the following facts that are of immediate practical interest.

Theorem 1. *Assume that the filter H in (5) (equivalently (6)) is such that its Fourier coefficients $h_n = \hat{H}(n)$ satisfy $0 \leq h_n \leq 1$. Then solutions exist for all $t \geq 0$ and are unique up to the choice of an initial condition. Moreover, we have*

$$\|u(t)\|^2 \leq \|u(0)\|^2 \exp(4t). \tag{15}$$

Proof. The first part of Theorem follows from Corrolary 1 and well known general results on existence of solutions of a (autonomous) system of first order ordinary differential equations solved with respect to derivative and with Lipschitz right-hand side (for example cf. [3]).

Let now $\langle .,. \rangle$ denote the scalar product corresponding to $\|.\|$. It follows from Corollary 1 that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 &= 2\langle\dot{u},u\rangle \leq 2\|\dot{u}\|\cdot\|u\|\\ &= 2\|H\star Mu\|\cdot\|u\| \leq 2\|Mu\|\cdot\|u\| \leq 4\|u\|^2.\end{aligned}$$

This implies (15) and completes the proof of Theorem. □

Arguments from the discussion of the 1D case at the end of the previous section show that the exponential estimate (15) cannot be improved. Naturally, solutions of the initial value problem do not always diverge. Their actual behavior continues to depend on the combination of choices of the initial condition and the filter. We address some of this type of questions in Section 4.

We now briefly digress from the discussion of the properties of the flow (5) to discuss an important side issue. Namely, in spite of all the benefits of nondiffusivity, there are instances when it is beneficial to introduce a controlled measure of smoothing into the flow, e.g. this may help suppress the so-called Gibbs phenomena, which may be present in a pre-compressed image, etc. As it turns out, perhaps somewhat surprisingly, this type of an effect can be obtained by replacing $H \star Mu$ with $H \star Mu - pu$ in the formula (5), where p is a real parameter. The idea behind it is quite clear. Consider the flow

$$\frac{\partial}{\partial t} u = H \star Mu - pu. \tag{16}$$

Using nothing more than the Lipschitz property of the operator $u \to H \star Mu$ we prove below that for a sufficiently large p a solution of (16) converges to the constant 0.

Proposition 2. *Let c denote the Lipschitz constant of the operator $u \to H \star Mu$ (e.g. $c = 2$ for the particular choices indicated above). When $p > c$ the flow (16) is a contraction. Moreover, if both $u = u(t)$ and $w = w(t)$ are solutions, then*

$$||u(t) - w(t)|| \leq ||u(0) - w(0)|| \exp{(c-p)t}.$$

In particular, $u(t) \to 0$ as $t \to \infty$.

Proof. We have

$$\begin{aligned} 1/2 \frac{d}{dt} ||u - w||^2 &= (\frac{d}{dt} u - \frac{d}{dt} w, u - w) \\ &= (H \star Mu - pu - H \star Mw + pw, u - w) \\ &\leq ||H \star Mu - H \star Mw|| \cdot ||u - w|| - p||u - w||^2 \\ &\leq (c - p)||u - w||^2. \end{aligned} \tag{17}$$

It follows that

$$\frac{d}{dt} \log ||u - w|| \leq c - p,$$

$$\text{i.e. } ||u(t) - w(t)|| \leq ||u(0) - w(0)|| \exp{(c-p)t}.$$

Selecting p larger than c obviously guarantees convergence as claimed. □

Here, the convergence is shown via an observation that the L_2-norm is diminishing during the evolution, which fact is of secondary importance in a finite dimensional (discrete) setting but would be quite disappointing in a continuous variable setting. A complete resolution of this problem would require us to find answers to some rather deep questions (cf. Section 5). Separately, it is perhaps worth reporting that experiments show that even when p is set lower than the constant c, we will still observe some (albeit weaker) regularizing effect. In particular, it may be expected that a more conventional regularization is to be expected as a result of this flow.

4. Remarks on Dynamics of the Flow

As it is well known, a more thorough analysis of a nonlinear dynamical system is in general nontrivial even in the case of really low-dimensional systems, e.g. in just three variables. Moreover, the property that a solution may both exponentially diverge and converge shows that the evolution process at hand is not controlled by any functional norm. An even more essential difficulty is in that the driving force is not a differential operator, so in particular it may not come from any functional by way of the Euler-Lagrange equations.

Experiment suggests that depending on the actual initial data, a solution of (5) either converges, diverges to the infinite horizon, or it approaches a cyclic orbit while we have never observed chaotic phenomena. On the other hand, experiment shows that there is always "convergence" in terms of optical appearance of solutions, which convinces us that it should be possible to prove convergence in some suitably defined weak sense. However, we do not pursue this direction at present.

The purpose of this section is to demonstrate the nature of evolution prescribed by the flow (5) by displaying simple examples when the median and the filters are especially easy to evaluate. As mentioned above, the natural context for studying the discrete version of the filtered median flow (FMF) is that of graph theory. In particular, the Lipschitz property of the median, as well as the existence and uniqueness results of the previous section carry over to this setting. Naturally, depending on the complexity of the underlying graph structure, details of the dynamical behavior of solutions may be more or less difficult to keep track of. However, the situation is quite clear when one considers the so-called bipartite graphs, cf. [2], while additionally assuming that the filter be given by the orthogonal projection to the highest harmonic.

Example 1. To better understand the form of solutions of the FMF, let us begin with an analysis of a low-dimensional situation when the signal is four-dimensional. Let us introduce notation

$$u(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$$

and let us define the neighborhood of each element as the complementary triplet of elements in the matrix so that in particular we have a median filter

$$\tilde{M}u = \begin{bmatrix} \mathcal{M}(b,c,d) & \mathcal{M}(a,c,d) \\ \mathcal{M}(a,b,d) & \mathcal{M}(a,b,c) \end{bmatrix}.$$

Let us fix the Fourier basis in the four-dimensional space as given by the following four matrices.

$$f_0 = .5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad f_1 = .5 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

$$f_2 = .5 \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad f_3 = .5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We want to understand convergence properties of the solutions to the initial value problem for a flow of type (5) with the initial condition

$$u(0) = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}.$$

First, consider the simplest case when

$$Hu = \langle u, f_3 \rangle f_3,$$

so that (5) becomes

$$\frac{d}{dt}u = \langle \tilde{M}u, f_3\rangle f_3. \tag{18}$$

We have

$$2\langle \tilde{M}u, f_3\rangle = \mathcal{M}(b,c,d) - \mathcal{M}(a,c,d) - \mathcal{M}(a,b,d) + \mathcal{M}(a,b,c), \tag{19}$$

so that (18) represents a nonlinear system of four differential equations. The actual value of the driving force (19) can be narrowed down more precisely since it depends only on the ordering of numbers a, b, c, d, e.g.

$$\text{when } a \le b \le c \le d \text{ we have } \langle \tilde{M}u, f_3\rangle = 0 \tag{20}$$

while

$$\text{for } a \le d \le b \le c \text{ we obtain } \langle \tilde{M}u, f_3\rangle = 2(b-d). \tag{21}$$

One checks directly that in the latter case, when the initial condition satisfies $a_0 \le d_0 \le b_0 \le c_0$ the solution is explicitly given by

$$a(t) = a_0 + .5(b_0 - d_0)(1 - e^{-t}), \quad b(t) = .5(b_0 + d_0) + .5(b_0 - d_0)e^{-t}, \tag{22}$$

$$c(t) = c_0 + .5(d_0 - b_0)(1 - e^{-t}), \quad d(t) = .5(b_0 + d_0) + .5(d_0 - b_0)e^{-t}. \tag{23}$$

In particular, $a + b + c + d =$const. As one sorts through all possibilities, one concludes that this is in fact typical in the following sense.

Fact 2. *Out of the 24 possible cases of ordering the elements a, b, c, d, the driving force in (18) is nonzero only for six of them—precisely when the two middle elements are contained in one row or one column and the two extreme elements are in the complementary row or column in the opposite order, e.g. $a < d < b < c$.*

When the driving force is nonzero, it is the double of the difference of the two middle elements, cf. (20)-(21). One checks directly that the solution, guarantied to be unique by Theorem 1, is given by (22)-(23) if $a_0 \le d_0 \le b_0 \le c_0$ and analogous formulas in the remaining five cases.

Proof. By inspection. □

Example 2. What happens when the high-pass filter is more inclusive? Consider for illustration the flow given by

$$\frac{d}{dt}u = \langle \tilde{M}u, f_1\rangle f_1 + \langle \tilde{M}u, f_2\rangle f_2 + \langle \tilde{M}u, f_3\rangle f_3. \tag{24}$$

One easily observes the following result.

Fact 3. *In all cases, regardless of the actual placement of the elements, the driving force in (24) equals the double of the difference of the two middle elements. The evolution, which is uniquely defined by the initial condition, obeys the same exponential pattern as in (22)-(23). In particular, the middle elements will converge to their mean and the extreme elements will come closer together by the same amount.*

Proof. By inspection. □

Note that even though the systems (18) and (24) are nonlinear, we have been able to give their solution in terms of combinations of exponentials (plus a selection-of-the-formula rule!) just as for a linear system. This is due to the fact that the median behaves nonlinearly only when ordering of coordinates changes and is linear everywhere "in-between". Moreover, the situation above is easy to analyze also due to the fact that the Fourier harmonics consist of ± 1 only.

Example 3. The observations we have just made will help us prove a general statement about the fully high-dimensional case when the filter is limited to the projection on the highest Fourier harmonic and the median operator is also simplified by allowing it to look up only the vertical and horizontal neighbors, see (26) below. Namely, let us introduce the following bi-periodic pattern with period n

$$\frac{1}{c}P = \begin{array}{cccccccc} & & \vdots & \vdots & \vdots & \vdots & & \\ & 1 & -1 & 1 & -1 & 1 & -1 & \\ \ldots & -1 & 1 & -1 & 1 & -1 & 1 & \ldots \\ \ldots & 1 & -1 & 1 & -1 & 1 & -1 & \ldots \\ & -1 & 1 & -1 & 1 & -1 & 1 & \\ & & \vdots & \vdots & \vdots & \vdots & & \end{array} \qquad (25)$$

where $c = 1/n$ so that $\|P\| = 1$. It is well known, cf. [2], that P represents the highest harmonic, i.e. an eigenfunction of the discrete Laplacian corresponding to the highest eigenvalue (which is 2 for one popular choice of normalization). Now the simplified median operator is given by

$$\breve{M}u_{i,j} = \mathcal{M}(u_{i-1,j}, u_{i+1,j}, u_{i,j-1}, u_{i,j+1}) \,. \tag{26}$$

Proposition 3. *Consider the flow*

$$\frac{d}{dt}u_{i,j} = \langle \breve{M}u, P\rangle P_{i,j}. \tag{27}$$

The solution of an initial value problem is explicitly given by the formula

$$u(t)_{i,j} = u(0)_{i,j} + (C_0/2c)(1 - e^{-kt})P_{i,j}, \tag{28}$$

where $k = 2c^2 = 2/n^2$ *and*

$$C_0 = \sum_{i,j:P_{i,j}=c} \breve{M}u(0)_{i,j} - \sum_{i,j:P_{i,j}=-c} \breve{M}u(0)_{i,j}. \tag{29}$$

Proof. Theorem 1 remains valid in this context and it guaranties that the solution we display is unique. Suppose that $u(t)$ is given by (28). We will now show that it satisfies (27). Let us first look at the right-hand side of (27). Observe that $\langle \breve{M}u(0), P\rangle = C_0c$, where C_0 is as in (29). Moreover, since in general

$$\breve{M}(u + \text{const}) = \breve{M}(u) + \text{const},$$

and since $\sum_{i,j:P_{i,j}=c} \breve{M}u(t)_{i,j}$ depends only on the values $u(t)$ assumes at points (i,j), where $P_{i,j} = -c$, we have

$$\sum_{i,j:P_{i,j}=c} \breve{M}u(t)_{i,j} = \sum_{i,j:P_{i,j}=c} \breve{M}u(0)_{i,j} - C_0(1 - e^{-kt})/2. \tag{30}$$

Analogously,

$$\sum_{i,j:P_{i,j}=-c} \breve{M}u(t)_{i,j} = \sum_{i,j:P_{i,j}=-c} \breve{M}u(0)_{i,j} + C_0(1 - e^{-kt})/2. \tag{31}$$

Subtracting identities (30)-(31) we obtain

$$\frac{1}{c}\langle \breve{M}u(t), P\rangle = \frac{1}{c}\langle \breve{M}u(0), P\rangle - C_0(1 - e^{-kt}) = C_0e^{-kt}, \tag{32}$$

so that the right-hand side of (27) satisfies

$$\langle \breve{M}u, P\rangle P_{i,j} = cC_0e^{-kt}P_{i,j}. \tag{33}$$

On the other hand, if $u(t)$ satisfies (28), then we obtain by way of direct differentiation that

$$\frac{d}{dt}u_{i,j} = k(C_0/2c)e^{-kt}P_{i,j}. \tag{34}$$

Comparing (33) and (34) we see that it suffices to set $k = 2c^2$ for (28) to satisfy the evolution equation. This completes the proof. □

We note that exponent k decreases to zero as $n \to \infty$. This is to be expected for two reasons: first, P is a purely discrete object that does not survive the limit $n \to \infty$ and secondly, even the median filter operator $\breve{M}$, or more strictly the local neighborhood it depends on the value at a point, would have to be rescaled in order to make sense in the limit. We want to point out that identities (30)-(31) indicate

$$\sum_{i,j:P_{i,j}=c} \breve{M}u(t)_{i,j} + \sum_{i,j:P_{i,j}=-c} \breve{M}u(t)_{i,j} = C_0,$$

so that the "integral" of $\breve{M}u$ is preserved during the evolution.

5. Summary and Future Directions

There remain many open questions as regards the properties of the Filtered Median Flow. In particular, we do not discuss its behavior in the presence of noise. Naturally, the main property displayed here is that the flow provides a regularizing effect while simultaneously enhancing the high frequency content of an image. This has many applications, and in fact the basic technique presented in this paper has been successfully applied to help provide solutions to some other well-known problems of image processing. Notably, it lead to an algorithm that allows to deblock an image while simultaneously sharpening it in the sense discussed in this article (some such methods and processes are proprietary to the *Pegasus Imaging Corporation*).

The way to achieving further substantial progress is through establishment and thorough understanding of a new type of relevant to images notion of regularity whose existence is indicated by the peculiar features of the various median-filter based techniques, and especially the Filtered

Median Flow. As we have seen, the FMF induces new types of regularizing effects in images. Let us try to see it in a more general context. Namely, I would like to suggest the following general point of view: Just as the Sobolev regularity paradigm is fundamental to a variety of nonlinear PDE and wavelet techniques, and the paradigm of Total Variation results in a bunch of more "singular" dynamical systems (Rudin-Osher, cf. [11]), so will this anticipated new type of regularity play a fundamental role for a broad family of existing and future median-filter based algorithms. Let me take this opportunity to somewhat elaborate this point below. I emphasize that it is not my intention to provide here anything like a brief summary of the history of Image Processing, but rather to share with the reader a personal perspective and to place the developments presented in this article in a broader context.

Typically, an image can be thought of as a collection of a bunch of smaller images. Sometimes a systematic error is present throughout the entire collection, but that is usually easy to correct. More often, however, images suffer from the presence of locally manifested deformations, contaminations and artifacts that are not spatially correlated. One of the great lessons that Image Processing has learned from Mathematical Physics is that quality of an image may be to some extent discussed within the framework of functional regularity, e.g. regularity in the sense of Sobolev. Accepting this paradigm, regularization/enhancement would be achieved by an iterative application of smoothing integral operators, i.e. diffusion. The major problem that gives impetus to this popular research direction is how to gain local control or assure local selectivity of such a process. The necessity of introducing feedbacks that would realize that particular requirement unavoidably leads to the introduction of nonlinearity to the diffusive process. Of course, the paradigm has a weakness as it turns out that Sobolev regularity is not the most relevant measure of image quality. It was later pointed out that a more relevant measure of regularity would be that of Total Variation. Subsequently, the Rudin-Osher flows were adapted as an "evolutionary motor" for artifact erasing processes. However, in practice one needs to regularize these processes, either by introducing the so-called viscosity or otherwise, which effectively makes these flows equivalent or very close to the mean-curvature flows. As a result of this, the regularized processes will introduce some dose of diffusion into an image. Attempts at providing a substitute (at least more) free of diffusion resulted in discrete machinery that does not allow generalization to the continuous domain setting. Needless to say, not all contributions to this research stream are explicitly stated in the formalism of Partial Differential Equations, viz the wavelet theory, but they could at least in principle be reformulated in that form, e.g. what one typically does in the wavelet domain amounts to (adaptive or not) construction of an integral operator.

Now, an independent stream largely kept alive by the practitioners of image processing and essentially lacking systematic theoretical background is that which comprises applying ad hoc filters - notably, the very successful median filter. The power of the median filter is in its robustness to noise and non-blurring behavior at an edge, but its very serious shortcoming is in that it over-regularizes the input image in some sense. Again, the main problem here is that we are lacking a rigorous description of the relevant notion of regularity as mentioned above. The Filtered Median Flow may be seen as an attempt to further control and exploit this anticipated hypothetical type of regularity while avoiding over-regularization. The most successful part of this effort, beyond its applicability to the erasure of artifacts, is that it seems to break the barrier toward phenomena that are beyond the reach of those methods that are inherently tied to the Sobolev or the Total Variation type regularity (which is not to say that these latter methods are not useful or beautiful within their range of applicability). I hope that a systematic development of the merger between the two streams mentioned above in continuation of the work on the Filtered Median Flow may help open a new and essentially different perspective on the problem of regularity of images.

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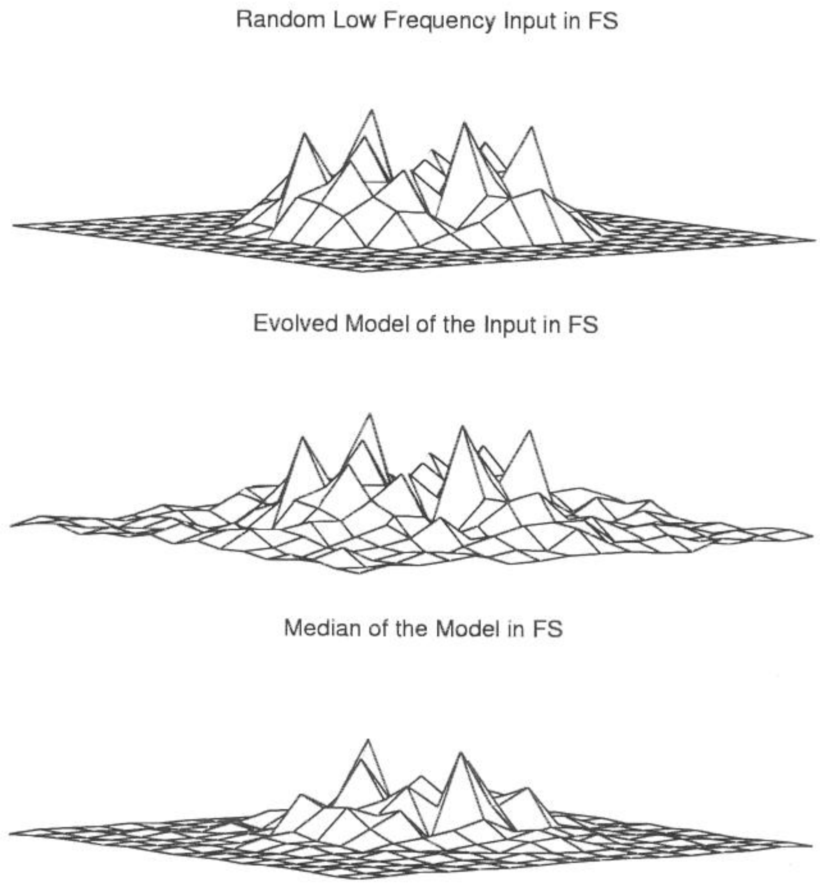

Figure 1: The result of an application of the flow (5) to a random low-frequency image as seen in the Fourier space (FS). It shows three frequency profiles: The model (obtained after 150 iterations) has lots and lots of high frequency energy while its median has no energy in the high pass.

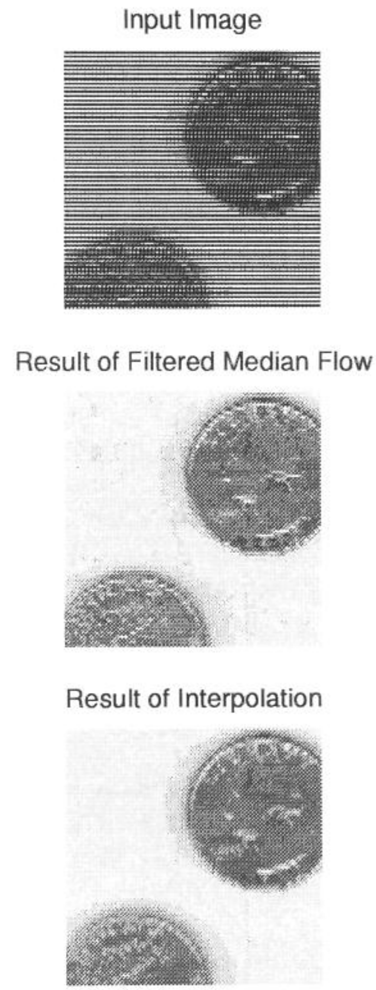

Figure 2: The result of an application of the flow (5) to an image, whose every second row is missing. The FMF has filled up the missing rows (in about 50 iterations) while keeping the existing information essentially intact. We emphasize that no mask or constraint of any sort have been applied. In spite of that, there has been no diffusion. We urge the reader to keep in mind that the difference in quality between the image obtained by an application of our algorithm and the result of plain interpolation is better visible on high-quality prints or directly in the ps-file.

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