Qualitative Analysis of Quantum Signals

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An approach to the analysis of quantum observables: The rich and beautiful discipline of Harmonic Analysis offers methods for the study of general qualities of functions and, in the context of applications, signals. The presently emerging reload of quantum science and technology prompts a rethink of some of its traditional techniques as, indeed, in the quantum realm signals morph into quantum states or observables. This creates a need to discuss the qualitative properties of linear operators as carriers of information. The problem comes to the fore quite explicitly in many ongoing pursuits, e.g. when discussing the dominant characteristics of quantum states pervading the Universe, [17], when considering conditions at which quantum computing may be essentially different than its classical counterpart, [13], [7], [5], or when extending the technological frontiers via such inventions as the quantum metamaterials, [15], [16].

The leading tool traditionally used toward qualitative examination of quantum states has been the Wigner transform, [4]. However, it is not always sufficient in and of itself. An alternative approach is based on representation of observables via the Q-transform proposed in [12]. In essence, the Q-transform establishes a linear correspondence between functions or distributions on a two-dimensional torus $\mathbb{T}$ on the one hand and operators in a fixed Hilbert space, say $\mathbb{H}$, on the other. It has the property of assigning self-adjoint operators to real-valued distributions. Moreover, the Q-transform is used to endow operator spaces with topology, e.g. the Sobolev space denoted H_Q^α consists precisely of those operators whose Q-transform counterpart falls into the classical Sobolev space $H^\alpha(\mathbb{T})$. Generally the Q-transform depends on the choice of an orthonormal basis in $\mathbb{H}$, we fix it as $(e_n)_{n\in\mathbb{Z}}$ for further use below. Most importantly, the notion of Sobolev regularity enables one to discuss states and observables in a manner appropriate for the discussion of quantum signals.

In the Heisenberg picture of quantum mechanics the state of the system remains static while any given quantum observable, say, $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ keeps evolving. In the case of a quantum system that is not effectively isolated from the influence of external environment, $\mathcal{A} = \mathcal{A}(t)$ satisfies the master equation in Lindblad form:

$$\partial_t \mathcal{A} = i[\mathcal{H}, \mathcal{A}] + \sum_j \{L_j^\dagger \mathcal{A} L_j - \frac{1}{2} L_j^\dagger L_j \mathcal{A} - \frac{1}{2} \mathcal{A} L_j^\dagger L_j\} \quad (1)$$

Here, the Hamiltonian $\mathcal{H} : \mathbb{H} \to \mathbb{H}$ is Hermitian and characterizes the system of interest, while the finite collection of operators $L_j : \mathbb{H} \to \mathbb{H}$ that need not be Hermitian characterize the influence of the environment. Many aspects of the basic theory concerning the master equation are already classical, [1], [2], [8]. However, an introduction of the Q-transform enables one to formulate regularity results, such as the following:

Theorem 1. [12] *Let $\alpha \geq 0$. Assume $L_j \in H_Q^\alpha$ for all j. Also assume that $\mathcal{C} \in H_Q^\alpha$ is self-adjoint. Denoting the orthogonal projection onto the line of e_n by Π_n we define the Hamiltonian*

$$\mathcal{H} = \sum_{n\in\mathbb{Z}} (an + b) \Pi_n + \mathcal{C} \quad (a, b \in \mathbb{R}).$$

Then, the solution of (1) with the initial condition $\mathcal{A}(0) \in H_Q^\alpha$ satisfies $\mathcal{A}(t) \in H_Q^\alpha$ for all times $t > 0$. Moreover, the dependence of solutions on their initial value is continuous in the α-norm.

This shows how the Q-transform enables a qualitative description of the propagation of *quantum signals* (observables or states). The result also suggests that some quantum processes are indeed *compressible*. This may be used to estimate resources needed to carry out any particular quantum information processing tasks.

Separately, it is easily seen that whenever $\alpha \geq 0$ the Q-transform endows $H^\alpha(\mathbb{T})$ with a nontrivial Lie bracket, giving an example of a nested family of infinite-dimensional Lie algebras.

The phenomenon of broadband redundancy: The core definition of the Q-transform incorporates the Fourier series. However, a broader perspective is gained when it is modified by replacing the trigonometric basis for $L_2[0, 1]$ with a basis of the form $(f_m)_{m\in\mathbb{Z}}$, where $f_0(x) \equiv 1$, while

$$f_m(x) = \sum_{n>0} a_n \exp 2\pi inmx \quad (m > 0),$$

and $f_{-m}(x) = \overline{f_m(x)}$. The change of basis transformation may be viewed as a matrix operator $D : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$. It has block structure subordinate to the decomposition $\ell_2(\mathbb{Z}) = \ell_2(-\mathbb{N}) \oplus \mathrm{span} f_0 \oplus \ell_2(\mathbb{N})$. The block corresponding to the third factor is representative and has the form:

$$
\begin{matrix}
a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
a_2 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
a_3 & \cdot & a_1 & \cdot & \cdot & \cdot & \cdot & \cdots \\
a_4 & a_2 & \cdot & a_1 & \cdot & \cdot & \cdot & \cdots \\
a_5 & \cdot & \cdot & \cdot & a_1 & \cdot & \cdot & \cdots \\
a_6 & a_3 & a_2 & \cdot & \cdot & a_1 & \cdot & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{matrix}
$$

Here, we have substituted dots for zeros to de-clutter the appearance of the matrix. Such matrices form a commutative algebra isomorphic to the algebra of formal Dirichlet series, [9]. In every particular instance the first essential problem is continuity of operator D. There is a convenient simple estimate of the operator norm, namely $\|D\| \leq \max\{1, \sum_n |a_n|\}$. If, however, the sequence (a_n) is not summable one needs to abandon ℓ_2 and instead work in the framework of weighted Hilbert spaces, alternatively Banach spaces, in order to obtain useful information, [10]. Moreover, to constitute a bona-fide change of basis D needs to be a linear homeomorphism. This means that together with the sequence (a_n) one needs to pay attention to its Dirichlet inverse which, indeed, defines D^{-1}. Once a homeomorphism D of this kind is in hand one readily obtains a corresponding generalized Q-transform, [12].

Of particular interest are operators D related to the Riemann zeta function ζ. They are obtained by taking $a_n = n^{-s}$, where $s = \sigma + i\tau$. We will denote such operators D_s. This choice of coefficients makes control of the norms of D_s and D_s^{-1} in $\ell_2(\mathbb{Z})$ an easy task; in particular these depend only on σ.

There is a well known phenomenon related to the nontrivial zeros of ζ: uniform distribution of the fractional parts of their ordinates $0 < \tau_1 \leq \tau_2 \leq \ldots$; it was first observed in [6], and well-honed strengthened versions were later given in [3], with further improvements in [14]. When applied in the context of D_s operators and/or the corresponding Q-transforms it has implications for the harmonic analysis of classical as well as quantum signals. To give an example, let us consider the sequence of operators $D_{\sigma+i\tau_n}$ and the resulting $Q^{[\sigma+i\tau_n]}$. Retaining the symbol Q for the original Q-transform (i.e. one based on the trigonometric basis) we have the following

Theore 2. [12] Let $\alpha > 1/2$ and $\sigma > \alpha + 1$. For $h \in H^{2\alpha}(\mathbb{T})$ we have

$$\left\| \frac{1}{N(T)} \sum_{\tau_n \leq T} Q^{[\sigma+i\tau_n]}h - Qh \right\|_0 \longrightarrow 0$$

as $T \to \infty$, and the rate of convergence depends on h only via its 2α-norm.

Here, $N(T) = \max\{n : \tau_n \leq T\}$, and $\|\ \|_0$ is the well-known Hilbert-Schmidt norm. It may be argued that the optimal convergence result should be much stronger, [11].

So it happens that functions $Q^{-1}[Q^{[\sigma+i\tau_n]}h]$ typically have more high-frequency contents than h itself. Therefore, the theorem points at the existence of large ensembles of quantum channels that, while being noisy on their own, suppress noise when set in unison.

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