Fractional Quantization of Hall Resistance as a Consequence of Mesoscopic Feedback

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Abstract. A nonlinear single-particle model is introduced, which captures the characteristic of systems in the quantum Hall regime. The model involves the magnetic Schrödinger equation with spatially variable magnetic flux density. The distribution of flux is prescribed via the postulates of the mesoscopic mechanics (MeM) introduced in my previous articles (cf. [9, 10]). The model is found to imply exact integer and fractional quantization of the Hall conductance. In fact, Hall resistance is found to be $R_H = (h/e^2)(M/N)$ at the filling factor value N/M. The assumed geometry of the Hall plate is rectangular. Special properties of the magnetic Schrödinger equation with the mesoscopic feedback loop allow us to demonstrate quantization of Hall resistance as a direct consequence of charge and flux quantization. I believe results presented here shed light at the overall status of the MeM in quantum physics, confirming its validity.

1. THE CONCEPTUAL FOUNDATION OF THE MODEL

In this section, I will present a derivation of the model from the principles of quantum mechanics and the postulates of the MeM. The impatient reader, who is interested in the modeling aspects only, may jump directly to the next section. However, I would like to emphasize that the underlying concepts have much broader consequences than this single result. To begin with, consider a system consisting of a two-dimensional electron gas confined to a surface S, say, a rectangular plate. Suppose now that a magnetic field is applied transversally to the surface. Let the corresponding vector potential be given as $\nabla_A = i\nabla + (e/\hbar)A$. We focus on those electronic properties of the system that can be captured by the single-particle Schrödinger equation

$$i\hbar\dot{\Psi} = H_A\Psi, \tag{1}$$

where the magnetic Hamiltonian is $H_A = -\frac{\hbar^2}{2m^*}\nabla_A^*\nabla_A$. The state vector Ψ is assumed to fully (or sufficiently) characterize the electronic state. In particular, the total free charge is given as

$$Q = e\int_S |\Psi|^2. \tag{2}$$

(The adopted convention is that the unit of $|\Psi|^2$ is $[m^{-2}]$, which in turn affects the definition of current density, etc.) In equilibrium, $\Psi = e^{-iEt/\hbar}\Psi_0$ where Ψ_0 is time-independent and satisfies the eigenvalue problem

$$H_A\Psi_0 = E\Psi_0. \tag{3}$$

At this point, we will use an assumption which is extrinsic to the classical quantum mechanics. Namely, the mesoscopic mechanics postulates (cf. [10], Second Postulate of the MeM) that the planar distribution of the magnetic flux is probabilistically described in terms of a certain transform K applied to the electronic state Ψ. Namely, the magnetic flux density is given as

$$B(x,y) = \Phi|K\Psi|^2(x,y), \quad (x,y) \in S, \tag{4}$$

where Φ is the total magnetic flux through the surface, i.e., the number of quanta of the magnetic field. The magnetic flux density function $B = B(x,y)$ affects the electronic states as it modifies the Hamiltonian H_A, thus closing the mesoscopic feedback loop. H_A is modified via the vector potential, which in a simply-connected domain satisfies $\nabla \times A = B(x,y)$. The particular distribution prescribed in (4) is the effect of an interaction of the magnetic field with the electronic structure.

The interaction is described via an operator evolution equation called the Mesoscopic Schrödinger equation

$$i\hbar\dot{K} = -KH_A - \tilde{B}^2(K^*)^{-1}, \tag{5}$$

which yields the transform K. The constant $\tilde{B}$ is different from zero when the magnetic field is switched on, but its exact value plays no explicit role in the analysis below. (We remark that the physical unit of $\tilde{B}^2$ is that of energy, while K is dimensionless.) The unique structure of solutions of equation (5) was described in [10]. Throughout this article, we focus attention on operator states (transforms) K of a special type, the equilibrium type, which satisfy the eigenvalue problem

$$KH_A + \tilde{B}^2(K^*)^{-1} = \nu K. \tag{6}$$

The constant ν is interpreted as the Fermi energy level. It is easily seen, cf. [9] or [10], that equilibrium states assume the form

$$K = U \circ \sum_{E_n < \nu} \frac{\tilde{B}}{(\nu - E_n)^{1/2}} \quad |\psi_n\rangle\langle\psi_n|. \tag{7}$$

The vectors $|\psi_n\rangle$ represent electronic eigenstates of H_A, corresponding to eigenvalues E_n. State vectors $|\psi_n\rangle$ are mutually orthogonal, which possibility is guarantied by the hermicity property of the Hamiltonian. (We remark that an arbitrary choice is involved if there are multiple eigenvalues.) The non-Abelian phase matrix U is unitary and we will assume here that in fact $U = I$ (identity). Let us briefly remark that the $U = I$ equilibrium solutions are interpreted within the MeM framework as representing the phase-correlated state, cf. [9] and [10] for details. Furthermore, we note that K may require normalization so as to guarantee that the probabilistic measure in (4) has total mass one. This is achieved by adjusting the constant $\tilde{B}$ if necessary. Now, with the K variable all set, recall that Ψ_0 is on the list of eigenstates. We assume that $E_0 = E < \nu$, which guaranties that Ψ_0 can be identified as one of the vectors in formula (7). Of course, if there are multiple eigenvalues, this latter circumstance may require a change of basis. The arrangements we have made guarantee that the magnetic feedback loop is switched on. Indeed, formula (4) yields

$$B(x, y) = \Phi|K\Psi|^2(x, y) = b|\Psi_0|^2, \tag{8}$$

where the constant b is such that $\Phi = \int_S B(x, y) = \int_S b|\Psi_0|^2 = bQ/e = bN$. Here N is the number of relevant (current carrying) charge quanta. In addition, let M denote the number of magnetic flux quanta, so that the total flux[1] $\Phi = Mh/e$. Quantization of flux and charge implies that b is a natural fraction expressed in units of the flux quantum, i.e.

$$b = \frac{M}{N}\frac{h}{e}. \tag{9}$$

The fraction N/M is known in the literature as the *filling factor*, cf. [1]. (The filling factor is typically discussed in the context of constant-magnetic-field Hamiltonian and the Landau levels.) Next we note that in a simply connected region, such as the rectangular surface S, the vector potential can be represented in a suitable gauge as $A = \varphi_x dy - \varphi_y dx$. Note that formula (8) is equivalent to

$$\varphi_{xx} + \varphi_{yy} = b|\Psi_0|^2. \tag{10}$$

In addition, the eigenvalue problem (3) expressed in terms of φ assumes the form

$$-\hbar^2(\partial_x^2 + \partial_y^2)\Psi_0 + 2ie\hbar(-\varphi_y\Psi_{0,x} + \varphi_x\Psi_{0,y}) + e^2(\varphi_x^2 + \varphi_y^2)\Psi_0 = 2m^*E\Psi_0. \tag{11}$$

Thus an application of the postulates of the MeM has lead us to a nonlinear system consisting of equations (10) and (11). Nonlinear systems are prominently present in the framework of quantum mechanics and (the classical) Maxwell theory. However, the novelty of the MeM is here manifested in the particular form of the nonlinearity. One might note that in the classical setting, a Poisson equation of the form (10) would yield the electric potential, yet here the variable φ is tied to the vector potential. In the remainder of this article, I will argue that this system of equations provides a relevant model for electronics of the QHE and FQHE regimes.

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[1]Some sources give $h/(2e)$ as the value of the magnetic flux quantum, which is what I have used in the older version of the manuscript. It has been pointed out to me that the correct value of the magnetic flux quantum is h/e, which is commonly used in QHE literature. We will adhere to this latter value throughout the article.

2. CALCULATION OF THE HALL RESISTANCE

As demonstrated above, the MeM prompts us to consider the magnetic Schrödinger equation (11) jointly with the Poisson-type equation (10). At this point, let us try to understand some properties of the nonlinear system of equations. To begin with, impose the Ansatz $\Psi_0(x,y) = e^{ikx}\chi(y)$, where $\chi = \chi(y)$ is a real function, and the unit of χ is $[1/m]$. Let us also simplify the gauge representation of vector potential A by asking that $\varphi = \varphi(y)$. In such a case, Eq. (10) implies

$$\varphi_y = b \int_0^y dy' \chi^2(y'). \tag{12}$$

Also, evaluating the Hamiltonian we obtain

$$2m^* e^{-ikx} H_A \Psi_0 = \hbar^2 k^2 \chi - \hbar^2 \chi_{yy} + 2e\hbar k \varphi_y \chi + e^2 \varphi_y^2 \chi = -\hbar^2 \chi_{yy} + (\hbar k + e\varphi_y)^2 \chi. \tag{13}$$

Thus, system (10)–(11) has been reduced to a nonlinear integro-differential eigenvalue problem, namely:

$$-\chi'' + \left(k + \frac{e}{\hbar}b \int_0^y dy' \chi^2(y')\right)^2 \chi = \frac{2m^* E}{\hbar^2}\chi. \tag{14}$$

It is convenient to introduce an auxiliary variable $w = k + \frac{e}{\hbar}b \int_0^y dy' \chi^2(y')$. Equation (14) becomes a pendulum-type system

$$w' = \frac{e}{\hbar}b\chi^2, \qquad \chi'' = \left(w^2 - \frac{2m^* E}{\hbar^2}\right)\chi. \tag{15}$$

In the next section, we discuss solutions of this equation so as to show that there exist physically significant wavefunctions satisfying the system (10), (11). First, however, let us calculate the resulting Hall resistance. Let us recall that the Hall plate S is a rectangle. Assume that the edges of S are parallel to the x and y-axes. The x-extent of the rectangle plays no explicit role in the analysis, but the y-extent figures more explicitly in our calculation. We will refer to the cut-offs localized at fixed y-coordinates as the 'left edge' and the 'right edge.' Now, let us note that the Hall potential, which is the difference of potentials between the two edges of the rectangle S, is directly read out of the Hamiltonian. (We will see in the next section that, in fact for certain solutions, the Hamiltonian exhibits broken symmetry.) More precisely, the Hall potential is directly derived from the reduced equation (14). It amounts to

$$V_H = \frac{\hbar^2}{2m^* e}(w^2(\text{right edge}) - w^2(\text{left edge})). \tag{16}$$

We shall see that this is the only possible interpretation. Indeed, a straightforward calculation shows that only the longitudinal (i.e. x-) component of the resulting current density is nonvanishing. In fact, it amounts to

$$j_x = \frac{e}{m^*}\Re\left\{\Psi_0^*\left(i\hbar\nabla + eA\right)\Psi_0\right\} = \frac{e\hbar}{m^*}\left(k + \frac{e}{\hbar}b \int_0^y dy' \chi^2(y')\right)\chi^2.$$

Thus, potential V_H is transversal to the current. It is now useful to represent the current density in the auxiliary variables. Namely,

$$j_x = \frac{e\hbar}{m^*}w\chi^2 = \frac{\hbar^2}{m^*}\frac{1}{b}ww' = \frac{\hbar^2}{m^*}\frac{1}{2b}(w^2)'. \tag{17}$$

Integrating and taking into account (16), we see that the total longitudinal current I_x is given as

$$I_x = \frac{e}{b}V_H. \tag{18}$$

In view of (9), we conclude that the Hall resistance $R_H = V_H/I_x$ is given by the following formula:

$$R_H = \frac{h}{e^2}\frac{M}{N}. \tag{19}$$

This is our main result.

3. EXISTENCE OF THE MESOSCOPIC FQHE STATES

We need to show that there exists a wavefunction Ψ_0, which (in pair with a matching φ) satisfies the system (10), (11) and is square-integrable in the rectangle S. In view of the reductions we have introduced in the previous section, it now suffices to show that equation (14) admits a bounded solution. This is what we will in fact observe. Remarkably, we will see that the solution conforms (although not strictly coincides) with the London penetration law as it applies to the magnetic flux density and the current density in superconducting materials, cf., e.g., [4]. Indeed, let us focus on the dynamical system (15). Differentiating the first equation, we observe that $w''/w' = 2\chi'/\chi$. Substituting into the second equation, we obtain an autonomous equation for w, namely:

$$\frac{w'''}{w'} = \frac{1}{2}\left(\frac{w''}{w'}\right)^2 + 2w^2 - 2\tilde{E}, \tag{20}$$

where $\tilde{E} = 2m^*E/\hbar^2$. The first equation in (15) shows that, as long as χ does not vanish, w is a monotone function. Thus we can assume without loss of generality that w' is a function of w, namely,

$$w' = f(w). \tag{21}$$

Equation (20) is now represented in the form

$$ff_{ww} + \frac{1}{2}f_w^2 = 2(w^2 - \tilde{E}). \tag{22}$$

A particular solution (of the form we need) is now easy to guess. Indeed, assume

$$f(w) = \alpha w^2 + \beta w + \gamma.$$

Substituting this into (22), we obtain $f = (\sqrt{2}/2)w^2 - \sqrt{2}\tilde{E}$. Next, substituting f into (21), we derive a particular solution of interest, i.e.

$$w = -\sqrt{2\tilde{E}}\,\frac{1 + Ce^{-2\sqrt{\tilde{E}}y}}{1 - Ce^{-2\sqrt{\tilde{E}}y}}, \tag{23}$$

where C is a constant such that $Cb > 0$. We now calculate via (15) that the wavefunction factor χ is given by

$$\chi(y) = \pm 2 \cdot 2^{1/4}\sqrt{\frac{\tilde{E}C\hbar}{eb}}\,\frac{e^{-\sqrt{\tilde{E}}y}}{1 - Ce^{-2\sqrt{\tilde{E}}y}}. \tag{24}$$

This is a bounded function on any right half-axis, and in particular between the two edges of the Hall plate. Recall that the wavefunction is determined as

$$\Psi = e^{-iEt/\hbar}e^{ikx}\chi(y).$$

Thus Ψ is square-integrable over the rectangle S, which is what we have intended to prove. In addition, note that constants C and $\tilde{E}$ are in fact constrained by condition (2). Suppose for example that the left edge of the plate is at $y = 0$, while the right edge is sufficiently far to the right so as to make the exponential factor $\exp(-\sqrt{\tilde{E}}y)$ negligible. The value of C is then easily estimated to yield

$$C \simeq 1 \Big/ \Big(1 + \frac{2\sqrt{2\tilde{E}}L_x}{M\pi}\Big).$$

Here L_x denotes the length of the plate along the x-direction. The charge density function $|\Psi|^2$ corresponding to this solution of the system (15) is localized at the edge of the plate, and its depletion toward the center is essentially indistinguishable quantitatively from the London penetration law. Here the depth of penetration is inversely proportional to the square root of the energy E. This localization property carries over to the current density, cf. formula (17), as well as the magnetic flux density, cf. (8). However, this property is not claimed as universal. Namely, let us recall that the variable w, which was introduced in the text following Eq. (14), satisfies $w(0) = k$. In view
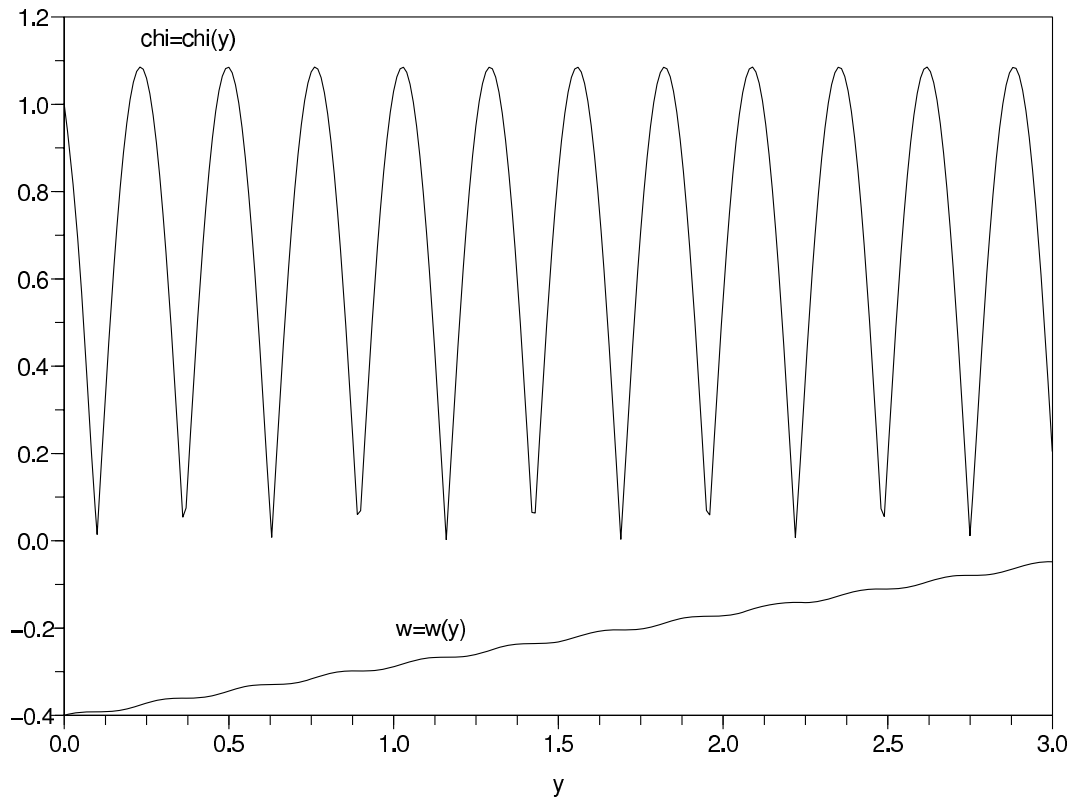
Fig. 1. A simulated solution pair $w(y)$ and $\chi(y)$, which is of a different type than the pair given in (23) and (24). In this case $\tilde{E} = 140.4$, $k = -0.4$, and $be/h = 1/5$. The two functions, which are here approximated by numerical simulation, satisfy equations (15) in the classical sense almost everywhere. Note that the pattern of successive concave humps will break at a large value of y—the point where the steadily increasing w overtakes the constant $\tilde{E}^{1/2}$. We consider this point to be beyond the plate's right edge. Recall that the corresponding magnetic field is $B = B(y) = b\chi(y)^2$. [Numerical simulations were performed with the Scilab software package of INRIA, cf. `www.scilab.org`.]

of (23), this leads to a relation between $\tilde{E}$ and k and imposes a constraint on the initial conditions satisfied by w and χ. At the same time, imposing different initial conditions leads to solutions of a different type, cf. Fig. 1. Arguably, wavefunctions constructed from such solutions are also physically admissible.[2] Fig. 1 gives evidence that, in this case, the London penetration law is not obeyed. In fact, charge and flux are somewhat concentrated along stripes running uniformly throughout the entire plate.

In this section, we have demonstrated that there are physically meaningful solutions of the set problem. As explained in the previous section, the electronic phenomenon they describe is characterized by fractionally quantized Hall resistance.

4. CLOSING COMMENTS

It is worthwhile to emphasize that the model introduced in this article does not make any direct references to material properties. It is derived from the magnetic Schrödinger equation, assuming a special form of charge-to-flux feedback loop. The loop has been postulated as relevant to planar electronic systems in my previous articles on the MeM, cf. [9, 10]. In this article, our discussion is limited to a single special (if important) case derived from a rich structure of the MeM. Separately,

[2]This issue can only be settled *a posteriori*, via further comparison of the model, in all its extensions, with experimental data. In the next section, we briefly discuss the possibility of enriching the model with additional elements of structure, which needs be done so as to better account for various properties of real materials.

we note that the particular solutions (FQHE states) discussed here may be too idealized to account for the celebrated plateaux observed in the Quantum Hall Effect and the Fractional Quantum Hall Effect. As we know, the plateaux occur at certain special values of the filling factor, thus distinguishing them from other theoretically admissible values. However, I would like to emphasize that the model enables one to incorporate quite directly additional elements of structure, including the lattice potential and impurity potentials (—the latter are believed to be particularly relevant to the QHE). In the same way, the model can be enriched by introducing the self-consistent Coulomb interaction potential (believed to be relevant to the FQHE). Further research is currently being conducted in order to analyze theoretically and numerically the effect that these perturbations will have on the properties of the model, some recent progress has been reported in [12]. In time we should be able to understand the full predictive strength of the model, including its ability to predict occurrence of the plateaux. Meanwhile, even as our knowledge of the properties of the model is still incomplete, it is already relevant to various mesoscopic modeling tasks.

I would also like to mention that I investigated the consequences of a similar mesoscopic feedback loop for the first time in [11]. That article discusses the *strongly correlated* case as captured by the formalism of a theory called the (fully) Nonlinear Maxwell Theory, cf. [8], which is in some sense dual to the MeM. The reader may wish to consult the differences and similarities of the two approaches and their consequences, e.g., equation (20) above has a companion of similar type presented in the other paper.

Experimental discovery of the quantum Hall effect was announced in [6], while the fractional quantum Hall effect was first announced in [13]. Over the years these phenomena have been discussed theoretically from many vantage points, cf., e.g., [7]; the literature is very abundant. The particular theme of spatially variable magnetic fields has been considered in the context of FQHE by many authors, e.g., [5, 2, 3]. More precisely, those articles examine the effect of randomly fluctuating fields. In comparison, here I have examined the postulate that the spatial distribution of the magnetic field depends in a certain way upon charge distribution. I believe there is now compelling evidence for the validity of this approach. I also believe that there may be more than one mathematically distinct and yet successful approaches to describing physical phenomena such as these. The history of quantum mechanics teaches us that establishing such complementary formalisms is actually extremely beneficial.

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