Study of Wave Propagation in Fiber-reinforced Elastic Solids Using Lie Symmetry and Conservation Law Analysis

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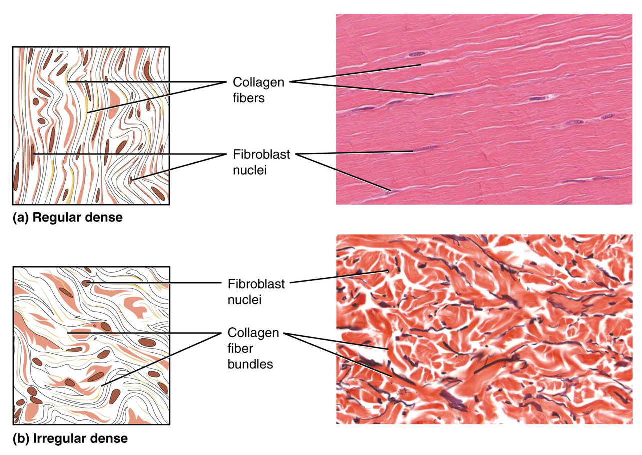
Motivation

Mooney-Rivlin elasticity equations are nonlinear coupled partial differential equations that are used to model various elastic materials. Models can be extended to account for fiber-reinforced materials. We study analytical properties of models of wave propagation in fiber-reinforced elastic solids using Lie symmetry and conservation law analysis.

Applications

Biological materials have been modeled as incompressible hyperelastic solids with anisotropic fiber bundles [1]. Material parameters are found for several internal organs in [2].

Biological Materials



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Arteries

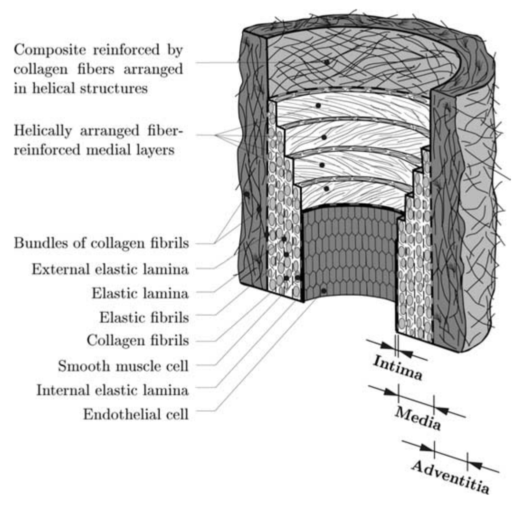
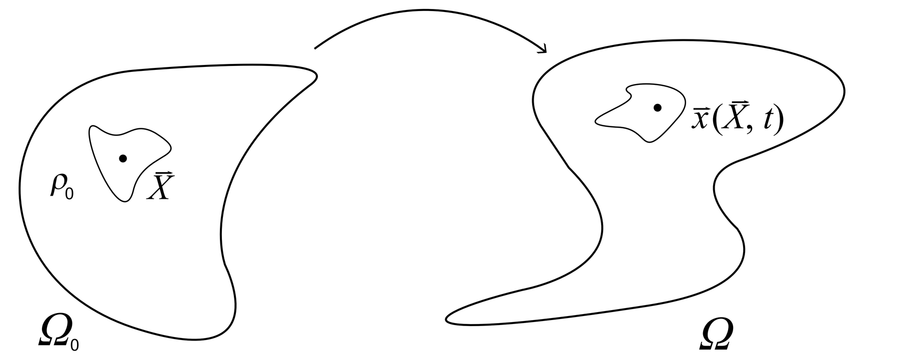


Diagram of arterial tissue from [1]

Theory of Incompressible Hyperelastic Solids

When an elastic solid undergoes a deformation, points $\vec{X}$ in the reference configuration Ω_0 at a time $t = 0$ are transformed to points $\vec{x}$ in the current configuration Ω at time t.



Here ρ_0 and ρ are mass densities in the reference and current configuration, respectively.

- The Jacobian matrix of the deformation is called the deformation gradient $\mathbf{F}$.

$$F^{ij}(\vec{X}, t) = \frac{\partial x^i}{\partial X^j}$$

- The material behaviour of a hyperelastic solid is described by the strain energy density $W(\mathbf{F})$.
- The first Piola-Kirchhoff stress $\mathbf{P}$ measures the stress within an incompressible hyperelastic solid with respect to undeformed area in the reference configuration:

$$P^{ij} = \rho_0 \frac{\partial W}{\partial F^{ij}} - p(F^{-1})^{ji}$$

where ρ_0 is mass density (assumed constant), and $p(\vec{X}, t)$ is hydrostatic pressure.

- For an incompressible Mooney-Rivlin material reinforced with a fiber bundle oriented along $\vec{A}$, the strain energy density takes the form

$$W = W_{iso}(I^1, I^2) + W_{aniso}(I^4) = a(I^1 - 3) + b(I^2 - 3) + c(I^4 - 3)^2, \quad a, b, c > 0,$$

where I^1 and I^2 are principal invariants under orthogonal transformations of the right Cauchy-Green stress tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and I^4 is a fiber-specific invariant.

$$I^1(\mathbf{C}) = \mathrm{Tr}(\mathbf{C}), \quad I^2(\mathbf{C}) = \frac{1}{2}\left(\mathrm{Tr}(\mathbf{C})^2 - \mathrm{Tr}(\mathbf{C}^2)\right) \qquad I^4 = \vec{A}^T \mathbf{C} \vec{A}$$

- The equations of motion can be derived from the incompressibility condition and moment balance as

$$\det \mathbf{F} = 1$$

$$\rho_0 x^i_{tt} = \sum_j \frac{\partial P^{ij}}{\partial X^j}$$

Lie Point Symmetries

Consider the differentiable function

$$f(x, y) = 0.$$

Suppose this equation undergoes a transformation of variables with parameter ϵ:

$$x^* = g(x, y, \epsilon), \qquad y^* = h(x, y, \epsilon).$$

This forms a symmetry transformation of the equation if it maps solutions into solutions; i.e.

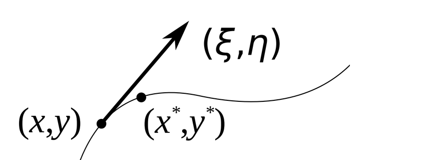
$$f(x^*, y^*) = 0 \quad \text{when} \quad f(t, x) = 0.$$

A local Lie point symmetry of a differential equation is a symmetry transformation which is a Lie group of point transformations. Consider the Taylor expansion of a Lie point transformation about $\epsilon = 0$:

$$x^* = g(x, y, \epsilon) \approx g(x, y, 0) + \epsilon\xi(x, y) + O(\epsilon^2),$$
$$y^* = h(x, y, \epsilon) \approx h(x, y, 0) + \epsilon\eta(x, y) + O(\epsilon^2).$$

The tangent vector field $(\xi(x, t), \eta(x, t))$ of the transformation is defined by the $O(\epsilon)$ terms, which form coefficients of the operator equivalent to the Lie group of point transformations, the infinitesimal generator.

$$\mathrm{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$$



The symmetry condition for Lie symmetries can thus be written in the form

$$\mathrm{X}f = 0 \quad \text{when} \quad f = 0.$$

Applications to Differential Equations

- Obtain the general solution of ODEs, and particular solutions of PDEs.
- New solutions can be generated from known ones through a Lie group of point transformations.

Traveling Wave Solutions through the Invariant Form Method

An important application of Lie symmetries is in seeking solutions to differential equations.

Example: consider traveling wave solutions for the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

This equation admits time and spatial translation Lie symmetries. Hence, it is also invariant under the linear combination of the equivalent infinitesimal generators.

$$\mathrm{X}^1 = \frac{\partial}{\partial t}, \quad \mathrm{X}^2 = \frac{\partial}{\partial x} \rightarrow \quad \bar{\mathrm{X}} = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}; \qquad t^* = t + \epsilon, \quad x^* = x + c\epsilon, \quad u^* = u.$$

Quantities invariant under the action of a Lie group of point transformations $\bar{\mathrm{X}}$ are

$$I = x - ct, \quad V = u.$$

To seek solutions invariant under $\bar{\mathrm{X}}$, one substitutes into the differential equation the invariant form $V(I)$, which is exactly the traveling wave *ansätze*

$$V(I) = u(x - ct).$$

Conservation Laws

A conservation law of a system of differential equations is a divergence expression that vanishes on solutions of the system. For example, the nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \left(\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial x^2} = 0$$

can be written in the divergence (conservation law) form

$$\mathrm{D}_t\left(\frac{\partial u}{\partial t}\right) - \mathrm{D}_x\left(\frac{1}{3}\left(\frac{\partial u}{\partial x}\right)^3\right) = 0.$$

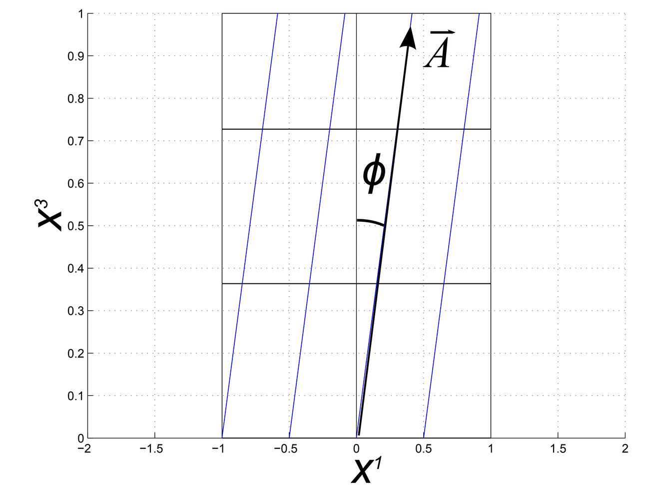
where D_t represents the total derivative with respect to t. Here, $\frac{\partial u}{\partial t}$ is the conserved density, while $-\frac{1}{3}\left(\frac{\partial u}{\partial x}\right)^3$ is the flux.

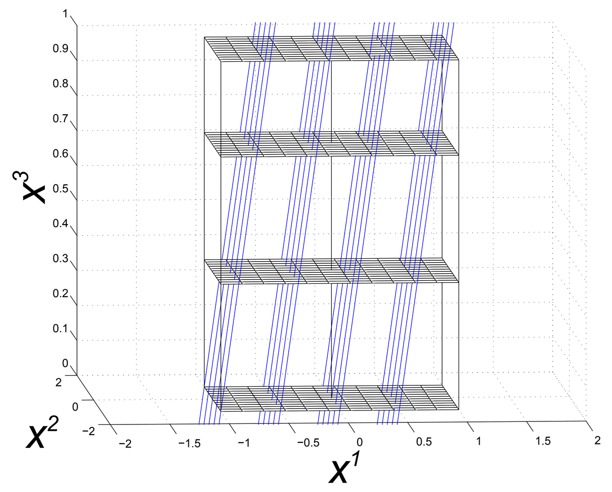
Applications to Differential Equations

- Conservation laws allow for a better understanding of underlying physical processes.
- Advanced numerical methods based on divergence forms have been developed.

Orientation of Fibers in Reference Configuration

The fiber bundle is oriented along the vector $\vec{A} = [\sin(\phi), 0, \cos(\phi)]^T$ at an angle ϕ to the X^3-axis in the X^1X^3-plane.



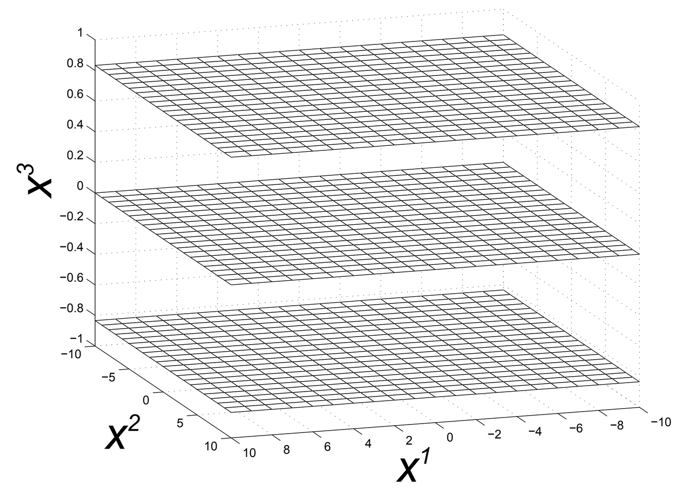


Solid with Sample Fiber Bundle (blue lines)

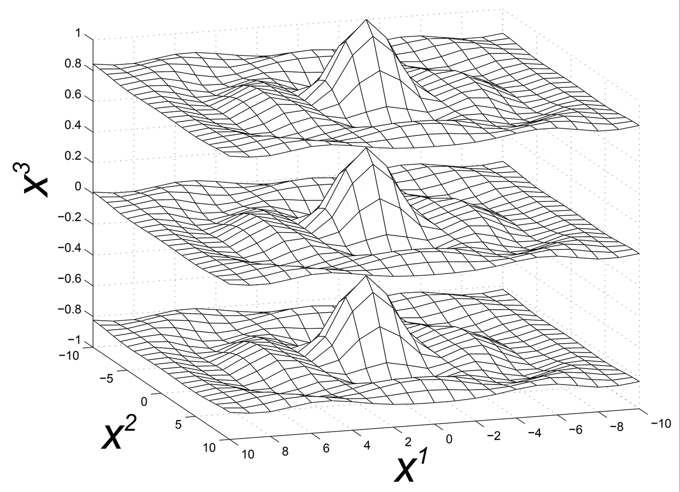
Case 1: Motion transverse to the X^1X^2-plane

Consider displacements transverse to the X^1X^2-plane, given by

$$\vec{x} = [X^1, X^2, X^3 + G\left(X^1, X^2, t\right)]^T, \quad p = p\left(X^1, X^2, t\right)$$



Reference Configuration



Actual Configuration

Lie Point Symmetries for Case 1

The equations of motion in Case 1 are invariant under the following Lie groups of point transformations.

Transformation	Symmetry Infinitesimal Generator
Time Translation	$Y^1 = \frac{\partial}{\partial t}$
Spatial Translations	$Y^2 = \frac{\partial}{\partial X^1}$, $Y^3 = \frac{\partial}{\partial X^2}$
Amplitude translation	$Y^4 = \frac{\partial}{\partial G}$
Time dependent amplitude translation	$Y^5 = t\frac{\partial}{\partial G}$
Scaling	$Y^6 = X^1\frac{\partial}{\partial X^1} + X^2\frac{\partial}{\partial X^2} + t\frac{\partial}{\partial t} + G\frac{\partial}{\partial G}$
Time dependent pressure translation	$Y^7 = F(t)\frac{\partial}{\partial p}$

The corresponding transformations for the above infinitesimal generators are

$$t^* = e^{\epsilon_6}t + \epsilon_1, \qquad (X^1)^* = e^{\epsilon_6}X^1 + \epsilon_2, \qquad (X^2)^* = e^{\epsilon_6}X^2 + \epsilon_3,$$

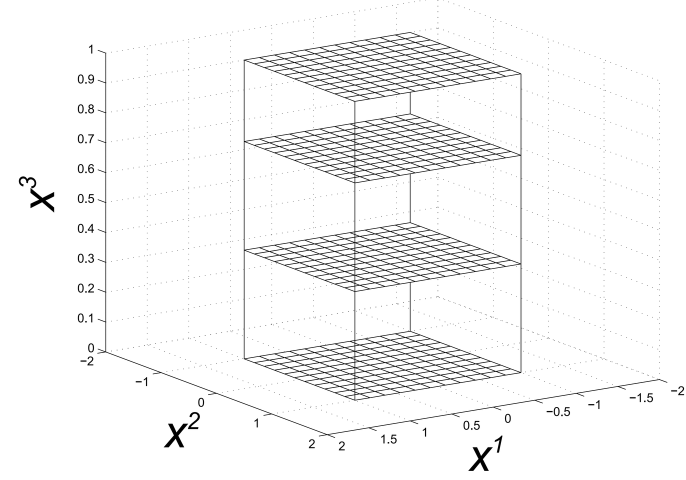
$$G^* = e^{\epsilon_6}G + \epsilon_4 + \epsilon_5(t + \epsilon_1), \qquad p^* = p + F(t),$$

where each ϵ_i is the transformation parameter corresponding to infinitesimal generator Y^i.

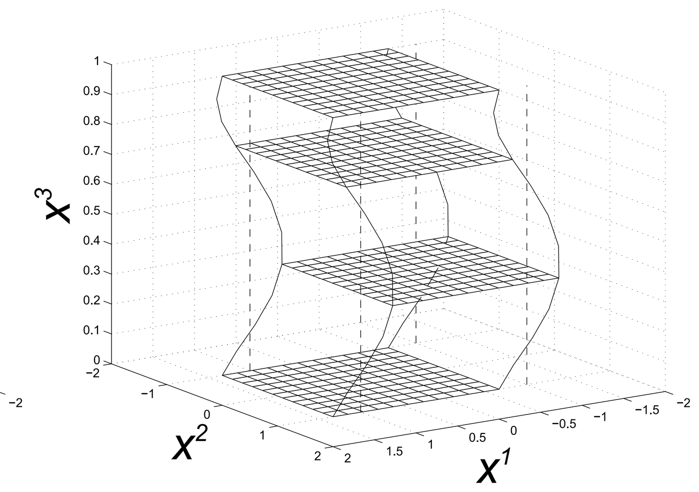
Case 2: Motion transverse to the X^3-axis

The second wave propagation *ansätze* is given by the displacement orthogonal to the X^3-axis, with coordinate dependence

$$\vec{x} = [X^1 + G^1\left(X^3, t\right), X^2 + G^2\left(X^3, t\right), X^3]^T, \quad p = p\left(X^3, t\right)$$



Reference Configuration



Actual Configuration

Lie Point Symmetries for Case 2

The equations of motion in Case 2 are invariant under the following Lie groups of point transformations.

Transformation	Symmetry Infinitesimal Generator
Time Translation	$Z^1 = \frac{\partial}{\partial t}$
Spatial Translation	$Z^2 = \frac{\partial}{\partial X^3}$
Amplitude translations	$Z^3 = \frac{\partial}{\partial G^1}$ and $Z^4 = \frac{\partial}{\partial G^2}$
Time dependent translations of dependent variables	$Z^5 = t\frac{\partial}{\partial G^1}$, $Z^6 = t\frac{\partial}{\partial G^2}$, $Z^7 = F(t)\frac{\partial}{\partial p}$
Fiber-affected rotations	$Z^8 = \left(\sin(\phi)X^3 + \cos(\phi)G^1\right)\frac{\partial}{\partial G^2} - \cos(\phi)G^2\frac{\partial}{\partial G^1}$
Scaling	$Z^9 = X^3\frac{\partial}{\partial X^3} + t\frac{\partial}{\partial t} + G^1\frac{\partial}{\partial G^1} + G^2\frac{\partial}{\partial G^2}$

The transformation corresponding to infinitesimal generator Z^8 is

$$t^* = t, \quad (X^3)^* = X^3, \quad p^* = p,$$

$$(G^1)^* = \cos(\sin(\phi)\epsilon_8)G^1 - \sin(\sin(\phi)\epsilon_8)G^2 + \tan\phi(\cos(\sin(\phi)\epsilon_8) - 1)X^3,$$

$$(G^2)^* = \cos(\sin(\phi)\epsilon_8)G^2 + \sin(\sin(\phi)\epsilon_8)G^1 + \tan\phi\sin(\sin(\phi)\epsilon_8).$$

Additional symmetry: for ϕ fixed such that

$$\sin^2(\phi) = \frac{1}{2}\left(1 \pm \sqrt{1 - 2(a+b)/e}\right),$$

the equations of Case 2 are also invariant under the transformations

$$Z^{10} = \cos(\phi)X^3\frac{\partial}{\partial X^3} + 2\cos(\phi)t\frac{\partial}{\partial t} - 2\cos(\phi)p\frac{\partial}{\partial p} - \sin(\phi)X^3\frac{\partial}{\partial G^1},$$

$$t^* = e^{2\cos(\phi)\epsilon_{10}}\, t,$$
$$(X^3)^* = e^{\cos(\phi)\epsilon_{10}}\, X^3,$$
$$(G^1)^* = G^1 + \tan(\phi)\left(1 - e^{\cos(\phi)\epsilon_{10}}\right) X^3,$$
$$(G^2)^* = G^2,$$
$$p^* = e^{-2\cos(\phi)\epsilon_{10}}\, p.$$

Conserved Quantities for Case 2

- **Conservation of Energy.** Conserved density = Kinetic + Potential =

$$\rho_0\left(\frac{1}{2}(G^1_t)^2 + \frac{1}{2}(G^2_t)^2 + (a+b)\left((G^1_3)^2 + (G^2_3)^2\right) + e\cos^2\phi\left(4(G^1_3)^2 + 4\cos(\phi)\sin(\phi)((G^1_3)^3 + G^1_3(G^2_3)^2)\right) + \cos^2\phi\left((G^1_3)^4 + (G^2_3)^4 + 2(G^1_3)^2(G^2_3)^2 - 4(G^1_3)^2\right)\right).$$

- **Conservation of Linear Momentum in Eulerian Frame.** Conserved densities: $\rho_0 G^1_t$ and $\rho_0 G^2_t$.
- **Conservation of Angular Momentum in Eulerian Frame.** Conserved density:

$$\rho_0\left(\cos(\phi)\left(G^1 G^2_t - G^2 G^1_t\right) + \sin(\phi)\left(X^3 G^2_t + 2(a+b)tG^2_3\right)\right).$$

- **Additional conserved densities:** $\rho_0\left(tG^1_t - G^1\right)$ and $\rho_0\left(tG^2_t - G^2\right)$.

Note: Subscript notation indicates partial differentiation.

Conclusions and Future Research

- Lie point symmetries and conservation laws have been classified in two *ansätze* for fiber-reinforced incompressible Mooney-Rivlin solids. Special cases have been isolated.
- Time and spatial translation Lie symmetries are admitted; traveling-wave solutions can be sought.
- Goal 1: Construct invariant solutions for particular symmetries.
- Goal 2: Study relations between symmetries and conservation laws.
- Goal 3: Seek additional conservation laws for each system, study physical meaning.

References

[1] G.A. Holzapfel, T.C. Gasser, and R.W. Ogden.
A new constitutive framework for arterial wall mechanics and a comparative study of material models.
J. Elasticity, 61:1–48, 2000.

[2] S. Umale, C. Deck, N. Bourdet, P. Dhumane, L. Soler, J. Marescaux, and R. Willinger.
Experimental mechanical characterization of abdominal organs: liver, kidney and spleen.
J. Mech. Behav. Biomed, 17:22–33, 2013.