Brownian Dynamics Modelling for the Narrow Escape Problem

Vaibhava Srivastava[1], Jason Gilbert, Mohamad Alwan and Alexei F. Cheviakov

UNIVERSITY OF SASKATCHEWAN

[1]Department of Mathematics, Indian Institute of Technology, Gandhinagar

The Mathematical Model for the Narrow Escape Problem

The **Narrow Escape Problem** concerns the calculation of the **mean first passage time** (MFPT) required for a **Brownian particle** confined in a bounded domain $\Omega \in \mathbb{R}^d$ (two or three dimensional spaces) to escape through one of finitely many small boundary windows, or **traps**. The domain boundary $\partial\Omega = \partial\Omega_r \cup \partial\Omega_a$ is almost entirely reflecting ($\partial\Omega_r$), except for traps, $\partial\Omega_a$).

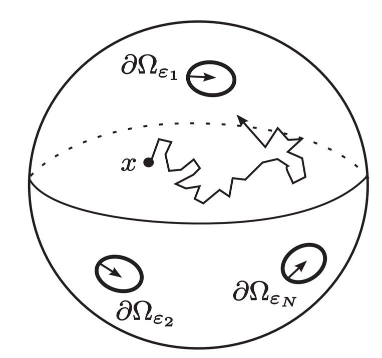
The MFPT $v(x)$ satisfies the mixed Dirichlet-Neumann problem

$$\triangle v = -\frac{1}{D}, \quad x \in \Omega; \qquad v = 0, \quad x \in \partial\Omega_a = \bigcup_{j=1}^{N} \partial\Omega_{\epsilon_j}; \qquad \partial_n v = 0, \quad x \in \partial\Omega_r, \tag{1}$$

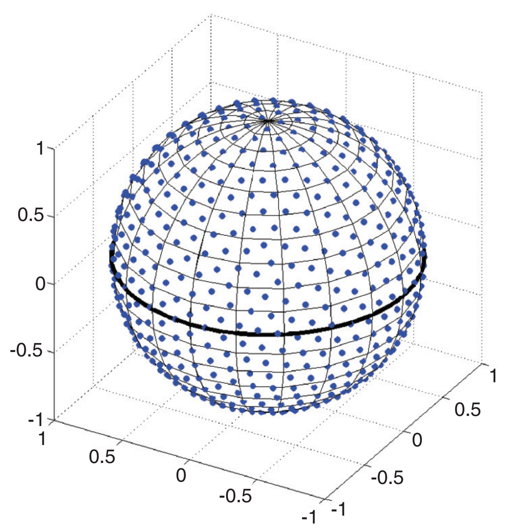
where D is constant **diffusivity**. An important integral characteristic of escape times from a domain with a prescribed trap arrangement is the **Average MFPT**:

$$\bar{v} = \frac{1}{|\Omega|}\int_{\Omega} v(x), dx = 0\,, \tag{2}$$

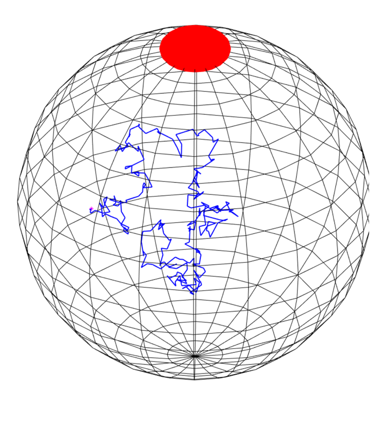
where $|\Omega|$ is the measure of the domain.



Schematics of the narrow Escape Problem in 3D

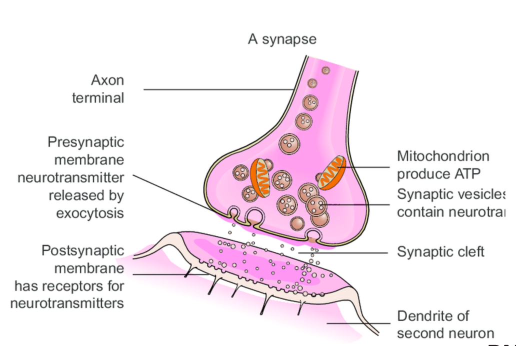


Sphere with 802 boundary traps

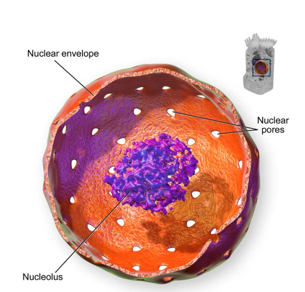


Dynamics of the Brownian particle in the sphere

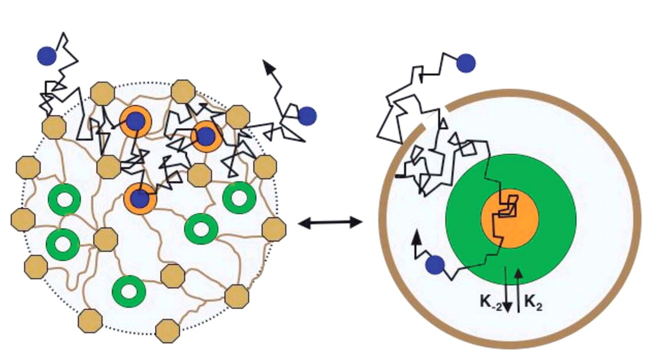
Areas of Application & Study



Receptor Trafficking in a Synaptic Membrane (Retrieved from https://www.researchgate.net)



RNA Transport from Cell Nucleus (Retrieved from Wikipedia)



Diffusion through Postsynaptic Membrane (Retrieved from https://www.pnas.org)

The Asymptotic MFPT for the Unit Sphere

Approximate asymptotic solutions of **(1)** have been obtained for various 3D domains WLOG, the physical problem (1) can be re-scaled :

- $\mathrm{diam}\,\Omega \sim 1$
- $D = 1$
- trap sizes $\sim \epsilon$, where $\epsilon \ll 1$
- well-separated traps: $|x_i - x_j| \gg \epsilon$

The MFPT $v(x)$ for the unit sphere with N non-equal traps was obtained in Ref. [1] using the method of matched asymptotic expansions, and is given by

$$v(x) = \bar{v} - \frac{|\Omega|}{DN\bar{c}}\sum_{j=1}^{N} c_j G_s(x; x_j) + \mathcal{O}(\epsilon \log \epsilon).$$

Correspondingly, the asymptotic average MFPT $\bar{v}$ is given by

$$\bar{v} = \frac{|\Omega|}{2\pi\epsilon DN\bar{c}}\left(1 + \epsilon\log\left(\frac{2}{\epsilon}\right)\frac{\sum_{j=1}^{N} c_j^2}{2N\bar{c}} + \frac{2\pi\epsilon}{N\bar{c}} p_c(x_1, \ldots, x_N) - \frac{\epsilon}{N\bar{c}}\sum_{j=1}^{N} c_j\kappa_j + \mathcal{O}(\epsilon^2\log\epsilon)\right).$$

Here $p_c(x_1, \ldots, x_N)$ is a 'repulsive potential' depending on the specific trap arrangement [1].

MATLAB Code for Brownian Simulations

- A MATLAB code was developed that can be to trace the trajectories of Brownian particles starting from a given point up to their escape through a trap
- Code takes into account boundary reflections.
- Parameters were chosen to match asymptotic MFPT results.

Parameters used in the code

- $d\tau = 6 \times 10^{-6}$
- $D = 1$
- $|d\vec{x}| \approx 0.0067$
- $\epsilon = 10^{-2}$

Specification of the machine used to run the simulations

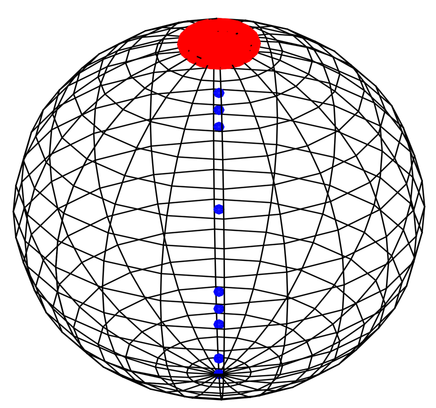
- OS: Red Hat Linux 7.6
- Memory: 128 GiB
- Processor: Intel Xeon(R) E5-2687W (3.10GHz) $\times$ 16

Comparison Between Asymptotic and Numerical Simulation Results

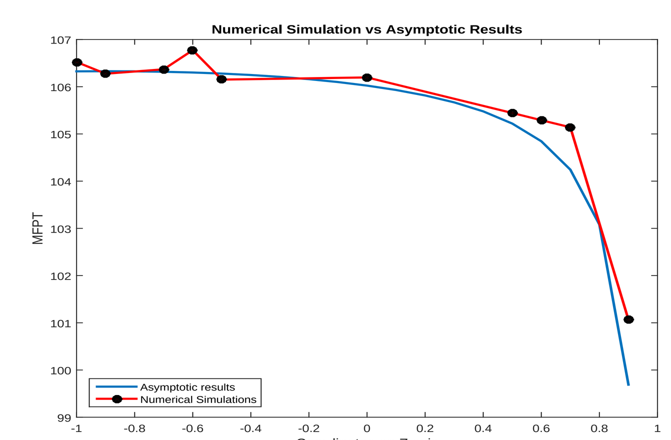
- Brownian numerically simulated MFPT is computed by averaging escape times of N Brownian particles launched from the same starting point:

$$v_N^B(x) = \frac{1}{N}\sum_{i=1}^{N} v_i.$$

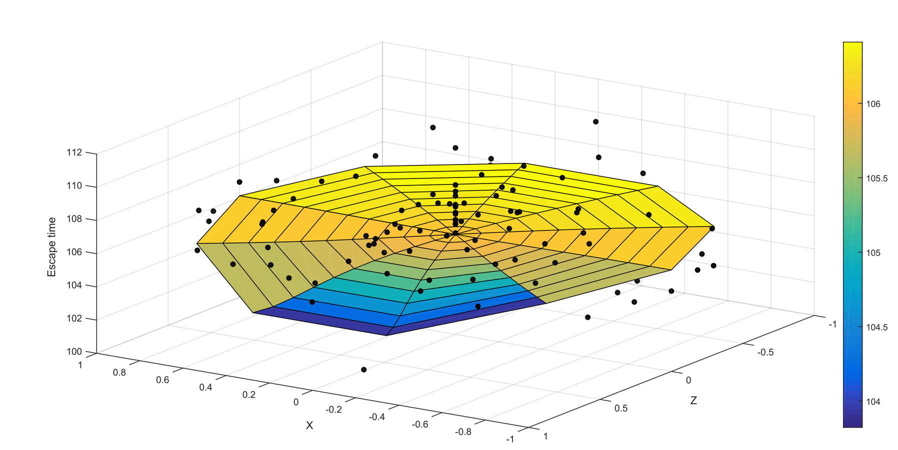
- Number of Brownian particle runs from each starting location: $N = 20000$



Brownian particle starting locations



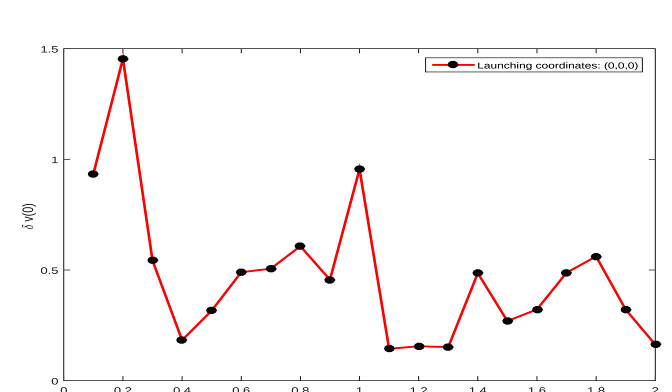
Averaged Escape Time vs Asymptotic Results for Trajectories of 4000 Brownian Particles Launched from Various Points (ϕ, r), $0 \le \phi \le 2\pi$ & $0 \le r \le 1$ with Each Tuple Specifying a Point of Launch



Averaged Brownian Escape Times: the Effect of Number of Launches

- N Brownian particles are launched from the sphere center, $v_N^B(0)$ is computed, and compared with the asymptotic value of $v(0)$:

$$\delta v(0) = \frac{|v(0) - v_N^B(0)|}{v(0)} \times 100\%.$$



Coordinates	Percentage Difference (%)
(0,0,-1)	0.0957
(0,0,-0.9)	0.1346
(0,0,-0.7)	0.0229
(0,0,-0.6)	0.0379
(0,0,-0.5)	0.1775
(0,0,0)	0.1633
(0,0,0.5)	0.3419
(0,0,0.6)	0.6044
(0,0,0.7)	1.1117
(0,0,0.9)	1.1960

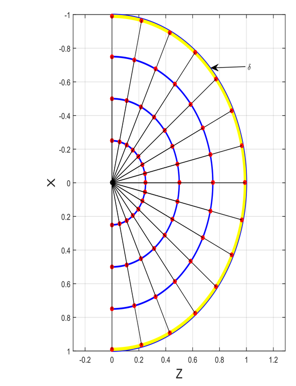
Table: Relative percentage error difference with respect to various launching coordinates for 20000 Brownian trajectories

Dynamics of Brownian Particle Near the Boundary

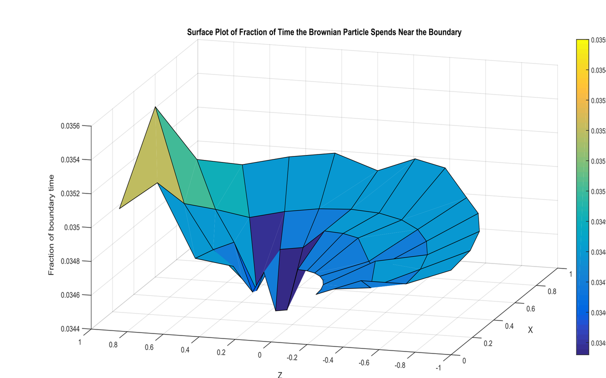
- **Observation**: MFPT PDE problem doesn't retain any information about the 3D trajectories of the Brownian particles.
- 4000 particles are launched from various positions inside the domain
- Parameters for tracing Brownian Dynamics near the boundary: $R = 1, \delta = 10^{-2}$.
- Relative time at the boundary, τ:

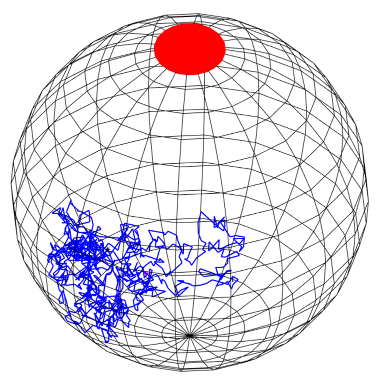
$$\tau = \frac{T_\delta}{T} \sim 3.5$$

- τ, when expressed as function of launching coordinates, $\tau(\phi, R)$ is constant $\sim 3.5\%$ throughout the domain.



Launching coordinates for the Brownian trajectories





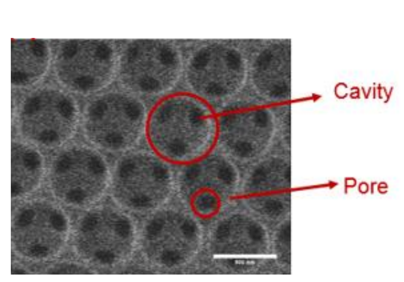
Dynamics of Brownian particle near the boundary

Discussion

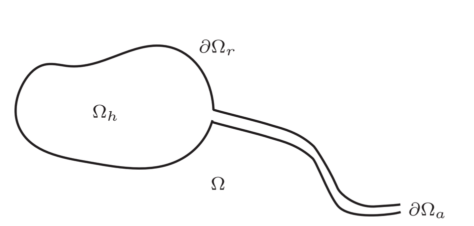
- Efficient and flexible, fully parallelized Matlab code was developed and tested; can be applied to study diffusion processes/average values as well as multiple other statistical characteristics for Brownian motion-based diffusion processes.
- A comprehensive study of results obtained from the simulations showed that averages of $\sim 10^4$ single-particle simulations are sufficient to closely match the asymptotic MFPT values for the unit sphere.
- Time spent by Brownian particles near the boundary was studied; it was shown that irrespective of initial conditions and relative trap location, time near the domain boundary remains about 3.5% of the particle's life time.

What Next

1. The developed code can be used to study the dynamics of Brownian particles in any 3D domains, for instance:
   - nanoparticle diffusion within **inverse opals** [2] and related man-made materials with cavities;
   - **domains with long necks** [3].
2. The code may be further optimized, and possibly improved by taking into account particle velocity-based simulated Brownian motion.



3D Model for Inverse Opal Structure.



Geometry Model of a Dendritic Spine with a Non-Straight Long Spine Neck, where Ω is the domain with a long neck, $\partial\Omega_r$ is the reflection Part, and $\partial\Omega_a$ is the absorbing Part.

References

[1] Cheviakov, A., Ward, M., Straube, R. (2010). An Asymptotic Analysis of the Mean First Passage Time for Narrow Escape Problems: Part II: The Sphere. Multiscale Modeling Simulation, 8(3), 836-870.

[2] Skaug, M., Wang, L., Ding, Y., Schwartz, D. (2015). Hindered Nanoparticle Diffusion and Void Accessibility in a Three-Dimensional Porous Medium. ACS Nano, 9(2), 2148-2156.

[3] Xiaofei Li. (2014). Matched asymptotic analysis to solve the narrow escape problem in a domain with a long neck. Journal Of Physics A: Mathematical And Theoretical, 47(50), 505202.