Exact Solutions of Ideal Plasma Equilibria

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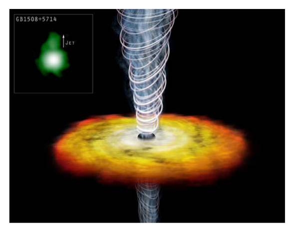
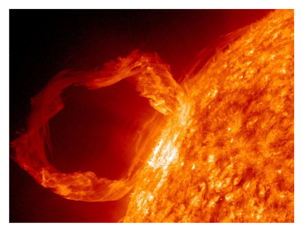
The Plasma Equilibrium Problem

Magnetohydrodynamics (MHD) is the description of the macroscopic behavior of plasmas. To describe the physical principles considered, the system of plasma equilibrium equations is used. Using this model, analytical solutions will be examined in order to better understand this "fourth state of matter".

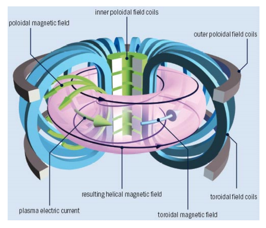
Physical Applications



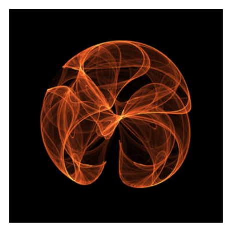
Astrophysical Jets



Solar Flares



Plasma confinement in Tokamak



Ball lightning

The Mathematical Model

The *System of isotropic ideal MHD equilibrium equations* is simply a combination of the Navier-Stokes equations in addition to Maxwell's equations (without the time dependence) and consists of the following:

$$\rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} = 0, \quad (1)$$

$$\operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0, \quad (2)$$

$$\operatorname{div} \mathbf{B} = 0, \quad (3)$$

$$\operatorname{div} \rho \mathbf{V} = 0. \quad (4)$$

Here $\mathbf{B}$ is the magnetic field, $\mathbf{V}$ is the plasma velocity, ρ plasma density, and μ is the magnetic permeability of free space.

We consider incompressible plasma flows satisfying

$$\operatorname{div} \mathbf{V} = 0. \quad (5)$$

There is no general method of obtaining solutions to the above *nonlinear* equations. To simplify, certain assumptions are made:

- $\mathbf{V} = 0$
- Scalar pressure
- Viscosity is negligible
- Plasma is a perfect conductor

This allows us to obtain *the system plasma equilibrium equations*:

$$\boxed{\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P, \ \ \operatorname{div} \mathbf{B} = 0.} \quad (6)$$

Equilibrium Topologies of Isotropic MHD

The *magnetic field lines* of a given magnetic field $\mathbf{B}(\mathbf{r})$ are defined as parametric curves $(x(t), y(t), z(t))$ that are solutions of:

$$\frac{dx}{dt} = B_1(x,y,z), \ \ \frac{dy}{dt} = B_2(x,y,z), \ \ \frac{dz}{dt} = B_3(x,y,z). \quad (7)$$

The same way, *plasma streamlines* are defined as curves tangent to the plasma velocity vector field $\mathbf{V}(\mathbf{r})$ [3].

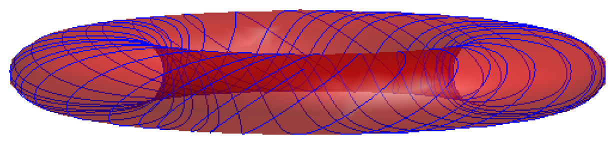
Consider $\operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0$ and $\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$. We have two cases:

1. $\mathbf{V} \neq 0$

Now $\operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0 \Rightarrow \mathbf{V} \times \mathbf{B} = \operatorname{grad} \Psi(\mathbf{r})$ which implies there exist magnetic surfaces on which $\mathbf{V}$ and $\mathbf{B}$ are tangent.

2. $\mathbf{V} = 0$

It follows from $\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$ that there again must exist surfaces of constant pressure (magnetic surfaces).



Magnetic surfaces form nested tori. Blue lines represent magnetic field lines.

As an additional note, the magnetic surfaces examined in this model are *generally* in the form of nested tori.

The Grad-Shafranov Equation

From the equations (6), one can obtain the *Grad-Shafranov equation* in cylindrical coordinates (r, ϕ, z) by imposing axial symmetry and independence of ϕ.

The *magnetic field* of the plasma has the form

$$\mathbf{B} = \frac{\Psi_z}{r} \mathbf{e_r} + \frac{I(\Psi)}{r} \mathbf{e}_\phi - \frac{\Psi_r}{r} \mathbf{e_z}.$$

The *Grad-Shafranov equation* reads

$$\boxed{\Psi_{rr} - \frac{\Psi_r}{r} + \Psi_{zz} + I(\Psi) I'(\Psi) = -\mu r^2 P'(\Psi).} \quad (8)$$

Here, $P(\Psi)$ is the plasma pressure and $I(\Psi)$ is an arbitrary function.

Lastly, one can see from (8) that surfaces $\Psi = const$ define magnetic surfaces.

Astrophysical Jets

In this application, $I(\Psi) = \alpha \Psi$ and $P(\Psi) = P_o - 2\beta^2 \Psi^2 / \mu$ where $P_o > (2\beta^2/\mu) \max(\Psi^2(r,z))$ and α and β are arbitrary constants. Under these assumptions, the Grad-Shafranov equation becomes linear:

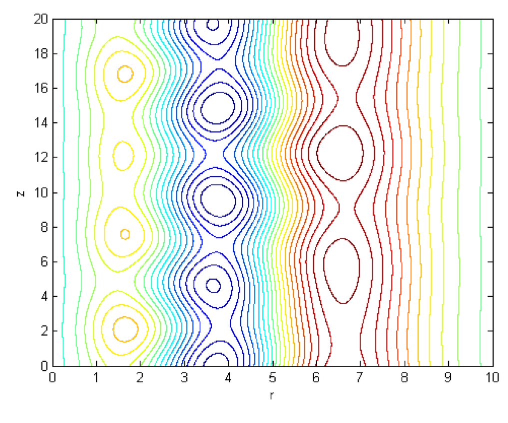
$$\Psi_{rr} - \frac{\Psi_r}{r} + \Psi_{zz} = -\alpha^2 \Psi + 4\beta r^2 \Psi. \quad (9)$$

Solutions to (9) are obtained for all of $\mathbb{R}^3$. Only solutions that have no singularities, are bounded for all z and tend to zero as $r \to \infty$ are considered.
And so the following exact solutions are obtained [2]:

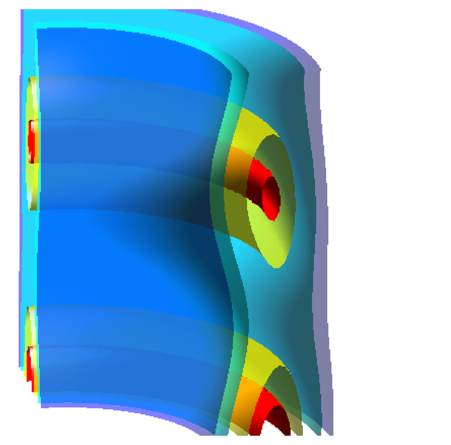
$$\Psi(r,z) = e^{-\beta r^2} \left(a_N L_N^*(2\beta r^2) + \sum_{n=1}^{N-1} (a_n \sin(\omega_n z) + b_n \cos(\omega_n z)) L_n^*(2\beta r^2) \right), \quad (10)$$

where $\omega_n = \sqrt{8\beta(N-n)}$ and the functions $L_n^*(x)$ are the generalized Laguerre polynomials:

$$L_n^*(x) = -\frac{x}{n!} e^x \frac{d^n}{dx^n} (e^{-x} x^{n-1}), \ \ L_0^*(x) = -1. \quad (11)$$



Quasi-periodic level curves $\Psi = const$ for $N = 3, \beta = 0.1, \alpha^2 = 24\beta$



Magnetic surfaces $\Psi = const.$ in the form of wavy cylinders and nested tori.

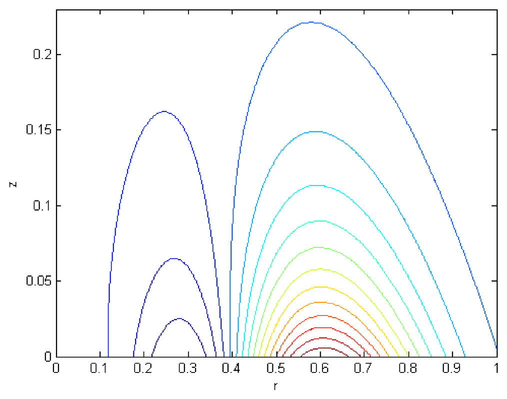
Solar Flares

When modeling solar flares, one can seek smooth solutions of equation (9) in the half space $z \geq 0$ which have no singularities and tend to zero both as $r \to \infty$ and as $z \to \infty$.

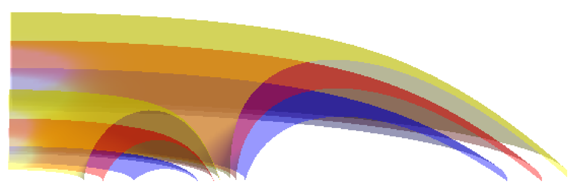
So for any α and $\beta > 0$, (9) has the exact solutions [2]:

$$\Psi(r,z) = e^{-\beta r^2} \sum_{n=N}^{N+m} c_n e^{-\kappa_n z} L_n^*(2\beta r^2) \quad (12)$$

where $N = \alpha^2/8\beta + 1$ and $\kappa_n = \sqrt{8\beta n - \alpha^2}$ and integer $m > 0$ is arbitrary.



Poloidal magnetic field lines for $N = 2, \beta = 10, \alpha = 0, m = 1$.



Cross section of magnetic surfaces.

The JFKO Equation

The *JFKO equation* (derived from (6)) describes helically symmetric plasma configurations. In cylindrical coordinates, solutions will depend on (r, ξ) where $\xi = z - \gamma\phi$. These solutions are invariant with respect to *the helical transformations*:

$$z \to z + \gamma h, \ \phi \to \phi + h, \ r \to r. \quad (13)$$

Using the above transformations, one can obtain the *JFKO equation*:

$$\boxed{\frac{\Psi_{\xi\xi}}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2 + \gamma^2} \Psi_r \right) + \frac{I(\Psi) I'(\Psi)}{r^2 + \gamma^2} + \frac{2\gamma I(\Psi)}{(r^2 + \gamma^2)^2} = -\mu P'(\Psi).} \quad (14)$$

The magnetic field is given by

$$\mathbf{B} = \frac{\Psi_\xi}{r} \mathbf{e_r} + B_1 \mathbf{e_z} + B_2 \mathbf{e}_\phi, \ \ B_1 = \frac{\gamma I(\Psi) - r\Psi_r}{r^2 + \gamma^2}, \ \ B_2 = \frac{r I(\Psi) + \gamma \Psi_r}{r^2 + \gamma^2} \quad (15)$$

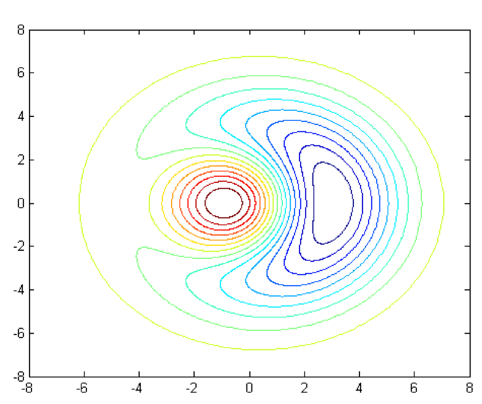
where $I(\Psi)$ and $P(\Psi)$ are arbitrary functions.
Assuming linear $I(\Psi)$ and quadratic $P(\Psi)$, Bogoyavlenskij [2] obtains the following exact solution to (14):

$$\Psi = e^{-\beta r^2} \left(a_N B_{0N}(x) + (a_{ln} \cos(l\xi/\gamma) + b_{ln} \sin(l\xi/\gamma)) r^l B_{ln}(x) \right), \quad (16)$$

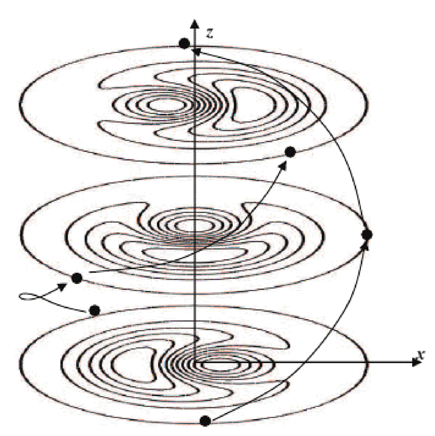
where $N, l \geq 1, n \geq 0, \ x = 2\beta r^2$ and

$$B_{ln}(x) = \frac{d^l}{dx^l} L_{l+n}(x) - k_{ln} x \frac{d^{l+1}}{dx^{l+1}} L_{l+n}(x). \quad (17)$$

Here, $L_m(x)$ are the Laguerre polynomials.



Section $z = 0$ of magnetic surfaces.



An illustration of helical symmetry

A Model of Ball Lightning as an Axially Symmetric Plasma Equilibrium

Ball lightning can be described as luminous spherical objects drifting through the air and vanishing either silently or with a bang [5]. To model this mysterious phenomenon as a plasma equilibrium, we consider the Grad-Shafranov equation in spherical coordinates (r, ϕ, θ):

$$\boxed{\left[\frac{\partial^2}{\partial r^2} + \frac{\sin(\theta)}{r^2} \frac{\partial}{\partial \theta} \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \right] + \lambda^2 \right] \Psi = \delta r^2 \sin^2(\theta)} \quad (18)$$

with conditions

$$\Delta \hat{\Psi} = 0 \text{ in } V^*, \quad \Psi = \hat{\Psi} = 0, \ \nabla \Psi = \nabla \hat{\Psi} \text{ on } \Gamma \quad (19)$$

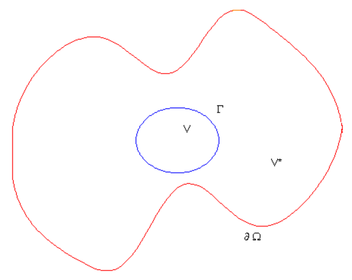


Diagram of plasma region V, vacuum region V^*, boundary Γ, and the outer wall $\partial\Omega$.

where $\hat{\Psi}$ is the function describing the region outside of V.

Solving (18) yields

$$\Psi = C W(r) \sin^2(\theta) \quad (20)$$

with

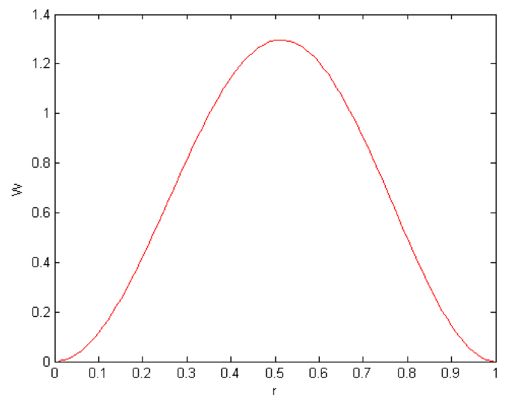
$$W(r) = \lambda r j_1(\lambda r) - \frac{j_1(\lambda R)}{\lambda R} (\lambda r)^2 \quad (21)$$

where j_1 is the spherical Bessel function of the first kind with n=1, and

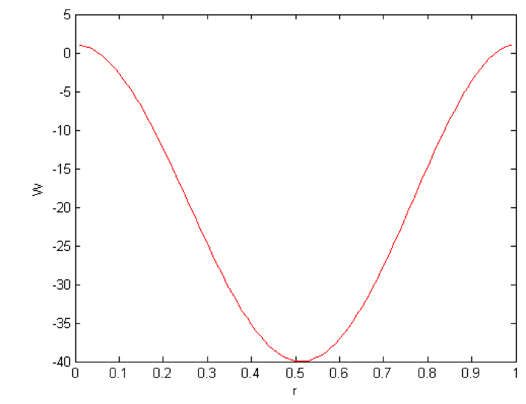
$$C = -\delta \frac{\lambda R}{\lambda^4 j_1(\lambda R)}. \quad (22)$$

Finally, *the magnetic field components for the fireball solution* become

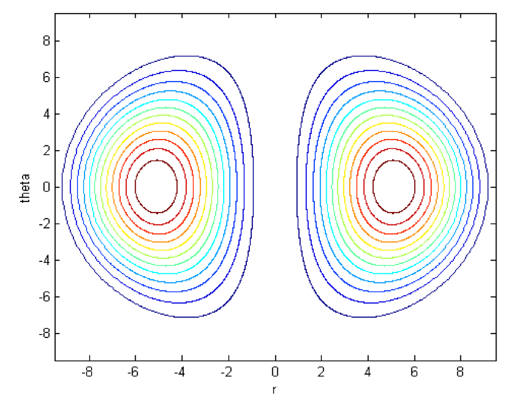
$$B_r = -2C \frac{W(r)}{r^2} \cos(\theta), \ \ B_\theta = C \frac{W'(r)}{r} \sin(\theta), \ \ B_\phi = C\lambda \frac{W(r)}{r} \sin(\theta) \quad (23)$$



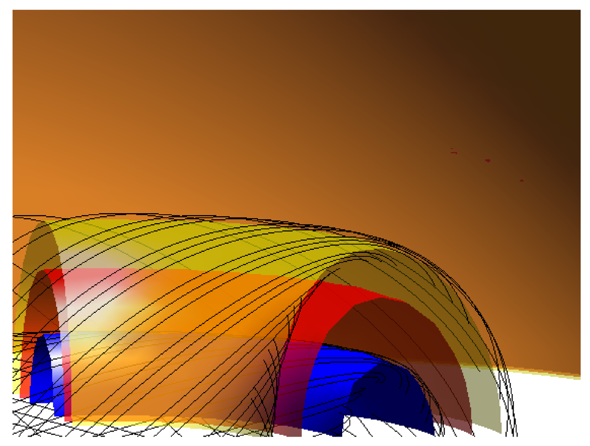
Radial dependence of Ψ



Radial dependence of pressure with $p_o = 1$.



Magnetic surfaces, $\lambda r = 5.76$, (first zero of j_1).



Cross section of magnetic surfaces. Black lines are magnetic field lines and orange surface is the plasma boundary.

Lastly, the *total magnetic energy* of the spherical plasma ball was calculated:

$$E = \iiint_{ball} \frac{B^2}{2} \, dV. \quad (24)$$

The above integral was evaluated and the following expression was found:

$$E = \frac{4\pi C^2}{3} \left(\frac{-\sin^2(\lambda R)}{R^2 \lambda^2} + \frac{1}{5} R^3 \lambda^4 j_1(\lambda R)^2 + \frac{(\lambda R)^2 - 1}{R} - 2R\lambda^2 j_1(\lambda R)(4 j_1(\lambda R) + \sin(\lambda R)) + 2\frac{\sin(\lambda R)\cos(\lambda R)}{R^2 \lambda} \right) \quad (25)$$

Conclusions

- Although the system of plasma equilibrium equations is simple-looking, this simplicity is deceptive. These equations comprise a complicated nonlinear system of partial differential equations with a nontrivial intrinsic geometry.
- Once axial or helical symmetry is imposed, the system simplifies significantly, to yield a single equation. Under certain conditions, this equation can become linear, and hence can be solved by conventional methods (i.e. separation of variables, integral transforms, etc.).

References

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