Exact Axially and Helically Symmetric Magnetohydrodynamic Equilibria

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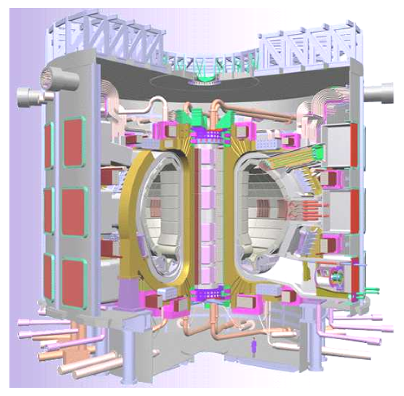
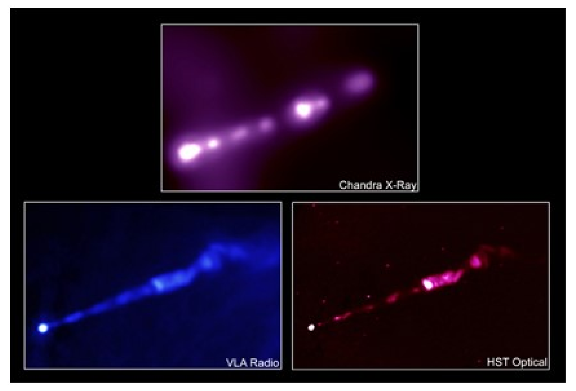
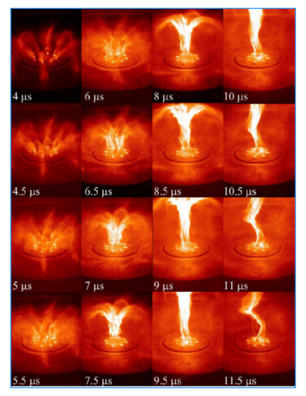
Applications of Magnetohydrodynamic



Plasma confinement in a tokamak



Astrophysical jets



Helical lab plasma jets

The Mathematical Model – Magnetohydrodynamics Equations

The **system of Isotropic Magnetohydrodynamic (MHD) equations** takes the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0,$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}), \quad \operatorname{div} \mathbf{B} = 0, \quad \mathbf{J} = \frac{1}{\mu} \operatorname{curl} \mathbf{B}.$$

$\mathbf{V}$: plasma velocity, $\mathbf{B}$: magnetic field, $\mathbf{J}$: current density, ρ: mass density, μ: magnetic permeability

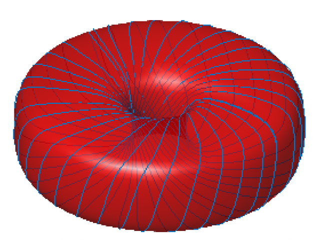
- An equation of state – incompressible plasmas: $\operatorname{div} \mathbf{V} = 0$.

Static Plasma Equilibrium Equations

- Equilibrium reduction: $\partial/\partial t = 0$.
- Static equilibrium reduction: $\mathbf{V} = 0$.
- Static plasma equilibrium equations:

$$\boxed{\operatorname{curl} \mathbf{B} \times \mathbf{B} = \operatorname{grad} P, \quad \operatorname{div} \mathbf{B} = 0.}$$

- Magnetic field lines of $\mathbf{B}(\mathbf{r})$ are defined as parametric curves $\mathbf{r}(s)$ solving $d\mathbf{r}/d\mathbf{s} = \mathbf{B}(\mathbf{r})$.
- For equilibrium MHD, magnetic field lines are tangent to magnetic surfaces $\psi = \text{const}$.
- Bounded magnetic surfaces are commonly tori.



Magnetic field lines on a magnetic surface

Axially Symmetric Plasma

- Use cylindrical coordinates (r, ϕ, z) and impose axial symmetry $\partial/\partial\phi = 0$.
- The static MHD equations collapse into a single PDE called the **Grad-Shafranov Equation**

$$\boxed{\Psi_{rr} - \frac{\Psi_r}{r} + \Psi_{zz} + I(\Psi)I'(\Psi) = -r^2 P'(\Psi).} \quad (1)$$

- Magnetic field and pressure:

$$\mathbf{B} = \frac{\Psi_z}{r}\mathbf{e_r} + \frac{I(\Psi)}{r}\mathbf{e}_\phi - \frac{\Psi_r}{r}\mathbf{e_z}, \quad P = P(\Psi), \quad (2)$$

were $I(\psi)$ and $P(\psi)$ are an arbitrary functions.

Exact Axially Symmetric Equilibrium Solutions: First Family

- For $I(\Psi) = \alpha\Psi$ and $P(\Psi) = P_0 - q^2\Psi^2$, (1) becomes linear.
- After separation of variables one arrives at separated solutions

$$\Psi^\omega(r,z) = (C_1 W_M(\eta, 1/2, qr^2) + C_2 W_W(\eta, 1/2, qr^2)) \sin(\omega z + C_3)$$

Where ω, q, C_i =const, and W_M, W_W are basis solutions of the **Whittaker ODE**

$$y''(s) + \left(-\frac{1}{4} + \frac{\eta}{s} + \frac{1/4 - \nu^2}{s^2}\right) y(s) = 0, \quad \nu = \frac{1}{2}, \quad \eta = \frac{\omega^2 - \alpha^2}{4q}.$$

- A general solution of (1) includes all linear combinations of particular solutions $\Psi(r,z)$, such as

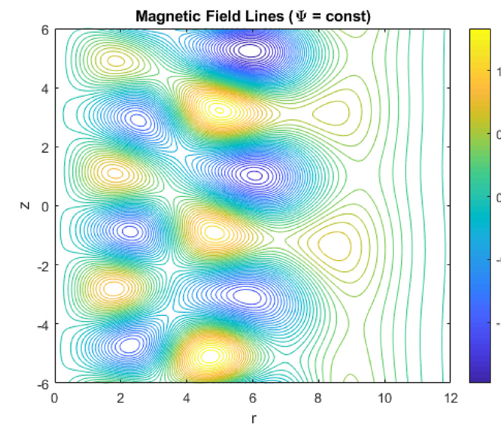
$$\Psi(r,z) = \int_{-\infty}^{\infty} C(\omega) W_M\left(\frac{\omega^2 - \alpha^2}{4q}, \frac{1}{2}, qr^2\right) \sin \omega z \, d\omega.$$

Exact Axially Symmetric Plasma Physical solutions: First Family

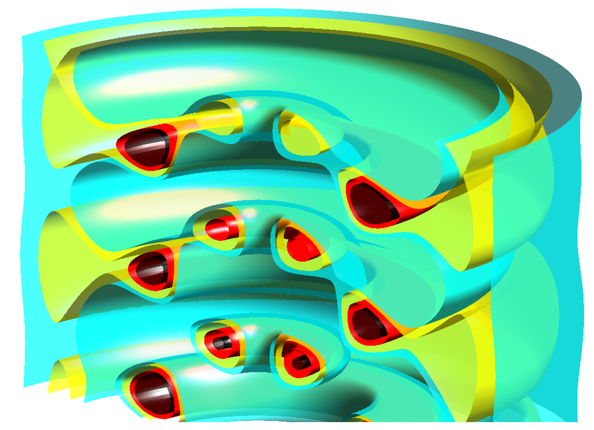
For integer values of η, Whittaker functions become polynomials multiplied by a Gaussian term [?]. These polynomials are related to the Laguerre polynomials. Exact solutions can be written as [?]

$$\Psi(r,z) = e^{-\beta r^2}\left(a_N L_N^*(2\beta r^2) + \sum_{n=1}^{N-1} (a_n \sin(\omega_n z + b_n)) L_n^*(2\beta r^2)\right),$$

where $\beta = q/\sqrt{2}$, L_n^* are the primitives of the Laguerre polynomials, and $\omega_n = \sqrt{8\beta(N-n)}$.



Quasi-periodic level curves $\Psi = \text{const}$ for $N = 3, \beta = 0.1, \alpha^2 = 24\beta$.



Magnetic surfaces in the form of wavy cylinders and nested tori.

Exact Axially Symmetric Equilibrium Solutions: Second Family

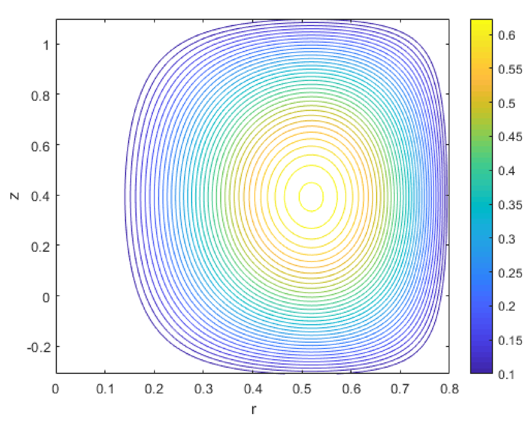
- For $I(\Psi) = \alpha\Psi$ and $P(\Psi) = P_0 + q^2\Psi^2$, (1) is also linear; one has separated solutions

$$\Psi^\omega(r,z) = (C_1 \mathcal{C}_F(0, \eta, qr^2) + C_2 \mathcal{C}_G(0, \eta, qr^2)) \sin(\omega z + C_3).$$

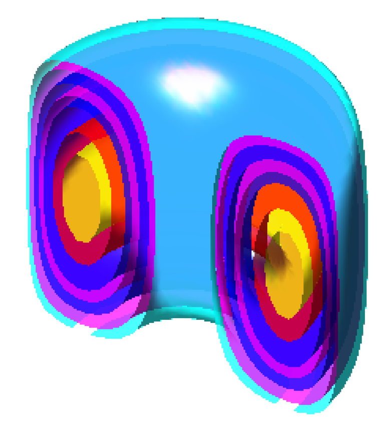
Here q, ω, and C_i = const, and $\mathcal{C}_F(L, \eta, x)$ and $\mathcal{C}_G(L, \eta, x)$ are Coulomb wave functions, the basis solutions of the Coulomb ODE

$$y''(x) + \left(1 - \frac{2\eta}{x} - \frac{L(L+1)}{x^2}\right) y(x) = 0,$$

where $L = 0$, and η is the same as before. General solution $\Psi(r,z)$ includes discrete and continuous combinations of $\Psi^\omega(r,z)$.



Sample level curves $\Psi = \text{const}$, $\omega = 2$, $C1 = \sqrt{2}$, $C2 = 0$, $C_3 = \pi/4$, $\alpha^2 = 25$, $q = \sqrt{3}$



Magnetic Surface

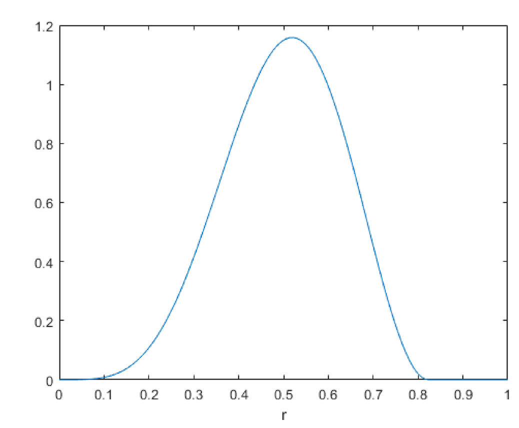
Positive-Pressure Axially Symmetric Plasma Configurations

- In order for plasma region surrounded by vacuum to have **positive pressure** and a **finite amount of magnetic energy** in each z-layer, a solution may be truncated beyond a given magnetic surface.
- Maxwell's electrodynamic equations provide the new boundary conditions: a boundary current sheet and can be expressed as

$$\mathbf{B} \times \mathbf{n} = \mu_0 \mathbf{K},$$

and $\mathbf{K}$ is the current density along the boundary.

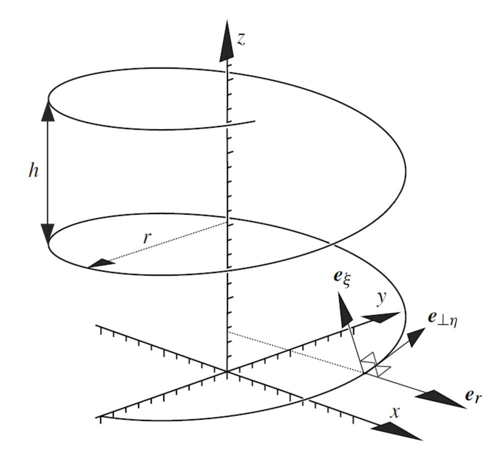
- Outside the plasma domain, $\mathbf{B} = \mathbf{P} = \mathbf{0}$.



Pressure in layer z = 0.46

Helically Symmetric Plasmas

- **Helical coordinates** (r, η, ξ) are defined using cylindricals: $\xi = az + b\phi$, $\eta = a\phi - bz/r^2$.
- MHD equations admit helical symmetry. Seek invariant solutions: $\partial/\partial\eta = 0$.
- The 4 PDEs collapse into the **JFKO equation** ($a = 1$, $b = -\gamma$):



$$\boxed{\frac{\Psi_{\xi\xi}}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2 + \gamma^2}\Psi_r\right) + \frac{I(\Psi)I'(\Psi)}{r^2 + \gamma^2} + \frac{2\gamma I(\Psi)}{(r^2 + \gamma^2)^2} = -\mu P'(\Psi).} \quad (3)$$

- Magnetic field and pressure: $\mathbf{B} = \frac{\Psi_\xi}{r}\mathbf{e}_r + \frac{I(\Psi)}{r}\mathbf{e}_\eta - \frac{\Psi_r}{r}\mathbf{e}_\xi$, $P = P(\Psi)$.

Exact Helically Symmetric Equilibrium Solutions: 1st family

For $I(\Psi) = \alpha\Psi$ and $P(\Psi) = P_0 - 2\beta^2\Psi^2$, (3) is **Linear**. After separation of variables:

$$\Psi(r,\xi) = e^{-r^2\beta}\left(C_1 r^b \mathcal{H}_C(a, b, -2, c, d, -r^2/\gamma^2) + C_2 r^{-b} \mathcal{H}_C(a, -b, -2, c, d, -r^2/\gamma^2)\right) \sin(\omega\xi + C_3),$$

where $a = 2\beta\gamma^2$, $b = \gamma\omega$, $c = \gamma^2(4\beta^2\gamma^2 - \alpha^2 + \omega^2)/4$, $d = 1 - \beta^2\gamma^4 + (\alpha^2 - \omega^2)\gamma^2/4 + \alpha\gamma/2$,
and α, β, ω, γ, C_i = const and $\mathcal{H}_C(a, b, 2, c, d, x)$ is a basis solution of the **confluent Heun ODE**

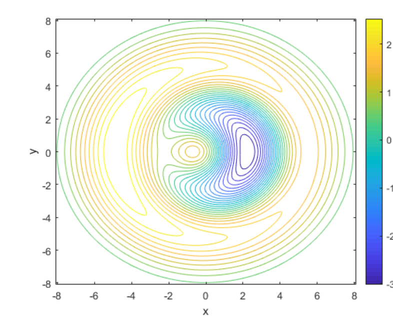
$$y'' - \frac{(-x^2 a + (-b + a - 4)x + b + 1)}{x(x-1)} y' - \frac{(((-b-4)a - 2c)x + (b+1)a - 3b - 2d - 2)}{2x(x-1)} y = 0.$$

Examples of Solutions: First Family

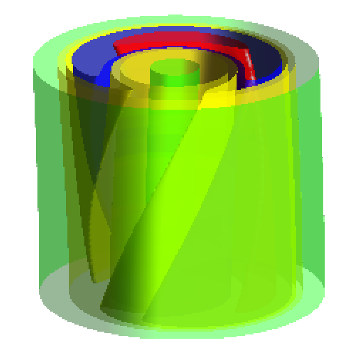
The confluent Heun functions with the 'right' parameters produce polynomials which again can be expressed in terms of Laguerre polynomials [?]. A solution can then be written as a linear combination of separated forms

$$\Psi_{mn}(r,\xi) = e^{-\beta r^2}\left(a_N B_{0N}(2\beta r^2) + r^m B_{mn}(2\beta r^2)(a_{mn} \sin(m\frac{\xi}{\gamma} + b_{mn})\right),$$

where $N, n, m \in \mathbb{N}$, $2N > 2n + m$, and $B_{mn}(x) = \frac{d^m}{dx^m} L_{m+n}(x) - \frac{m + 2\beta\gamma^2 - \alpha\gamma}{4\beta\gamma^2 n} x \frac{d^{m+1}}{dx^{m+1}} L_{m+n}(x)$..



Sample level curves $\Psi = \text{const}$

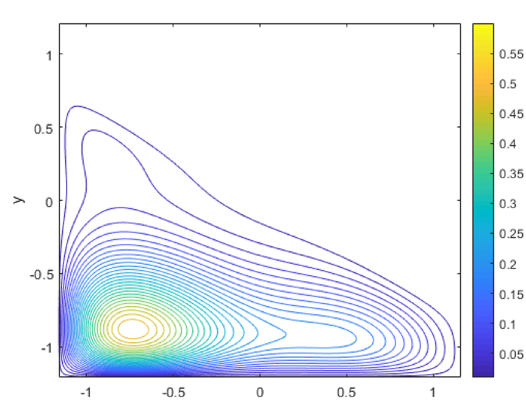


Exact helically Symmetric Plasma: 2nd family

Again, the linear case of JFKO (3) with the positive pressure is obtained when $I(\Psi) = \alpha\Psi$, $P(\Psi) = P_0 + 2\beta^2\Psi^2$. In particular, one obtains real separated solutions

$$\Psi(r,\xi) = C_1 e^{-ir^2\beta} r^b \mathcal{H}_C(ia, b, -2, c, d, -r^2/\gamma^2) \sin(\omega\xi + C_2),$$

where $\mathcal{H}_C$ is the confluent Heun function, and α, β, ω, γ, $C_i \in \mathbb{R}$.



Sample level curves $\Psi = \text{const}$.

Conclusions

Summary

- **Exact solutions** of nonlinear physical equations like MHD are highly important; hard to obtain.
- In axially and helically symmetric reductions, the model is drastically modified to yield single PDEs (GS, JFKO).
- Even Linear cases of GS, JFKO correspond to nonlinear MHD equations. Using linear methods (separation of variables), one can find physically meaningful exact solutions corresponding to static plasma equilibria.
- **New results:**
  - use special functions (Whittacker, Coulomb, Heun) to extend the set of known exact solutions;
  - Find **regular closed-form exact solutions** for physical cases of plasma surrounded by vacuum, with **positive internal pressure**. This is achieved using boundary conditions with current sheets.

Future Work

- Use **Galas-Bogoyavlenskij transformations** [?] to obtain exact dynamic localized plasma equilibria ($\mathbf{V} \neq \mathbf{0}$) from new solutions.
- Analyze stability of the new exact solutions.
- Take into account other effects including anisotropy, viscosity and conductivity in non-ideal plasmas.

References

[1] D. Biskamp, *Nonlinear Magnetohydrodynamics*. (Cambridge University Press, United Kingdom, 1993).

[2] O. I. Bogoyavlenskij, *Exact Global Plasma Equilibria*. Mathematical Preprint #1999-2, (Kingston, Ontario, Canada, 1999).