Mathematical Modelling of the Dynamics of a Viscous Fluid with Gas Bubbles

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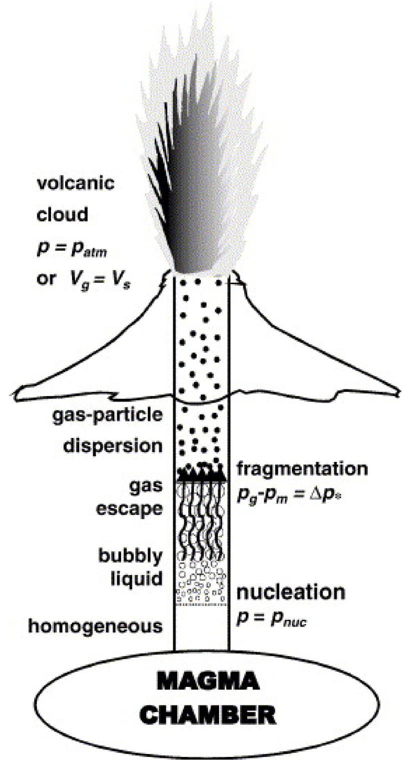
Motivation & Application Areas

Bubble flow: a two-phase flow; small bubbles are dispersed or suspended in liquid continuum.

- General interest: bubbles change flow dynamics by increasing or decreasing local turbulence.
- A specific application: bubble regime of laminar magma flow in a volcanic conduit.

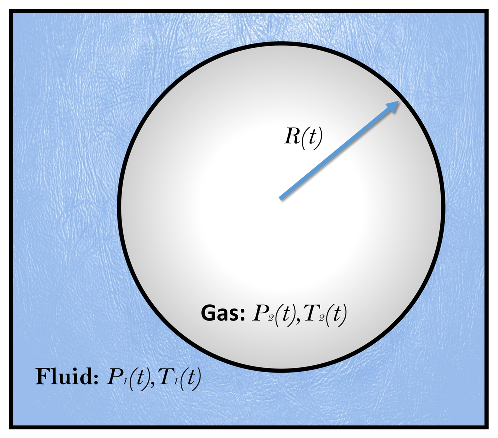
Retrieved July 27, 2015 from http://www.pitt.edu/ cejones/GeoImages/

Vesicular basalt; retrieved July 27, 2015 from http://www.sciencebuzz.org/buzz-tags/magmatic-differentiation



A schematic of magma flow regimes in a volcanic conduit [2].

Dynamics of a Single Gas Bubble



$R(t)$ is the radius of the bubble; $P=P_1(t)$, $P_2(t)$ denote the pressure of fluid and gas; $T_1(t)$, $T_2(t)$, the temperature of fluid and gas; P_0, T_0 and R_0 are the initial values of gas pressure, temperature, and the gas bubble radius; ρ_1 is the mass density of the liquid; ν is the fluid kinematic viscosity coefficient; γ is the adiabatic constant; χ is the gas thermal conduction coefficient; Nu is the dimensionless Nusselt number (relative thickness of the thermal layer).

Gas bubble dynamics equations:

- The Rayleigh equation: single bubble radius dynamics [3]

$$P_2 = P + \rho_1\left(RR_{tt} + \frac{3}{2}R_t^2 + \frac{4\nu}{3R}R_t\right). \tag{1a}$$

- Pressure equation, from conservation of energy and assumption that Nusselt number is constant:

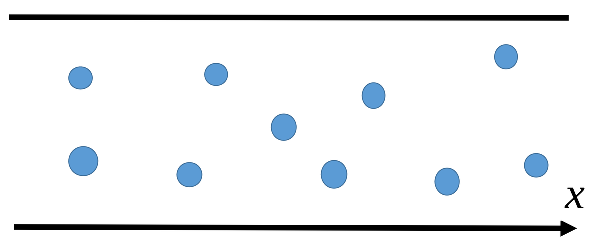
$$(P_2)_t + \frac{3\gamma P_2}{R}R_t + \frac{3\chi\,\mathrm{Nu}\,(\gamma-1)}{2R^2}(T_2 - T_1) = 0, \qquad \mathrm{Nu} = \text{const}. \tag{1b}$$

- Ideal gas relationship:

$$T_2 = \frac{T_0 P_2}{P_0}\left(\frac{R}{R_0}\right)^3. \tag{1c}$$

- Derivatives are denote by subscripts: $\partial P_2/\partial t = (P_2)_t$, etc.

Dynamics of a Fluid with Gas Bubbles



Main assumptions:

- Quantities depend on space and time ($R=R(x,t)$, $P=P(x,t)$, etc.).
- Fluid temperature is constant: $T_1 = T_0$.
- The bubble radius is a small deviation from its average:

$$R(x,t) = R_0 + \eta(x,t). \tag{2}$$

From these assumptions and (1), the following PDE is obtained (cf. [1]):

$$\begin{aligned} P - P_0 &+ \frac{1}{R_0}\eta P + \frac{3\gamma\kappa}{R_0}P\eta_t + \kappa P_t + \frac{\rho_1(3R_0^2+4\nu\kappa)}{3R_0}\eta_{tt} + \frac{\rho_l(6R_0^2-4\nu\kappa)}{3R_0^2}\eta\eta_{tt} \\ &+ \frac{\rho_1(8\nu\kappa(3\gamma-1)+9R_0^2)}{6R_0^2}\eta_t^2 + \frac{4\nu\rho_1}{3R_0}\eta_t + \frac{2P_0}{R_0}\eta - \frac{3P_0}{R_0^2}\eta^2 \\ &+ \rho_1\kappa R_0\eta_{ttt} - \frac{3\kappa\gamma}{R_0^2}\eta\eta_t P + \rho_1\kappa\eta\eta_{ttt} + \rho_l\kappa(3\gamma+4)\eta_t\eta_{tt} = 0, \end{aligned} \tag{3}$$

where

$$\kappa = \text{const} = \frac{2R_0^2P_0}{3\chi\,\mathrm{Nu}\,(\gamma-1)T_0}.$$

Density of the fluid mixture:

$$\rho = \frac{\rho_1}{1 - X + \widehat{V}\rho_1}, \tag{4}$$

where $\widehat{V}$ is the relative volume of gas (gas volume per unit mass of mixture) and X is the relative mass content of gas (mass of gas per unit mass of mixture).

Fundamental equations of fluid dynamics:

The standard Euler equations that describe fluid dynamics are also used:

$$\rho_t + (\rho u)_x = 0, \tag{5}$$

$$\rho(u_t + uu_x) + P_x = 0. \tag{6}$$

A Dimensionless Model for the Fluid with Gas Bubbles

We recast the model (2) – (6) into the dimensionless form, using the substitution

$$t = \frac{\ell}{c_0}t', \quad x = \ell x', \quad u = c_0u', \quad \eta = R_0\eta', \quad P = P_0P' + P_0, \tag{7}$$

where ℓ is the characteristic wavelength, and $c_0 = \sqrt{\frac{3P_0}{\mu R_0}}$ is the characteristic wave speed. The primes on the new variables will be dropped for convenience.

Making the substitutions, we arrive at the following **dimensionless model for a fluid with gas bubbles:**

$$\eta_t - \frac{\rho_0}{\mu R_0}u_x + u\eta_x + \eta u_x - \frac{2\mu_1R_0}{\mu}\eta\eta_t - \frac{2R_0\mu_1}{\mu}u\eta\eta_x - \frac{R_0\mu_1}{\mu}u_x\eta^2 = 0, \tag{8a}$$

$$-\frac{\rho_0}{\mu R_0}(u_t + uu_x) + \eta u_t - \frac{1}{3}P_x + \eta uu_x - \frac{R_0\mu_1}{\mu}\eta^2u_t - \frac{R_0\mu_1}{\mu}\eta^2uu_x = 0, \tag{8b}$$

$$\begin{aligned} P &+ \kappa_1P_t + \eta P + 3\gamma\kappa_1\eta_tP + (\beta_1+\beta_2)\eta_{tt} + (2\beta_2-\beta_1)\eta\eta_{tt} + \left((3\gamma-1)\beta_1 + \tfrac{3}{2}\beta_2\right)\eta_t^2 \\ &+ \lambda\eta_t + 3\eta - 3\eta^2 + \omega\eta_{ttt} + \omega\eta\eta_{ttt} + \omega(3\gamma+4)\eta_t\eta_{tt} - 3\kappa_1\gamma\eta\eta_tP - 3\kappa_1\gamma\eta\eta_t = 0, \end{aligned} \tag{8c}$$

where

$$\beta_1 = \frac{4\nu\kappa\rho_lc_0^2}{3P_0l^2}, \quad \beta_2 = \frac{\rho_lc_0^2R_0^2}{P_0l^2}, \quad \kappa_1 = \frac{\kappa c_0}{l}, \quad \omega = \kappa_1\beta_2, \quad \lambda = \frac{\beta_1}{\kappa_1} + 3\gamma\kappa_1,$$

$$\rho_0 = \frac{\rho_1}{1 - X_0 + V_0\rho_1}, \quad \mu = \frac{3\rho_1^2V_0}{R_0(1 - X_0 + V_0\rho_1)^2}, \quad \mu_1 = \frac{6\rho_1^2V_0(2\rho_1V_0 - 1 + X_0)}{R_0^2(1 - X_0 + V_0\rho_1)^3},$$

and V_0, X_0 are the average relative volume and mass content of gas.

An Asymptotic Approximation

Rescale the independent variables:

$$\xi = \varepsilon^m(x - t), \quad \tau = \varepsilon^{m+1}t, \quad 0 < \varepsilon \ll 1, \quad m > 0. \tag{9}$$

- ξ: a large-scale moving spatial variable ($\xi \sim 1$ when $x \sim \varepsilon^{-m} \gg 1$)
- τ: 'slow time'.

Assume a standard asymptotic expansion of flow parameters near the equilibrium:

$$u = \varepsilon u_1 + \varepsilon^2u_2, \quad \eta = \varepsilon\eta_1 + \varepsilon^2\eta_2, \quad P = \varepsilon P_1 + \varepsilon^2P_2. \tag{10}$$

- (9), (10) are substituted in (8).
- Various m can be chosen.
- Coefficients at different powers of ε must vanish independently.
- Obtain **a single PDE on** $P_1(\xi,\tau)$: a scaled, dimensionless pressure perturbation.

Case A: $m=1$

In this case, we arrive at the **Burgers' equation**

$$P_{1\tau} + A_1P_1P_{1\xi} + B_1P_{1\xi\xi} = 0, \tag{11}$$

where

$$A_1 = \frac{1}{3}\left(\frac{\mu R_0}{\rho_0} - \frac{\mu_1R_0}{\mu} + 2\right), \quad B_1 = \left(\frac{\kappa_1}{2} - \frac{\kappa_1\gamma}{2} - \frac{1}{6}\frac{\beta_1}{\kappa_1}\right).$$

A change of variables

$$\xi = \left(\frac{B_1^2}{A_1}\right)^{\frac{1}{3}}x, \quad \tau = -\left(\frac{B_1}{A_1^2}\right)^{\frac{1}{3}}t, \quad P_1(\xi,\tau) = -\left(\frac{B_1}{A_1^2}\right)^{\frac{1}{3}}u(x,t) \tag{12}$$

maps (11) into the standard form

$$u_t + uu_x = u_{xx}. \tag{13}$$

Case B: $m=\frac{1}{2}$

In this case, we arrive at the PDEs

$$P_{1\tau} + A_2P_1P_{1\xi} + B_2P_{1\xi\xi\xi} = 0, \qquad C_2P_{1\xi} = 0, \tag{14}$$

where

$$A_2 = \frac{1}{3}\left(\frac{\mu R_0}{\rho_0} - \frac{\mu_1R_0}{\mu} + 2\right), \quad B_2 = \frac{1}{6}(\beta_1 + \beta_2), \quad C_2 = \left(3\gamma\kappa_1^2 - 3\kappa_1^2 + \beta_1\right). \tag{15}$$

When $C_2 = 0$, a change of variables

$$\xi = \left(\frac{6B_2^2}{A_2}\right)^{\frac{1}{5}}x, \quad \tau = \left(\frac{216B_2}{A_2^3}\right)^{\frac{1}{5}}t, \quad P_1(\xi,\tau) = \left(\frac{216B_2}{A_2^3}\right)^{\frac{1}{5}}u(x,t) \tag{16}$$

maps (14) into a canonical **Korteweg - de Vries (KdV) equation**

$$u_t + 6uu_x + u_{xxx} = 0. \tag{17}$$

Other Cases and Further Work

Physical motivations were not considered in the choosing of m. Cases $m=1$ and $m=\frac{1}{2}$ work well with the Taylor expansion (10). Some other cases of m that were tried yielded results that were similar (differed by a term) to the cases $m=1$ and $m=\frac{1}{2}$, while other cases were degenerate.

In particular, in both Cases (A) and (B), a determining equation was found that gave the relationship between P_1 and η_1 as:

$$P_1 + 3\eta_1 = 0 \tag{18}$$

- Physically, η_1 is the dimensionless scaled radius perturbation of the gas bubbles. The meaning of the relationship (18) is that higher pressure leads to lower bubble radius.
- It remains to systematically study the case of general m and its compatibility with more general asymptotic expansions (10), which will possibly lead to more general nonlinear PDE models.

Traveling Wave Solutions of the Burgers Equation

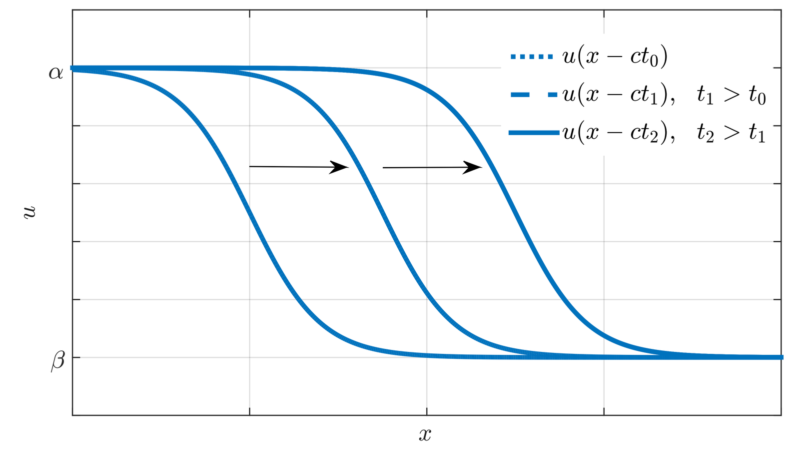
One can obtain particular solutions of (13) and (17) using the **traveling wave ansatz:** $u(x,t) = g(z)$ **where** $z = x - ct$. This approach is based on *symmetries of PDEs*, and is useful in many cases, since a PDE can be reduced to a much simpler ODE. For example, the Burgers equation reduces to the following ODE:

$$-cg'(z) + g(z)g'(z) - g''(z) = 0. \tag{19}$$

The latter can be solved by integrating twice:

$$u(x - ct) = g(z) = \frac{\alpha + \beta e^{\frac{1}{2}(\alpha-\beta)z}}{1 + e^{\frac{1}{2}(\alpha-\beta)z}}, \tag{20}$$

where α and β are arbitrary constants, and $c = \frac{1}{2}(\alpha + \beta)$. Assuming $\alpha > \beta$, one observes that α is the top of the waveform and β is its bottom.



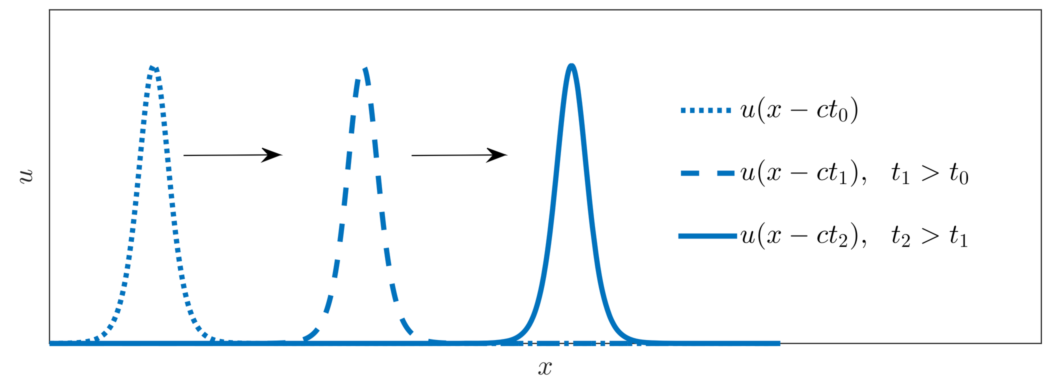
Traveling wave solution for the Burgers equation (13).

One-Soliton Solution of the KdV equation

We use travelling wave ansatz for the KdV, reducing it to an ODE. The latter admits the well-known solitary wave-type exact solutions

$$u(x - ct) = g(z) = \frac{c}{2\cosh^2\left(-\frac{\sqrt{c}}{2}z\right)}, \tag{21}$$

where c, the wave's speed, is an arbitrary constant. (21) is known as a **single-soliton travelling wave solution** for the KdV equation.



Travelling wave soliton solution of the KdV equation.

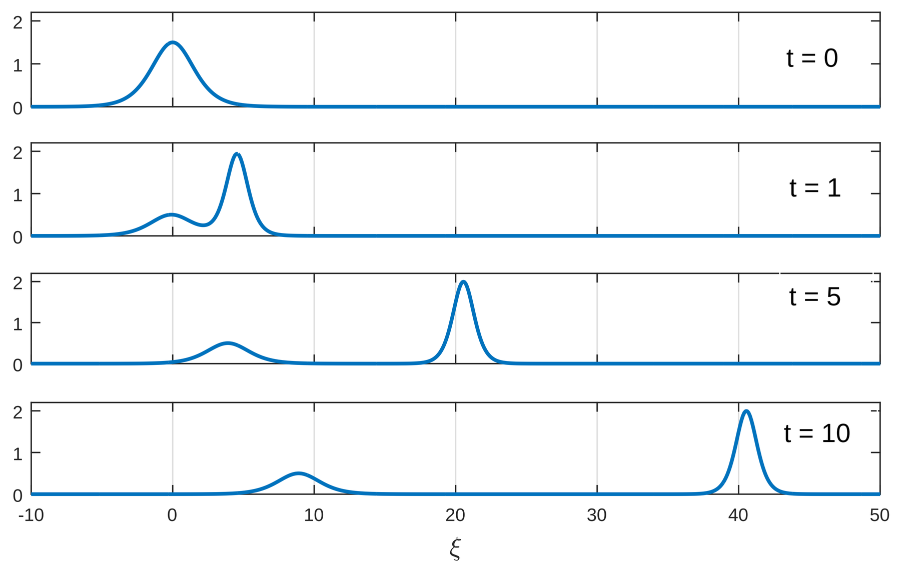
Multi-Soliton Solutions of the KdV equation

Multi-soliton solutions exist for (17); a two-soliton solution, for example, is given by

$$u(x,t) = -2\frac{\left(\cosh\left(k_1\left(4k_1^2t - x\right)\right)\right)^2k_1^2k_2^2 - \left(\cosh\left(k_1\left(4k_1^2t - x\right)\right)\right)^2k_2^4 + \left(\cosh\left(k_2\left(4k_2^2t - x\right)\right)\right)^2k_1^4 - \left(\cosh\left(k_2\left(4k_2^2t - x\right)\right)\right)^2k_1^2k_2^2 - k_1^4 + k_1^2k_2^2}{\left(\cosh\left(k_1\left(4k_1^2t - x\right)\right)k_2\cosh\left(k_2\left(4k_2^2t - x\right)\right) - \sinh\left(k_2\left(4k_2^2t - x\right)\right)k_1\sinh\left(k_1\left(4k_1^2t - x\right)\right)\right)^2}, \tag{22}$$

where k_1 and k_2 are constants.

- KdV is an **integrable equation.**
- Nonlinear solitary waves interact like particles.
- Higher waves have higher speeds. Increasing c in (21) will increase the wave's speed and height.



Evolution of a two soliton solution when $k_1 = \frac{1}{2}$ and $k_2 = 1$.

Conclusions & Future Work

Boat following a solitary wave
Retrieved Aug 4, 2015 from http://www.maplesoft.com/

- Solitons on water surface are straightforward to reproduce in shallow constant-depth channels.
- Stable solitary wave-type arise for various nonlinear models, including fluid dynamics, plasma physics, and nonlinear optics, and are observed in laboratory and numerical experiments.
- It has been shown that the classical Burgers' and KdV equations arise in the context of bubble flow dynamics.
- **Ongoing work:** consider more general asymptotic expansions of the form (9), (10) to model a wider range of physical situations for the bubble flows.

References

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