Shear Cylindrical Waves in the Framework of Finite Fiber-Reinforced Hyper- and Viscoelasticity: Applications to Blood Vessel Models

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Applications

Finite Elasticity has broad applications in the understanding and development of elastic materials, such as biological tissues (blood vessels, arteries, skin, etc.) as well as man made materials. It developed, largely, starting in the 1940's with the rubber industry, but also has significant biomedical applications [3].

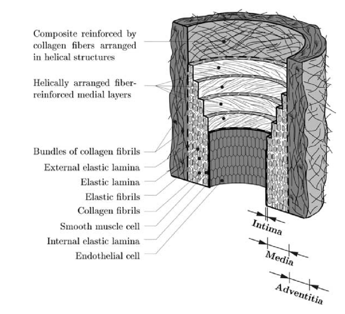
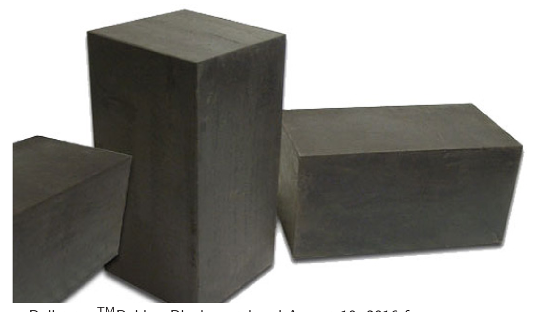


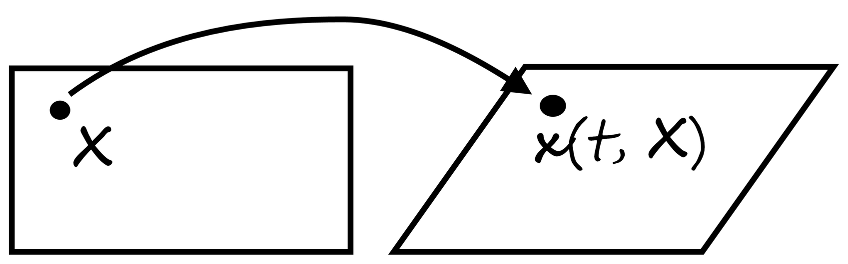
Diagram of an artery [4]



Bulletstop™ Rubber Blocks; retrieved August 10, 2016 from http://www.atsusa.biz/live-fire-shoothouses/ballistic-rubber-blocks.php

Incompressible Finite Elasticity

The field of finite elasticity provides a framework for analyzing large deformations of solid objects with elastic properties, such as arteries.



- $\mathbf{F}$: Deformation Gradient Tensor
- $\mathbf{P}$: Piola-Kirchoff Stress Tensor
- p: Hydrostatic Pressure
- ρ: Density
- W: Strain Energy Function

$$\mathbf{F} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial y}{\partial X} & \frac{\partial z}{\partial X} \\ \frac{\partial x}{\partial Y} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Y} \\ \frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} & \frac{\partial z}{\partial Z} \end{pmatrix}$$

$$\mathbf{P} = \rho_0 \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-T}$$

Equations of motion are given by:

$$\rho \mathbf{x}_{tt} = \nabla \mathbf{P}.$$

For incompressible materials, there is an additional condition:

$$J = det(\mathbf{F}) = 1.$$

To reduce the scope of the problem and examine specific types of motions, we can place restrictions on the class of deformations under consideration in the definitions of $x(t, X, Y, Z)$, $y(t, X, Y, Z)$, and $z(t, X, Y, Z)$.

Research Model

The goal of this project is to develop a model of how pulsing blood flow interacts with arteries.

Energy Models: We study deformations using two different forms of strain energy models.

Hyperelastic:

$$W_h = \underbrace{a(I_1-3)+b(I_2-3)}_{\text{Relates to object deformation}} + \underbrace{q_1(I_4-1)^2+q_2(I_6-1)^2+K_3I_8^2+K_4I_8}_{\text{Relates to fiber deformation}} \quad (1)$$

Viscoelastic:

$$W_v = W_h + \underbrace{\frac{\mu_1}{4}J_2(I_1-3)}_{\text{Object Deformation}} + \underbrace{\frac{\mu_2}{2}J_9(I_4-1)^2}_{\text{Fiber Deformation}} \quad (2)$$

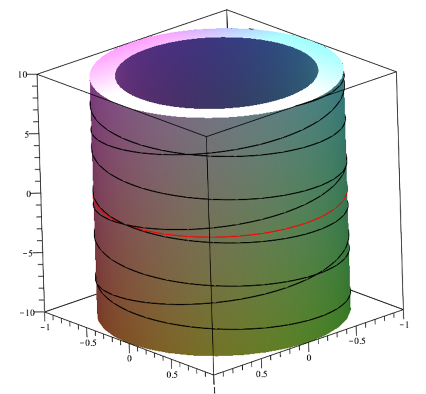
Here, J_2 and J_9 are also related to deformation rates of change in time.

Artery Model:

Restrict the deformation class to:

$$r = R$$
$$\phi = \Phi$$
$$z = Z + G(t, R)$$

where $G(t, R)$ describes large displacements.
Also require $p = p(t, R)$.



Depiction of artery model. Black lines indicate helical fibers.

Methods

To derive equations of motion from the full set of incompressible finite elasticity equations, for the specified displacement type, we used Maple symbolic software. In the hyperelastic case, where the wave equation is linear (see below), the problems can be solved using standard separation of variables technique, and the exact solution is given by a Fourier series in terms of Bessel functions. In the viscoelastic case, the nonlinear damped wave equation was solved numerically in Matlab using the method of lines.

(A) Hyperelastic Case

In the hyperelastic case, we examined the effects of blood flow acting as a driven boundary with a hyperelastic strain energy model, (1) for the artery material. Note that the density term, ρ has been absorbed into a rescaling of the hydrostatic pressure.

Equations Of Motion:

$$x, y: 0 = -\frac{1}{R}\left(2bG_R^2 + p_R R + 4\cos^2(\beta)\cos(2\beta)\left(K_3\cos^2(2\beta) + \frac{1}{2}K_4\right)\right). \quad (3a)$$

$$z: G_{tt} = 2(a+b)\left(G_{RR} + \frac{1}{R}G_R\right). \quad (3b)$$

- Displacement function, $G(t, R)$, is found using (3b).
- Hydrostatic pressure, $p(t, R)$, can be found using (3a).
- Simplify (3b) using $\alpha = 2(a+b)$.

$$G_{tt} = \alpha\left(G_{RR} + \frac{1}{R}G_R\right) \quad (4)$$

Interestingly, despite the complexity of the initial problem, this result is linear and all of the *fiber influence has vanished.* To better understand the motion described by (4), we solved for Dirichlet and Neumann boundary conditions and plotted the results.

Case (A): Results

Exact Solution

We found an exact solution to (4) using the standard separation of variables technique.

$$G(t, R) = (C_1 sin(\omega t) + C_2 cos(\omega t))\left(C_3 J_0\left(\frac{\omega R}{\sqrt{\alpha}}\right) + C_4 Y_0\left(\frac{\omega R}{\sqrt{\alpha}}\right)\right),$$

where C_i are arbitrary constants, J_0 and Y_0 are Bessel functions, and ω is a constant. The initial/boundary condition problem can also be solved using the standard Fourier methods.

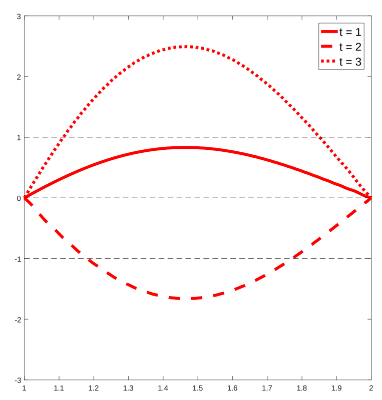
Numerical Solution

Dirichlet Boundary Conditions:

Initial Conditions:
- $G(0, R) = 0$
- $G_t(0, R) = 0$

Boundary Conditions:
- $G(t, R_1) = \sin(\omega t)$
- $G(t, R_2) = 0$

Dimensionless Parameters:
- $R_1 = 1$
- $R_2 = 2$
- $\alpha = 1$
- $\omega = \pi$



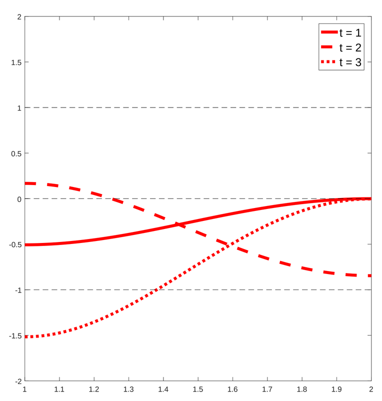
At resonant frequencies, the oscillations grow in time

Neumann Boundary Conditions:

Initial Conditions:
- $G(0, R) = 0$
- $G_t(0, R) = 0$

Boundary Conditions:
- $G_R(t, R_1) = \sin(\omega t)$
- $G_R(t, R_2) = 0$

Dimensionless Parameters:
- $R_1 = 1$
- $R_2 = 2$
- $\alpha = 1$
- $\omega = \pi$



With Neumann boundary conditions, the displacements can shift off-axis.

(B) Viscoelastic Case

For the viscoelastic case, we repeat the analysis using (2) for the strain energy. As with the hyperelastic case, the density has been absorbed into a rescaling of the pressure, and we use $\alpha = 2(a+b)$. Again, we get one equation for pressure (not shown) and one for the displacement.

Equations Of Motion:

$$z: G_{tt} = \alpha\left(G_{RR} + \frac{1}{R}G_R\right) + \mu_1 G_R\left(G_R\left(G_{tRR} + \frac{1}{R}G_{tR}\right)(2G_R^2+1) + 2G_{tR}G_{RR}(4G_R^2+1)\right)$$

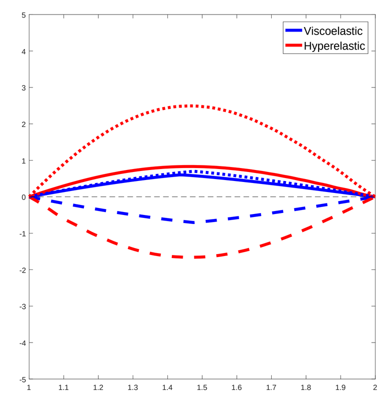
Case (B): Results

Dirichlet Boundary Conditions:

Initial Conditions:
- $G(0, R) = 0$
- $G_t(0, R) = 0$

Boundary Conditions:
- $G(t, R_1) = \sin(\omega t)$
- $G(t, R_2) = 0$

Dimensionless Parameters:
- $R_1 = 1$
- $R_2 = 2$
- $\alpha = 1$
- $\omega = \pi$
- $\mu_1 = 0.1$



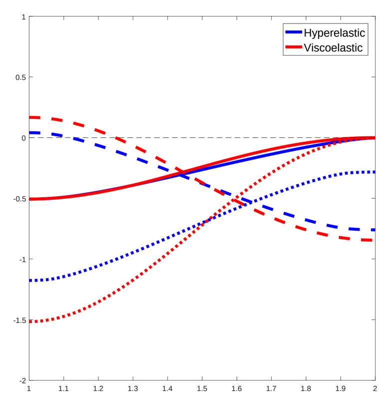
At resonant frequencies, the oscillations grow in time

Neumann Boundary Conditions:

Initial Conditions:
- $G(0, R) = 0$
- $G_t(0, R) = 0$

Boundary Conditions:
- $G_R(t, R_1) = \sin(\omega t)$
- $G_R(t, R_2) = 0$

Dimensionless Parameters:
- $R_1 = 1$
- $R_2 = 2$
- $\alpha = 1$
- $\omega = \pi$
- $\mu_1 = 0.1$



With Neumann boundary conditions, the displacements can shift off-axis.

Conclusions and Future Work

Results: Surprisingly, our highly complicated PDE system yields rather simple wave equations that can even be solved exactly in some cases.

Hyperelastic Model:
- Equation of motion is *linear.*
- Helical fibers *do not* affect z displacement.

Viscoelastic Model:
- Equation of motion is *nonlinear.*
- Fiber effects are manifested through an additional nonlinear *damping* term.

Future Work:
- Analyze other types of motions compatible with incompressibility
  - twisting modes;
  - radial expansion/contraction modes.
- Generalize to compressible models.
- Study conservation law structure; implement advanced numerical methods
- Determine material parameters applicable to realistic modeling situations

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