Nonlinear Dynamics of Viscous Fluids with Gas Bubbles

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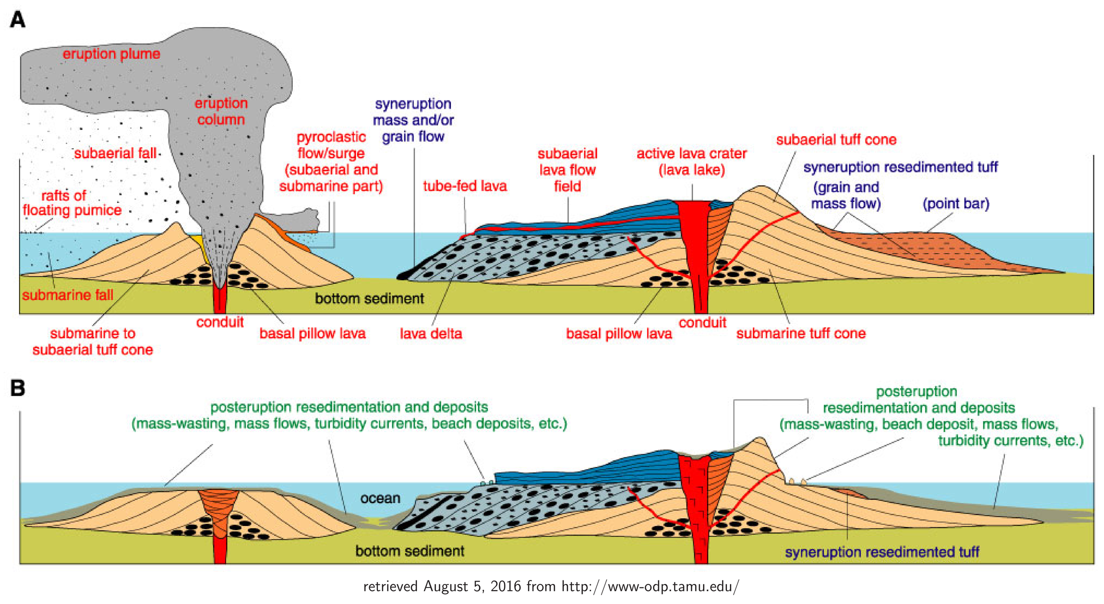
Motivation & Examples of Applications

Viscous fluid flows with gas bubbles: multiple applications, e.g.,

- laminar magma flows in volcanic conduits and subaerial lava flow fields;
- oil and freon flows; other industrial processes.

An eruption; retrieved August 3, 2016 from http://thewatchers.adorraeli.com

Vesicular basalt; retrieved July 27, 2015 from http://www.sciencebuzz.org



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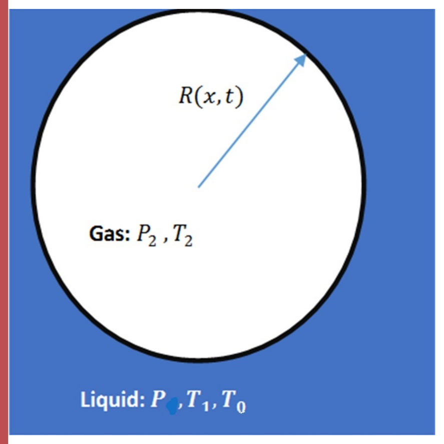
The Mathematical Model



The **state variables** of the system are:

- $P = P(x,t)$: pressure of the mixture.
- $P_2 = P_2(x,t)$: pressure of the gas.
- $\rho = \rho(x,t)$: density of the mixture.
- $u = u(x,t)$: velocity of the mixture.
- $R = R(x,t)$: radius of the gas bubbles.

Constant parameters:

- μ: dynamic viscosity of the mixture,
- ρ_l: density of the liquid,
- γ: ratio of the specific heats,
- h: heat transfer coefficient,
- X: mass of the gas per 1 kg of mixture,
- T_0: temperature of the liquid,
- N: number of bubbles in the mixture.

The first two equations are standard 1-D **Naiver-Stokes equations** for mass conservation and momentum conservation:

$$\rho_t + (\rho u)_x = 0 \tag{1}$$

$$(\rho u)_t + (\rho u^2 + P)_x - \mu u_{xx} = 0 \tag{2}$$

The **Rayleigh-Plesset equation** describes bubble dynamics:

$$P = P_2 - \rho_l\left(RR_{tt} - \frac{3}{2}(R_t)^2 - \frac{4\mu}{3R}R_t\right) \tag{3}$$

The heat transfer equation, from conservation of energy using Newton's law of cooling to describe heat transfer through the bubble's surface:

$$(P_2)_t + 3\gamma P_2\frac{R_t}{R} + \frac{3(\gamma-1)h}{R}(T_g - T_0) = 0 \tag{4}$$

Ideal gas relationship to relate the pressure of the gas to the temperature of the gas:

$$T_g = \frac{P_2 T_0}{P_{20}}\frac{R^3}{R_0{}^3} \tag{5}$$

Density-bubble radius relation, from the conservation of mass:

$$(1 - X + \frac{4}{3}\pi N R^3 \rho_l)\rho - \rho_l = 0 \tag{6}$$

Derivatives are denoted by subscripts: $\partial f/\partial t = f_t$, etc.

Non-Dimensionalization of the Model

Rescale the physical variables using typical values:

$$P = A_p\tilde{P}, \quad \rho = \rho_1\tilde{\rho}, \quad x = L\tilde{x}, \quad u = v_0\tilde{u}$$

$$t = \frac{v_0}{L}\tilde{t}, \quad R = R_0\tilde{R}, \quad T = A_T\tilde{T},$$

A_p: characteristic pressure, rho_1: density of the liquid, L: characteristic length, v_0: characteristic speed, R_0: initial bubble radius, A_T: characteristic temperature.

The dimensionless system:

$$\tilde\rho_t + (\tilde\rho\tilde u)_x = 0 \tag{7}$$

$$(\tilde\rho\tilde u)_t + (\tilde\rho\tilde u^2)_x + Eu\tilde P_x - \frac{1}{Re}\tilde u_{xx} = 0 \tag{8}$$

$$\tilde P = \tilde P_2 - \frac{\delta^2}{Eu}\tilde R\tilde R_{tt} - \frac{3\delta^2}{2Eu}(\tilde R_t)^2 - \frac{4}{3(Eu)(Re)}\frac{\tilde R_t}{\tilde R} \tag{9}$$

$$(\tilde P_2)_t + 3\gamma\tilde P_2\frac{\tilde R_t}{\tilde R} + \frac{(\gamma-1)W}{(Eu)\delta\tilde R}\left(\tilde T_2 - \tilde T_0\right) = 0 \tag{10}$$

$$(1 - X + B\tilde R^3)\tilde\rho - 1 = 0 \tag{11}$$

Fundamental dimensionless parameters:

$$Eu = \frac{A_p}{\rho_1 v_0{}^2}, \quad Re = \frac{\rho_1 v_0 L}{\mu}, \quad \delta = \frac{R_0}{L}$$

$$B = \frac{4}{3}\pi\rho_1 N R_0{}^3, \quad W = \frac{3A_T h}{v_0\rho_1}$$

The Asymptotic Approximation

Change of the independent variables:

$$\xi = \epsilon^\alpha(x - a_0 t), \quad \tau = \epsilon^{\alpha+1}t, \quad 0 < \varepsilon \ll 1;$$
$$\frac{\partial}{\partial x} = \epsilon^\alpha\frac{\partial}{\partial\xi}, \quad \frac{\partial}{\partial t} = a_0\epsilon^{\alpha+1}\frac{\partial}{\partial\tau} - \epsilon^\alpha\frac{\partial}{\partial\xi}, \quad \alpha, a_0 > 0. \tag{12}$$

- ξ: a large-scale moving wave variable ($\xi \sim 1$ when $x \sim \varepsilon^{-\alpha} \gg 1$).
- τ: 'slow time'.

Assume a standard asymptotic expansion of flow parameters near the equilibrium:

$$u = \epsilon u^{(1)} + \epsilon^2 u^{(2)} + ..., \quad R = R_0 + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + ..., \quad \rho = \rho_0 + \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)} + ...,$$
$$P = P_0 + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + .., \quad P_2 = P_{20} + \epsilon P_2^{(1)} + \epsilon^2 P_2^{(2)} + .., \tag{13}$$

- (12), (13) are substituted in (7)–(11) (tildes omitted).
- Various α can be chosen.
- Coefficients at different powers of ε must vanish independently $\Rightarrow$ **parameter relationships**.
- Obtain **a single PDE** for $\rho^{(1)}(\xi,\tau)$, describing **small perturbations of the equilibrium state**.

Case A: $\alpha = 1$

In this case, we arrive at the classical **Burgers' equation**

$$\boxed{\rho_\tau^{(1)} + A\rho^{(1)}\rho_\xi^{(1)} + B\rho_{\xi\xi}^{(1)} = 0,} \tag{14}$$

where

$$A = \frac{a_0}{BR_0^3\rho_0^2}, \quad B = -\left(\frac{1}{2Re\rho_0} + \frac{2a_0^2}{9EuReP_0} + \frac{R_0P_0Eu\delta a_0^2}{T_0W}\right), \quad a_0{}^2 = \frac{Eul\,P_0}{BR_0{}^3\rho_0{}^2}.$$

A change of variables

$$\xi = \left(\frac{B^2}{A}\right)^{\frac{1}{3}}x, \quad \tau = -\left(\frac{B}{A^2}\right)^{\frac{1}{3}}t, \quad \rho^{(1)}(\xi,\tau) = -\left(\frac{B}{A^2}\right)^{\frac{1}{3}}v(x,t) \tag{15}$$

maps (14) into the standard form

$$\boxed{v_t + vv_x = v_{xx}.} \tag{16}$$

Case B: $\alpha = \frac{1}{2}$

In this case, we arrive at the **Korteweg-de Vries equation**

$$\boxed{\rho_\tau^{(1)} + A\rho^{(1)}\rho_\xi^{(1)} + C\rho_{\xi\xi\xi}^{(1)} = 0,} \tag{17}$$

where

$$A = \frac{a_0}{BR_0^3\rho^2}, \quad C = \frac{R_0{}^2a_0{}^3\delta^2}{6Eu\,P_0}, \quad a_0{}^2 = \frac{Eu\,P_0}{BR_0{}^3\rho_0{}^2}. \tag{18}$$

- This case has another equation to be satisfied:

$$\frac{1}{Re}\rho_{\xi\xi}^{(1)} = 0, \tag{19}$$

hence the dynamic viscosity must vanish: $\mu = 0$.

A scaling-type change of variables

$$\xi = \left(216\frac{C}{A^3}\right)^{\frac{1}{2}}x, \quad \tau = \left(6\frac{C}{A}\right)^{\frac{1}{2}}t \tag{20}$$

maps (17) into a canonical form

$$\boxed{v_t + 6\,vv_x + v_{xxx} = 0.} \tag{21}$$

Traveling Wave Solutions of the Burgers Equation

One can obtain particular solutions of (16) and (21) using the **traveling wave ansatz:**

$$v(x,t) = g(z), \quad z = x - ct, \quad c = \text{const}.$$

- The approach is based on *symmetries* of the PDEs at hand, and is useful since the PDEs are reduced to much simpler ODEs.
- For example, the **Burgers' equation** reduces to the following ODE:

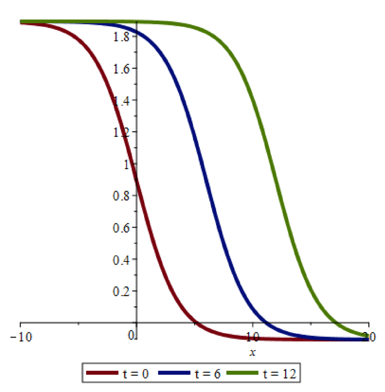
$$-cg'(z) + g(z)g'(z) - g''(z) = 0. \tag{22}$$

The latter can be solved by integrating twice:

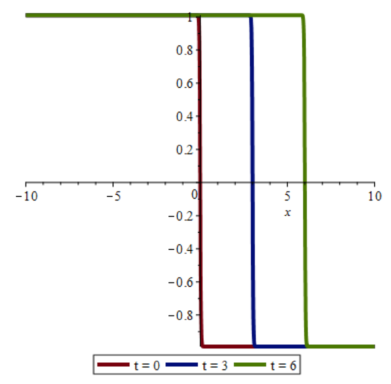
$$\boxed{v(x-ct) = g(z) = -\tanh\left(\frac{z + C_2}{4\sqrt{c^2 - C_1}}\right) + \frac{\sqrt{c^2 - C_1}}{c}} \tag{23}$$

where C_1 and C_2 are arbitrary constants.

- Sample front-type traveling wave solutions:



Traveling wave solution for the Burgers equation (16).



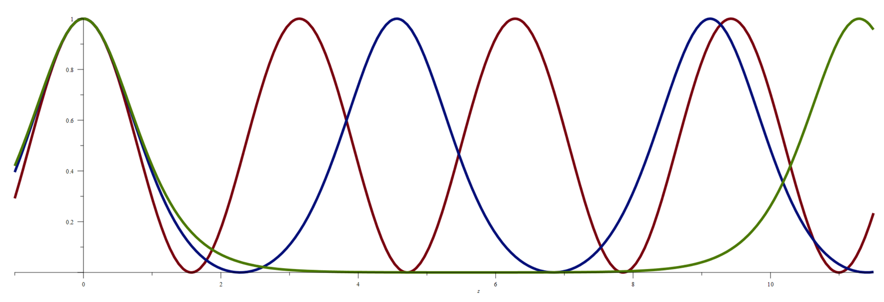
Weak solution of the Burgers equation (16) with $C = 0$ (shock wave).

Solutions of the KdV equation

We use **traveling wave ansatz** for the **Korteweg - de Vries equation**, reducing it to an ODE. The latter admits the well-known wave-type exact solutions

$$\boxed{v(x-ct) = g(z) = 2k^2\left(\frac{c}{2k^2-1}\right)\text{cn}\left(\frac{1}{2}\left(\frac{c}{2k^2-1}\right)^{\frac{1}{2}}z, k\right)^2} \tag{24}$$

where $\text{cn}(z,k)$ denotes a **Jacobi elliptic cosine function** with the parameter k, $0 < k < 1$, and c is the wave speed.

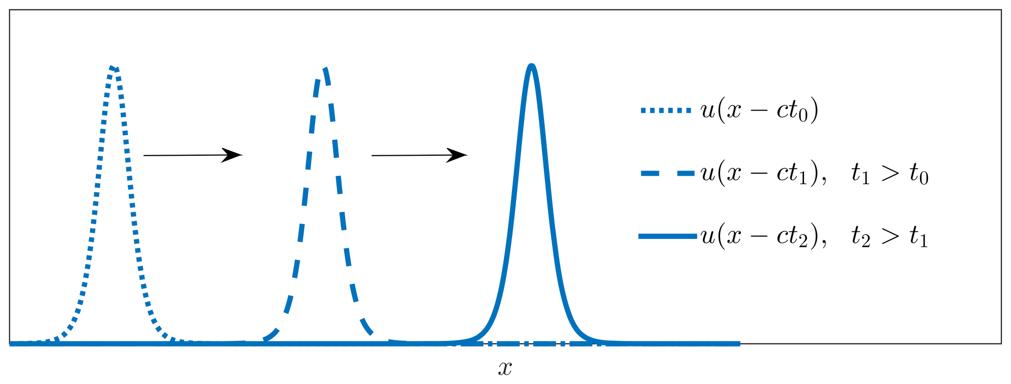


Sample plots of $v(z) = \text{cn}(z,k)^2$; blue: $k = 0$, red: $k = 0.9$, green: $k = 0.9999$.

(24) is a **cnoidal traveling wave solution** for the KdV equation. , and k are arbitrary constants. For $k = 1$, (24) becomes

$$v(x-ct) = 2c\left(\text{sech}\left(\frac{\sqrt{c}}{2}(x-ct)\right)\right)^2 \tag{25}$$

which is a **solitary** (single-soliton) **traveling wave solution**:



Traveling wave soliton solution of the KdV equation.

Multi-Soliton Solutions of the KdV equation

The **Korteweg - de Vries equation** (21) is *exactly solvable (integrable)*.

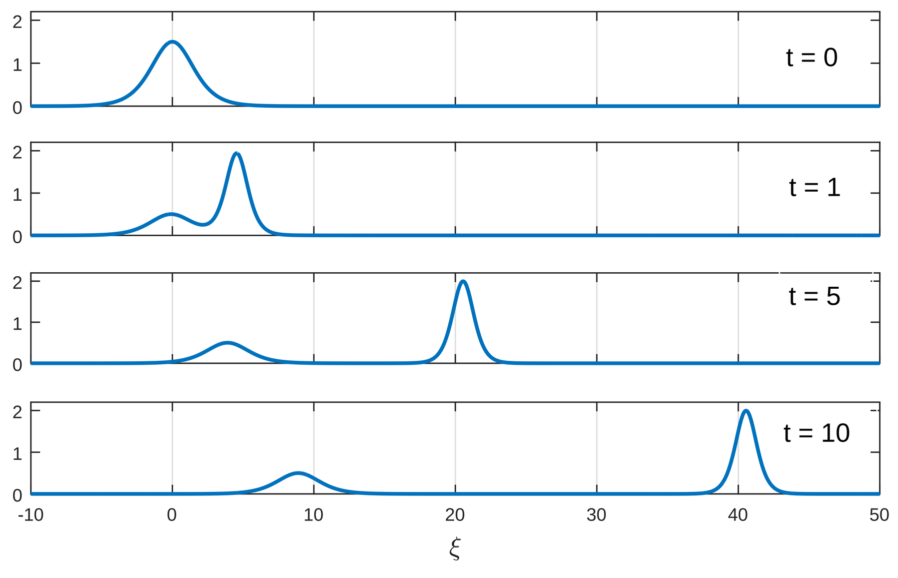
- Nonlinear PDE whose solutions can be exactly and precisely specified for a wide class of initial data.
- **multi-soliton solutions** can be systematically constructed.

Example of an **exact two-soliton solution**:

$$u(x,t) = -2\frac{\left(\left(\cosh\left(k_1\left(4k_1{}^2t - x\right)\right)\right)^2k_2{}^2 + \left(\cosh\left(k_2\left(4k_2{}^2t - x\right)\right)\right)^2k_1{}^2 - k_1{}^2\right)\left(k_1{}^2 - k_2{}^2\right)}{\left(\cosh\left(k_1\left(4k_1{}^2t - x\right)\right)k_2\cosh\left(k_2\left(4k_2{}^2t - x\right)\right) - \sinh\left(k_2\left(4k_2{}^2t - x\right)\right)k_1\sinh\left(k_1\left(4k_1{}^2t - x\right)\right)\right)^2}, \tag{26}$$

where k_1 and k_2 are constants.

- Two soliton waves scatter elastically.
- After the collision, solitons regain their original shape and velocity.
- The only difference is a slight change in the position they would have reached without the collision.
- Higher waves have higher speeds. Increasing c in (24) will increase the wave's speed and height.



Evolution of a two soliton solution when $k_1 = \frac{1}{2}$ and $k_2 = 1$.

Conclusions & Future Work

- An extended model compared to [1].
- The Su-Gardner-type [2] asymptotic analysis for the complex bubble fluid model leads, under different assumptions, to **Burgers'** and **Korteweg-de Vries** equations.
- The classical, well-understood **Burgers'** and **Korteweg-de Vries** equations arise in various nonlinear models, including fluid dynamics, plasma physics, and nonlinear optics, and are observed in laboratory and numerical experiments.
- First-order perturbations of all parameters in the approximate solution are related *linearly*. In particular, in both Cases (A) and (B), one has:

$$P^{(1)} + 3R^{(1)} = 0; \tag{27}$$

thus higher pressure leads to lower bubble radius.

- **Ongoing work:** Asymptotic analysis around non-constant equilibrium solution.
- **Future work direction 1:** Systematic study the case of general α and its compatibility with general asymptotic expansions (13).
- **Future work direction 2:** Study vertical flows through the addition of z-dependence (gravity term); apply to the study of magmas in volcanic conduits; extend the model as required.

References

[1] N.A. Kudryashov and D.I. Sinelshchikov.
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[2] CH Su and CS Gardner.
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Journal of Mathematical Physics, 10(3):536–539, 1969.