# Symmetry Properties of a Family of Benjamin-Bona-Mahony-type Equations

Alexei Cheviakov

Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada

Yanenko 100 conference 2021

August 25, 2021

# Collaborators


- **D. Dutykh**, Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France
- **A. Assylbekuly**, Department of Mathematics, Novosibirsk State University, Novosibirsk, Russia

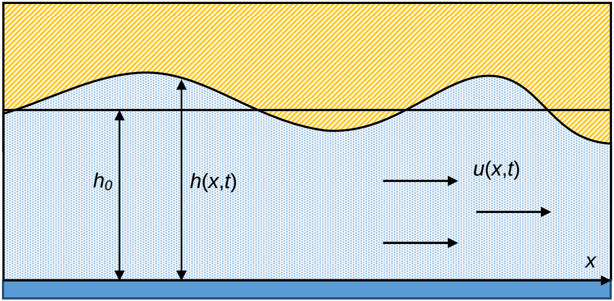

- Symmetries (point & local) and conservation laws (multiplier method) have been computed in Maple using GeM software [A.C. 2004-2021]

# Outline

# Outline

# Shallow water waves and the Korteweg-de Vries (KdV) equation

- Long waves in shallow water:




- Starting from 2D incompressible, irrotational Euler equations
- $A$: typical wave amplitude; $\lambda$: typical wave length
- Surface elevation: $h = h_0 + \eta(x,t)$; horizontal velocity: $u(x,t)$
- Wave speed: $c_0 = \sqrt{gh_0}$
- Small parameters: $\varepsilon = A/h_0$, $\delta = h_0/\lambda$

- The Boussinesq regime: $\varepsilon \sim \delta^2 \ll 1$
- Dimensional KdV:

$$\eta_t + c_0\,\eta_x + \frac{3}{2}\frac{c_0}{h_0}\,\eta\eta_x + \frac{c_0^2 h_0^2}{6}\,\eta_{xxx} = 0$$

- The non-dimensionalizing scaling transformation:

$$\begin{aligned} &t = \frac{\lambda}{c_0} t^*, \quad x = \lambda x^*, \quad z = h_0 z^*, \quad h = h_0 h^*, \quad \eta = A\eta^*, \\ &v = \varepsilon c_0 v^*, \quad w = \varepsilon\delta c_0 w^*, \quad p = \varepsilon\rho c_0^2 p^*, \quad \bar{u} = \varepsilon c_0 \bar{u}^* \end{aligned} \tag{Sc}$$

- Dimensionless KdV:

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\,\eta^*\eta^*_{x^*} + \frac{\delta^2}{6}\,\eta^*_{x^*x^*x^*} = 0$$

- Canonical form (further transformed $x, t, u$):

$$u_t + 6uu_x + u_{xxx} = 0$$

- A well-known S-integrable model with rich analytic structure

- The BBM equation can be obtained e.g. from the Korteweg-de Vries equation through a lower-order approximation

$$\eta^*_{t^*} = -\eta^*_{x^*} + O(\varepsilon, \delta^2)$$

- Dimensional BBM:

$$\eta_t + c_0\,\eta_x + \frac{3}{2}\frac{c_0}{h_0}\,\eta\eta_x - \frac{h_0^2}{6}\,\eta_{xxt} = 0$$

- Dimensionless BBM:

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\eta^*\eta^*_{x^*} - \frac{1}{6}\delta^2\eta^*_{x^*x^*t^*} = 0$$

- Canonical form:

$$\boxed{u_t + u_x + uu_x - u_{xxt} = 0}$$

- No Lax pair is known. Solitary wave interactions are inelastic
- Believed to be neither S-integrable nor C-integrable
- Has exactly three local conservation laws [Olver 1979]

- The BBM: $\boxed{u_t + u_x + uu_x - u_{xxt} = 0}$
- BBM is also known as *the regularized long wave equation*
- Originally derived by Peregrine (thus satisfying the Arnold Principle: "If a notion bears a personal name, then this name is not the name of the discoverer.")
- Unlike the KdV, the Benjamin-Bona-Mahony equation has a more physical dispersion relation:

$$\omega = \frac{c_0 k}{1 + h_0^2 k^2/6}, \qquad c = \frac{c_0}{1 + h_0^2 k^2/6}$$

- Cf. for the KdV:

$$\omega = c_0\, k \left(1 - \frac{1}{6} h_0^2 k^2\right), \qquad c = c_0 \left(1 - \frac{1}{6} h_0^2 k^2\right)$$

- Full water wave theory:

$$\omega = c_0 \sqrt{\frac{k}{h_0}\, \tanh k h_0}, \qquad c = c_0 \sqrt{\frac{\tanh k h_0}{k h_0}}$$

# Symmetries of the BBM

- The BBM: $\boxed{u_t + u_x + uu_x - u_{xxt} = 0}$

- Point symmetries:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\,\partial_t - (u + 1)\,\partial_u$$

- Not Galilei-invariant as it stands!

- Considering *approximate point symmetries* $X = X^{(0)} + \varepsilon X^{(1)} + \mathcal{O}(\varepsilon^2)$, the BBM admits a Galilei-type generator

$$X_G = \partial_u + \frac{3}{2}\varepsilon\, t\, \partial_x,$$

with the corresponding global group

$$t^* = t, \quad x^* = x + \frac{3}{2} a\,\varepsilon\, t, \quad u^* = u + a$$

# Local conservation laws of the BBM

- The BBM: $\boxed{u_t + u_x + uu_x - u_{xxt} = 0}$
- Local CLs [Olver 1979]:

$$\mathcal{D}_t(u - u_{xx}) + \mathcal{D}_x\left(u + \frac{u^2}{2}\right) = 0,$$

$$\mathcal{D}_t\left(\tfrac{1}{2}(u^2 + u_x^2)\right) + \mathcal{D}_x\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 - u\,u_{xt}\right) = 0,$$

$$\mathcal{D}_t\left(\tfrac{1}{2}u^2 + \tfrac{1}{6}u^3\right) +$$
$$\mathcal{D}_x\left(\tfrac{1}{8}u^4 + \tfrac{1}{2}(u - u_{xt} + 1)u^2 - u\,u_{xt} + \tfrac{1}{2}(u_{xt}^2 - u_t^2)\right) = 0.$$

- Corresponding global conserved quantities:

$$M(t) = \int_a^b (u - u_{xx})\,\mathrm{d}x, \qquad \frac{\mathrm{d}M}{\mathrm{d}t} = 0,$$

$$E(t) = \int_a^b \frac{1}{2}(u^2 + u_x^2)\,\mathrm{d}x, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0,$$

$$\mathcal{H}(t) = \frac{1}{2}\int_a^b \left(u^2 + \frac{1}{3}u^3\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0$$

(momentum; kinetic energy; Hamiltonian)

- The BBM: $\boxed{u_t + u_x + uu_x - u_{xxt} = 0}$
- Hamiltonian formulation:

$$u_t = \mathbb{J}\,\frac{\delta\,\mathcal{H}}{\delta\,u},$$

$$\mathbb{J} = \left(\mathbb{1} - \partial_x^2\right)^{-1} \cdot (-\partial_x), \qquad \mathcal{H} = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + \frac{1}{3}\,u^3\right) \mathrm{d}x$$

- Lagrangian formulation: the Lagrangian density

$$L[w] = -\frac{1}{2}\left(\frac{1}{3}w_x^3 + w_x^2 + w_x w_t + w_{xx} w_{xt}\right)$$

yields the potential BBM equation $(u = w_x)$

$$w_{xt} + w_{xx} + w_x w_{xx} - w_{xxxt} = 0$$

# Outline

# A Galilei-invariant BBM equation

- Dimensionless BBM:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{1}{6}\delta^2\eta_{xxt} = 0$$

- With the next term:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{1}{6}\delta^2\eta_{xxt} - \frac{1}{4}\varepsilon\delta^2\eta\eta_{xxx} = 0$$

- The two are equivalent in the Boussinesq regime
- A rescaled form (iBBM):

$$\boxed{u_t + u_x + uu_x - u_{xxt} - uu_{xxx} = 0}$$

- iBBM is exactly Galilei-invariant!
- Point symmetries: $\mathrm{X}_1 = \partial_t,\ \ \mathrm{X}_2 = \partial_x,\ \ \mathrm{X}_3 = t\,\partial_x\ +\ \partial_u$

# A Galilei-invariant BBM equation

- Dimensionless BBM:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{1}{6}\delta^2\eta_{xxt} = 0$$

- With the next term:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{1}{6}\delta^2\eta_{xxt} - \frac{1}{4}\varepsilon\delta^2\eta\eta_{xxx} = 0$$

- The two are equivalent in the Boussinesq regime
- A rescaled form (iBBM):

$$\boxed{u_t + u_x + uu_x - u_{xxt} - uu_{xxx} = 0}$$

- Has peakon solutions $u(x,t) = C_1 \; + \; C_2\, e^{|x-ct|}$
- Belongs to the $b-$family (with $b = 0$) of equations

$$u_t + 2\kappa u_x + (b+1)uu_x - bu_x u_{xx} - u_{xxt} - uu_{xxx} = 0$$

that include the integrable Camassa-Holm ($b = 2$) and Degasperis-Procesi ($b = 3$) equations.

- iBBM: $\boxed{u_t + u_x + uu_x - u_{xxt} - uu_{xxx} = 0}$

- Two local CLs:

$$\begin{aligned} \mathcal{D}_t(u - u_{xx}) + \mathcal{D}_x\left(\frac{1}{2}(u^2 + u_x^2) + u(1 - u_{xx})\right) &= 0, \\ \mathcal{D}_t\left(e^{u - u_{xx}}\right) + \mathcal{D}_x\left(u\, e^{u - u_{xx}}\right) &= 0 \end{aligned}$$

- Global conserved quantities:

$$M(t) = \int_a^b (u - u_{xx})\,\mathrm{d}x, \qquad \frac{\mathrm{d}M}{\mathrm{d}t} = 0,$$

$$P(t) = \int_a^b e^{u - u_{xx}}\,\mathrm{d}x, \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = 0$$

- No conservation of energy $E(t)$!

# Outline

# A Galilei-invariant and energy-preserving BBM-type model: eBBM

- Dimensionless BBM:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{1}{6}\delta^2\eta_{xxt} = 0$$

- With the next-order term:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{1}{6}\delta^2\eta_{xxt} - \frac{1}{4}\varepsilon\delta^2\eta\eta_{xxx} = 0$$

- With an extra term of the same higher order:

$$\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{1}{6}\delta^2\eta_{xxt} - \frac{1}{4}\varepsilon\delta^2\eta\eta_{xxx} - \frac{1}{2}\varepsilon\,\delta^2\eta_x\eta_{xx} = 0$$

  (equivalent in the Boussinesq regime; follows from energy CL multiplier)

- A rescaled form (eBBM):

$$\boxed{u_t + u_x + uu_x - u_{xxt} - uu_{xxx} - 2u_x u_{xx} = 0}$$

- Same point symmetries as for the iBBM. Galilean invariance is preserved
- Energy-preserving; has additional CLs (below)

# Local conservation laws of the eBBM

- Shared with the BBM: momentum; kinetic energy

$$M(t) = \int_a^b (u - u_{xx})\,\mathrm{d}x, \qquad \frac{\mathrm{d}M}{\mathrm{d}t} = 0,$$

$$E(t) = \int_a^b \frac{1}{2}(u^2 + u_x^2)\,\mathrm{d}x, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

- A conserved quantity related to the Hamiltonian structure:

$$\mathcal{N}(t) = \int_a^b \left(\frac{1}{3}u^3 + (u-1)u_x^2\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t} = 0$$

- An additional conserved quantity $\sim$ "center of mass theorem":

$$\mathcal{C}(t) = \int_a^b \left(\left(t\left(1+\frac{1}{2}u\right) - x\right)(u - u_{xx})\right)\mathrm{d}x$$

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{B}t$$

$$\mathcal{A} = \int_a^b x\,(u - u_{xx})\,\mathrm{d}x, \quad \mathcal{B} = \int_a^b \left(1+\frac{1}{2}u\right)(u - u_{xx})\,\mathrm{d}x, \quad \mathcal{A}_0 = \mathrm{const.}$$

# Outline

- Consider a one-parameter A-family of equations

$$\boxed{u_t + u_x + uu_x - u_{xxt} - A\left(u\, u_{xxx} + 2u_x\, u_{xx}\right) \;=\; 0,} \quad A = \mathrm{const}$$

- Reduces to the BBM when $A = 0$, to the eBBM when $A = 1$
- The $A$-term has the highest asymptotic order $\mathcal{O}(\varepsilon^2) \sim \mathcal{O}(\delta^4)$ in the Boussinesq regime $\varepsilon \sim \delta^2$
- Shares/generalizes multiple properties of the BBM and the eBBM

# A-family: symmetries, CLs, Hamiltonian structure

- A-family: $\boxed{u_t + u_x + uu_x - u_{xxt} - A\,(u\,u_{xxx} + 2u_x\,u_{xx}) \;=\; 0}$
- Three point symmetries:

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad \mathrm{X}_3 = (A-1)t\partial_t \; + At\partial_x + (1 + (1-A)u)\partial_u$$

- Conserved quantities:

$$M(t) = \int_a^b (u - u_{xx})\,\mathrm{d}x, \qquad \frac{\mathrm{d}M}{\mathrm{d}t} = 0,$$

$$E(t) = \int_a^b \frac{1}{2}(u^2 + u_x^2)\,\mathrm{d}x, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

$$\mathcal{N}_A(t) = \int_a^b \left(\frac{1}{3}u^3 + (Au - 1)\,u_x^2\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{N}_A}{\mathrm{d}t} = 0$$

- A linear combination of $\mathcal{N}_A(t)$ and $E(t)$ defines a Hamiltonian

$$\mathcal{H}_A(t) = \frac{1}{2}\int_a^b \left(u^2 + \frac{1}{3}u^3 + Auu_x^2\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{H}_A}{\mathrm{d}t} = 0,$$

which yields the same Hamiltonian form of the whole A-family as for the original BBM above.

- A-family: $\boxed{u_t + u_x + uu_x - u_{xxt} - A\,(u\,u_{xxx} + 2u_x\,u_{xx}) \;=\; 0}$
- Potential variable: $u = w_x$
- The potential A-family:

$$w_{tx} \;+\; w_{xx} \;+\; w_x w_{xx} \;-\; w_{xxxt} \;-\; A(\;ww_{xxxx} \;+\; 2w_{xx}w_{xxx}\;) \;=\; 0$$

- A Lagrangian formulation holding for all $A$:

$$L[w] = -\frac{1}{2}\left(\frac{1}{3}w_x^3 + w_x^2 + w_x w_t + w_{xx}w_{xt} + A\;w_x\;w_{xx}^2\right), \quad \frac{\delta\,L}{\delta\,w} = 0.$$

# The potential A-family: Lagrangian structure; symmetries

- A-family: $\boxed{u_t + u_x + uu_x - u_{xxt} - A\,(u\,u_{xxx} + 2u_x\,u_{xx}) \;=\; 0}$

- Potential variable: $u = w_x$

- The potential A-family:

$$w_{tx} \;+\; w_{xx} \;+\; w_x w_{xx} \;-\; w_{xxxt} \;-\; A(\; ww_{xxxx} \;+\; 2w_{xx}w_{xxx}\;) \;=\; 0$$

- Symmetries of the potential A-family, $\hat{\mathrm{X}} = \zeta(x, t, w, w_x, w_{xx}, w_{xxt}, w_{xxx})\,\partial_w$:

$$\zeta_1 = w_t, \quad \zeta_2 = w_x, \quad \zeta_3 = x - (A-1)(w + tw_t) - Atw_x, \quad \zeta_F = F(t),$$

$$\zeta_4 = 2(x - tw_x) + (A-1)^2\; t\;(w_{xx}^2 + 2w_x w_{xxx})$$
$$\qquad + (A-1)\left(2w + t(2w_{xxt} + w_{xx}^2 + 2w_x w_{xxx} - w_x^2)\right),$$

$$\zeta_5 = \; w_x^2 \; - 2w_{xxt} - A\,(w_{xx}^2 + 2w_x w_{xxx})$$

- $\zeta_4$: a nonlocal symmetry of the A-family.

# Outline

- A-family: $\boxed{u_t + u_x + uu_x - u_{xxt} - A\left(u\,u_{xxx} + 2u_x\,u_{xx}\right) \;=\; 0}$
- $A = 1/3$ : the eBBM(1/3) equation

$$\boxed{u_t + u_x + uu_x - u_{xxt} - \frac{1}{3}\,u\,u_{xxx} - \frac{2}{3}u_x\,u_{xx} \;=\; 0}$$

- The eBBM(1/3) equation has a hierarchy of higher-order symmetries

$$\hat{\mathrm{X}}_1 \;=\; \frac{u_x \;-\; u_{xxx}}{\left(2(u \;-\; u_{xx}) \;+\; 3\right)^{3/2}}\,\partial_u, \quad \hat{\mathrm{X}}_2 \;=\; \frac{A[u]}{2\left(2(u \;-\; u_{xx}) \;+\; 3\right)^{9/2}}\,\partial_u, \quad \ldots$$

  and similar conserved quantities of increasing orders.

- It can be shown that a time scaling $t = 3\tau$, $\tau \to t$ maps the eBBM(1/3) into the Camassa-Holm equation with $\kappa = 3/2$:

$$u_t \;+\; 3u_x \;+\; 3uu_x \;-\; 2u_xu_{xx} \;-\; u_{xxt} \;-\; uu_{xxx} \;=\; 0$$

# Outline

# Numerical investigations

- Simple traveling waves $u = u(x - ct)$ for BBM, iBBM, eBBM, and eBBM(1/3):

$$(1-c)u' + cu''' + \left(\tfrac{1}{2}\,u^2\right)' = 0,$$

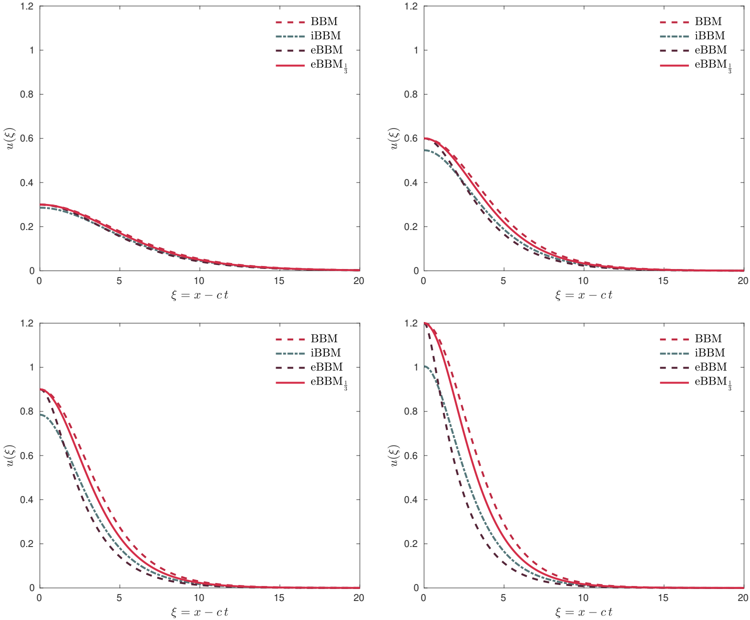
$$(1-c)u' + cu''' + \left(\tfrac{1}{2}\,u^2\right)' - uu''' = 0,$$

$$(1-c)u' + cu''' + \left(\tfrac{1}{2}\,u^2\right)' - uu''' - 2u'u'' = 0,$$

$$(1-c)u' + cu''' + \left(\tfrac{1}{2}\,u^2\right)' - \tfrac{1}{3}\,uu''' - \tfrac{2}{3}\,u'u'' = 0$$

- Sample $c = 1.1, 1.2, 1.3, 1.4$; numerical Petviashvili iteration

# Numerical investigations

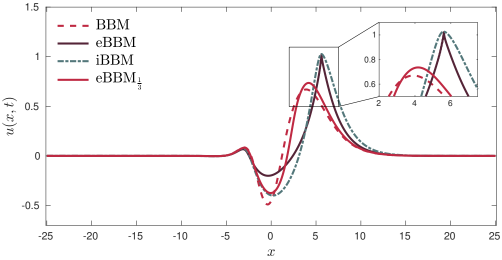


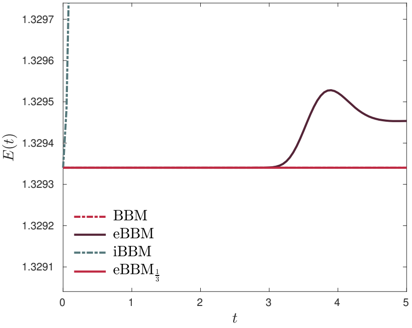
- Bump evolution:

$$u(x,0) \;=\; e^{-\frac{1}{2}\,x^2}$$

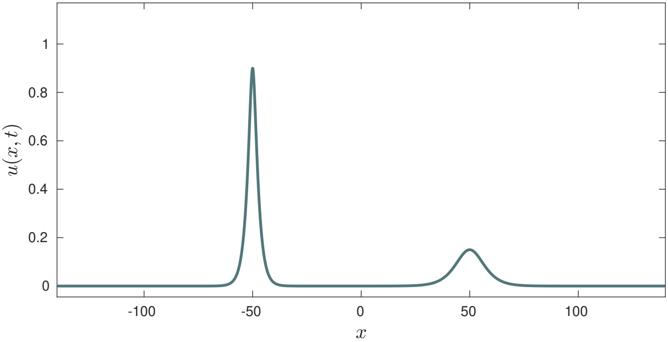
- Numerical method: Fourier-type pseudo-spectral method with periodic boundary conditions [D.D.]

- Energy $E(t)$ is conserved for BBM, eBBM(1/3), and eBBM; for the latter, loss of solution regularity leads to $\sim 0.015\%$ growth around $t = 3$.



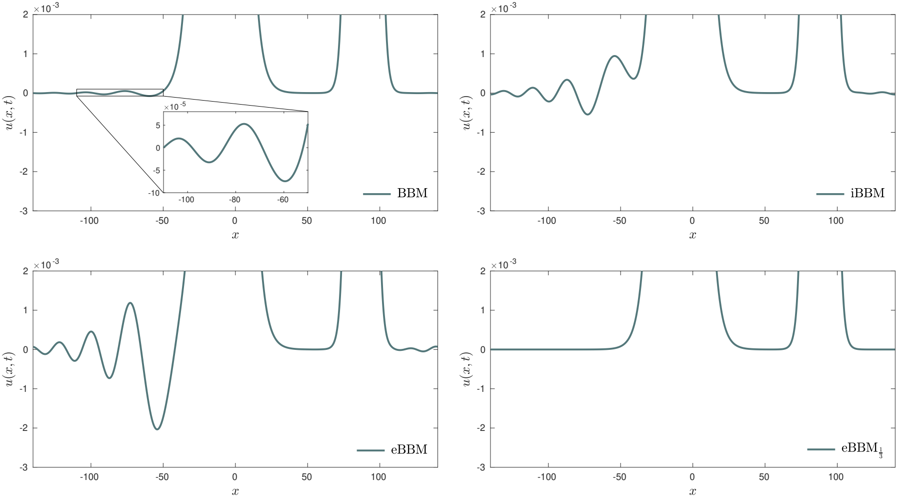



- Solitary wave collisions: initial condition




- Inelastic for BBM, iBBM, eBBM
- Elastic for eBBM(1/3)

# Solitary wave collisions



# Outline

# Discussion

- A Galilei-invariant (iBBM) extension of the Benjamin-Bona-Mahony (BBM) equation is presented, asymptotically equivalent to the BBM in the Boussinesq approximation.
- A further Galilei-invariant and energy-preserving (eBBM) extension is derived based on local conservation law multipliers.
- The one-parameter A-family of PDEs is found, which includes both the BBM and eBBM as special cases, and admits multiple common analytical properties, including symmetries, conservation laws, Lagrangian and Hamiltonian structures.
- For $A = 1/3$, the eBBM(1/3) equation has advanced symmetry/conservation law structure with rational fractional-order generators/multipliers. The eBBM(1/3) is shown to be related to the Camassa-Holm equation.
- Numerical investigations show existence of solitary waves for all models, the expected elastic solitary wave collisions for eBBM(1/3) (CH), and inelastic wave interactions for BBM, iBBM, and eBBM models.
- Techniques similar to ones considered here can be applied to other Boussinesq-type equations; many systems of this kind lack the Galilean invariance and/or energy conservation.

Benjamin, T. B., Bona, J. L., and Mahony, J. J. (1972).
Model equations for long waves in nonlinear dispersive systems.
*Philos. Trans. Royal Soc. London Ser. A* 272, 47–78.

Olver, P. J. (1979).
Euler operators and conservation laws of the BBM equation.
*Math. Proc. Camb. Phil. Soc.* 85, 143–160.

Cheviakov, A., Dutykh, D., and Assylbekuly, A. (2021).
On Galilean Invariant and Energy Preserving BBM-Type Equations.
*Symmetry*, 13 (5), 878.

**Thank you for your attention!**