Conservation laws, similarity reductions and exact solutions for helically symmetric incompressible fluid flows

Alexey Shevyakov

(*Alt. English spelling: Alexei Cheviakov*)

University of Saskatchewan, Saskatoon, Canada

May 2020

Outline

Collaborators

- **M. Oberlack**, Chair of Fluid Dynamics, TU Darmstadt, Germany
- **O. Kelbin, D. Dierkes**, Ph.D. students, TU Darmstadt, Germany

Goals of this talk

- Conservation laws (CL)
- Euler and Navier-Stokes (NS) equations of fluid flow: some results
- Helical invariance: applications and formulas
- Reduction of Euler and NS systems under helical invariance
- General and additional CLs
- New exact solutions of helically invariant NS

Euler and Navier-Stokes equations

Constant-density incompressible fluid flow equations

Equations of gas/fluid dynamics

$$\rho_t + \nabla(\rho \mathbf{u}) = 0,$$

$$\rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) = -\nabla p + \mu \,\Delta \mathbf{u},$$

Closure / Equation of state.

- Independent variables: $t, \mathbf{x} = (x, y, z)$.
- Dependent variables: $\rho(t, \mathbf{x})$, $p(t, \mathbf{x})$, $u^i(t, \mathbf{x})$, $i = 1, 2, 3$.
- Navier-Stokes equations when $\mu \neq 0$.
- Euler equations in the inviscid case $\mu = 0$.

Constant-density incompressible fluid flow equations

Navier-Stokes equations for a fluid with constant density

$$\nabla \cdot \mathbf{u} = 0,$$

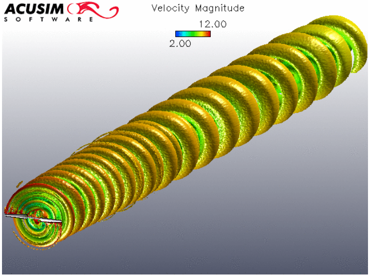
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \, \Delta \mathbf{u} = 0.$$

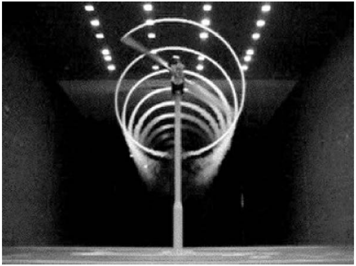
- Constant density (WLOG can assume $\rho = 1$) $\rightarrow$ conservation of mass, div-free $\mathbf{u}$.
- Inviscid case: $\nu = \mu\rho = 0$ (Euler equations).

Helical flows: examples

Examples of Helical Flows in Nature

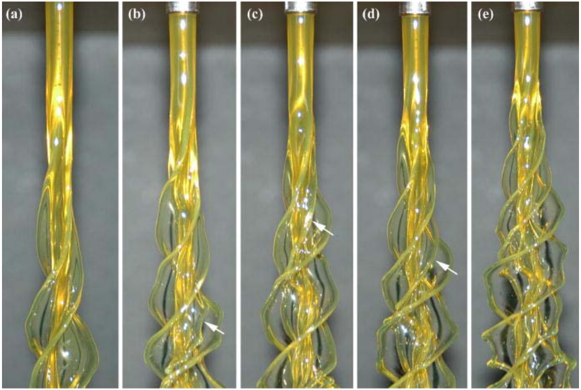
- Wind turbine wakes in aerodynamics [*Vermeer, Sorensen & Crespo, 2003*]





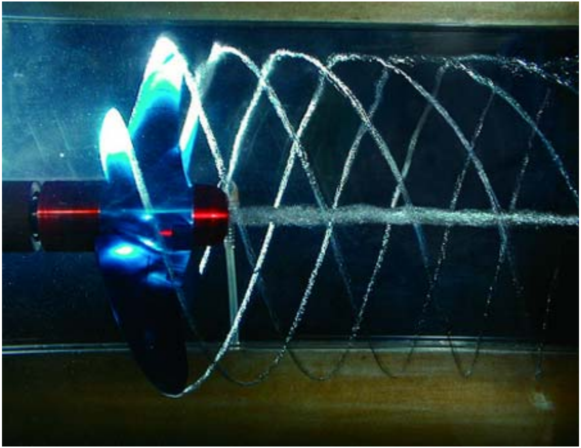
Examples of Helical Flows in Nature

- Helical instability of rotating viscous jets [*Kubitschek & Weidman, 2007*]



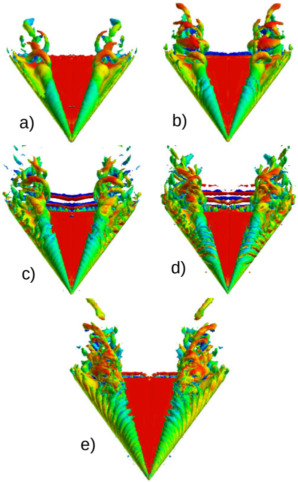
Examples of Helical Flows in Nature

- Helical water flow past a propeller



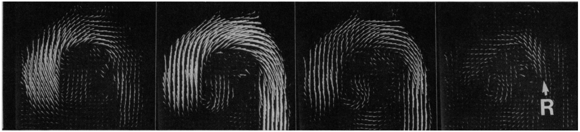
Examples of Helical Flows in Nature

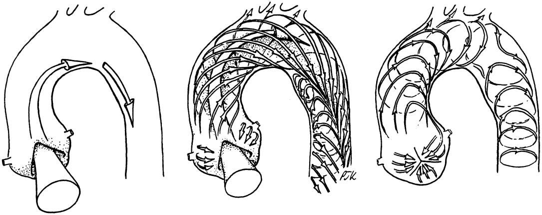
- Wing tip vortices, in particular, on delta wings [*Mitchell, Morton & Forsythe, 1997*]



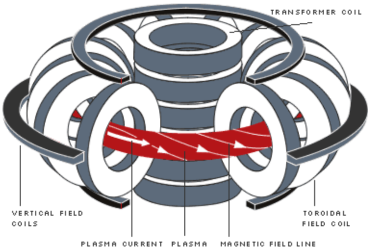
Examples of Helical Flows in Nature

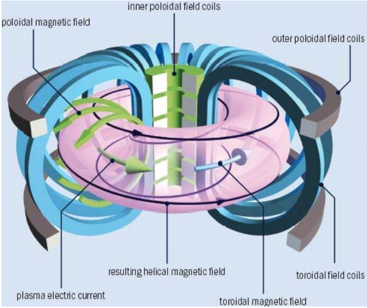
- Helical blood flow patterns in the aortic arch [*Kilner et al, 1993*]





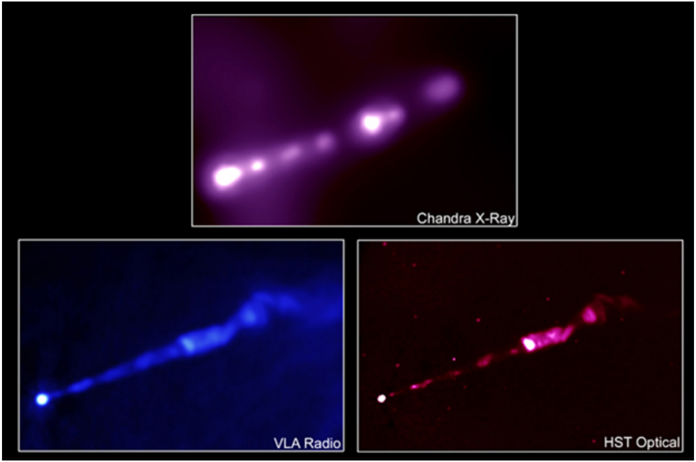
- Helical plasma flows in tokamaks





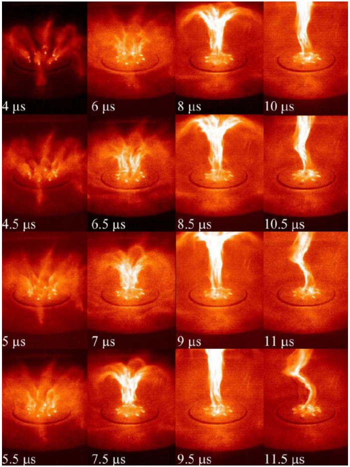
Examples of Helical Flows in Nature

- Helical plasma structures in astrophysics



Examples of Helical Flows in Nature

- Collimated helical plasma jet formation in a plasma discharge



Results overview: two papers

Paper 1: Conservation laws of NS and Euler equations under helical symmetry

O. Kelbin, A. Cheviakov, and M. Oberlack (2013)
New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. *JFM* 721, 340-366.

Helically-invariant fluid dynamics equations

- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions: zero velocity component in symmetric direction.

Additional conservation laws – systematic construction (multiplier method)

- Three-component Euler:
  - Generalized momenta. Generalized helicity. Additional vorticity CLs.
- Three-component Navier-Stokes:
  - Additional CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - Additional CLs in viscous and inviscid case, for plane and axisymmetric flows.

Paper 2: Conservation laws of NS and Euler equations under helical symmetry

D. Dierkes, A. Cheviakov, and M. Oberlack (2020, Physics of Fluids, accepted)
New similarity reductions and exact solutions for helically symmetric viscous flows.

The new v-equation for Galilei-invariant helical flows

- Full helically-invariant Navier-Stokes equations, invariant with respect to the Galilei group

$$G^4: \quad r \to r, \quad t \to t, \quad \xi \to \xi + \varepsilon t, \quad p \to p,$$
$$u^r \to u^r, \quad u^\xi \to u^\xi + \varepsilon B(r), \quad u^\eta \to u^\eta - \varepsilon \frac{b}{ar} B(r).$$

- Such solutions satisfy the new v-equation

$$v_{rt} + \left(\frac{v\, v_r}{r}\right)_r - 2\frac{v_r^2}{r} - \nu\left[v_{rrr} + \frac{v_r}{r^2} - \frac{v_{rr}}{r}\right] = 0.$$

Exact solutions of helically invariant Navier-Stokes equations

- The v-equation: exact Galilei-invariant solutions.
- Beltrami flow ansatz: exact linearization, families of separated solutions.

Conservation laws of dynamic PDEs

Notation

- Independent variables: (x, t), or (t, x, y, z), or $z = (z^1, ..., z^n)$.
- Dependent variables: $u(x, t)$, or generally $v = (v^1(z), ..., v^m(z))$.
- Derivatives:

$$\frac{d}{dt} w(t) = w'(t); \qquad \frac{\partial}{\partial x} u(x, t) = u_x; \qquad \frac{\partial}{\partial z^k} v^p(z) = v^p_k.$$

- All derivatives of order p: $\partial^p v$.
- A differential function:

$$H[v] = H(z, v, \partial v, \ldots, \partial^k v)$$

- A total derivative of a differential function: the chain rule

$$\mathrm{D}_i H[v] = \frac{\partial H}{\partial z^i} + \frac{\partial H}{\partial v^\alpha} v^\alpha_i + \frac{\partial H}{\partial v^\alpha_j} v^\alpha_{ij} + \ldots.$$

Local and global conservation laws

- A system of differential equations (PDE or ODE) $G[v] = 0$:

$$G^\sigma(z, v, \partial v, \dots, \partial^{q_\sigma} v) = 0, \quad \sigma = 1, \dots, M.$$

- The basic notion:

A local conservation law:

A divergence expression

$$\boxed{\mathrm{D}_i \Phi^i[v] = 0}$$

vanishing on solutions of $G[v] = 0$. Here $\Phi = (\Phi^1[v], \dots, \Phi^n[v])$ is the flux vector.

Local and global conservation laws – PDEs

- For time-dependent PDEs, the meaning of a local conservation law is that the rate of change of some "total amount" is balanced by a boundary flux.

- **(1+1)-dimensional PDEs:** $v = v(x,t)$, only one CL type.

  Local form:

$$\boxed{\mathrm{D}_t T[v] + \mathrm{D}_x \Psi[v] = 0.}$$

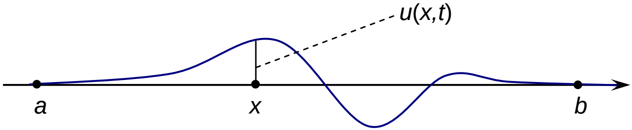
  Global form:

$$\boxed{\frac{d}{dt}\int_a^b T[v]\,dx = -\Psi[v]\Big|_a^b.}$$

Local and global conservation laws – PDE example

(1+1)-dimensional linear wave equation:

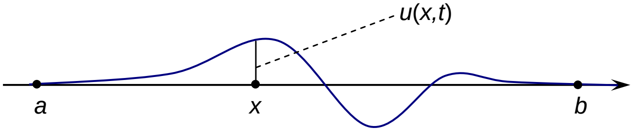
$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad c^2 = \tau/\rho, \quad a < x < b \text{ or } -\infty < x < \infty.$$



Local and global conservation laws – PDE example

(1+1)-dimensional linear wave equation:

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad c^2 = \tau/\rho, \quad a < x < b \ \text{ or } \ -\infty < x < \infty.$$



- A local CL – momentum conservation: $\mathrm{D}_t(\rho u_t) - \mathrm{D}_x(\tau u_x) = 0$.
- Global form:

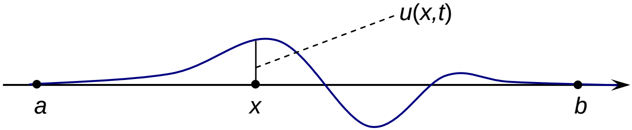
$$\frac{d}{dt} m = \frac{d}{dt} \int_a^b \rho u_t \, dx = \tau u_x \Big|_a^b.$$

- $dm/dt = 0$ for zero Neumann BCs $\to$ the momentum is conserved, $m = \mathrm{const}$.
- (E.g., a finite perturbation of an infinite string.)

Local and global conservation laws – PDE example

(1+1)-dimensional linear wave equation:

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad c^2 = \tau/\rho, \quad a < x < b \ \text{ or } \ -\infty < x < \infty.$$



- A local CL – energy conservation: $\mathrm{D}_t \left(\frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2} \right) - \mathrm{D}_x(\tau u_t u_x) = 0.$
- Global form:

$$\frac{d}{dt} E = \frac{d}{dt} \int \left(\frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2} \right) dx = \tau u_t u_x \Big|_a^b.$$

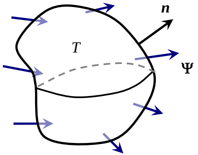
- For which BCs is $E = \mathrm{const}$?

- **(3+1)-dimensional PDEs:** $R[v] = 0, \;\; v = v(t, x, y, z)$.

- Local form: $\boxed{D_t T[v] + \mathrm{Div}\, \mathbf{\Psi}[v] = 0} \qquad \Leftrightarrow \qquad \boxed{\mathrm{D}_i \Phi^i[v] = 0}$

- Global form: $\boxed{\dfrac{d}{dt} \displaystyle\int_{\mathcal{V}} T \, dV = - \oint_{\partial \mathcal{V}} \mathbf{\Psi} \cdot d\mathbf{S}}$

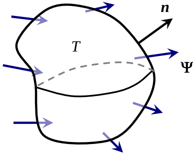
- Holds for all solutions $v(t, x, y, z)$, in some physical domain $\mathcal{V}$.



- **Example:** conservation of mass, gas/fluid dynamics.

- Local form: $\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \quad$ (A).

- Global form: $\dfrac{d}{dt} M = \dfrac{d}{dt} \displaystyle\int_{\mathcal{V}} \rho \, dV = - \oint_{\partial \mathcal{V}} \rho \mathbf{u} \cdot d\mathbf{S}$.



- Note: conservation laws are coordinate-independent (i.e., the divergence form (A) is invariant).

Material conservation laws

- For incompressible flows with velocity field $\mathbf{u}$, $\operatorname{div}\mathbf{u} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} T \equiv \mathrm{D}_t T + \mathbf{u}\cdot\nabla T = \mathrm{D}_t T + \operatorname*{div}_{x,y,\ldots}\Big(T\mathbf{u}\Big) = 0.$$

- T is conserved in a domain $\mathcal{V}(t)$ moving with the flow:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} T\, dV = 0.$$

- **Example:** conservation of mass in an incompressible flow:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = \mathrm{D}_t\rho + \mathbf{u}\cdot\nabla\rho = 0;$$

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_{\mathcal{V}(t)} \rho\, dV = 0.$$

Applications of Conservation Laws

Applications to ODEs

- Constants of motion:

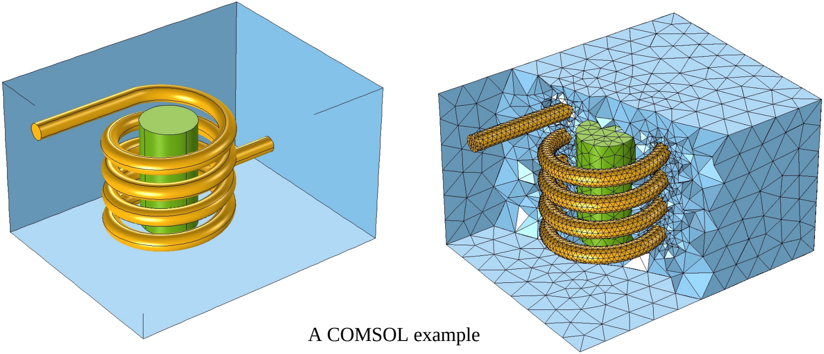
$$\mathrm{D}_t T[v] = 0 \quad \Rightarrow \quad T[v] = \mathrm{const.}$$

- Reduction of order / integration.

Applications to PDEs

$$D_t T[v] + \mathrm{Div}\,\mathbf{\Psi}[v] = 0$$

- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Divergence forms of PDEs for analysis: existence, uniqueness, stability, Fokas method.
- Weak solutions.
- Potentials, stream functions, etc.
- An infinite number of CLs may indicate integrability/linearization.
- Numerical methods: divergence forms of PDEs (finite-element, finite volume); constants of motion.



A COMSOL example

Coordinate invariance of CLs

Coordinate invariance of CLs

Given PDE system:

- Variables: $v = (v^1(z), ..., v^m(z))$, $z = (z^1, ..., z^n)$
- PDEs: $G[v] = 0$
- Local CL: $\mathrm{D}_{z^i}\Phi^i[v] = 0$

Point transformation: $(z, v(z)) \to (y, u(y))$

$$\begin{array}{rcll} y^i & = & y^i(z, v), & i = 1, \ldots, n, \\ u^\mu & = & u^\mu(z, v), & \mu = 1, \ldots, m, \end{array}, \qquad \frac{Du}{Dv} \neq 0.$$

Transformed PDE system:

- PDEs: $S[u(y)] = 0$
- Divergence expressions: $\mathrm{D}_{z^i}\Phi^i[v] = \mathrm{J} \cdot \mathrm{D}_{y^j}\Psi^j[u]$, $\quad \mathrm{J} = \dfrac{D(y^1, \ldots, y^n)}{D(z^1, \ldots, z^n)}$.
- Local CL: $\mathrm{D}_{y^j}\Psi^j[u] = 0$

Systematic computation of conservation laws: the direct (multiplier) method

The idea of the direct (multiplier) CL construction method

Independent and dependent variables of the problem:
$z = (z^1, ..., z^n), \quad v = v(z) = (v^1, ..., v^m).$

Definition

The Euler operator with respect to an arbitrary function v^j:

$$\mathrm{E}_{v^j} = \frac{\partial}{\partial v^j} - \mathrm{D}_i \frac{\partial}{\partial v^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial v^j_{i_1 \ldots i_s}} + \cdots, \quad j = 1, \ldots, m.$$

Theorem

The equations

$$\mathrm{E}_{v^j} F[v] \equiv 0, \quad j = 1, \ldots, m$$

hold for arbitrary $v(z)$ *if and only if* F *is a divergence:*

$$F[v] \equiv \mathrm{D}_i \Phi^i$$

for some functions $\Phi^i = \Phi^i[v]$.

The direct (multiplier) method

Given:

- A system of M DEs $G^{\sigma}[v] = 0, \quad \sigma = 1, \ldots, M.$
- Variables: $z = (z^1, ..., z^n), \quad v = (v^1(z), ..., v^m(z)).$

The direct (multiplier) method

Given:

- A system of M DEs $G^\sigma[v] = 0, \quad \sigma = 1, \ldots, M$.
- Variables: $z = (z^1, ..., z^n), \quad v = (v^1(z), ..., v^m(z))$.

The direct (multiplier) method

1. Specify the dependence of multipliers: $\Lambda_\sigma[v] = \Lambda_\sigma(z, v, \partial v, ...)$.
2. Solve the set of determining equations $\mathrm{E}_{v^j}(\Lambda_\sigma[v] G^\sigma[v]) \equiv 0, \quad j = 1, \ldots, m$, for <u>arbitrary</u> $v(z)$, to find all sets of multipliers.
3. Find the corresponding fluxes $\Phi^i[v]$ satisfying the identity
$$\Lambda_\sigma[v] G^\sigma[v] \equiv \mathrm{D}_i \Phi^i[v].$$
4. For each set of fluxes, on solutions, get a local conservation law
$$\boxed{\mathrm{D}_i \Phi^i[v] = 0.}$$
5. Implemented in GeM module for Maple (A.C. – see my web page)

Completeness of the multiplier method

Extended Kovalevskaya form

A PDE system $G[v] = 0$ is in *extended Kovalevskaya form* with respect to an independent variable z^j, if the system is solved for the highest derivative of each dependent variable with respect to z^j, i.e.,

$$\frac{\partial^{s_\sigma}}{\partial (z^j)^{s_\sigma}} v^\sigma = G^\sigma(z, v, \partial v, \ldots, \partial^k v), \quad 1 \leq s_\sigma \leq k, \quad \sigma = 1, \ldots, m,$$

where all derivatives with respect to z^j appearing in the right-hand side of each PDE above are of lower order than those appearing on the left-hand side.

Completeness of the multiplier method

Extended Kovalevskaya form

A PDE system $G[v]=0$ is in *extended Kovalevskaya form* with respect to an independent variable z^j, if the system is solved for the highest derivative of each dependent variable with respect to z^j, i.e.,

$$\frac{\partial^{s_\sigma}}{\partial (z^j)^{s_\sigma}} v^\sigma = G^\sigma(z, v, \partial v, \ldots, \partial^k v), \quad 1 \leq s_\sigma \leq k, \quad \sigma = 1, \ldots, m,$$

where all derivatives with respect to z^j appearing in the right-hand side of each PDE above are of lower order than those appearing on the left-hand side.

Theorem [*M. Alonso (1979)*]

Let $G[v]=0$ be a PDE system in the extended Kovalevskaya form. Then every its local CL equivalence class has a representative in the characteristic form,

$$\Lambda_\sigma[v] G^\sigma[v] \equiv \mathrm{D}_i \Phi^i[v] = 0,$$

such that $\{\Lambda_\sigma[v]\}$ do not involve the leading derivatives or their differential consequences.
[Hence one can safely use nonsingular multipliers!]

Completeness of the multiplier method

Extended Kovalevskaya form

A PDE system $G[v] = 0$ is in *extended Kovalevskaya form* with respect to an independent variable z^j, if the system is solved for the highest derivative of each dependent variable with respect to z^j, i.e.,

$$\frac{\partial^{s_\sigma}}{\partial (z^j)^{s_\sigma}} v^\sigma = G^\sigma(z, v, \partial v, \ldots, \partial^k v), \quad 1 \le s_\sigma \le k, \quad \sigma = 1, \ldots, m,$$

where all derivatives with respect to z^j appearing in the right-hand side of each PDE above are of lower order than those appearing on the left-hand side.

Example

The KdV equation

$$R[u] = u_t + u u_x + u_{xxx} = 0$$

has the extended Kovalevskaya form with respect to t ($u_t = \ldots$) or x ($u_{xxx} = \ldots$).

Completeness of the multiplier method

- For systems in the extended Kovalevskaya form, the multiplier method is complete (to any fixed order of derivatives).
- The multiplier method does not predict maximum CL order.
- For systems in a solved form but not in the extended Kovalevskaya form, multipliers may involve leading derivatives/their differential consequences.
- In practice, even if the extended Kovalevskaya form exists for a given system, it may be too complex to work with.
- One may use the multiplier method on non-Kovalevskaya systems to get partial CL results.

Conservation laws of Euler and NS equations in 3+1 dimensions

Navier-Stokes equations for a constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \, \Delta \mathbf{u} = 0. \qquad \text{(A)}$$

- No higher-order CLs [Gusyatnikova & Yumaguzhin (1989)].

The **complete list** of local CLs of (A) (e.g., [Batchelor (2000); A.C. and M. Oberlack (2014)]):

- Generalized continuity equation: $\nabla \cdot (k(t)\,\mathbf{u}) = 0$

Navier-Stokes equations for a constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu\, \Delta \mathbf{u} = 0. \qquad \text{(A)}$$

- No higher-order CLs [Gusyatnikova & Yumaguzhin (1989)].

The **complete list** of local CLs of (A) (e.g., [Batchelor (2000); A.C. and M. Oberlack (2014)]):

- Generalized momentum in $x-$direction (same for y, z):

$$\frac{\partial}{\partial t}(f(t)u^1) + \frac{\partial}{\partial x}\Big((u^1 f(t) - xf'(t))u^1 + f(t)(p - \nu u^1_x)\Big)$$
$$+\frac{\partial}{\partial y}\Big((u^1 f(t) - xf'(t))u^2 - \nu f(t) u^1_y\Big)$$
$$+\frac{\partial}{\partial z}\Big((u^1 f(t) - xf'(t))u^3 - \nu f(t) u^1_z\Big) = 0$$

Navier-Stokes equations for a constant-density fluid

$$\nabla\cdot\mathbf{u}=0,\qquad \mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\nu\,\Delta\mathbf{u}=0.\qquad\text{(A)}$$

- No higher-order CLs [Gusyatnikova & Yumaguzhin (1989)].

The **complete list** of local CLs of (A) (e.g., [Batchelor (2000); A.C. and M. Oberlack (2014)]):

- Angular momentum in $x-$direction (same for y,z):

$$\begin{aligned}&\frac{\partial}{\partial t}(zu^2-yu^3)+\frac{\partial}{\partial x}\Big((zu^2-yu^3)u^1+\nu(yu^3_x-zu^2_x)\Big)\\&+\frac{\partial}{\partial y}\Big((zu^2-yu^3)u^2+zp+\nu(yu^3_y-zu^2_y-u^3)\Big)\\&+\frac{\partial}{\partial z}\Big((zu^2-yu^3)u^3-yp+\nu(yu^3_z-zu^2_z+u^2)\Big)=0\end{aligned}$$

(Angular momentum vector: $\mathbf{P}=\mathbf{r}\times\mathbf{u}$.)

Euler equations, constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0. \qquad \text{(B)}$$

- Classical local CLs (below) known for a long time.
- No upper limit for the CL order has been established to date.

Local CLs of Euler equations (B) (e.g., [Batchelor (2000); A.C. and M. Oberlack (2014)]):

- Generalized continuity equation: $\nabla \cdot (k(t)\,\mathbf{u}) = 0$.
- Generalized momentum in x, y, z (same as NS with $\nu = 0$).
- Angular momentum in x, y, z (same as NS with $\nu = 0$).

Euler equations, constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0. \qquad \text{(B)}$$

- Classical local CLs (below) known for a long time.
- No upper limit for the CL order has been established to date.

Local CLs of Euler equations (B) (e.g., [Batchelor (2000); A.C. and M. Oberlack (2014)]):

- Conservation of kinetic energy:

$$\frac{\partial}{\partial t} K + \nabla \cdot \Big((K + p)\, \mathbf{u} \Big) = 0, \qquad K = \frac{1}{2} |\mathbf{u}|^2.$$

Euler equations, constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0. \qquad \text{(B)}$$

- Classical local CLs (below) known for a long time.
- No upper limit for the CL order has been established to date.

Local CLs of Euler equations (B) (e.g., [Batchelor (2000); A.C. and M. Oberlack (2014)]):

- Conservation of helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega};$$

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0,$$

where $E = \frac{1}{2}|\mathbf{u}|^2 + p$ is total energy density,

and $\boldsymbol{\omega} = \mathrm{curl}\, \mathbf{u}$ is vorticity.

Conservation laws of Euler equations in 3+1 dimensions

Euler equations, constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0. \qquad \text{(B)}$$

- Classical local CLs (below) known for a long time.
- No upper limit for the CL order has been established to date.

Local CLs of Euler equations (B) (e.g., [Batchelor (2000); A.C. and M. Oberlack (2014)]):

- Euler equations in vorticity formulation: $\nabla \cdot \mathbf{u} = 0$, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, hence

$$\nabla \cdot \boldsymbol{\omega} = 0, \qquad \boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = 0.$$

- Three components of vorticity $\boldsymbol{\omega}$ are conserved.

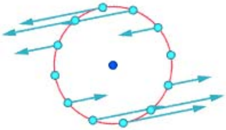
Euler classical two-component plane flow:

- Two-component, Cartesian 2D Euler equations:

$$\begin{aligned}
&(u^x)_x + (u^y)_y = 0,\\
&(u^x)_t + u^x (u^x)_x + u^y (u^x)_y = -p_x,\\
&(u^y)_t + u^x (u^y)_x + u^y (u^y)_y = -p_y,\\
&u^z = 0.
\end{aligned}$$

- Scalar vorticity equation: $\omega^x = \omega^y = 0$, $\omega^z = -(u^x)_y + (u^y)$,

$$(\omega^z)_t + u^x (\omega^z)_x + u^y (\omega^z)_y = 0.$$



Euler equations in 2+1 dimensions; conservation of enstrophy

Euler classical two-component plane flow:

- Two-component, Cartesian 2D Euler equations:

$$\begin{aligned} &(u^x)_x + (u^y)_y = 0,\\ &(u^x)_t + u^x(u^x)_x + u^y(u^x)_y = -p_x,\\ &(u^y)_t + u^x(u^y)_x + u^y(u^y)_y = -p_y,\\ &u^z = 0. \end{aligned}$$

- Scalar vorticity equation: $\omega^x = \omega^y = 0$, $\omega^z = -(u^x)_y + (u^y)_x$,

$$(\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y = 0.$$

Enstrophy Conservation

- Enstrophy: $\mathcal{E} = |\boldsymbol{\omega}|^2 = (\omega^z)^2$.
- Material conservation law: $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathrm{D}_t\,\mathcal{E} + \mathrm{D}_x\,(u^x\mathcal{E}) + \mathrm{D}_y\,(u^y\mathcal{E}) = 0.$
- Was commonly known to hold for plane flows, $(2+1)$-dimensions.

Helical invariance and helical reduction of Euler and NS equations

Navier-Stokes equations for a constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\,\Delta\mathbf{u} = 0. \qquad \text{(A)}$$

- A symmetry – translations in z: $z \to z + z_0$ (similarly in x and y, as well as t).
- Symmetry reduction: $p, p^i(t, x, y, z) \;\to\; p, u^i(t, x, y)$.
- In case of *additional time independence*, for Euler equations ($\nu = 0$), get a single PDE

  $$\xi_{xx} + \xi_{yy} = -I(\xi)I'(\xi) - P'(\xi),$$

  where $\xi = \xi(x, y)$ is the stream function,

  $$\mathbf{u} = -\xi_y\mathbf{e}_x + \xi_x\mathbf{e}_y + I(\xi)\mathbf{e}_z, \quad p = p(\xi),$$

  and $I(\xi)$ and $p(\xi)$ are arbitrary functions.

Navier-Stokes equations for a constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\,\Delta\mathbf{u} = 0. \qquad \text{(A)}$$

- A symmetry – rotations around the z-axis (translations in cylindrical angle φ): $\varphi \to \varphi + \varphi_0$.
- Symmetry reduction: $p, u^i(t, x, y, z) \;\to\; p, u^i(t, r, z)$.
- In case of additional time independence, for Euler equations ($\nu = 0$), get a single PDE – Grad-Safranov (Bragg-Hawthorne) equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi),$$

where $\psi = \psi(r, z)$ is the stream function,

$$\mathbf{u} = \frac{\psi_z}{r}\mathbf{e}_r + \frac{I(\psi)}{r}\mathbf{e}_\varphi - \frac{\psi_r}{r}\mathbf{e}_z, \quad p = p(\psi),$$

and $I(\psi)$ and $p(\psi)$ are arbitrary functions.

Navier-Stokes equations for a constant-density fluid

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \, \Delta \mathbf{u} = 0. \qquad \text{(A)}$$

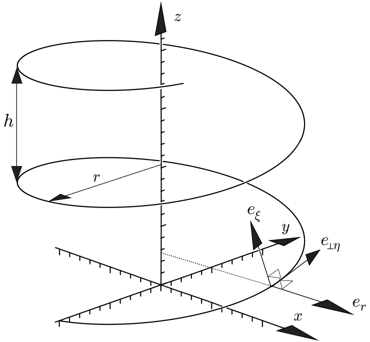
- A symmetry – combination of rotations in $x - y$ plane and translations in z.
- Cylindrical coordinates: (r, φ, z). Helical coordinates: (r, η, ξ):

$$\xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \qquad a, b = \text{const}, \quad a^2 + b^2 > 0.$$

- Symmetry reduction: $p, u^i(t, x, y, z) \;\rightarrow\; p, u^i(t, r, \xi)$.
- In case of additional time independence, for Euler equations ($\nu = 0$), get a single PDE – JFKO equation (similar to Bragg-Hawthorne).

Additional CLs for helically symmetric Euler and NS equations

Helical coordinates

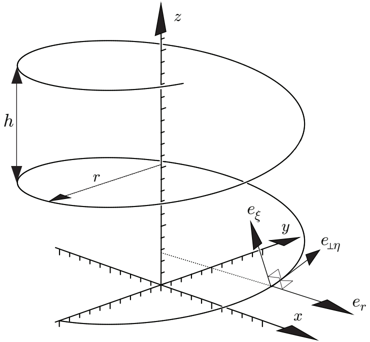


Helical Coordinates

- Cylindrical coordinates: (r, φ, z). **Helical coordinates:** (r, η, ξ)

$$\xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \qquad a, b = \text{const}, \quad a^2 + b^2 > 0.$$

Helical coordinates

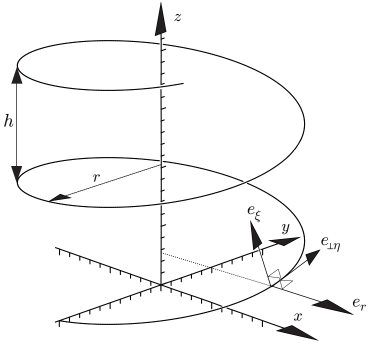


Orthogonal Basis

$$\mathbf{e}_r = \frac{\nabla r}{|\nabla r|}, \quad \mathbf{e}_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad \mathbf{e}_{\perp\eta} = \frac{\nabla_\perp \eta}{|\nabla_\perp \eta|} = \mathbf{e}_\xi \times \mathbf{e}_r.$$

- Scaling factors: $H_r = 1, H_\eta = r, H_\xi = B(r), \qquad B(r) = \dfrac{r}{\sqrt{a^2 r^2 + b^2}}.$

Helical coordinates

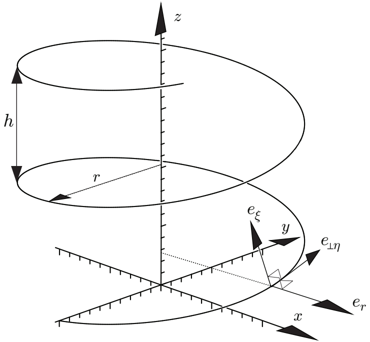


Vector expansion

$$\mathbf{u} \;=\; u^r\mathbf{e}_r + u^\varphi\mathbf{e}_\varphi + u^z\mathbf{e}_z \;=\; u^r\mathbf{e}_r + u^\eta\mathbf{e}_{\perp\eta} + u^\xi\mathbf{e}_\xi.$$

$$u^\eta = \mathbf{u}\cdot\mathbf{e}_{\perp\eta} = B\left(au^\varphi - \frac{b}{r}u^z\right), \qquad u^\xi = \mathbf{u}\cdot\mathbf{e}_\xi = B\left(\frac{b}{r}u^\varphi + au^z\right).$$

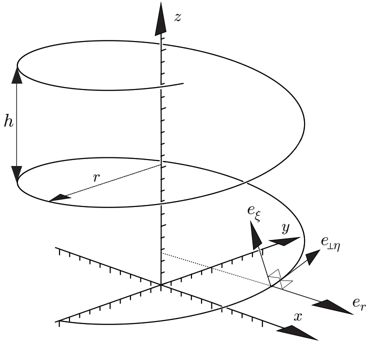
Helical coordinates



Helical invariance: generalizes axal and translational invariance

- Helical coordinates: $r, \quad \xi = az + b\varphi, \quad \eta = a\varphi - bz/r^2.$
- **General helical symmetry:** $f = f(r, \xi), \quad a, b \neq 0.$
- **Axial:** $a = 1, \; b = 0.$ $\quad$ ***z*-Translational:** $a = 0, \; b = 1.$

Helical coordinates



Details:

O. Kelbin, A. Cheviakov, and M. Oberlack (2013)
New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. *JFM* 721, 340-366.

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

Helically invariant Navier-Stokes equations

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

Continuity:

$$\frac{1}{r} u^r + (u^r)_r + \frac{1}{B} (u^\xi)_\xi = 0$$

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

r-momentum:

$$(u^r)_t + u^r (u^r)_r + \frac{1}{B} u^\xi (u^r)_\xi - \frac{B^2}{r}\left(\frac{b}{r}u^\xi + a u^\eta\right)^2 = -p_r$$
$$+ \nu \left[\frac{1}{r}(r(u^r)_r)_r + \frac{1}{B^2}(u^r)_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2bB}{r^2}\left(a(u^\eta)_\xi + \frac{b}{r}(u^\xi)_\xi\right)\right]$$

Helically invariant Navier-Stokes equations

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

η-momentum:

$$(u^\eta)_t + u^r (u^\eta)_r + \frac{1}{B} u^\xi (u^\eta)_\xi + \frac{a^2 B^2}{r} u^r u^\eta$$
$$= \nu \left[\frac{1}{r}(r(u^\eta)_r)_r + \frac{1}{B^2}(u^\eta)_{\xi\xi} + \frac{a^2 B^2 (a^2 B^2 - 2)}{r^2} u^\eta + \frac{2abB}{r^2} \left((u^r)_\xi - \left(B u^\xi \right)_r \right) \right]$$

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

ξ-momentum:

$$\begin{aligned}(u^\xi)_t + u^r (u^\xi)_r + \frac{1}{B} u^\xi (u^\xi)_\xi + \frac{2abB^2}{r^2} u^r u^\eta + \frac{b^2 B^2}{r^3} u^r u^\xi = -\frac{1}{B} p_\xi \\ + \nu \left[\frac{1}{r}(r(u^\xi)_r)_r + \frac{1}{B^2}(u^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2} u^\xi + \frac{2bB}{r} \left(\frac{b}{r^2}(u^r)_\xi + \left(\frac{aB}{r} u^\eta \right)_r \right) \right]\end{aligned}$$

Navier-Stokes equations, vorticity formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times \, (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

Navier-Stokes equations, vorticity formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

Vorticity definition:

$$\omega^r = -\frac{1}{B}(u^\eta)_\xi,$$

$$\omega^\eta = \frac{1}{B}(u^r)_\xi - \frac{1}{r}\left(r u^\xi\right)_r - \frac{2abB^2}{r^2}u^\eta + \frac{a^2B^2}{r}u^\xi,$$

$$\omega^\xi = (u^\eta)_r + \frac{a^2B^2}{r}u^\eta$$

Helically invariant vorticity formulation

Navier-Stokes equations, vorticity formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

r-vorticity:

$$\begin{aligned}(\omega^r)_t + u_r(\omega^r)_r + \frac{1}{B} u^\xi (\omega^r)_\xi &= \omega^r (u^r)_r + \frac{1}{B}\omega^\xi (u^r)_\xi \\ &+ \nu \left[\frac{1}{r}(r(\omega^r)_r)_r + \frac{1}{B^2}(\omega^r)_{\xi\xi} - \frac{1}{r^2}\omega^r - \frac{2bB}{r^2}\left(a(\omega^\eta)_\xi + \frac{b}{r}(\omega^\xi)_\xi \right) \right]\end{aligned}$$

Navier-Stokes equations, vorticity formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times \left(\boldsymbol{\omega} \times \mathbf{u}\right) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

η-vorticity:

$$\begin{aligned}
&(\omega^\eta)_t + u^r(\omega^\eta)_r + \frac{1}{B} u^\xi (\omega^\eta)_\xi \\
&\quad - \frac{a^2 B^2}{r}(u^r \omega^\eta - u^\eta \omega^r) + \frac{2abB^2}{r^2}(u^\xi \omega^r - u^r \omega^\xi) = \omega^r (u^\eta)_r + \frac{1}{B}\omega^\xi (u^\eta)_\xi \\
&\quad + \nu \left[\frac{1}{r}(r(\omega^\eta)_r)_r + \frac{1}{B^2}(\omega^\eta)_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2}\omega^\eta + \frac{2abB}{r^2}\left((\omega^r)_\xi - \left(B\omega^\xi\right)_r \right) \right]
\end{aligned}$$

Helically invariant vorticity formulation

Navier-Stokes equations, vorticity formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

ξ-vorticity:

$$\begin{aligned}(\omega^\xi)_t + u^r(\omega^\xi)_r + \frac{1}{B} u^\xi (\omega^\xi)_\xi \\ + \frac{1 - a^2B^2}{r}(u^\xi \omega^r - u^r \omega^\xi) = \omega^r (u^\xi)_r + \frac{1}{B}\omega^\xi (u^\xi)_\xi \\ + \nu \left[\frac{1}{r}(r(\omega^\xi)_r)_r + \frac{1}{B^2}(\omega^\xi)_{\xi\xi} + \frac{a^4B^4 - 1}{r^2}\omega^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}(\omega^r)_\xi + \left(\frac{aB}{r}\omega^\eta \right)_r \right) \right]\end{aligned}$$

For helically symmetric flows:

- Seek local conservation laws

$$\frac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{\Phi} \equiv \frac{\partial T}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{1}{B}\frac{\partial \Phi^\xi}{\partial \xi} = 0$$

using divergence expressions

$$\frac{\partial \Gamma^1}{\partial t} + \frac{\partial \Gamma^2}{\partial r} + \frac{\partial \Gamma^3}{\partial \xi} = r\left[\frac{\partial}{\partial t}\left(\frac{\Gamma^1}{r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Gamma^2}{r}\right) + \frac{1}{B}\frac{\partial}{\partial \xi}\left(\frac{B}{r}\Gamma^3\right)\right] = 0,$$

i.e.,

$$T \equiv \frac{\Gamma^1}{r}, \quad \Phi^r \equiv \frac{\Gamma^2}{r}, \quad \Phi^\xi \equiv \frac{B}{r}\Gamma^3.$$

- 1st-order multipliers in primitive variables.
- 0th-order multipliers in vorticity formulation.

Conservation laws for helically symmetric Euler flows: $\nu = 0$

Primitive variables - EP1 - kinetic energy

$$T = K, \quad \Phi^r = u^r(K+p), \quad \Phi^\xi = u^\xi(K+p), \qquad K = \frac{1}{2}|\mathbf{u}|^2.$$

Primitive variables - EP2 - z-momentum

$$T = B\left(-\frac{b}{r}u^\eta + au^\xi\right) = u^z, \quad \Phi^r = u^r u^z, \quad \Phi^\xi = u^\xi u^z + aBp.$$

Primitive variables - EP3 - z-angular momentum

$$T = rB\left(au^\eta + \frac{b}{r}u^\xi\right) = ru^\varphi, \quad \Phi^r = ru^r u^\varphi, \quad \Phi^\xi = ru^\xi u^\varphi + bBp.$$

Primitive variables - EP4 - generalized momenta/angular momenta (NEW)

$$T = F\left(\frac{r}{B}u^\eta\right), \quad \Phi^r = u^r F\left(\frac{r}{B}u^\eta\right), \quad \Phi^\xi = u^\xi F\left(\frac{r}{B}u^\eta\right),$$

where $F(\cdot)$ is an arbitrary function.

Vorticity formulation - EV1 - conservation of helicity

Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi.$$

The conservation law:

$$\begin{aligned} T &= h, \\ \Phi^r &= \omega^r \left(E - (u^\eta)^2 - \left(u^\xi\right)^2 \right) + u^r \left(h - u^r \omega^r \right), \\ \Phi^\xi &= \omega^\xi \left(E - (u^r)^2 - (u^\eta)^2 \right) + u^\xi \left(h - u^\xi \omega^\xi \right), \end{aligned}$$

where

$$E = \frac{1}{2}|\mathbf{u}|^2 + p = \frac{1}{2}\left((u^r)^2 + (u^\eta)^2 + \left(u^\xi\right)^2 \right) + p$$

is the total energy density. In vector notation:

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0.$$

Vorticity formulation - EV2 - generalized helicity (NEW)

Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi .$$

$$\frac{\partial}{\partial t}\left(h\, H\left(\frac{r}{B}u^\eta\right)\right)+\nabla\cdot\left[H\left(\frac{r}{B}u^\eta\right)[\mathbf{u}\times\nabla E+(\boldsymbol{\omega}\times\mathbf{u})\times\mathbf{u}]+Eu^\eta\mathbf{e}_{\perp\eta}\times\nabla H\left(\frac{r}{B}u^\eta\right)\right]=0$$

for an arbitrary function $H = H(\cdot)$.

Conservation laws for helically symmetric Euler flows: $\nu = 0$

Vorticity formulation - EV3 - vorticity conservation laws (NEW)

$$
\begin{aligned}
T &= \frac{Q(t)}{r}\omega^{\varphi},\\
\Phi^{r} &= \frac{1}{r}\left(Q(t)[u^{r}\omega^{\varphi} - \omega^{r}u^{\varphi}] + Q'(t)u^{z}\right),\\
\Phi^{\xi} &= -\frac{aB}{r}\left(Q(t)\left[u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}\right] + Q'(t)u^{r}\right),
\end{aligned}
$$

where $Q(t)$ is an arbitrary function.

Vorticity formulation - EV4 - vorticity conservation law (NEW)

$$
\begin{aligned}
T \quad &= -rB\left(a^{3}\omega^{\eta} - \frac{b^{3}}{r^{3}}\omega^{\xi}\right),\\
\Phi^{r} \quad &= -2a^{2}u^{r}u^{z} - a^{3}Br\left(u^{r}\omega^{\eta} - u^{\eta}\omega^{r}\right) + \frac{Bb^{3}}{r^{2}}\left(u^{r}\omega^{\xi} - u^{\xi}\omega^{r}\right),\\
\Phi^{\xi} \quad &= a^{3}B\left[(u^{r})^{2} + (u^{\eta})^{2} - (u^{\xi})^{2} + r\left(u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}\right)\right] + \frac{2a^{2}bB}{r}u^{\eta}u^{\xi}.
\end{aligned}
$$

Vorticity formulation - EV5 - vorticity conservation law (NEW)

$$T = -\frac{B}{r^2}\left(\frac{b^2 r^2}{B^2}\omega^\xi + a^3 r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2 r^2}{B^2}\omega^\xi + \frac{a^3 r^4}{B}\omega^z\right),$$

$$\begin{aligned}\Phi^r =& a^3 rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right)\\ &- \frac{a^4 r^4 + a^2 r^2 b^2 + b^4}{r\sqrt{a^2 r^2 + b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right),\\ \Phi^\xi =& -a^3 bB\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4 rBu^\eta u^\xi.\end{aligned}$$

Vorticity formulation - EV6 - vorticity conservation law (NEW)

$$\nabla \cdot \mathbf{\Phi} = 0, \quad \Phi^r = N\omega^r - \frac{1}{B}N_\xi u^\eta, \quad \Phi^\xi = N\omega^\xi,$$

for an arbitrary $N(t, \xi)$.

- Generalization of the obvious divergence expression $\nabla \cdot (G(t)\boldsymbol{\omega}) = 0$.

Conservation laws for helically symmetric viscous (NS) flows

Primitive variables - NSP1 - z-momentum.

$$T = u^z, \quad \Phi^r = u^r u^z - \nu (u^z)_r, \quad \Phi^\xi = u^\xi u^z + aBp - \frac{\nu}{B}(u^z)_\xi.$$

Primitive variables - NSP2 - generalized momentum (NEW)

$$T = \frac{r}{B} u^\eta,$$

$$\begin{aligned}
\Phi^r &= \frac{r}{B} u^r u^\eta - \nu \left[-2aB \left(a u^\eta + 2\frac{b}{r} u^\xi \right) + \left(\frac{r}{B} u^\eta \right)_r \right] \\
&= \frac{r}{B} u^r u^\eta - \nu \left[-2a u^\varphi + \left(\frac{r}{B} u^\eta \right)_r \right],
\end{aligned}$$

$$\Phi^\xi = \frac{r}{B} u^\eta u^\xi - \nu \frac{1}{B} \left[\frac{2abB^2}{r} u^r + \left(\frac{r}{B} u^\eta \right)_\xi \right].$$

Vorticity formulation - NSV1 - family of vorticity conservation laws (NEW)

$$T = \frac{Q(t)}{r} B\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) = \frac{Q(t)}{r}\omega^\varphi,$$

$$\Phi^r = \frac{1}{r}\Big\{Q(t)\left[u^r B\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) - \omega^r B\left(au^\eta + \frac{b}{r}u^\xi\right)\right] + Q'(t)B\left(-\frac{b}{r}u^\eta + au^\xi\right)$$
$$-Q(t)\nu\left[\frac{aB}{r}\omega^\eta + \frac{b^2 B}{r\left(a^2 r^2 + b^2\right)}\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) + B\left(a\omega^\eta_r + \frac{b}{r}\omega^\xi_r\right)\right]\Big\},$$

$$\Phi^\xi = -\frac{B}{r}\Big\{aQ(t)\left[u^\eta\omega^\xi - u^\xi\omega^\eta\right] + aQ'(t)u^r$$
$$+\frac{Q(t)}{r^3}\nu\left[\frac{r^3}{B}\left(a\omega^\eta_\xi + \frac{b}{r}\omega^\xi_\xi\right) + 2br\omega^r\right]\Big\},$$

for an arbitrary function $Q(t)$.

Vorticity formulation - NSV2 - vorticity conservation law (NEW)

$$T = \quad -rB\left(a^3\omega^\eta - \frac{b^3}{r^3}\omega^\xi\right),$$

$$\Phi^r = \quad -\frac{B}{r^2}\left(a^3r^3\left(u^r\omega^\eta - u^\eta\omega^r\right) - b^3\left(u^r\omega^\xi - u^\xi\omega^r\right)\right) - 2a^2Bu^r\left(-\frac{b}{r}u^\eta + au^\xi\right)$$
$$-\frac{B}{r^2}\nu\left[\frac{r^2}{B^2}\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) - r^3\left(a^3\omega^\eta_r - \frac{b^3}{r^3}\omega^\xi_r\right) + abB^2r\left(\frac{b^3}{r^3}\omega^\eta + a^3\omega^\xi\right)\right],$$

$$\Phi^\xi = \quad a^3B\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + \frac{2a^2bB}{r}u^\eta u^\xi$$
$$+\frac{2a^2bB}{r}\nu\left[\left(1 - \frac{b^2}{a^2r^2}\right)\omega^r + \frac{r^2}{2a^2bB}\left(a^3\omega^\eta_\xi - \frac{b^3}{r^3}\omega^\xi_\xi\right)\right].$$

Vorticity formulation - NSV3 - vorticity conservation law (NEW)

$$T = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + a^3r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + \frac{a^3r^4}{B}\omega^z\right),$$

$$\begin{aligned}\Phi^r = \; & a^3rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right) \\ & -\frac{a^4r^4 + a^2r^2b^2 + b^4}{r\sqrt{a^2r^2+b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right) \\ & +\nu\left[4a^3B\left(au^\eta + \frac{b}{r}u^\xi\right) - a^3brB(\omega^\eta)_r + \frac{B}{r^3}\left(b^4 - a^4r^4 - \frac{a^6r^6}{a^2r^2+b^2}\right)\omega^\xi\right. \\ & \left.\quad + \frac{B}{r^2}\left(a^4r^4 + a^2r^2b^2 + b^4\right)(\omega^\xi)_r + \frac{ab}{B}\left(2 + \frac{a^4r^4}{(a^2r^2+b^2)^2}\right)\omega^\eta\right],\end{aligned}$$

$$\begin{aligned}\Phi^\xi = \; & -a^3bB\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4rBu^\eta u^\xi \\ & +\nu\left[\frac{1}{r^2}\left(a^4r^4 + a^2r^2b^2 + b^4\right)(\omega^\xi)_\xi - a^3br(\omega^\eta)_\xi - \frac{4a^3bB}{r}u^r + \frac{2b^4B}{r^3}\omega^r\right].\end{aligned}$$

Generalized enstrophy for inviscid plane flow (known)

$$T = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

Some conservation laws for two-component flows

Generalized enstrophy for inviscid plane flow (known)

$$T = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

Generalized enstrophy for inviscid axisymmetric flow (NEW)

$$T = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right)$$

for arbitrary $S(\cdot)$.

Some conservation laws for two-component flows

Generalized enstrophy for inviscid plane flow (known)

$$T = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

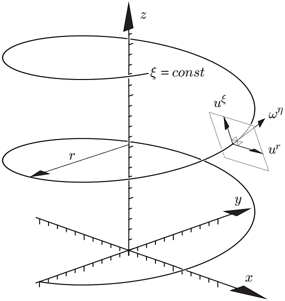
$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

Generalized enstrophy for inviscid axisymmetric flow (NEW)

$$T = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right)$$

for arbitrary $S(\cdot)$.

- Several additional new conservation laws for **plane** and **axisymmetric**, inviscid and viscous flows (details in paper).



Generalized enstrophy for general inviscid helical 2-component flow (NEW)

$$T = T\left(\frac{B}{r}\omega^\eta\right), \quad \Phi^r = u^r\, T\left(\frac{B}{r}\omega^\eta\right), \quad \Phi^\xi = u^\xi\, T\left(\frac{B}{r}\omega^\eta\right),$$

for an arbitrary $T(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\, T\left(\frac{B}{r}\omega^\eta\right) \equiv \mathrm{D}_t T + \mathbf{u}\cdot\nabla T = 0.$$

Helical CLs: results and open problems

Helically-invariant equations

- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

New conservation laws

- Three-component Euler:
  - Generalized momenta. Generalized helicity. Additional vorticity CLs.
- Three-component Navier-Stokes:
  - New CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - New CLs in viscous and inviscid case, for plane and axisymmetric flows.

Open problems

- Understand the nature of the new CLs.
- Explore the usefulness of the new CLs for numerical simulation and analysis (e.g., computing stability conditions for equilibria).

Exact solutions for helically invariant NS equations: Galilei symmetry

Paper 2: Conservation laws of NS and Euler equations under helical symmetry

D. Dierkes, A. Cheviakov, and M. Oberlack (2019, JFM, submitted)
New similarity reductions and exact solutions for helically symmetric viscous flows.

- Few exact closed-form solutions to Navier-Stokes equations are available, only for special settings.
- Helical flows: important in nature and applications.
- Time-dependent numerical solvers:
  Discontinuous Galerkin [F. Kummer, M. Oberlack *et al*] with helical symmetry capability.
- Need any sample exact helically symmetric solutions to test numerics, for local physical understanding etc.
- Local or global regularity in space and time is acceptable.

Helically invariant NS; their point symmetries

$$\frac{1}{r}u^r + u^r_r + \frac{1}{B}u^\xi_\xi = 0,$$

$$u^r_t + u^r u^r_r + \frac{1}{B}u^\xi u^r_\xi - \frac{B^2}{r}\left(\frac{b}{r}u^\xi + au^\eta\right)^2 = -p_r$$
$$+ \nu\left[\frac{1}{r}(ru^r_r)_r + \frac{1}{B^2}u^r_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2bB}{r^2}\left(au^\eta_\xi + \frac{b}{r}u^\xi_\xi\right)\right],$$

$$u^\eta_t + u^r u^\eta_r + \frac{1}{B}u^\xi u^\eta_\xi + \frac{a^2B^2}{r}u^r u^\eta$$
$$= \nu\left[\frac{1}{r}(ru^\eta_r)_r + \frac{1}{B^2}u^\eta_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2}u^\eta + \frac{2abB}{r^2}\left(u^r_\xi - \left(Bu^\xi\right)_r\right)\right],$$

$$u^\xi_t + u^r u^\xi_r + \frac{1}{B}u^\xi u^\xi_\xi + \frac{2abB^2}{r^2}u^r u^\eta + \frac{b^2B^2}{r^3}u^r u^\xi = -\frac{1}{B}p_\xi$$
$$+ \nu\left[\frac{1}{r}(ru^\xi_r)_r + \frac{1}{B^2}u^\xi_{\xi\xi} + \frac{a^4B^4-1}{r^2}u^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}u^r_\xi + \left(\frac{aB}{r}u^\eta\right)_r\right)\right],$$

- Point symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \xi}, X_3 = f(t)\frac{\partial}{\partial p}, \quad X_4 = t\frac{\partial}{\partial \xi} - \frac{b}{ar}B\frac{\partial}{\partial u^\eta} + B\frac{\partial}{\partial u^\xi}.$$

Helically invariant NS; their point symmetries

$$\frac{1}{r}u^r + u^r_r + \frac{1}{B}u^\xi_\xi = 0,$$

$$u^r_t + u^r u^r_r + \frac{1}{B}u^\xi u^r_\xi - \frac{B^2}{r}\left(\frac{b}{r}u^\xi + au^\eta\right)^2 = -p_r$$
$$+\nu\left[\frac{1}{r}(ru^r_r)_r + \frac{1}{B^2}u^r_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2bB}{r^2}\left(au^\eta_\xi + \frac{b}{r}u^\xi_\xi\right)\right],$$

$$u^\eta_t + u^r u^\eta_r + \frac{1}{B}u^\xi u^\eta_\xi + \frac{a^2B^2}{r}u^r u^\eta$$
$$= \nu\left[\frac{1}{r}(ru^\eta_r)_r + \frac{1}{B^2}u^\eta_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2}u^\eta + \frac{2abB}{r^2}\left(u^r_\xi - \left(Bu^\xi\right)_r\right)\right],$$

$$u^\xi_t + u^r u^\xi_r + \frac{1}{B}u^\xi u^\xi_\xi + \frac{2abB^2}{r^2}u^r u^\eta + \frac{b^2B^2}{r^3}u^r u^\xi = -\frac{1}{B}p_\xi$$
$$+\nu\left[\frac{1}{r}(ru^\xi_r)_r + \frac{1}{B^2}u^\xi_{\xi\xi} + \frac{a^4B^4-1}{r^2}u^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}u^r_\xi + \left(\frac{aB}{r}u^\eta\right)_r\right)\right],$$

- Solutions invariant with respect to Galilei symmetry X_4:

$$u^r = u^r(r,t), \quad u^\xi = F^\xi(r,t)\xi + G^\xi(r,t), \quad u^\eta = F^\eta(r,t)\xi + G^\eta(r,t), \quad p = p(r,t).$$

Helically invariant NS; their point symmetries

$$\frac{1}{r}u^r + u^r_r + \frac{1}{B}u^\xi_\xi = 0,$$

$$u^r_t + u^r u^r_r + \frac{1}{B}u^\xi u^r_\xi - \frac{B^2}{r}\left(\frac{b}{r}u^\xi + a u^\eta\right)^2 = -p_r$$
$$+\,\nu\left[\frac{1}{r}(r u^r_r)_r + \frac{1}{B^2}u^r_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2bB}{r^2}\left(a u^\eta_\xi + \frac{b}{r}u^\xi_\xi\right)\right],$$

$$u^\eta_t + u^r u^\eta_r + \frac{1}{B}u^\xi u^\eta_\xi + \frac{a^2B^2}{r}u^r u^\eta$$
$$= \nu\left[\frac{1}{r}(r u^\eta_r)_r + \frac{1}{B^2}u^\eta_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2}u^\eta + \frac{2abB}{r^2}\left(u^r_\xi - \left(B u^\xi\right)_r\right)\right],$$

$$u^\xi_t + u^r u^\xi_r + \frac{1}{B}u^\xi u^\xi_\xi + \frac{2abB^2}{r^2}u^r u^\eta + \frac{b^2B^2}{r^3}u^r u^\xi = -\frac{1}{B}p_\xi$$
$$+\,\nu\left[\frac{1}{r}(r u^\xi_r)_r + \frac{1}{B^2}u^\xi_{\xi\xi} + \frac{a^4B^4-1}{r^2}u^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}u^r_\xi + \left(\frac{aB}{r}u^\eta\right)_r\right)\right],$$

- Using this ansatz, and denoting $v(r,t) = r\,u^r(r,t)$, arrive at the v-equation

$$\boxed{v_{rt} + \left(\frac{v\,v_r}{r}\right)_r - 2\frac{v_r^2}{r} - \nu\left[v_{rrr} + \frac{v_r}{r^2} - \frac{v_{rr}}{r}\right] = 0.}$$

Solutions of the v-equation

The v-equation

$$v_{rt} + \left(\frac{v\, v_r}{r}\right)_r - 2\frac{v_r^2}{r} - \nu\left[v_{rrr} + \frac{v_r}{r^2} - \frac{v_{rr}}{r}\right] = 0.$$

- Solve using it symmetries: scaling and translation $Y_1 = r\dfrac{\partial}{\partial r} + 2t\dfrac{\partial}{\partial t}$, $Y_2 = \dfrac{\partial}{\partial t}$.
- Similarity variable: $s = \dfrac{r}{\sqrt{4\nu\,(t+t_0)}}$.
- Symmetry ansatz: $v = v(s)$.
- ODE: $s^3 v'' + 2s\,(v')^2 + s^2 v' - 2svv'' + 2vv' + \nu\left[2s^2 v''' - 2sv'' + 2v'\right] = 0.$
- **Solution family 1**: $v(r,t) = Ae^{-\frac{r^2}{4\nu(t+t_0)}}$, with free constant parameters A and t_0.
- **Solution family 2**: $v(r,t) = g(t) - \dfrac{r^2}{2\,(t+t_0)}$, where $g(t)$ is an arbitrary time-dependent function.

Solution family 1: details

Solution family 1

$$v(r,t) = Ae^{-\frac{r^2}{4\nu(t+t_0)}}$$

- In physical variables:

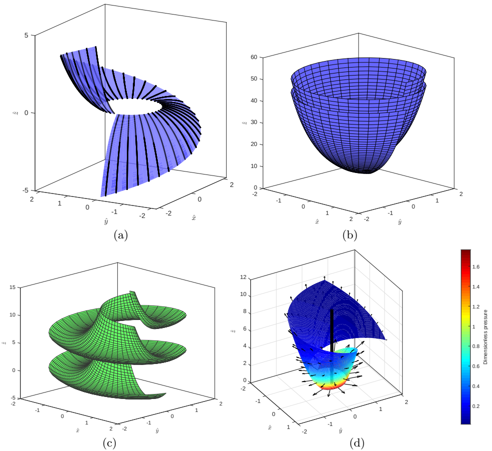
$$u^r = \frac{A}{r}\, e^{-\frac{r^2}{4\nu(t+t_0)}}, \qquad u^\eta = -\frac{AbB\xi}{2\nu ar\,(t+t_0)}\, e^{-\frac{r^2}{4\nu(t+t_0)}},$$

$$u^\xi = \frac{AB\xi}{2\nu\,(t+t_0)}\, e^{-\frac{r^2}{4\nu(t+t_0)}}, \qquad p = -\frac{A^2}{2r^2}\, e^{-\frac{r^2}{2\nu(t+t_0)}} + f(t),$$

where $f(t)$ is an arbitrary function of time.

- Singular on the axis $r = 0$, regular elsewhere.

Solution family 1 plots



(a) The streamlines emanating from the circle $z = 0$, $r = 1$.
(b) The velocity magnitude isosurface $|\mathbf{u}| = 10$, plotted for $0 \le \phi \le 4\pi$, $\xi \ge 0$.
(c) The vorticity magnitude isosurface $|\boldsymbol{\omega}| = 2$, plotted for $0 \le \phi \le 4\pi$, $\xi \ge 0$.
(d) the helical coordinate rectangle $\eta = -6$, $0.5 \le r \le 2$, $0 \le \xi \le 2\pi$ in the physical space, with velocity vectors and pressure p color map.

NS exact solutions II: exact linearization, Beltrami-type solutions

- The momentum equation in the NS model is often written in the form

$$\mathbf{u}_t + (\mathrm{curl}\,\mathbf{u}) \times \mathbf{u} + \nabla P - \nu\, \nabla^2 \mathbf{u} = 0,$$

where the modified pressure is given by $P = p + \dfrac{1}{2}|\mathbf{u}|^2$.

- *Beltrami flow* ansatz of vorticity and velocity collinearity: $\boldsymbol{\omega} \equiv \mathrm{curl}\,\mathbf{u} = \vartheta \mathbf{u}$.

- Remaining *linear* PDEs: $\mathrm{curl}\,\mathbf{u} = \vartheta \mathbf{u}$, plus the NS equations

$$\frac{1}{r} u^r + (u^r)_r + \frac{1}{B}(u^\xi)_\xi = 0,$$

$$(u^r)_t = -P_r + \nu \left[\frac{1}{r}(r(u^r)_r)_r + \frac{1}{B^2}(u^r)_{\xi\xi} - \frac{1}{r^2} u^r - \frac{2bB}{r^2} \left(a(u^\eta)_\xi + \frac{b}{r}(u^\xi)_\xi \right) \right],$$

$$(u^\eta)_t = \nu \left[\frac{1}{r}(r(u^\eta)_r)_r + \frac{1}{B^2}(u^\eta)_{\xi\xi} + \frac{a^2 B^2 (a^2 B^2 - 2)}{r^2} u^\eta + \frac{2abB}{r^2} \left((u^r)_\xi - \left(B u^\xi \right)_r \right) \right],$$

$$(u^\xi)_t = -\frac{1}{B} P_\xi + \nu \left[\frac{1}{r}(r(u^\xi)_r)_r + \frac{1}{B^2}(u^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2} u^\xi + \frac{2bB}{r} \left(\frac{b}{r^2}(u^r)_\xi + \left(\frac{aB}{r} u^\eta \right)_r \right) \right].$$

Exact linearization of helical NS and Beltrami-type solutions

- Separation of variables ansatz: $f(t,r,\xi) = T(t)\,R(r)\,\Xi(\xi)$.

- Separated solutions:

$$u^r = e^{-\nu Q^2 t}(K_1 \cos\lambda\xi + K_2 \sin\lambda\xi)R_1(r),$$

$$u^\xi = e^{-\nu Q^2 t}(K_3 \cos\lambda\xi + K_4 \sin\lambda\xi)R_2(r),$$

$$u^\eta = e^{-\nu Q^2 t}(K_5 \cos\lambda\xi + K_6 \sin\lambda\xi)R_3(r),$$

$$\vartheta = Q = \mathrm{const},$$

$$P = e^{-\nu Q^2 t}(K_7 \cos\lambda\xi + K_8 \sin\lambda\xi)R_p(r)$$

- Helical variable ξ-periodicity requirement: $\lambda = \lambda_n = n/b,\ n = 0, 1, 2, \ldots$.

- Derive ODE on $R_1(r)$:

$$\frac{\mathrm{d}^2 R_1}{\mathrm{d}r^2} + \frac{B^2}{r}\left(\frac{3b^2}{r^2} + a^2\right)\frac{\mathrm{d}R_1}{\mathrm{d}r} - \left(\frac{\lambda^2}{B^2} + \frac{2a^2B^2 - 1}{r^2} - \vartheta^2 - \frac{2ab\vartheta B^2}{r^2}\right)R_1 = 0.$$

- ODE on $R_1(r)$:

$$\frac{\mathrm{d}^2 R_1}{\mathrm{d}r^2} + \frac{B^2}{r}\left(\frac{3b^2}{r^2} + a^2\right)\frac{\mathrm{d}R_1}{\mathrm{d}r} - \left(\frac{\lambda^2}{B^2} + \frac{2a^2B^2 - 1}{r^2} - \vartheta^2 - \frac{2ab\vartheta B^2}{r^2}\right)R_1 = 0.$$

- confluent Heun ODE:

$$Y''(z) + \frac{\alpha z^2 + (\beta - \alpha + \gamma + 2)z + \beta + 1}{z(z-1)}Y'(z) + \frac{((\beta + \gamma + 2)\alpha + 2\delta)z - (\beta + 1)\alpha + (\gamma + 1)\beta + 2\eta + \gamma}{2z(z-1)}Y(z) = 0,$$

- ODE on $R_1(r)$ solution: $R_1(r) = R_{1\,n}(r) = C_1 r^{n-1} H_{C+} + C_2 r^{-n-1} H_{C-}$, where

$$H_{C+} = H_C\left(\alpha, \beta, \gamma, \delta, \eta, -a^2r^2/b^2\right), \quad H_{C-} = H_C\left(\alpha, -\beta, \gamma, \delta, \eta, -a^2r^2/b^2\right)$$

are confluent Heun functions with parameters

$$\alpha = 0, \quad \beta = b\lambda_n = n, \quad \gamma = -2, \quad \delta = \frac{a^2n^2 - \vartheta^2 b^2}{4a^2},$$

$$\eta = \frac{a^2(4 - n^2) + \vartheta b(2a + \vartheta b)}{4a^2}.$$

- Dimensionless solutions:

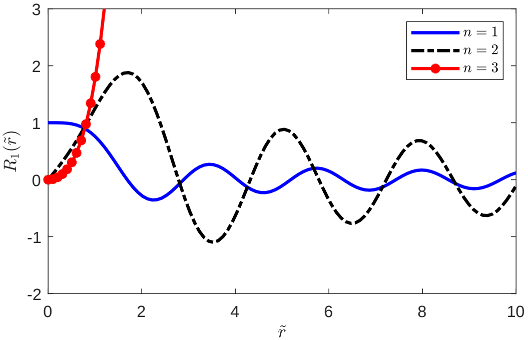
$$\tilde{u}^r_n = e^{-\tilde{t}} \left(\tilde{C}_{1\,n}\, \tilde{r}^{\,n-1} H_{C+} + \tilde{C}_{2\,n}\, \tilde{r}^{\,-n-1} H_{C-} \right) \sin(n\tilde{\xi} + \psi_n),$$

$$\tilde{u}^\xi_n = e^{-\tilde{t}}\; \tilde{B} \left[\tilde{C}_{1\,n} \left(\tilde{r}^{\,n-2} H_{C+} - \frac{2}{n} \tilde{r}^{\,n} H'_{C+} \right) - \tilde{C}_{2\,n} \left(\tilde{r}^{\,-n-2} H_{C-} + \frac{2}{n} \tilde{r}^{\,-n} H'_{C-} \right) \right] \cos(n\tilde{\xi} + \psi_n),$$

$$\tilde{u}^\eta_n = e^{-\tilde{t}}\; \frac{\gamma \tilde{B}}{n} \left(\tilde{C}_{1\,n} \tilde{r}^{\,n-1} H_{C+} + \tilde{C}_{2\,n} \tilde{r}^{\,-n-1} H_{C-} \right) \cos(n\tilde{\xi} + \psi_n),$$

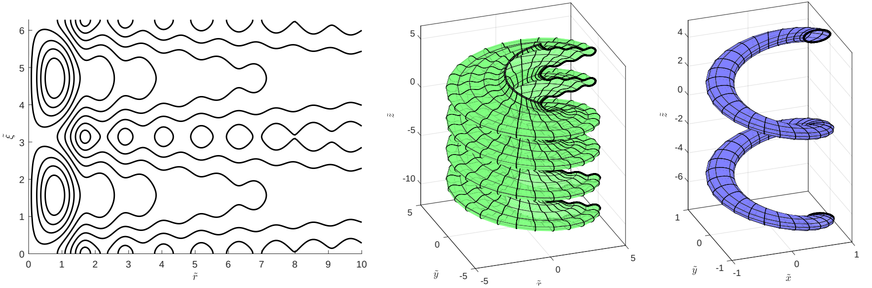
$$\tilde{p}_n = p_{0\,n} - \frac{1}{2} \left(|\tilde{u}^r_n|^2 + |\tilde{u}^\xi_n|^2 + |\tilde{u}^\eta_n|^2 \right).$$

Exact linearization of helical NS and Beltrami-type solutions



An illustration of the radial part $R_{1\,n}(\tilde{r})$ of the velocity component $\tilde{u}^r_n$ of the Beltrami solution for $n = 1, 2, 3$, $\tilde{C}_{1\,n} = 1$, $\tilde{C}_{2\,n} = 0$, $\gamma = -3$.

Beltrami-type solutions: illustration for $n = 1$



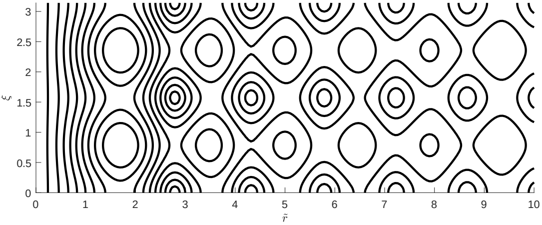
Level surfaces $|\tilde{\mathbf{u}}|^2 = \mathrm{const}$ (equivalently, $\tilde{p} = \mathrm{const}$, $|\tilde{\boldsymbol{\omega}}|^2 = \mathrm{const}$, or $\tilde{h} = \mathrm{const}$) for the exact dimensionless Beltrami solution for $n = 1$, $C_1 = 1$, $C_2 = 0$, $\psi = -\pi/2$.

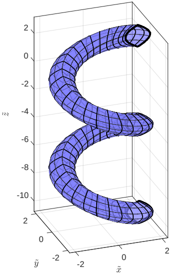
(a) A cross-section of level surfaces plot $|\tilde{\mathbf{u}}|^2 = \mathrm{const}$, for one period $0 \leq \tilde{\xi} \leq 2\pi$.

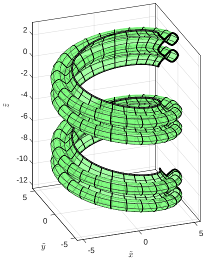
(b) A connected component of the level surface $|\tilde{\mathbf{u}}|^2 = 0.4$.

(c) A connected component of the level surface $|\tilde{\mathbf{u}}|^2 = 2.6$.

Beltrami-type solutions: illustration for $n = 2$





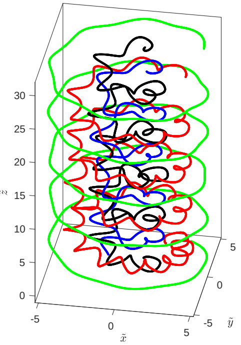


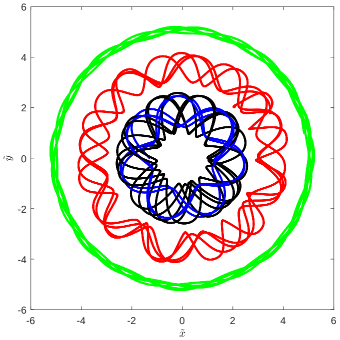
Level surfaces $|\tilde{\mathbf{u}}|^2 = \mathrm{const}$ (equivalently, $\tilde{p} = \mathrm{const}$, $|\tilde{\boldsymbol{\omega}}|^2 = \mathrm{const}$, or $\tilde{h} = \mathrm{const}$) for the exact dimensionless Beltrami solution for $n = 2$, $C_1 = 1$, $C_2 = 0$, $\psi = -\pi/2$.

(a) A cross-section of level surfaces plot $|\tilde{\mathbf{u}}|^2 = \mathrm{const}$, for one period $0 \leq \tilde{\xi} \leq 2\pi$.

(b) A connected component of the level surface $|\tilde{\mathbf{u}}|^2 = 3.54$.

(c) A connected component of the level surface $|\tilde{\mathbf{u}}|^2 = 0.97$.





Four sample streamlines for the exact dimensionless Beltrami solution for $n = 2$, $C_1 = 1$, $C_2 = 0$, emanating from various points in the plane $z = 1$.

(a) Side view. (b) Top view.

Conclusions

Part 1: Conservation laws of NS and Euler equations under helical symmetry

Helically-invariant fluid dynamics equations

- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions: zero velocity component in symmetric direction.

Additional conservation laws – systematic construction (multiplier method)

- Three-component Euler:
  - Generalized momenta. Generalized helicity. Additional vorticity CLs.
- Three-component Navier-Stokes:
  - Additional CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - Additional CLs in viscous and inviscid case, for plane and axisymmetric flows.

Part 2: Conservation laws of NS and Euler equations under helical symmetry

The new v-equation for Galilei-invariant helical flows

- Full helically-invariant Navier-Stokes equations, invariant with respect to the Galilei group

$$G^4: \quad r \to r, \quad t \to t, \quad \xi \to \xi + \varepsilon t, \quad p \to p,$$

$$u^r \to u^r, \quad u^\xi \to u^\xi + \varepsilon B(r), \quad u^\eta \to u^\eta - \varepsilon \frac{b}{ar} B(r).$$

- Such solutions satisfy the new v-equation

$$v_{rt} + \left(\frac{v\, v_r}{r}\right)_r - 2\frac{v_r^2}{r} - \nu \left[v_{rrr} + \frac{v_r}{r^2} - \frac{v_{rr}}{r}\right] = 0.$$

Exact solutions of helically invariant Navier-Stokes equations

- The v-equation: exact Galilei-invariant solutions.
- Beltrami flow ansatz: exact linearization, families of separated solutions, regular, with interesting geometry.

Some references

Batchelor, G.K. (2000).
An Introduction to Fluid Dynamics, Cambridge University Press.

Cheviakov, A.F. (2007).
GeM software package for computation of symmetries and conservation laws of differential equations. *Comput. Phys. Commun.* **176**, 48–61.

Bluman, G., Cheviakov, A., and Anco, S. (2010)
Applications of Symmetry Methods to Partial Differential Equations. Springer.

Kelbin, O., Cheviakov, A.F., and Oberlack, M. (2013).
New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. J. Fluid Mech. **721**, 340–366.

Cheviakov, A.F., and Oberlack, M. (2014).
Generalized Ertel's theorem and infinite hierarchies of conserved quantities for three-dimensional time-dependent Euler and Navier Stokes equations. J. Fluid Mech. **760**, 368–386.

Dierkes, D., Cheviakov, A.F., and Oberlack, M. (2020, Phys. Fluids, accepted)
New similarity reductions and exact solutions for helically symmetric viscous flows.

Some references

Batchelor, G.K. (2000).
An Introduction to Fluid Dynamics, Cambridge University Press.

Cheviakov, A.F. (2007).
GeM software package for computation of symmetries and conservation laws of differential equations. *Comput. Phys. Commun.* **176**, 48–61.

Bluman, G., Cheviakov, A., and Anco, S. (2010)
Applications of Symmetry Methods to Partial Differential Equations. Springer.

Kelbin, O., Cheviakov, A.F., and Oberlack, M. (2013).
New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. J. Fluid Mech. **721**, 340–366.

Cheviakov, A.F., and Oberlack, M. (2014).
Generalized Ertel's theorem and infinite hierarchies of conserved quantities for three-dimensional time-dependent Euler and Navier Stokes equations. J. Fluid Mech. **760**, 368–386.

Dierkes, D., Cheviakov, A.F., and Oberlack, M. (2020, Phys. Fluids, accepted)
New similarity reductions and exact solutions for helically symmetric viscous flows.

Thank you for your attention!