Conservation laws of differential equations: computation, connections, and applications

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Outline

Notation, etc.

- Independent variables: (x, t), or (t, x, y, z), or $z = (z^1, ..., z^n)$.
- Dependent variables: $u(x, t)$, or generally $v = (v^1(z), ..., v^m(z))$.
- Derivatives:

$$\frac{d}{dt} w(t) = w'(t); \qquad \frac{\partial}{\partial x} u(x, t) = u_x; \qquad \frac{\partial}{\partial z^k} v^p(z) = v^p_k.$$

- All derivatives of order p: $\partial^p v$.
- A differential function:

$$H[v] = H(z, v, \partial v, \ldots, \partial^k v)$$

- A total derivative of a differential function: the chain rule

$$\mathrm{D}_i H[v] = \frac{\partial H}{\partial z^i} + \frac{\partial H}{\partial v^\alpha} v^\alpha_i + \frac{\partial H}{\partial v^\alpha_j} v^\alpha_{ij} + \ldots.$$

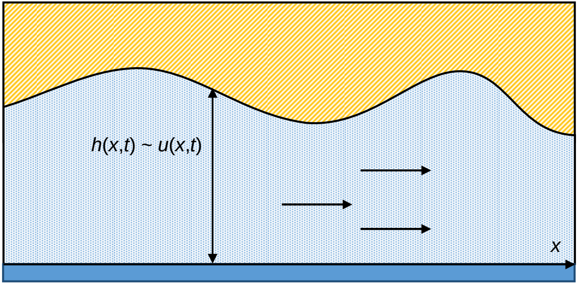
Notation, etc.

- A PDE Example: the KdV (Korteweg-de Vries) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

for the dimensionless fluid depth $u = u(x, t)$ of long surface waves on shallow water:

$$G[u] = u_t + uu_x + u_{xxx} = 0.$$



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- $J^k(x, t|u)$: the k-th order jet space with coordinates $x, t, u, \partial u, \ldots, \partial^k u$.
- The solution manifold $\mathcal{E}$ in $J^k(x, t|u)$ is defined by the DE(s)+differential consequences to order k:

$$G[u] = 0, \quad \mathrm{D}_x G[u] = 0, \quad \mathrm{D}_t G[u] = 0, \ldots$$

- Statements are often formulated for differential functions defined in $J^k(x, t|u)$.

Local and global conservation laws

Local and global conservation laws

- System of differential equations (PDE or ODE) $G[v] = 0$:

$$G^\sigma(z, v, \partial v, \ldots, \partial^{q_\sigma} v) = 0, \quad \sigma = 1, \ldots, M.$$

- The fundamental notion –

A local divergence-type conservation law:

A divergence expression

$$\boxed{\mathrm{D}_i \Phi^i[v] = 0}$$

vanishing on solutions of $G[v]$. Here $\mathbf{\Phi} = (\Phi^1[v], \ldots, \Phi^n[v])$ is the flux vector.

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ODE: A constant of motion (conserved quantity):

$$v = v(t), \qquad \mathrm{D}_t T[v] = 0 \quad \Rightarrow \quad T[v] = \mathrm{const}.$$

- E.g. $v'' + 2vv' = 5$:

$$\mathrm{D}_t(v' + v^2 - 5t) = 0 \;\Rightarrow\; v' + v^2 - 5t = C = \mathrm{const}.$$

Local and global conservation laws

- For PDEs, the meaning of a local conservation law is different: the total amount of "density" is "conserved" in another sense.

- **(1+1)-dimensional PDEs:** $v = v(x,t)$, only one CL type.

  Local form:

  $$\boxed{\mathrm{D}_t T[v] + \mathrm{D}_x \Psi[v] = 0.}$$

  Global form:

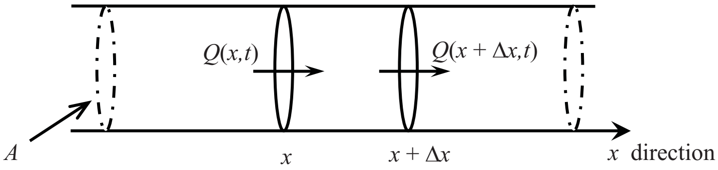
  $$\boxed{\frac{d}{dt}\int_a^b T[v]\,dt = -\Psi[v]\Big|_a^b.}$$

- **Multidimensional PDE systems:** several different CL types.

Conservation principles to derive model DEs.

- Continuity equation – gas/fluid flow:

$$\rho_t + (\rho v)_x = 0, \qquad \rho = \rho(x,t), \qquad v = v(x,t).$$

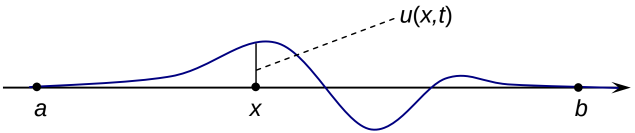


- Global form:

$$\frac{d}{dt} m = \frac{d}{dt} \int_x^{x+\Delta x} \rho \, dx = (\rho v)\Big|_x^{x+\Delta x}.$$

(1+1)-dimensional linear wave equation:

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad c^2 = \tau/\rho, \quad a < x < b \text{ or } -\infty < x < \infty.$$



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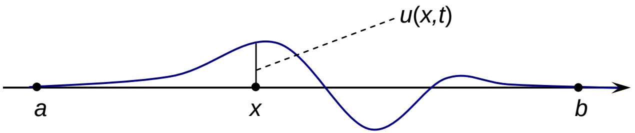


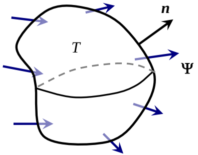
- A local CL – energy conservation: $\mathrm{D}_t \left(\frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2} \right) - \mathrm{D}_x(\tau u_t u_x) = 0.$

- Global form:

$$\frac{d}{dt}E = \frac{d}{dt} \int \left(\frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2} \right) dx = \tau u_t u_x \Big|_a^b .$$

E.g., for Dirichlet BCs $u|_{x=a,b}$, $E = \mathrm{const}$.

- **(3+1)-dimensional PDEs:** $v = v(t, x, y, z)$.
- Local form: $\boxed{D_t T[v] + \mathrm{Div}\, \mathbf{\Psi}[v] = 0} \quad \Leftrightarrow \quad \boxed{\mathrm{D}_i \Phi^i[v] = 0}$
- Global form: $\boxed{\dfrac{d}{dt}\displaystyle\int_{\mathcal{V}} T\, dV = -\oint_{\partial\mathcal{V}} \mathbf{\Psi} \cdot d\mathbf{S}}$
- Holds for all solutions $v(t, x, y, z) \in \mathcal{E}$, in some physical domain $\mathcal{V}$.



- In 3D, CLs of other types on static and moving domains can exist.

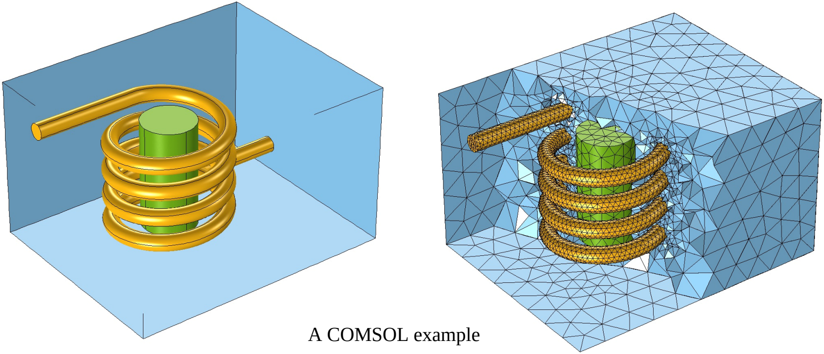
Applications

Applications to ODEs

- Constants of motion.
- Reduction of order / integration.

Applications to PDEs

- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Analysis of solution behaviour: existence, uniqueness, stability.
- Potentials, stream functions, etc.
- An infinite number of CLs may indicate integrability/linearization.
- Conserved PDEs forms for finite volume/discontinuous Galerkin/special numerical methods.
- Conservation law-preserving numerical methods.
- Numerical method testing.



A COMSOL example

CLs with no physical content?

Example: (1+1)-dimensional linear wave equation

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t).$$

Trivial conservation laws:

- Density/flux vanishes on solutions (Type I, vanishing density/flux).
  For example,
  $$\mathrm{D}_t(u_{tt} - c^2 u_{xx}) + \mathrm{D}_x\left(2u\,[u_{ttx} - c^2 u_{xxx}]\right) = 0.$$
- Holds as an identity for any $u(x,t)$ (Type II, null divergence).
  For example,
  $$\mathrm{D}_t(x + u_x) + \mathrm{D}_x(2t - u_t) \equiv 0.$$
- A combination thereof.

Trivial and equivalent local conservation laws

Example: (1+1)-dimensional linear wave equation

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t).$$

Equivalent conservation laws:

- Differ by a trivial one.
  For example,
  $$\mathrm{D}_t(u_t) - \mathrm{D}_x(c^2 u_x) = 0$$
  and
  $$\mathrm{D}_t(u_t + x) - \mathrm{D}_x(c^2 u_x - 1) = 0$$
  describe the same physical quantity.
- Natural to study equivalence classes of CLs.
- Linear space $CL(G)$ of all CLs of a system $G[v] = 0 \rightarrow$ a factor space of equivalence classes.
- It is of interest to determine a basis of CLs in the factor space.

Example: (1+1)-dimensional linear wave equation

$$u_{tt} = c^2 u_{xx}, \quad u = u(x, t).$$

- Same ideas for multi-dimensional models.

Characteristic form of a CL

Characteristic form of a CL

- What is an "algebraic handle" to compute divergence-type CLs

$$\mathrm{D}_i \Phi^i[v] = 0$$

of a DE system $G^\sigma[v] = 0$, $\sigma = 1, \ldots, M$?

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Hadamard lemma for differential functions

A smooth differential function $Q[v]$ vanishes on solutions of a *totally nondegenerate* PDE system $G^\sigma[v] = 0$ if and only if it has the form, for all v,

$$Q[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v] \, \mathrm{D}_k G^\sigma[v] + \ldots .$$

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- Off of solution set, for all v:

$$\mathrm{D}_i \Phi^i[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v]\, \mathrm{D}_k G^\sigma[v] + \ldots.$$

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$$\mathrm{D}_i \Phi^i[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v]\, \mathrm{D}_k G^\sigma[v] + \ldots.$$

- An equivalent CL:

$$\mathrm{D}_i \tilde{\Phi}^i[v] = \tilde{\Lambda}_\sigma[v] G^\sigma[v].$$

Characteristic form of a CL

A characteristic form of a local CL:

$$\boxed{\mathrm{D}_i \Phi^i[v] = \Lambda_\sigma[v] G^\sigma[v].}$$

- $\Phi^i[v]$: fluxes.
- $\Lambda_\sigma[v]$: multipliers.
- There is "usually" a 1:1 correspondence between sets of (nontrivial) multipliers and the respective (nontrivial) local CLs.

How many local CLs?

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- **Possibility I:** a finite number. For example:

Theorem (Ibragimov, 1985)

For any $(1+1)$-dimensional even-order scalar evolution equation

$$u_t = F(x, t, u, \partial_x u, \dots, \partial_x^{2k} u), \qquad u = u(x, t),$$

the flux and the density of local CLs

$$\mathrm{D}_t T[u] + \mathrm{D}_x \Psi[u] = 0$$

(up to equivalence) depend only on x, t, u and derivatives of u with respect to x, and the maximal order of a derivative in the CL density T is k.

How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?
- **Possibility I:** a finite number. For example:

A nonlinear diffusion equation

$$u_t = (u^2 u_x)_x, \qquad u = u(x,t).$$

Two local CLs only:

$$\mathrm{D}_t(u) - \mathrm{D}_x(u^2 u_x) = 0,$$

$$\mathrm{D}_t(xu) + \mathrm{D}_x\left(\frac{u^3}{3} - xu^2 u_x\right) = 0.$$

How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?
- **Possibility I:** a finite number. For example:

Constant-density Navier-Stokes equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} \, p + \nu \, \Delta \mathbf{u}.$$

CLs [Gusyatnikova & Yumaguzhin, 1989]:

- Continuity (generalized): $\nabla \cdot (k(t) \, \mathbf{u}) = 0$.
- Momentum (generalized): $\mathrm{D}_t(f(t) u^1) + \mathrm{D}_x(\ldots) + \mathrm{D}_y(\ldots) + \mathrm{D}_z(\ldots) = 0$; same for y, z.
- Angular momentum: $\mathrm{D}_t(zu^2 - yu^3) + \mathrm{D}_x(\ldots) + \mathrm{D}_y(\ldots) + \mathrm{D}_z(\ldots) = 0$; same for y, z.

How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?
- **Possibility II:** an infinite countable set. E.g., CLs of an integrable equation.

Example: the KdV

$$u_t + uu_x + u_{xxx} = 0, \qquad u = u(x,t).$$

A hierarchy of local CLs:

$$\mathrm{D}_t(u) + \mathrm{D}_x\left(\tfrac{1}{2}u^2 + u_{xx}\right) = 0,$$

$$\mathrm{D}_t\left(\tfrac{1}{2}u^2\right) + \mathrm{D}_x\left(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2\right) = 0,$$

$$\mathrm{D}_t\left(\tfrac{1}{6}u^3 - \tfrac{1}{2}u_x^2\right) + \mathrm{D}_x\left(\tfrac{1}{8}u^4 - uu_x^2 + \tfrac{1}{2}(u^2u_{xx} + u_{xx}^2) - u_x u_{xxx}\right) = 0,$$

$$\vdots$$

How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?
- **Possibility III:** an infinite CL family involving arbitrary functions.
  E.g., linear/linearizable equations, etc.

Example:

- A linear heat equation $u_t = a^2 u_{xx}, \quad u = u(x,t)$.
- Local CLs: $\Lambda(x,t)(u_t - u_{xx}) = \mathrm{D}_t T + \mathrm{D}_x \Psi = 0$.
- The multiplier $\Lambda(x,t)$ is any solution of the adjoint linear PDE $\Lambda_t = -a^2 \Lambda_{xx}$.
- E.g., $\Lambda(x,t) = e^{a^2 t} \sin x$, then $\mathrm{D}_t \left(e^{a^2 t}\, u \sin x \right) + \mathrm{D}_x \left(a^2 e^{a^2 t} [u \cos x - u_x \sin x] \right) = 0$.

- Existence of a "large" CL family is a necessary condition of invertible linearization (e.g., Bluman, Anco & Wolf, 2008).

How to compute CLs?

The idea of the direct construction method

Independent and dependent variables of the problem:
$z = (z^1, ..., z^n), \quad v = v(z) = (v^1, ..., v^m).$

Definition

The Euler operator with respect to an arbitrary function v^j:

$$\mathrm{E}_{v^j} = \frac{\partial}{\partial v^j} - \mathrm{D}_i \frac{\partial}{\partial v^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial v^j_{i_1 \ldots i_s}} + \cdots, \quad j = 1, \ldots, m.$$

Theorem

The equations

$$\mathrm{E}_{v^j} F[v] \equiv 0, \quad j = 1, \ldots, m$$

hold for arbitrary $v(z)$ if and only if

$$F[v] \equiv \mathrm{D}_i \Phi^i$$

for some functions $\Phi^i = \Phi^i[v]$.

The direct construction method

Given:

- A system of M DEs $G^{\sigma}[v] = 0, \quad \sigma = 1, \ldots, M.$
- Variables: $z = (z^1, ..., z^n), \quad v = (v^1(z), ..., v^m(z)).$

Given:

- A system of M DEs $G^\sigma[v] = 0, \quad \sigma = 1, \ldots, M$.
- Variables: $z = (z^1, ..., z^n), \quad v = (v^1(z), ..., v^m(z))$.

The Direct CL Construction Method

1. Specify the dependence of multipliers: $\Lambda_\sigma = \Lambda_\sigma[z, v, \partial v, ...]$.
2. Solve the set of determining equations $\mathrm{E}_{v^j}(\Lambda_\sigma[v] G^\sigma[v]) \equiv 0, \quad j = 1, \ldots, m$, for arbitrary $v(z)$, to find all sets of multipliers.
3. Find the corresponding fluxes $\Phi^i[V]$ satisfying the identity

$$\Lambda_\sigma[v] G^\sigma[v] \equiv \mathrm{D}_i \Phi^i[v].$$

4. For each set of fluxes, on solutions, get a local conservation law

$$\boxed{\mathrm{D}_i \Phi^i[v] = 0.}$$

A detailed example

Consider a nonlinear telegraph system for $v^1 = u(x,t)$, $v^2 = v(x,t)$:

$$G^1[u,v] = v_t - (u^2+1)u_x - u = 0,$$
$$G^2[u,v] = u_t - v_x = 0.$$

Multiplier ansatz: $\Lambda_1 = \Lambda_1(x,t,u,v)$, $\Lambda_2 = \Lambda_2(x,t,u,v)$.

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Determining equations:

$$\mathrm{E}_u\left[\Lambda_1(x,t,u,v)(v_t - (u^2+1)u_x - u) + \Lambda_2(x,t,u,v)(u_t - v_x)\right] \equiv 0,$$

$$\mathrm{E}_v\left[\Lambda_1(x,t,u,v)(v_t - (u^2+1)u_x - u) + \Lambda_2(x,t,u,v)(u_t - v_x)\right] \equiv 0.$$

Euler operators:

$$\mathrm{E}_u = \frac{\partial}{\partial u} - \mathrm{D}_x\frac{\partial}{\partial u_x} - \mathrm{D}_t\frac{\partial}{\partial u_t},$$
$$\mathrm{E}_v = \frac{\partial}{\partial v} - \mathrm{D}_x\frac{\partial}{\partial v_x} - \mathrm{D}_t\frac{\partial}{\partial v_t}.$$

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Split determining equations:

$$\Lambda_{2v} - \Lambda_{1u} = 0, \qquad \Lambda_{2u} - (u^2+1)\Lambda_{1v} = 0,$$
$$\Lambda_{2x} - \Lambda_{1t} - u\Lambda_{1v} = 0, \qquad (u^2+1)\Lambda_{1x} - \phi_t - u\Lambda_{1u} - \Lambda_1 = 0.$$

A detailed example

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Multiplier ansatz: $\Lambda_1 = \Lambda_1(x,t,u,v)$, $\Lambda_2 = \Lambda_2(x,t,u,v)$.

Solution: five sets of multipliers $(\Lambda_1, \Lambda_2) =$

0	1
t	$x - \frac{1}{2}t^2$
1	$-t$
$e^{x+\frac{1}{2}u^2+v}$	$ue^{x+\frac{1}{2}u^2+v}$
$e^{x+\frac{1}{2}u^2-v}$	$-ue^{x+\frac{1}{2}u^2-v}$

A detailed example

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Multiplier ansatz: $\Lambda_1 = \Lambda_1(x,t,u,v)$, $\Lambda_2 = \Lambda_2(x,t,u,v)$.

Resulting five conservation laws:

$$\mathrm{D}_t u - \mathrm{D}_x v = 0,$$

$$\mathrm{D}_t[(x - \tfrac{1}{2}t^2)u + tv] + \mathrm{D}_x[(\tfrac{1}{2}t^2 - x)v - t(\tfrac{1}{3}u^3 + u)] = 0,$$

$$\mathrm{D}_t[v - tu] + \mathrm{D}_x[tv - (\tfrac{1}{3}u^3 + u)] = 0,$$

$$\mathrm{D}_t[e^{x+\frac{1}{2}u^2+v}] + \mathrm{D}_x[-ue^{x+\frac{1}{2}u^2+v}] = 0,$$

$$\mathrm{D}_t[e^{x+\frac{1}{2}u^2-v}] + \mathrm{D}_x[ue^{x+\frac{1}{2}u^2-v}] = 0.$$

- To obtain **further conservation laws**, extend the multiplier ansatz...

Example of use of the GeM package for Maple for the KdV.

- Use the module: `read("d:/gem32_12.mpl"):`
- Declare variables: `gem_decl_vars(indeps=[x,t], deps=[U(x,t),V(x,t)]);`
- Declare the PDEs:

```
gem_decl_eqs([diff(V(x,t),t)=(U(x,t)^2+1)*diff(U(x,t),x)+U(x,t),
    diff(U(x,t),t)= diff(V(x,t),x)],
    solve_for=[diff(V(x,t),t), diff(U(x,t),t)]);
```

- Generate determining equations:

```
det_eqs:=gem_conslaw_det_eqs([x,t,U(x,t),V(x,t)]):
```

- Reduce the overdetermined system:

```
CL_multipliers:=gem_conslaw_multipliers();
simplified_eqs:=DEtools[rifsimp](det_eqs, CL_multipliers, mindim=1);
```

- Solve determining equations:

```
multipliers_sol:=pdsolve(simplified_eqs[Solved]);
```

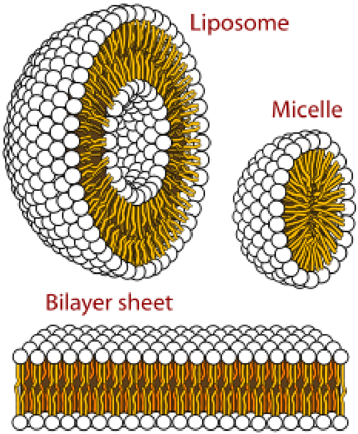
- Obtain corresponding conservation law fluxes/densities:

```
gem_get_CL_fluxes(multipliers_sol, method=*****);
```

Computational examples

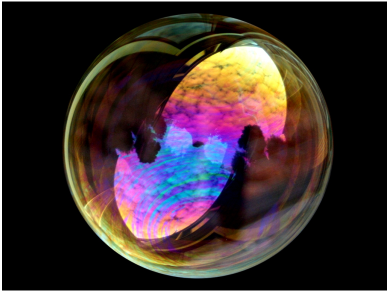
Surfactants - Applications

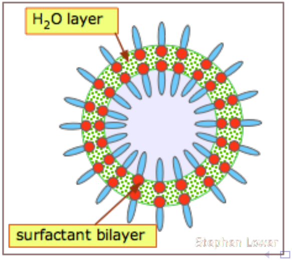
- Surfactant molecules adsorb at phase separation interfaces.
- Can form micelles, double layers, etc.

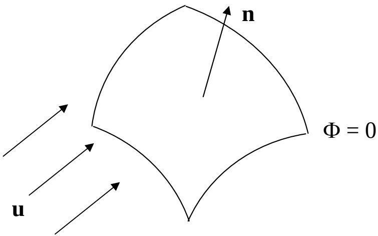


Surfactants - Applications

- Soap bubbles...

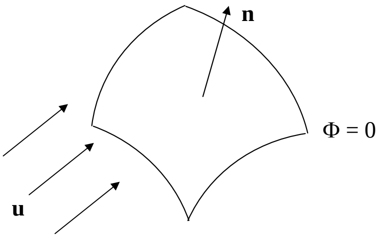






Parameters

- Surfactant concentration $c = c(\mathbf{x}, t)$.
- Flow velocity $\mathbf{u}(\mathbf{x}, t)$.
- Two-phase interface: phase separation surface $\Phi(\mathbf{x}, t) = 0$.
- Unit normal: $\mathbf{n} = -\dfrac{\nabla \Phi}{|\nabla \Phi|}$.

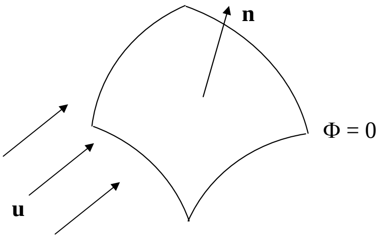


Surface gradient

- Surface projection tensor: $p_{ij} = \delta_{ij} - n_i n_j$.
- Surface gradient operator: $\nabla^s = \mathbf{p} \cdot \nabla = (\delta_{ij} - n_i n_j) \dfrac{\partial}{\partial x^j}$.
- Surface Laplacian:

$$\Delta^s F = (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left((\delta_{ik} - n_i n_k) \frac{\partial F}{\partial x^k} \right).$$

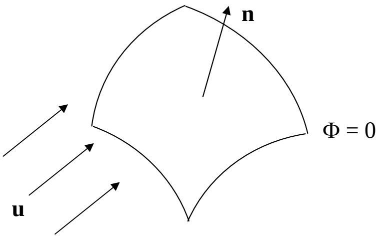
Surfactant Transport Equations



Governing equations

- Incompressibility condition: $\nabla \cdot \mathbf{u} = 0.$
- Fluid dynamics equations: Euler or Navier-Stokes.
- Interface transport by the flow: $\Phi_t + \mathbf{u} \cdot \nabla \Phi = 0.$
- Surfactant transport equation:

$$c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha(\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left((\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.$$



Fully conserved form?

$$c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha(\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left((\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.$$

- Can the surfactant transport equation be written in the conserved form?

Surfactant Transport Equations: CLs

Governing equations ($\alpha \neq 0$)

$$G^1 = \frac{\partial u^i}{\partial x^i} = 0,$$

$$G^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0,$$

$$G^3 = c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha(\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left((\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.$$

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Multipliers:

$$\Lambda^1 = \Phi \mathcal{F}(\Phi) \left| \nabla \Phi \right|^{-1} \left(\frac{\partial}{\partial x^j} \left(c \frac{\partial \Phi}{\partial x^j} \right) - c n_i n_j \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \right),$$

$$\Lambda^2 = -\mathcal{F}(\Phi) \left| \nabla \Phi \right|^{-1} \left(\frac{\partial}{\partial x^j} \left(c \frac{\partial \Phi}{\partial x^j} \right) - c n_i n_j \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \right),$$

$$\Lambda^3 = \mathcal{F}(\Phi) |\nabla \Phi|,$$

where $\mathcal{F} = \mathcal{F}(\Phi)$ is an arbitrary sufficiently smooth function.

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An infinite CL family:

$$\boxed{\mathrm{D}_t \left(c\, \mathcal{F}(\Phi)\, |\nabla \Phi| \right) + \mathrm{D}_i \left(A^i\, \mathcal{F}(\Phi)\, |\nabla \Phi| \right) = 0,}$$

where

$$A^i = c u^i - \alpha \left((\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right), \quad i = 1, 2, 3.$$

Euler equations of inviscid fluid flow:

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

Euler equations: a CL study

Euler equations of inviscid fluid flow:

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

CL Multiplier ansatz [Oberlack & C., 2014]:

Λ_σ, $\sigma = 1, 2, 3, 4$, are functions of 45 variables

$$\begin{aligned}
&t, x, y, z, \quad u^1, u^2, u^3, p, \quad u^1_y, u^1_z, \quad u^2_x, u^2_y, u^2_z, \quad u^3_x, u^3_y, u^3_z, \quad p_t, p_x, p_y, p_z,\\
&u^1_{yy}, u^1_{yz}, u^1_{zz}, \quad u^2_{xx}, u^2_{xy}, u^2_{xz}, u^2_{yy}, u^2_{yz}, u^2_{zz}, \quad u^3_{xx}, u^3_{xy}, u^3_{xz}, u^3_{yy}, u^3_{yz}, u^3_{zz},\\
&p_{tt}, p_{tx}, p_{ty}, p_{tz}, p_{xx}, p_{xy}, p_{xz}, p_{yy}, p_{yz}, p_{zz}.
\end{aligned}$$

Euler equations: a CL study

Euler equations of inviscid fluid flow:

$$\nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

Computed CLs:

- Continuity (generalized): $\nabla \cdot (k(t)\,\mathbf{u}) = 0$.
- Momentum (generalized): $\mathrm{D}_t(f(t)u^1) + \mathrm{D}_x(\ldots) + \mathrm{D}_y(\ldots) + \mathrm{D}_z(\ldots) = 0$; same for y, z.
- Angular momentum: $\mathrm{D}_t(zu^2 - yu^3) + \mathrm{D}_x(\ldots) + \mathrm{D}_y(\ldots) + \mathrm{D}_z(\ldots) = 0$; same for y, z.
- Kinetic energy: $\mathrm{D}_t(K) + \ldots = 0, \quad K = \frac{1}{2}\,|\mathbf{u}|^2$.
- Helicity: $\mathrm{D}_t(h) + \ldots = 0, \quad h = \mathbf{u} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \mathrm{curl}\,\mathbf{u}$.

- Linear overdetermined system of 58,273 determining equations on the unknown Λ_σ.
- Additional special CLs arise in symmetry-reduced settings.

Global and local conservation laws...

For a DE system $G[v]=0$:

- The solution manifold $\mathcal{E}$ is a geometric object.

- CLs reflect its properties, and are coordinate-independent. In particular,

$$\mathrm{D}_{(z^*)^i}(\Phi^*)^i[v^*] = J\,\mathrm{D}_i\Phi^i[v] = 0$$

  after a change of variables

$$\begin{aligned} (z^*)^i &= f^i(z,v), \qquad i=1,\dots,n,\\ (v^*)^k &= g^k(z,v), \qquad k=1,\dots,m. \end{aligned}$$

- CLs have a characteristic form: $\boxed{\mathrm{D}_i\Phi^i[v] = \Lambda_\sigma[v]G^\sigma[v].}$

- CLs can be systematically computed (the direct method and Maple implementation).

- The direct method is complete, within a chosen ansatz.

Variational systems and Noether's 1st theorem

- Local symmetries and local conservation laws of DE systems are closely related.
- A specific well-known relationship: Noether's 1st theorem for variational DE systems.

- System of differential equations (PDE or ODE) $G[v] = 0$:

$$G^\sigma(z, v, \partial v, \ldots, \partial^{q_\sigma} v) = 0, \quad \sigma = 1, \ldots, M.$$

- Independent and dependent variables: $z = (z^1, ..., z^n)$, $v = v(z) = (v^1, ..., v^m)$.

- A point symmetry: a change of variables

$$\begin{aligned} (z^*)^i &= f^i(z, v), \quad & i = 1, \ldots, n, \\ (v^*)^k &= g^k(z, v), \quad & k = 1, \ldots, m \end{aligned}$$

mapping solutions to solutions.

- A Lie group of point symmetries: a symmetry group with parameter(s) a

$$\begin{aligned} (z^*)^i &= f^i(z, v; a) = z^i + a\xi^i(z, v) + O(a^2), \quad & i = 1, \ldots, n, \\ (v^*)^k &= g^k(z, v; a) = v^k + a\eta^k(z, v) + O(a^2), \quad & k = 1, \ldots, m. \end{aligned}$$

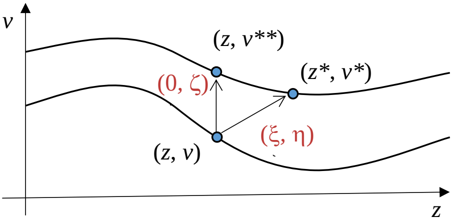
- A corresponding Lie algebra of infinitesimal generators:

$$\mathrm{X} = \xi^i(z, v)\frac{\partial}{\partial z^i} + \eta^k(z, v)\frac{\partial}{\partial v^k}.$$

- Evolutionary form of a Lie point symmetry:

$$\hat{\mathrm{X}} = \zeta^k[v]\frac{\partial}{\partial v^\mu},$$

$$\begin{aligned} (z^{**})^i &= z^i, & i &= 1, \dots, n, \\ (v^{**})^k &= v^k + a\zeta^k[v] + O(a^2), & k &= 1, \dots, m. \end{aligned}$$



Example 1: translations

A translation

$$x^* = x + C, \quad t^* = t, \quad u^* = u \quad (C \in \mathbb{R})$$

leaves the KdV equation invariant:

$$u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^* x^* x^*}.$$

Example 2: scalings

A scaling

$$x^* = \alpha x, \quad t^* = \alpha^3 t, \quad u^* = \alpha^{-2} u \quad (\alpha \in \mathbb{R})$$

also leaves the KdV equation invariant:

$$u_t + uu_x + u_{xxx} = 0 = \alpha^5 \left(u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^* x^* x^*} \right).$$

Action integral

$$J[v] = \int_\Omega \mathcal{L}(z, v, \partial v, \ldots, \partial^k v)\, dz.$$

Principle of extremal action

- Variation of v: $v(z) \to v(z) + \delta v(z)$; $\delta v(z) = \varepsilon w(z)$; $\delta v(z)\big|_{\partial\Omega} = 0$.
- Variation of action: $\delta J \equiv J[v + \varepsilon w] - J[v] = o(\varepsilon) \quad \Rightarrow$
- Euler-Lagrange equations:

$$G^\sigma[v] = \mathrm{E}_{v^\sigma}(\mathcal{L}[v]) = 0, \qquad \sigma = 1, \ldots, m.$$

- $\#$ equations $= \#$ unknowns.

Variational principles

- Example: Wave equation for $u(x,t)$

$$\mathcal{L} = P - K = \frac{1}{2}\tau {u_x}^2 - \frac{1}{2}\rho {u_t}^2.$$

$$\mathrm{E}_u = \frac{d}{du} - \mathrm{D}_t \frac{d}{du_t} - \mathrm{D}_x \frac{d}{du_x}.$$

$$\mathrm{E}_u\mathcal{L} = \rho(u_{tt} - c^2 u_{xx}) = 0, \qquad c^2 = \tau/\rho.$$

- Philosophical rather than physical!
- The vast majority of models do not have a variational formulation.
- Mathematically, related to the self-adjointness of linearization (coordinate-dependent!)
- It remains an open problem how to determine whether a given system has a variational formulation.

Noether's 1st theorem

- A variational symmetry: preserves the action integral.

Theorem

Given:

1. *a PDE system* $G[v] = 0$*, following from a variational principle;*
2. *a local variational symmetry in an evolutionary form:*

$$(z^i)^* = z^i, \quad (v^k)^* = v^k + a\,\zeta^k[v] + O(a^2).$$

Then the given DE system has a **local conservation law** $\mathrm{D}_i\Phi^i[v] = 0$*.*
In particular,

$$\mathrm{D}_i\Phi^i[v] = \Lambda_\sigma[v]R^\sigma[v],$$

where the multipliers are the evolutionary symmetry components:

$$\boxed{\Lambda_\sigma[v] = \zeta^\sigma[v].}$$

Example: wave equation

- **Equation:** $u_{tt} = c^2 u_{xx}, \quad u = u(x,t).$
- **Time translation symmetry:**

$$\begin{array}{ll} t^* = t + a, & \xi^t = 1; \\ x^* = x, & \xi^x = 0, \\ u^* = u, & \eta = 0, \end{array}$$

- **Evolutionary symmetry component**: $\zeta = -u_t;$
- **Multiplier**: $\Lambda = \zeta = -u_t;$
- **Conservation law (Energy):**

$$\Lambda R = -u_t(u_{tt} - c^2 u_{xx}) = -\left[D_t \left(\frac{u_t^2}{2} + c^2 \frac{u_x^2}{2} \right) - D_x \left(c^2 u_t u_x \right) \right] = 0.$$

Noether's 1st theorem and CL computation?

Noether's 1st theorem – summary

- The system $G[v] = 0$ may or may not be variational.
- Lie symmetries can be systematically computed. For variational models, some of them are variational (preserve the action).
- Evolutionary components $\zeta[v]$ of symmetry generators satisfy linearized equations.
- CL multipliers satisfy adjoint linearized equations and extra conditions.
- For a variational system, linearization is self-adjoint.

  Then evolutionary variational symmetry components = CL multipliers.
- Noether's theorem is insightful, but not general nor efficient way to compute CLs.
- The direct CL construction method is general; it is a practical shortcut even for variational DE systems.

Different types of CLs in 3D

PDE models in three spatial dimensions

General classical physical systems in 3D:

- Independent variables: coordinates $x = (x^1, x^2, x^3) \in \Omega$, and possibly time t.
- Dependent variables: $v = v(t, \mathbf{x})$ or $v(x)$; $m \geq 1$ scalars.
- PDEs: $G^\sigma[v] = 0$, $\sigma = 1, \ldots, M$.

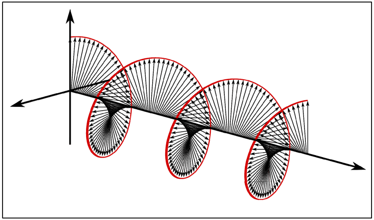
Typical applications:

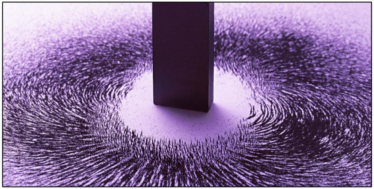
- Nonlinear mechanics, elasticity, viscoelasticity, plasticity
- Fluid mechanics
- Electromagnetism
- Wave propagation; problems, diffusion, etc.

Example: Microscopic Maxwell's equations in Gaussian units

$$\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{B}_t + c \operatorname{curl} \mathbf{E} = 0,$$

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \qquad \mathbf{E}_t - c \operatorname{curl} \mathbf{B} = -4\pi \mathbf{J}.$$





Example: Navier-Stokes fluid dynamics equations

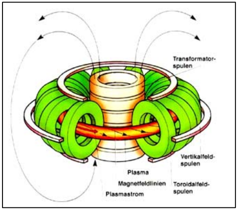
$$\rho_t + \operatorname{div} \rho \mathbf{u} = 0,$$

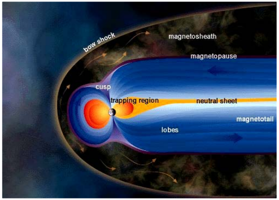
$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\operatorname{grad}\, p + \mu\, \Delta \mathbf{u}.$$

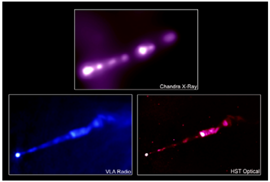
Example: Ideal magnetohydrodynamics (MHD) equations

$$\rho_t + \operatorname{div} \rho \mathbf{u} = 0, \qquad \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\frac{1}{\mu}\mathbf{B}\times \operatorname{curl}\mathbf{B} - \operatorname{grad}\, p,$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{u}\times\mathbf{B}), \qquad \operatorname{div}\mathbf{B} = 0.$$







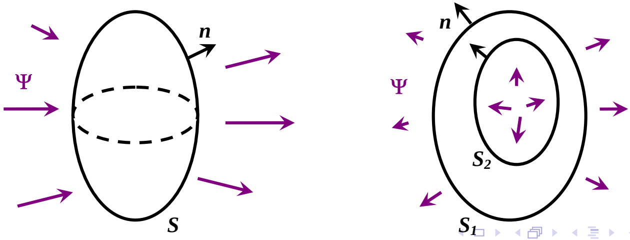
Applications:

- Time-independent models.
- Differential constraints, e.g., $\operatorname{div} \mathbf{B} = 0$, $\operatorname{curl} \mathbf{u} = 0$...

1A. Spatial divergence/topological flux conservation laws

- Local form: $\boxed{\operatorname{Div} \boldsymbol{\Psi}[v] = 0.}$
- Global form in $\mathcal{V}$, $\partial\mathcal{V} = \mathcal{S}$: $\boxed{\oint_{\mathcal{S}} \boldsymbol{\Psi}[v] \cdot d\mathbf{S}\big|_{\mathcal{E}} = 0.}$ (Gauss thm.)
- Global form when $\partial\mathcal{V} = \mathcal{S}_1 \cup \mathcal{S}_2$:

$$\boxed{\oint_{\mathcal{S}_1} \boldsymbol{\Psi}[v]|_{\mathcal{E}} \cdot d\mathbf{S} = \oint_{\mathcal{S}_2} \boldsymbol{\Psi}[v]|_{\mathcal{E}} \cdot d\mathbf{S}.}$$



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Examples:

- Incompressible flow: $\mathrm{div}\, \mathbf{u} = 0.$
- Absence of magnetic sources: $\mathrm{div}\, \mathbf{B} = 0.$

1B. Spatial curl/topological circulation conservation laws

- Local form: $\boxed{\mathrm{Curl}\,\boldsymbol{\Psi}[v]|_{\mathcal{E}} = 0.}$
- Global form in $\mathcal{S}$, $\partial\mathcal{S} = \mathcal{C}$: $\boxed{\int_{\mathcal{C}} \boldsymbol{\Psi}[v]\cdot d\boldsymbol{\ell} = 0.}$
- Global form, $\partial\mathcal{S} = \mathcal{C}_1 \cup \mathcal{C}_2$:

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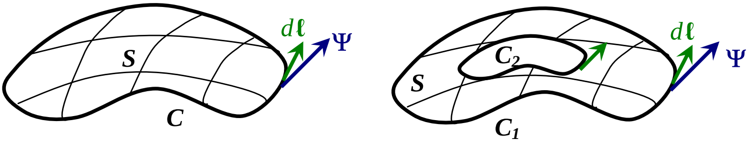

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Examples:

- Irrotational flow: $\mathrm{curl}\,\mathbf{u} = 0$.
- Equilibrium MHD–magnetic equation: $\mathrm{curl}\,(\mathbf{u}\times\mathbf{B}) = 0$
  $\Rightarrow$ circulation condition:

$$\forall\mathcal{S}\subset\Omega,\quad \int_{\partial\mathcal{S}} (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell} = 0.$$

2A. Volumetric conservation laws:

- A global volumetric conservation law of a given 3D PDE model, for $\mathcal{V} \subset \Omega$:

$$\boxed{\frac{d}{dt} \int_{\mathcal{V}} T \, dV = - \oint_{\partial \mathcal{V}} \mathbf{\Psi} \cdot d\mathbf{S},}$$

  holding for all solutions $v(t, \mathbf{x}) \in \mathcal{E}$.

- Local formulation: a continuity equation

$$\boxed{D_t T[v] + \mathrm{Div}\, \mathbf{\Psi}[v] = 0, \qquad v \in \mathcal{E}.}$$

- Scalar *conserved density*: $T = T[v]$, vector *spatial flux*: $\mathbf{\Psi} = \mathbf{\Psi}[v]$.

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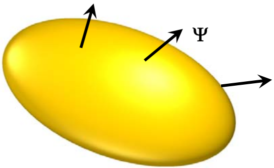
holding for all solutions $v(t, \mathbf{x}) \in \mathcal{E}$.

- Physical meaning: the rate of change of the volume quantity

$$\int_{\mathcal{V}} T[v] \, dV$$

is balanced by the surface flux

$$\oint_{\partial \mathcal{V}} \mathbf{\Psi}[v] \cdot d\mathbf{S}.$$



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Conservation of electromagnetic energy:

$$\tfrac{1}{2}\partial_t \left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) + c \operatorname{div}(\mathbf{E} \times \mathbf{B}) = 0.$$

2B. Surface-flux conservation laws:

- A global surface-flux conservation law of a given 3D PDE model:

$$\frac{d}{dt}\int_{\mathcal{S}} \mathbf{T}\cdot d\mathbf{S} = -\oint_{\partial\mathcal{S}} \mathbf{\Psi}\cdot d\boldsymbol{\ell}, \qquad v\in\mathcal{E}.$$

- Local formulation: a vector PDE

$$D_t\,\mathbf{T}[v] + \mathrm{Curl}\;\mathbf{\Psi}[v] = 0, \qquad v\in\mathcal{E}.$$

- $\mathcal{S}\subseteq\Omega$ is a fixed bounded surface.
- Vector *conserved flux density*: $\mathbf{T}=\mathbf{T}[v]$; vector *spatial circulation flux*: $\mathbf{\Psi}=\mathbf{\Psi}[v]$.
- Local form: three related scalar divergence-type CLs.

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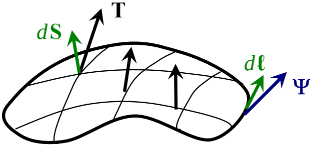
- Local formulation: a vector PDE

$$D_t\,\mathbf{T}[v] + \mathrm{Curl}\ \mathbf{\Psi}[v] = 0, \qquad v\in\mathcal{E}.$$



- Physical meaning: rate of change of the surface quantity

$$\int_{\mathcal{S}} \mathbf{T}[v]\cdot d\mathbf{S}$$

is balanced by the circulation

$$\oint_{\partial\mathcal{S}} \mathbf{\Psi}[v]\cdot d\boldsymbol{\ell}.$$

Example: microscopic Maxwell's equations in Gaussian units

$$\operatorname{div} \mathbf{B} = 0, \qquad \boxed{\mathbf{B}_t + c \operatorname{curl} \mathbf{E} = 0,}$$

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \qquad \mathbf{E}_t - c \operatorname{curl} \mathbf{B} = -4\pi \mathbf{J}.$$

Magnetic flux conservation: a global surface-flux conservation law (Faraday's law)

$$\frac{d}{dt} \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = -c \oint_{\partial \mathcal{S}} \mathbf{E} \cdot d\boldsymbol{\ell}.$$

Example: ideal magnetohydrodynamics (MHD) equations

$$\rho_t + \operatorname{div} \rho\mathbf{u} = 0,$$

$$\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\frac{1}{\mu}\mathbf{B}\times\operatorname{curl}\mathbf{B} - \operatorname{grad} p,$$

$$\operatorname{div}\mathbf{B} = 0,$$

$$\boxed{\mathbf{B}_t = \operatorname{curl}(\mathbf{u}\times\mathbf{B}).}$$

Conserved flux density, spatial circulation flux:

$$\mathbf{T} = \mathbf{B}, \qquad \mathbf{\Psi} = \mathbf{B}\times\mathbf{u}.$$

The global form of the surface-flux conservation law

$$\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{S} = -\oint_{\partial\mathcal{S}} (\mathbf{B}\times\mathbf{u})\cdot d\boldsymbol{\ell}$$

describes the time evolution of the total magnetic flux through a given fixed surface $\mathcal{S}$.

- A similar CL holds for non-ideal (resistive, viscous) plasmas.

2C. Circulatory conservation laws:

- A global circulatory conservation law of a given 3D PDE model:

$$\boxed{\frac{d}{dt}\int_{\mathcal{C}} \mathbf{T}\cdot d\ell = -\Psi\big|_{\partial\mathcal{C}}, \qquad v \in \mathcal{E}.}$$

- Local local circulatory conservation law:

$$\boxed{D_t\,\mathbf{T}[v] + \mathrm{Grad}\;\Psi[v] = 0, \qquad v \in \mathcal{E}.}$$

- $\mathcal{C} \subseteq \Omega$ is a fixed simple curve.
- Vector *conserved circulation density*: $\mathbf{T} = \mathbf{T}[v]$; vector *spatial boundary flow*: $\Psi = \Psi[v]$.
- Local form: three related scalar divergence-type CLs.

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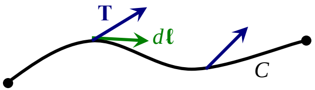


- Physical meaning: rate of change of the line integral quantity

$$\int_{\mathcal{C}} \mathbf{T}\cdot d\boldsymbol{\ell}$$

is balanced by the flow through the ends of the curve.

Example: irrotational barotropic gas flow.

$$\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0,$$

$$\mathbf{u}_t + (\mathrm{curl}\, \mathbf{u}) \times \mathbf{u} + \mathrm{grad}\; f = 0, \qquad f = f_{\mathrm{bar}} = \frac{|\mathbf{u}|^2}{2} + \int \frac{p'(\rho)}{\rho}\, d\rho.$$

- Irrotational: $\mathrm{curl}\, \mathbf{u} = 0.$
- Barotropic: $p = p(\rho), \quad \Rightarrow \boxed{\mathbf{u}_t + \mathrm{grad}\; f = 0.}$
- Circulatory conservation law over an arbitrary static curve $\mathcal{C}$:

$$\frac{d}{dt} \int_{\mathcal{C}} \mathbf{u} \cdot d\boldsymbol{\ell} = -f|_{\partial \mathcal{C}}.$$

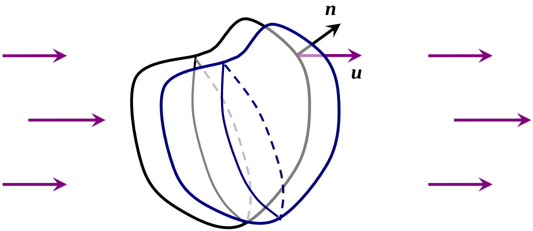
- For closed curves, $\partial \mathcal{C} = \emptyset$:

$$\frac{d}{dt} \oint_{\mathcal{C}} \mathbf{u} \cdot d\boldsymbol{\ell} = 0,$$

conservation of a global velocity circulation around a static closed path.

CLs on moving domains

- Flow velocity: $\mathbf{u}(t, \mathbf{x})$.
- A moving material domain consists of the same material points.



Moving volumetric conservation laws:

- A moving volumetric conservation law of a given 3D PDE model:

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}(t)} T[\mathbf{u},v]\,dV = -\oint_{\partial\mathcal{V}(t)} \boldsymbol{\Upsilon}[\mathbf{u},v]\cdot d\mathbf{S},}$$

holding for all solutions $v = v(t,\mathbf{x}) \in \mathcal{E}$, for a volume $\mathcal{V}(t) \in \Omega$ transported by the flow.

Local formulation:

- Leibniz's rule for moving domains:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} T[\mathbf{u},v]\,dV = \int_{\mathcal{V}(t)} D_t T[\mathbf{u},v]\,dV + \oint_{\partial\mathcal{V}(t)} T[\mathbf{u},v]\,\mathbf{u}\cdot d\mathbf{S}$$

- Local form:

$$\boxed{D_t\, T[\mathbf{u},v] + \mathrm{Div}\left(\boldsymbol{\Upsilon}[\mathbf{u},v] + T[\mathbf{u},v]\mathbf{u}\right) = 0.}$$

Moving volumetric CL example: helicity

- Constant-density fluid flow:

$$\operatorname{div} \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \operatorname{grad} f = 0, \qquad f = \frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho}.$$

- The fluid helicity: $h \equiv \mathbf{u} \cdot \boldsymbol{\omega}$.
- Helicity dynamics equation: $h_t + \operatorname{div}(\boldsymbol{\omega} \cdot \operatorname{grad} f + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0$.
- Moving volumetric CL, local form:

$$\boxed{D_t\, T[\mathbf{u}, v] + \operatorname{Div}(\boldsymbol{\Upsilon}[\mathbf{u}, v] + T[\mathbf{u}, v]\mathbf{u}) = 0, \qquad v \in \mathcal{E}.}$$

$$T = h = \mathbf{u} \cdot \boldsymbol{\omega}, \qquad \boldsymbol{\Upsilon} = (f - |\mathbf{u}|^2)\, \boldsymbol{\omega}.$$

- Global form:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} h\, dV = - \oint_{\partial \mathcal{V}(t)} (f - |\mathbf{u}|^2)\, \boldsymbol{\omega} \cdot d\mathbf{S}.$$

Material conservation laws

- A material conservation law: a moving volumetric CL with a vanishing spatial flux, $\boldsymbol{\Upsilon}[\mathbf{u}, v]|_{\mathcal{E}} = 0.$ of a given 3D PDE model, for $\mathcal{V} \subset \Omega$:

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}(t)} T[\mathbf{u}, v]\, dV = -\oint_{\partial\mathcal{V}(t)} \boldsymbol{\Upsilon}[\mathbf{u}, v] \cdot d\mathbf{S} = 0.}$$

- Local formulation:

$$\boxed{D_t T[\mathbf{u}, v] + \mathrm{Div}(T[\mathbf{u}, v]\mathbf{u}) = 0.}$$

- A well-known expression for incompressible flows $\mathrm{div}\,\mathbf{u} = 0$:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} T[\mathbf{u}, v] \;=\; 0,} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \;\equiv\; D_t + \mathbf{u} \cdot \mathrm{Grad}$$

Material conservation laws: example

The continuity equation in gas/fluid dynamics:

$$\boxed{\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0,}$$

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p = \mu\, \Delta \mathbf{u} + \rho \mathbf{g}.$$

Conservation of mass in a moving material domain :

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \rho\, dV = 0.$$

3. Time-dependent CLs on moving domains

- In a similar way, moving surface-flux and moving circulatory CLs in material domains arise.
- Material CLs arise in a similar manner.

CLs in 3D: overview

Conservation laws in 3D: overview

- PDE systems in (3+1) dimensions can have 8 different kinds of CLs:
  - 2 time-independent/topological.
  - 3 time-dependent (fixed domains).
  - 3 time-dependent (moving domains) (also material CLs).
- Each has a local and a global form.
- Common framework, clear physical meaning.
- Each kind is locally given by divergence expression(s) $\Rightarrow$ systematic computation.
- Physical examples are readily available.

Talk summary

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- CLs are coordinate-independent; they can be obtained systematically through the Direct construction method.
- Symbolic software for such computations exists.
- For variational models, Noether's theorem gives useful insights in symmetry-CL relations. These relations are, however, known in a more general setting.

What was left out...

We did not discuss:

- Multiple computational aspects; multiplier dependencies; singular multipliers; etc.
- CL triviality and equivalence questions.
- 2nd Noether's theorem.
- Useful tricks and techniques to get CLs "cheap".
- Higher-order & nonlocal symmetries. Nonlocal CLs.
- Integrability, linearization, ...

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Thank you for your attention!