Symbolic Computation of Symmetries and First Integrals in Dynamical Systems

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Outline

1. Motivation
2. Symmetries of ODE/PDE, applications, computation
3. First integrals of ODEs, conservation laws of PDEs, applications, computation
4. Conclusions

Motivation

- Exact solutions of ODE, PDE models are required where possible.
- Admitted Lie groups of point symmetries can reduce order of ODEs, without loss of solutions.
- First integrals (FI, constants of motion) also lead to direct integration of ODEs.
- For PDEs, point symmetries lead to reductions, interesting particular solutions, mappings between solutions.
- Conservation laws (CL) for PDEs yield global conserved quantities, and are highly useful in analysis.
- Local symmetries and FI/CLs are related.

Motivation (ctd.)

- Symmetries, as well as CL/FI, can be systematically computed. For nontrivial models, these computations are, however, computationally demanding. Pencil/paper computations usually not realistic.

- Maple: a great symbolic package for DEs. It has built-in Symm/CL routines, but they are slow and not flexible.

- This talk: examples of the use of GeM module for Maple to compute symmetries, FI, CL for ODEs and PDEs.

Notation

- Independent variables: $\mathbf{x} = (x^1, x^2, ..., x^n)$ or $(t, x^1, x^2, ...)$ or $(t, x, y, ...)$.
- Dependent variables: $\mathbf{u} = (u^1(\mathbf{x}), u^2(\mathbf{x}), ..., u^m(\mathbf{x}))$ or $(u(\mathbf{x}), v(\mathbf{x}), ...)$.
- Ordinary derivatives: $\dfrac{dy(x)}{dx} = y'(x)$.
- Partial derivatives: $\dfrac{\partial u^k}{\partial x^m} = u^k_{x^m} = u^k_m$.
- All p^{th}-order partial derivatives: $\partial^p \mathbf{u}$.
- A *differential function*: a function on the jet space, $F[\mathbf{u}] = F(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^p\mathbf{u})$.
- A *total derivative* of a differential function: a basic chain rule

$$\mathrm{D}_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{ii_1i_2} \frac{\partial}{\partial u^\mu_{i_1i_2}} + \cdots .$$

Outline

Consider a general DE system $R^{\sigma}[\mathbf{u}] = \mathbf{R}^{\sigma}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^{k}\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$

- A one-parameter Lie group of point transformations (the global group):

$$
\begin{aligned}
(x^*)^i &= f^i(\mathbf{x}, \mathbf{u}; \varepsilon) = x^i + \varepsilon \xi^i(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2), \quad i = 1, \ldots, n, \\
(u^*)^\mu &= g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon) = u^\mu + \varepsilon \eta^\mu(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2), \quad \mu = 1, \ldots, m.
\end{aligned}
$$

- Infinitesimal generator: $\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u}) \dfrac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u}) \dfrac{\partial}{\partial u^\mu}.$

- Infinitesimal components:

$$
\xi^i(\mathbf{x}, \mathbf{u}) = \frac{\partial f^i(\mathbf{x}, \mathbf{u}; \varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad \eta^\mu(\mathbf{x}, \mathbf{u}) = \frac{\partial g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}.
$$

- Global group recovery:

$$
(x^*)^i = f^i(\mathbf{x}, \mathbf{u}; \varepsilon) = e^{\varepsilon \mathrm{X}}\, x^i, \quad (u^*)^\mu = g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon) = e^{\varepsilon \mathrm{X}}\, u^\mu.
$$

Point symmetries

Consider a general DE system $R^\sigma[\mathbf{u}] = \mathbf{R}^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$

- A one-parameter Lie group of point transformations (the global group):

$$
\begin{aligned}
(x^*)^i &= f^i(\mathbf{x}, \mathbf{u}; \varepsilon) = x^i + \varepsilon\xi^i(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2), \quad i = 1, \ldots, n,\\
(u^*)^\mu &= g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon) = u^\mu + \varepsilon\eta^\mu(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2), \quad \mu = 1, \ldots, m.
\end{aligned}
$$

- Infinitesimal generator: $\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u})\dfrac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u})\dfrac{\partial}{\partial u^\mu}.$

- k^{th} prolongation:

$$
\mathrm{X}^{(k)} = \mathrm{X} + \eta_i^{(1)\,\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u})\frac{\partial}{\partial u_i^\mu} + \cdots + \eta_{i_1\ldots i_k}^{(k)\,\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u})\frac{\partial}{\partial u_{i_1\ldots i_k}^\mu}.
$$

- A group-invariant differential function $F[\mathbf{u}]$: $\mathrm{X}^{(\infty)}F \equiv 0$.

- Infinitesimal criterion of invariance of a DE system under the Lie group action:

$$
\mathrm{X}^{(k)}R^\alpha(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u})\Big|_{\mathbf{R}[\mathbf{u}]=0} = 0, \quad \alpha = 1, \ldots, N.
$$

Consider a general DE system $R^\sigma[\mathbf{u}] = \mathbf{R}^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$

- For an ODE, an admitted one-parameter Lie group of point symmetries can be used to reduce ODE order by one, using differential invariants or canonical coordinates.
- A solvable m-parameter Lie algebra of point symmetries can be used to reduce ODE order by m.
- Reduction of order: using canonical coordinates or differential invariants.

Example:

$$y''(x) = 0,$$

admitting an 8-parameter symmetry group, maximal for 2nd-order ODEs. [Maple file]

- For an ODE of order n, one can have at most $(n+4)$-parameter Lie group of point symmetries.

Another example: the Blasius equation (first-order boundary layer theory for the Navier-Stokes equations):

$$y''' + \frac{1}{2} y y'' = 0.$$

It admits a 2-parameter symmetry group, with Lie algebra [Maple file]

$$\mathrm{X}_1 = \frac{\partial}{\partial x}, \quad \mathrm{X}_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Since every 2-dimensional Lie algebra is solvable, the order can be reduced by 2, to get a 1st-order ODE on $V(U)$:

$$V'(U) = \frac{V}{U}\,\frac{U + V + 1/2}{2U - V}.$$

As a result, the general solution of the Blasius equation can be written in quadratures [Bluman & Kumei].

Symmetries of a PDE model:

- A PDE arising in a model of a flame front propagating upwards in a vertical channel [M. Ward & A.C. (2007)]:

$$u_t = \epsilon^2(u_{xx} + u_{yy}) + u \log u,$$

where $u = u(x, y, t)$, and ϵ is a parameter.

- The admitted 7-parameter Lie algebra of point symmetry generators [Maple]

$$\mathrm{X}_1 = \frac{\partial}{\partial x}, \quad \mathrm{X}_2 = \frac{\partial}{\partial y}, \quad \mathrm{X}_3 = \frac{\partial}{\partial t}, \quad \mathrm{X}_4 = e^t u \frac{\partial}{\partial u}, \quad \mathrm{X}_5 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$
$$\mathrm{X}_6 = e^t \frac{\partial}{\partial x} - \frac{e^t x}{2\epsilon^2} u \frac{\partial}{\partial u}, \quad \mathrm{X}_7 = e^t \frac{\partial}{\partial y} - \frac{e^t y}{2\epsilon^2} u \frac{\partial}{\partial u}.$$

- An invariant solution w.r.t. $\mathrm{X}_3, \mathrm{X}_6, \mathrm{X}_7$: an all-space Gaussian bell equilibrium

$$u^\infty(\mathbf{x}; \mathbf{x}_0) = \exp\left(1 - \frac{|\mathbf{x} - \mathbf{x}_0|^2}{4\epsilon^2}\right), \quad \mathbf{x} \in \mathbb{R}^2;$$

a spike of width $\sim \epsilon$ about $\mathbf{x}_0$.

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Local conservation laws of DEs

- A system of differential equations (PDE or ODE) $\mathbf{R}[\mathbf{u}] = 0$:

$$R^\sigma[\mathbf{u}] = \mathbf{R}^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$$

- The basic notion:

A local conservation law:

A divergence expression

$$\boxed{\mathrm{D}_i \Phi^i[\mathbf{u}] = 0}$$

vanishing on solutions of $\mathbf{R}[\mathbf{u}] = 0$. Here $\Phi = (\Phi^1[\mathbf{u}], \ldots, \Phi^n[\mathbf{u}])$ is the flux vector.

Local and global CL form – PDEs

- For time-dependent PDEs, the meaning of a local conservation law is that the rate of change of some "total amount" is balanced by a boundary flux.

- **(1+1)-dimensional PDEs:** $\mathbf{u} = \mathbf{u}(x, t)$, only one CL type.

  Local CL form:

$$\boxed{\mathrm{D}_t T[\mathbf{u}] + \mathrm{D}_x \Psi[\mathbf{u}] = 0.}$$

  $T[\mathbf{u}]$: CL density; $\Psi[\mathbf{u}]$: CL flux.

  Global CL form:

$$\boxed{\frac{d}{dt}\int_a^b T[\mathbf{u}]\, dx = -\Psi[\mathbf{u}]\Big|_a^b.}$$

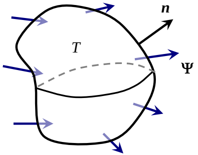
Local and global CL form – PDEs

- **(3+1)-dimensional PDEs:** $R[\mathbf{u}] = 0, \ \ \mathbf{u} = \mathbf{u}(t, x, y, z).$

- Local CL form: $\boxed{D_t T[\mathbf{u}] + \mathrm{Div}\, \boldsymbol{\Psi}[\mathbf{u}] = 0} \quad \Leftrightarrow \quad \boxed{\mathrm{D}_i \Phi^i[\mathbf{u}] = 0}$

- Global CL form: $\boxed{\dfrac{d}{dt}\int_{\mathcal{V}} T[\mathbf{u}]\, dV = -\oint_{\partial\mathcal{V}} \boldsymbol{\Psi}[\mathbf{u}] \cdot d\mathbf{S}}$

- Holds for all solutions $\mathbf{u}(t, x, y, z)$, for $\mathcal{V} \subset \Omega$, in some physical domain Ω.



The idea of the direct (multiplier) CL construction method

Independent and dependent variables of the problem:
$\mathbf{x} = (x^1, ..., x^n)$, $\mathbf{u}(\mathbf{x}) = (u^1, ..., u^m)$.

Definition

The Euler operator with respect to an arbitrary function u^j:

$$\mathrm{E}_{u^j} = \frac{\partial}{\partial u^j} - \mathrm{D}_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial u^j_{i_1 \ldots i_s}} + \cdots , \quad j = 1, \ldots, m.$$

Theorem

The equations

$$\mathrm{E}_{u^j} F[\mathbf{u}] \equiv 0, \quad j = 1, \ldots, m$$

hold for arbitrary $\mathbf{u}(\mathbf{x})$ *if and only if* F *is a divergence expression*

$$F[\mathbf{u}] \equiv \mathrm{D}_i \Phi^i$$

for some functions $\Phi^i = \Phi^i[\mathbf{u}]$.

Given:

- A system of M DEs $R^\sigma[\mathbf{u}] = 0, \quad \sigma = 1, \ldots, M$.

The direct (multiplier) method

1. Specify the dependence of multipliers: $\Lambda_\sigma[\mathbf{u}] = \Lambda_\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots)$.
2. Solve the set of determining equations $\mathrm{E}_{u^j}(\Lambda_\sigma[\mathbf{u}]R^\sigma[\mathbf{u}]) \equiv 0, \quad j = 1, \ldots, m$, for arbitrary $\mathbf{u}(\mathbf{x})$, to find all sets of multipliers.
3. Find the corresponding fluxes $\Phi^i[\mathbf{u}]$ satisfying the identity
$$\Lambda_\sigma[\mathbf{u}]R^\sigma[\mathbf{u}] \equiv \mathrm{D}_i\Phi^i[\mathbf{u}].$$
4. For each set of fluxes, on solutions, get a local conservation law
$$\boxed{\mathrm{D}_i\Phi^i[\mathbf{u}] = 0.}$$
5. Implemented in GeM module for Maple (on my web page)

Applications to ODEs

- First integrals (constants of motion):

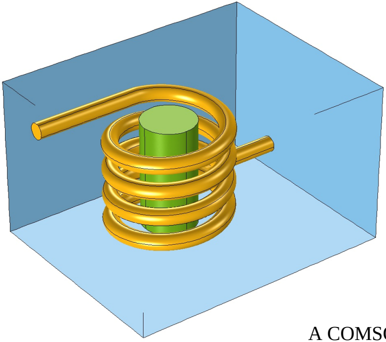
$$\mathrm{D}_t T[\mathbf{u}] = 0 \quad \Rightarrow \quad T[\mathbf{u}] = \mathrm{const}.$$

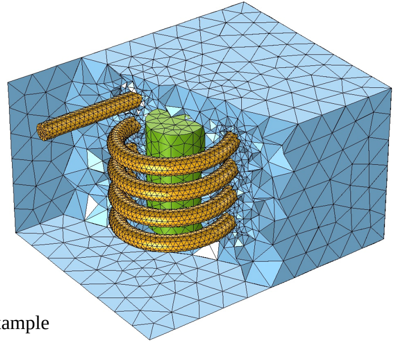
- Reduction of order / integration.

Applications to PDEs

$$D_t T[\mathbf{u}] + \mathrm{Div}\, \mathbf{\Psi}[\mathbf{u}] = 0$$

- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Divergence forms of PDEs for analysis: existence, uniqueness, stability, Fokas method.
- Weak solutions.
- Potentials, stream functions, etc.
- An infinite number of CLs may indicate integrability/linearization.
- Numerical methods: divergence forms of PDEs (finite-element, finite volume); constants of motion.





A COMSOL example

ODE example 1: harmonic oscillator, mass-spring system [Maple file]

$$m\ddot{x}(t) + kx(t) = 0; \quad k, m = \mathrm{const.}$$

- Seek multipliers $\Lambda = \Lambda(\dot{x})$, find $\Lambda = C\dot{x}$.
- Conservation law:

$$\frac{d}{dt}\left(\frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2}\right) = 0.$$

- First integral:

$$E = \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2} = \mathrm{const.}$$

ODE FI example 2: predator-prey model

ODE example 2: Lotka-Volterra predator-prey ODE system [Maple file]

$$x' = \alpha x - \beta xy, \qquad y' = \delta xy - \gamma y.$$

Here $x = x(t)$=number of prey, $y = y(t)$=number of predator, and $\alpha, \beta, \gamma, \delta = \mathrm{const}$.

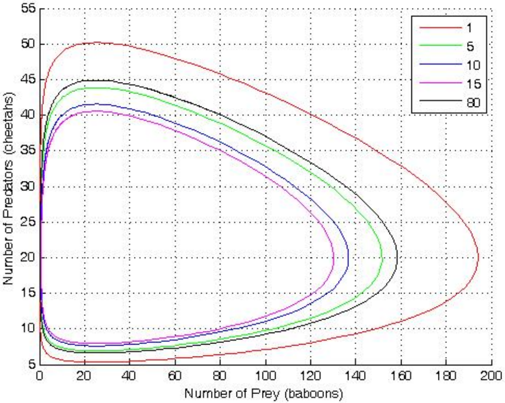
- Seek CL multipliers: $\Lambda_1 = \Lambda_1(x)$, $\Lambda_2 = \Lambda_2(y)$.
- Find $\Lambda_1 = C(d - g/x)$, $\Lambda_2 = C(b - a/y)$.
- Conservation law:
$$\frac{d}{dt}\left(\delta x - \gamma \ln x + \beta y - \alpha \ln y\right) = 0.$$
- First integral:
$$V(t) = \delta x - \gamma \ln x + \beta y - \alpha \ln y = \mathrm{const}.$$

ODE example 2: Lotka-Volterra predator-prey ODE system [Maple file]

$$x' = \alpha x - \beta xy, \qquad y' = \delta xy - \gamma y.$$

Here $x = x(t)$=number of prey, $y = y(t)$=number of predator, and $\alpha, \beta, \gamma, \delta = \mathrm{const}$.

- Trajectories: cycles $V(t) = \mathrm{const}$.



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LETTER TO THE EDITORS

DO HARES EAT LYNX?

To test a recently developed predator-prey model against reality, I chose the well-known Canadian hare-lynx system. A measure of the state of this system for the last 200-odd years is available in the fur catch records of the Hudson Bay Company (MacLulich 1937; Elton and Nicholson 1942). Although the accuracy of these data is questionable (see Elton and Nicholson 1942 for a full discussion), they represent the only long-term population record available to ecologists.

The model I tested is

$$dH/dt = H(r_H + C_{HL}L + S_H H + I_H H^2), \quad (1a)$$

$$dL/dt = L(r_L + C_{LH}H + S_L L + I_L L^2), \quad (1b)$$

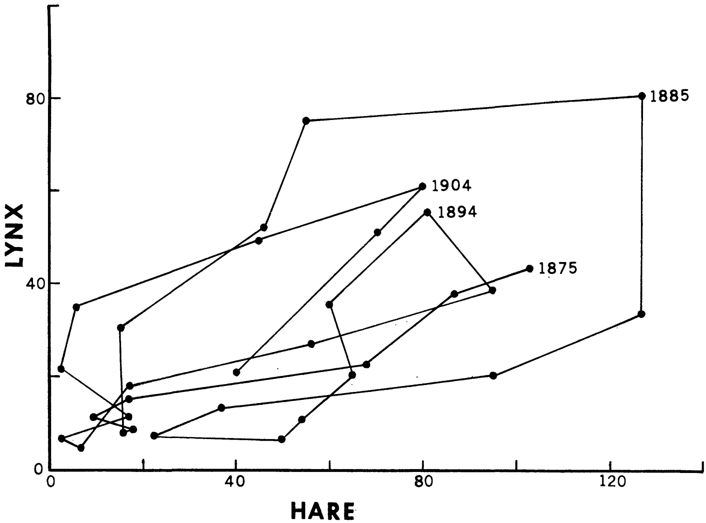


FIG. 1.—Yearly states of the Canadian lynx-hare system from 1875 to 1906. The numbers on the axes represent the numbers of the respective animals in thousands.

ODE example 3: a nonlinear ODE arising in symmetry classification

$$K'''(x) = \frac{-2\left(K''(x)\right)^2 K(x) - \left(K'(x)\right)^2 K''(x)}{K(x)K'(x)}.$$

[Maple file]

- Seek multipliers: $\Lambda = \Lambda(x, K, K')$.
- Find three multipliers:

$$\Lambda_1 = \frac{K}{(K')^2}, \quad \Lambda_2 = \frac{xK}{(K')^2}, \quad \Lambda_3 = \frac{K \ln K}{(K')^2}.$$

- Three FIs:

$$\frac{KK''}{(K')^2} = M_1, \quad \frac{K(K' + xK'') - x(K')^2}{(K')^2} = M_2, \quad \frac{ln(K)(KK'' - (K')^2)}{(K')^2} = M_3.$$

- General solution (after redefining the constants):

$$K(x) = c_1(x + c_2)^{c_3}.$$

CL computation for PDE systems

[Maple/GeM]:

- Can compute CLs of several PDEs, with multi-component unknowns $\mathbf{u}(\mathbf{x})$ depending on several scalar variables.
- Examples: Euler & Navier-Stokes, nonlinear mechanics, integrable equations/higher-order CLs.
- New results have been obtained for various models.

PDE example 1: the "short pulse" equation [Schäfer & Wayne (2004)], a model of ultra-short optical pulses in nonlinear media

$$u_{tx} = u + 6uu_x^2 + 3u^2 u_{xx}.$$

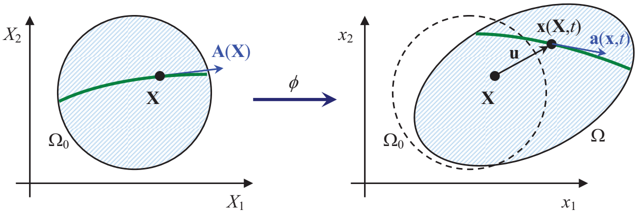
Here $u = u(t,x)$. [Maple file]

- This is an integrable equation [2× Sakovich (2005)] – admits a Lax pair, a recursion operator, an infinite hierarchy of higher-order symmetries and CLs; related to the sine-Gordon equation.
- Seek CL multipliers depending on up to 3rd derivatives of u:

$$\Lambda = \Lambda(t, x, u, u_t, u_x, \ldots, u_{xxx}).$$

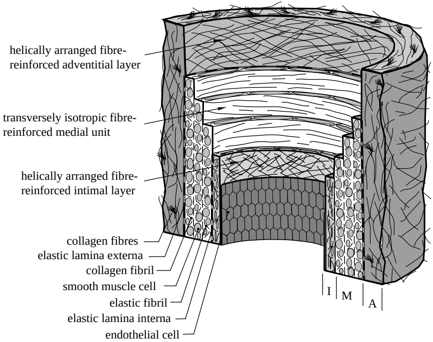
- Find three multipliers.
- Three CLs in this ansatz, non-polynomial form.

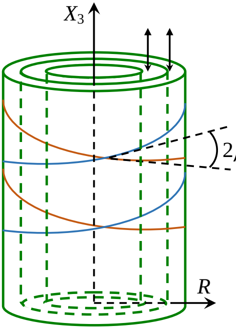
PDE CL example 2: Mooney-Rivlin incompressible hyperelasticity



- Lagrangian coordinates $\mathbf{X}$, actual (Eulerian) coordinates $\mathbf{x} = \phi(\mathbf{X}, t)$.
- Deformation gradient: $\mathbb{F}(\mathbf{X}, t) = \mathrm{grad}_{(\mathbf{X})}\, \phi(\mathbf{X}, t)$; Jacobian: $J = \det \mathbb{F} > 0$.
- Density: $\rho(\mathbf{X}, t) = \rho_0(\mathbf{X})/J$.
- Isotropic + anisotropic elastic energy density: $W = W_{\mathrm{iso}} + W_{\mathrm{aniso}}$.
- The Piola-Kirchhoff stress tensor: $\mathbb{P} = -p\, \mathbb{F}^{-T} + \rho_0 \dfrac{\partial W}{\partial \mathbb{F}}$.
- Equations of motion: $\rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(\mathbf{X})}\, \mathbb{P} + \mathbf{Q}, \quad J = 1$.

PDE CL/Symm example: displacements in fiber-reinforced hyperelastic material





- Z-displacements $G(t;X)$ for a fiber-reinforced elastic solid, a Cartesian analog: $G_{tt} = (\alpha + 3\beta G_x^2)G_{xx}$ [A.C. & J.-F. Ganghoffer (2016)].
- Dimensionless: $\boxed{u_{tt} = (1 + u_x^2)u_{xx}.}$ Lagrangian: $\mathcal{L} = \dfrac{1}{2}(u_x^2 - u_t^2) + \dfrac{1}{12}u_x^4.$

PDE CL/Symm example: displacements in fiber-reinforced hyperelastic material

- PDE: $\boxed{u_{tt} = (1 + u_x^2)u_{xx}.}$

- Noether's theorem $\rightarrow$ variational symmetries in evolutionary form

$$\hat{\mathrm{X}} = \zeta(x, t, u, \ldots)\frac{\partial}{\partial u},$$

  must match CL multipliers: $\Lambda = \zeta$.

- 1st-order local symmetries in evolutionary form: $\zeta = \zeta(x, t, u, u_x, u_t)$ [Maple file]

$$\zeta_1 = 1, \quad \zeta_2 = t, \quad \zeta_3 = u_x, \quad \zeta_4 = u_t, \quad \zeta_5 = u_x u_t, \quad \zeta_6 = -xu_x - tu_t + u.$$

- 1st-order CL multiplies: $\Lambda = \Lambda(x, t, u, u_x, u_t)$ [Maple file]

$$\Lambda_1 = 1, \quad \Lambda_2 = t, \quad \Lambda_3 = u_x, \quad \Lambda_4 = u_t, \quad \Lambda_5 = u_x u_t.$$

- ζ_6 corresponds to a scaling $t^* = Ct$, $x^* = Cx$, $u^* = Cu$, which is not a variational symmetry.

CLs of the linear wave equation?

- Linear wave equation: $\boxed{u_{tt} = u_{xx},}$ introduced by d'Alembert in 1747.
- Linear $\rightarrow$ infinite CL family (multipliers solve the adjoint linear PDE).
- Some basic CLs:

$$
\begin{aligned}
&M_1 = 1, && \mathrm{D}_t\,(u_t) - \mathrm{D}_x\,(u_x) = 0,\\
&M_2 = u_x, && \mathrm{D}_t\,(u_t u_x) - \mathrm{D}_x\left(\tfrac{u_t^2+u_x^2}{2}\right) = 0,\\
&M_3 = u_t, && \mathrm{D}_t\left(\tfrac{u_t^2+u_x^2}{2}\right) - \mathrm{D}_x\,(u_t u_x) = 0,\\
&M_4 = t, && \mathrm{D}_t\,(tu_t - u) - D_y x\,(tu_x) = 0,\\
&M_5 = x, && \mathrm{D}_t\,(xu_t) - \mathrm{D}_x\,(xu_x - u) = 0,\\
&M_6 = xu_x + tu_t, && \mathrm{D}_t\left(xu_t u_x + \frac{t}{2}(u_t^2 + u_x^2)\right) - \mathrm{D}_x\left(tu_t u_x + \frac{x}{2}(u_t^2 + u_x^2)\right) = 0,\\
&M_7 = tu_x + xu_t, && \mathrm{D}_t\left(tu_t u_x + \frac{x}{2}(u_t^2 + u_x^2)\right) - \mathrm{D}_x\left(xu_t u_x + \frac{t}{2}(u_t^2 + u_x^2)\right) = 0.
\end{aligned}
$$

- The full set of local CLs has not been classified to date.
- (2019) R. Popovych, A.C.: complete CL classification, using the second canonical form $w_{\xi\eta} = 0$.

Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

A. Cheviakov and M. Oberlack (2014)
Generalized Ertel's theorem and infinite hierarchies of conserved quantities for three-dimensional time-dependent Euler and Navier-Stokes equations. *JFM* 760: 368-386.

- seek CLs to second-order multipliers, depending on up to 45 variables,

$$t, x, y, z, \quad u^1, u^2, u^3, p, \quad u^1_y, u^1_z, \quad u^2_x, u^2_y, u^2_z, \quad u^3_x, u^3_y, u^3_z, \quad p_t, p_x, p_y, p_z,$$
$$u^1_{yy}, u^1_{yz}, u^1_{zz}, \quad u^2_{xx}, u^2_{xy}, u^2_{xz}, u^2_{yy}, u^2_{yz}, u^2_{zz}, \quad u^3_{xx}, u^3_{xy}, u^3_{xz}, u^3_{yy}, u^3_{yz}, u^3_{zz},$$
$$p_{tt}, p_{tx}, p_{ty}, p_{tz}, p_{xx}, p_{xy}, p_{xz}, p_{yy}, p_{yz}, p_{zz}.$$

Example: CLs of Euler equations

Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div}\mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\operatorname{grad} p.$$

1. Conservation of generalized momentum.

$$\Lambda_1 = f(t)u^1 - xf'(t), \quad \Lambda_2 = f(t), \quad \Lambda_3 = \Lambda_4 = 0;$$

$$\frac{\partial}{\partial t}(f(t)u^1) + \frac{\partial}{\partial x}\Big((u^1 f(t) - xf'(t))u^1 + f(t)p\Big)$$
$$+\frac{\partial}{\partial y}\Big((u^1 f(t) - xf'(t))u^2\Big) + \frac{\partial}{\partial z}\Big((u^1 f(t) - xf'(t))u^3\Big) = 0,$$

with analogous expressions holding for y- and the z-directions.

Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

2. Conservation of the angular momentum.

$$\Lambda_1 = u^2_z - u^3_y, \quad \Lambda_2 = 0, \quad \Lambda_3 = z, \quad \Lambda_4 = -y;$$

$$\frac{\partial}{\partial t}(zu^2 - yu^3) + \frac{\partial}{\partial x}\Big((zu^2 - yu^3)u^1\Big)$$
$$+\frac{\partial}{\partial y}\Big((zu^2 - yu^3)u^2 + zp\Big) + \frac{\partial}{\partial z}\Big((zu^2 - yu^3)u^3 - yp\Big) = 0.$$

with cyclic permutations for y- and the z-directions.

Constant-density Euler equations

$$\rho = \mathrm{const}, \quad \mathrm{div}\,\mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\,\mathrm{grad}\; p.$$

3. Conservation of the kinetic energy.

$$\Lambda_1 = K + p, \quad [\Lambda_2, \Lambda_3, \Lambda_4] = \mathbf{u};$$

$$\frac{\partial}{\partial t}K + \nabla\cdot\Big((K+p)\,\mathbf{u}\Big) = 0, \qquad K = \tfrac{1}{2}\,|\mathbf{u}|^2.$$

Example: CLs of Euler equations

Constant-density Euler equations

$$\rho = \mathrm{const}, \quad \mathrm{div}\,\mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\,\mathrm{grad}\; p.$$

4. Generalized continuity equation.

$$\Lambda_1 = k(t), \quad \Lambda_2 = \Lambda_3 = \Lambda_4 = 0;$$

$$\nabla \cdot (k(t)\,\mathbf{u}) = 0.$$

Constant-density Euler equations

$$\rho = \mathrm{const}, \quad \mathrm{div}\, \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\,\mathrm{grad}\; p.$$

5. Conservation of helicity.

$$\Lambda_1 = 0, \quad [\Lambda_2, \Lambda_3, \Lambda_4] = \boldsymbol{\omega} = \mathrm{curl}\, \mathbf{u};$$

$$h = \mathbf{u} \cdot \boldsymbol{\omega}; \quad E = K + p, \quad K = \tfrac{1}{2}\, |\mathbf{u}|^2;$$

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0.$$

Example: CLs of NS and Euler equations under helical symmetry

Kelbin, O., Cheviakov, A.F., and Oberlack, M. (2013)
New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. *JFM* 721, 340-366.

Helically-invariant equations

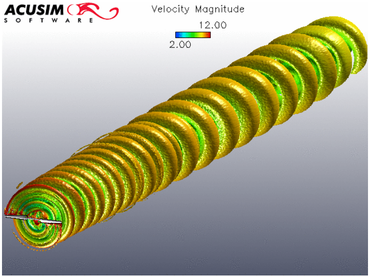
- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

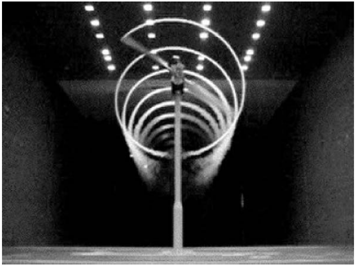
Additional conservation laws – through direct construction

- Three-component Euler:
  - Generalized momenta. Generalized helicity. Additional vorticity CLs.
- Three-component Navier-Stokes:
  - Additional CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - Additional CLs in viscous and inviscid case, for plane and axisymmetric flows.

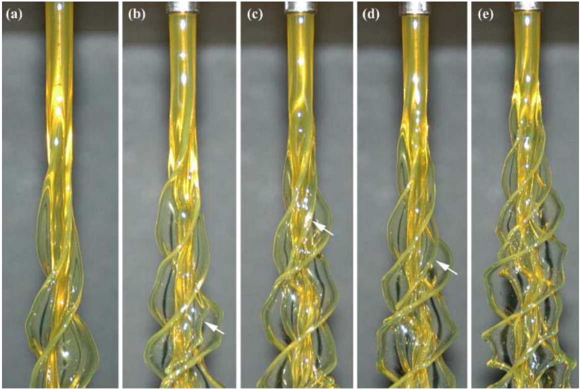
Example: CLs of NS and Euler equations under helical symmetry

- Wind turbine wakes in aerodynamics [Vermeer, Sorensen & Crespo, 2003]

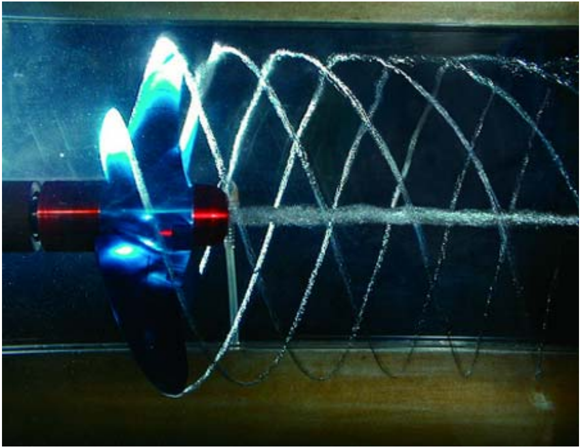


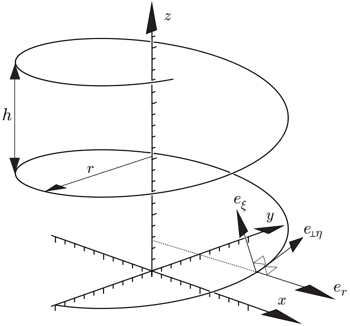


- Helical instability of rotating viscous jets [Kubitschek & Weidman, 2007]



- Helical water flow past a propeller





Helical Coordinates

- Helical coordinates: (r, η, ξ);

$$\xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \qquad a, b = \text{const}, \quad a^2 + b^2 > 0.$$

- Helical invariance: $f = f(r, \xi), \quad a, b \neq 0.$
- Axial: $a = 1,\ b = 0.$ z-Translational: $a = 0,\ b = 1.$

Summary:

- Simple, systematic computation of point and higher-order symmetries of ODE/PDE in Maple/GeM; global group.
- Similarly, Lie groups of equivalence transformations can be computed.
- Systematic computation of FIs for ODE, CLs for PDE in Maple/GeM: direct (multiplier) method.
- Symbolic software capable of working with multiple PDEs with many dependent/independent variables.
- Classification of symm/FI/CLs for families of DEs using Maple/rifsimp.

Work to do:

- Computation of invariants, differential invariants.
- Lie group structure.
- Canonical coordinates, invariant reduction.
- Object-oriented approach; parallelization for heavy computations.

- GeM for Maple: a symmetry/CL symbolic computation package. https://math.usask.ca/~shevyakov/gem/
- A.C. & M. Ward. (2007) A two-dimensional metastable flame-front and a degenerate spike-layer problem. *Interfaces and Free Boundaries* 9 (4), 513–547.
- T. Schäfer & C. Wayne (2004) Propagation of ultra-short optical pulses in cubic nonlinear media. *Physica D* 196, 90–105.
- A.C. & J.-F. Ganghoffer (2016) One-dimensional nonlinear elastodynamic models and their local conservation laws with applications to biological membranes. *JMBBM* 58, 105–121.

Thank you for your attention!