New exact plasma equilibria with axial and helical symmetry

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Outline

1. MHD equations
2. Exact solutions with axial symmetry
3. Exact solutions with helical symmetry
4. Dynamic solutions
5. Discussion

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MHD equations

Ideal incompressible magnetohydrodynamics equations

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\ \rho \mathbf{V} = 0,$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \rho \mathbf{V} \times \mathrm{curl}\, \mathbf{V} + \frac{1}{\mu} \mathrm{curl}\, \mathbf{B} \times \mathbf{B} - \mathrm{grad}\, P - \rho\, \mathrm{grad}\, \frac{\mathbf{V}^2}{2},$$

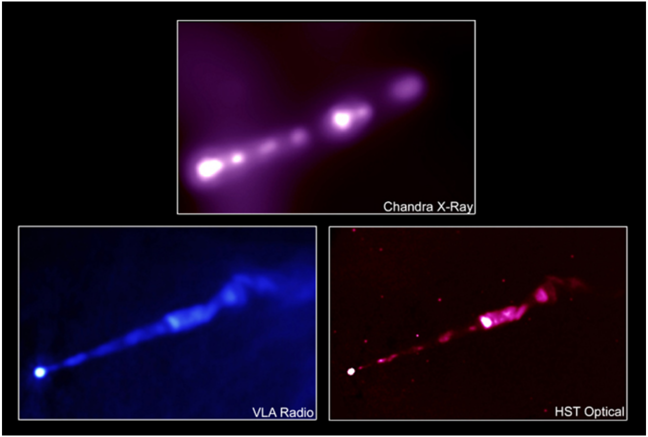
$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{V} \times \mathbf{B}),$$

$$\mathrm{div}\, \mathbf{B} = 0, \quad \mathrm{div}\, \mathbf{V} = 0$$

Notation

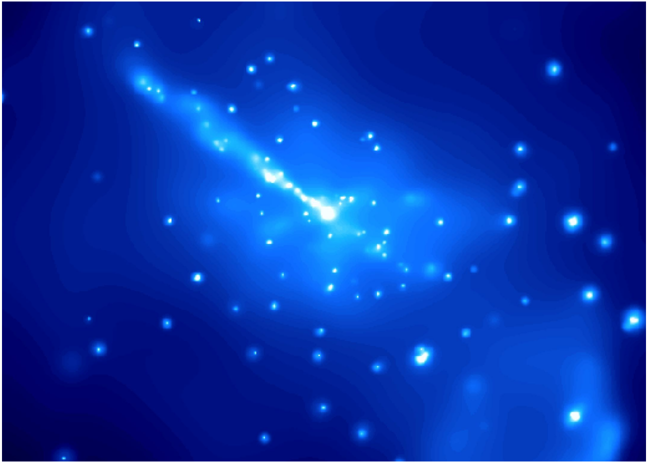
1. ρ plasma density
2. $\mathbf{B}$ magnetic induction vector
3. $\mathbf{V}$ plasma velocity vector
4. P scalar pressure
5. μ magnetic permeability coefficient

Applications

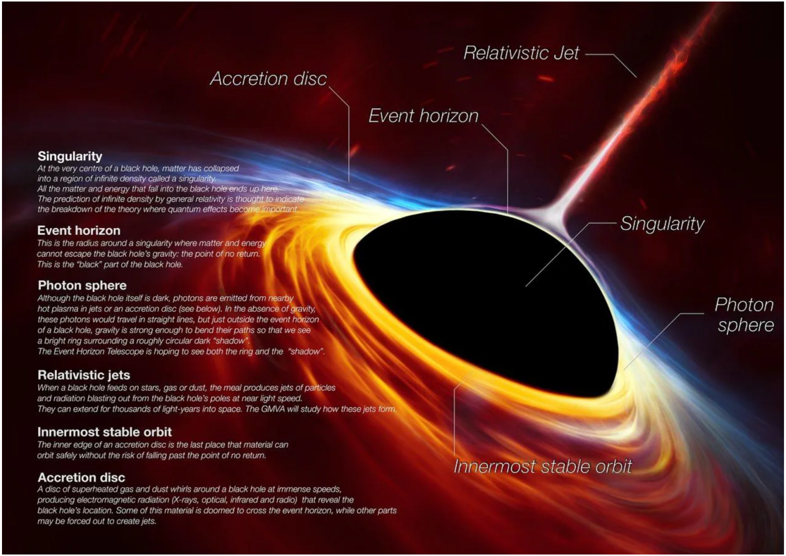


- Messier 87 supergiant elliptical galaxy jet, $\sim 5 \times 10^3$ light years

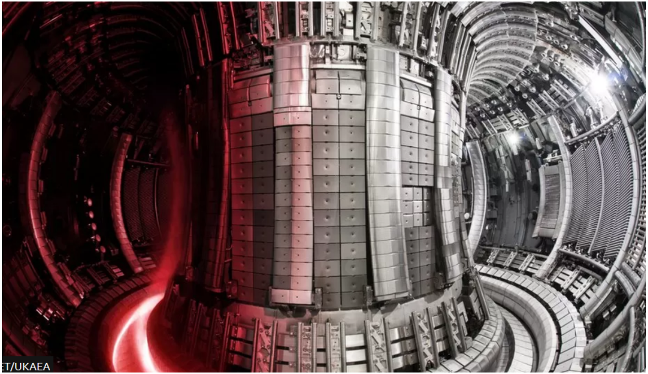
Applications



- Centaurus A: a relativistic jet from a supermassive black hole, $\sim 10^6$ light years


Relativistic Jet
Accretion disc
Event horizon
Singularity
At the very centre of a black hole, matter has collapsed into a region of infinite density called a singularity. All the matter and energy that fall into the black hole ends up here. The prediction of infinite density by general relativity is thought to indicate the breakdown of the theory where quantum effects become important.
Event horizon
This is the radius around a singularity where matter and energy cannot escape the black hole's gravity: the point of no return. This is the "black" part of the black hole.
Photon sphere
Although the black hole itself is dark, photons are emitted from nearby hot plasma in jets or an accretion disc (see below). In the absence of gravity, these photons would travel in straight lines, but just outside the event horizon of a black hole, gravity is strong enough to bend their paths so that we see a bright ring surrounding a roughly circular dark "shadow". The Event Horizon Telescope is hoping to see both the ring and the "shadow".
Relativistic jets
When a black hole feeds on stars, gas or dust, the meal produces jets of particles and radiation blasting out from the black hole's poles at near light speed. They can extend for thousands of light-years into space. The GMVA will study how these jets form.
Innermost stable orbit
The inner edge of an accretion disc is the last place that material can orbit safely without the risk of falling past the point of no return.
Accretion disc
A disc of superheated gas and dust whirls around a black hole at immense speeds, producing electromagnetic radiation (X-rays, optical, infrared and radio) that reveal the black hole's location. Some of this material is doomed to cross the event horizon, while other parts may be forced out to create jets.
Singularity
Photon sphere
Innermost stable orbit

Applications



- Plasma discharge in the UK JET reactor

Dynamic MHD equilibria in 3D

$$\operatorname{div} \rho \mathbf{V} = 0,$$

$$\rho \mathbf{V} \times \operatorname{curl} \mathbf{V} + \frac{1}{\mu} \operatorname{curl} \mathbf{B} \times \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} = 0,$$

$$\operatorname{curl} (\mathbf{V} \times \mathbf{B}) = 0,$$

$$\operatorname{div} \mathbf{B} = \operatorname{div} \mathbf{V} = 0$$

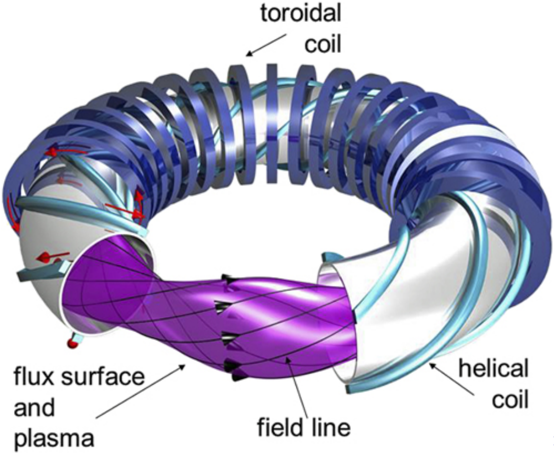
Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

MHD equilibrium topology

- Magnetic field lines/ streamlines go to infinity
- Magnetic field lines/ streamlines are tangent to *magnetic flux surfaces*
  - topological tori
  - constant pressure or flux function



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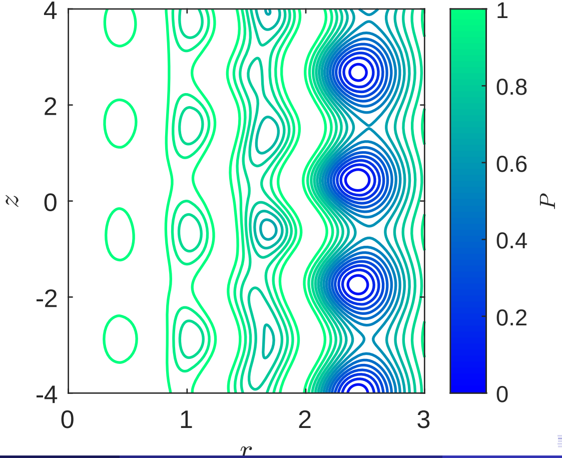


Physical requirements

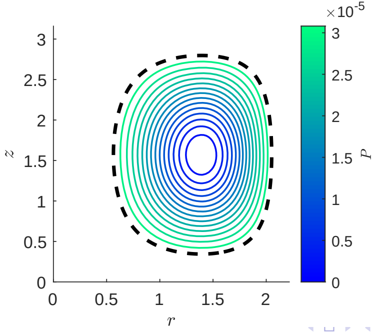
- Bounded kinetic and magnetic energy: total in $\mathcal{V}$, or per-layer

$$\int_{\mathcal{V}} |\mathbf{V}(\mathbf{x})|^2 \, dV \,, \qquad \int_{\mathcal{V}} |\mathbf{B}(\mathbf{x})|^2 \, dV \quad < \quad +\infty$$

- Pressure asymptotics: vacuum or ambient medium

$$P \to P_0 = \mathrm{const} \quad \text{as} \quad |\mathbf{x}| \to +\infty \quad \text{or on } \partial\mathcal{V}$$

- Current sheet for a "truncated" configuration: $\mathbf{K} = \dfrac{\mathbf{B}}{\mu} \times \mathbf{n}$ on $\partial\mathcal{V}$



Outline

Axial symmetry

Static MHD equilibria

$$\operatorname{div}\mathbf{B} = 0,$$
$$\operatorname{curl}\mathbf{B}\times\mathbf{B} = \mu \operatorname{grad} P$$

- Rotational symmetry $X = \partial/\partial\varphi$

- Magnetic field and pressure:

$$\mathbf{B} = B^r(r,z)\mathbf{e}_r + B^\varphi(r,z)\mathbf{e}_\varphi + B^z(r,z)\mathbf{e}_z, \qquad P = P(r,z)$$

Axial symmetry

Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Reduction to one PDE on one dependent variable (magnetic flux function):

$$\mathbf{B} = \frac{\psi_z}{r}\mathbf{e}_r + \frac{I(\psi)}{r}\mathbf{e}_\phi - \frac{\psi_r}{r}\mathbf{e}_z, \qquad P = P(\psi)$$

- Grad-Shafranov equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi)$$

- $I(\psi)$, $P(\psi)$ "free"
- Linear cases ($P(\psi)$ quadratic, $I(\psi)$ linear) have been considered by Bogoyavlenskij, Kaiser & Lortz, and others; interesting solutions were obtained

Linear homogeneous G-S

- Grad-Shafranov equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi)$$

- Linear ansatz:

$$P(\psi) = P_0 + b\psi + \frac{1}{2}a\psi^2, \qquad I(\psi) = \alpha\psi$$

- Linear homogeneous case: $b = 0$,

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + \frac{\partial^2\psi}{\partial z^2} + (\alpha^2 + ar^2)\psi = 0$$

- Separation of variables:

$$Z'' = \lambda Z, \qquad R'' - \frac{1}{r}R' + (\alpha^2 + ar^2 + \lambda)R = 0$$

- Z-solutions for stretched configurations:

$$Z = C_3 \sin(kz) + C_4 \cos(kz)$$

- R-ODE:

$$R'' - \frac{1}{r}R' + (\alpha^2 + ar^2 + \lambda)R = 0$$

- Substitution $s = qr^2$, $R(r) = S(s)$ gives

$$S'' + \left(-\frac{1}{4} + \frac{\alpha^2 - k^2}{4qs}\right) S = 0,$$

which is related to the Whittaker ODE

$$y''(s) + \left(-\frac{1}{4} + \frac{\delta}{s} + \frac{1/4 - \nu^2}{s^2}\right) y(s) = 0$$

when $\delta = \alpha^2 - k^2/4q$, $\nu = 1/2$.

The first family of axially symmetric solutions

- Separated solutions of the GS:

$$\psi_k(r,z) = \left(C_1 W_M\left(\delta, \frac{1}{2}, qr^2\right) + C_2 W_W\left(\delta, \frac{1}{2}, qr^2\right)\right)(C_3 \sin kz + C_4 \cos kz)$$

- Globally regular if and only if $\delta \in \mathbb{N}$; regularity is not necessary under proper BCs

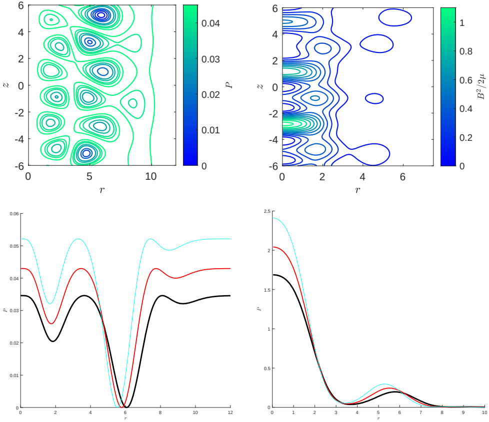
- A general linear combination: $\Psi(r,z) = \int_{-\infty}^{\infty} \psi_k(r,z)\, dk$

The first family of axially symmetric solutions

- **Example** [Bogoyavlenskij solutions]: pressure and magnetic energy

$$\psi(r,z) = e^{-\beta r^2}\left(a_N L_N^*(2\beta r^2) + \sum_{n=1}^{N-1} a_n \sin(\omega_n z + b_n) L_n^*(2\beta r^2) \right)$$

- Externally supported

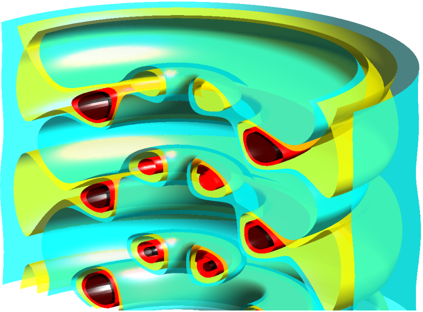


The first family of axially symmetric solutions

- **Example** [Bogoyavlenskij solutions]: pressure and magnetic energy

$$\psi(r,z) = e^{-\beta r^2}\left(a_N L_N^*(2\beta r^2) + \sum_{n=1}^{N-1} a_n \sin(\omega_n z + b_n) L_n^*(2\beta r^2)\right)$$

- Magnetic surfaces:

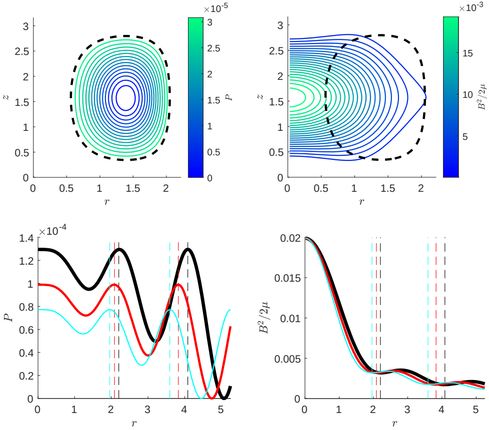


The first family of axially symmetric solutions

- **Example 2**: Whittaker functions

$$\psi_k(r,z) = \left(C_1 W_M\left(\delta, \frac{1}{2}, qr^2\right) + C_2 W_W\left(\delta, \frac{1}{2}, qr^2\right)\right)(C_3 \sin kz + C_4 \cos kz)$$

- Also externally supported



The second family of axially symmetric solutions

- **Second axial family**
- Pressure: $P(\psi) = P_0 + \frac{1}{2} q^2 \psi^2$, positive inside $\mathcal{V}$, zero outside $\mathcal{V}$
- R-ODE:

$$R'' + \left(-\frac{1}{4} + \frac{k^2 - \alpha^2}{4qx} i \right) R = 0$$

- Whittaker functions of complex index $\rightarrow$ Coulomb ODE:

$$y''(s) + \left(1 - \frac{2\sigma}{s} - \frac{L(L+1)}{s^2} \right) y(s) = 0$$

- Radial solution component:

$$R(r) = C_1 \, \mathcal{C}_F \left(0, -\delta, \frac{q}{2} r^2 \right) + C_2 \, \mathcal{C}_G \left(0, -\delta, \frac{q}{2} r^2 \right)$$

in terms of Coulomb special functions.

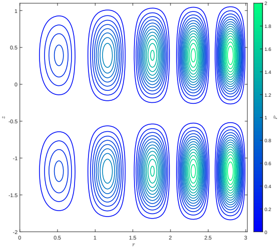
The second family of axially symmetric solutions

- Separated solutions of the GS:

$$\psi_k(r,z) = \left(C_1\,\mathcal{C}_F\left(0, -\delta, \frac{q}{2}r^2\right) + C_2\,\mathcal{C}_G\left(0, -\delta, \frac{q}{2}r^2\right)\right)(C_3 \sin kz + C_4 \cos kz)$$

- A general linear combination: $\Psi(r,z) = \int_{-\infty}^{\infty} \psi_k(r,z)\, dk$

- **Example:** magnetic surfaces



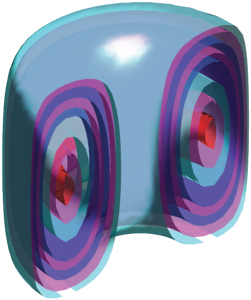
The second family of axially symmetric solutions

- Separated solutions of the GS:

$$\psi_k(r,z) = \left(C_1\, \mathcal{C}_F \left(0, -\delta, \frac{q}{2} r^2\right) + C_2\, \mathcal{C}_G \left(0, -\delta, \frac{q}{2} r^2\right)\right) (C_3 \sin kz + C_4 \cos kz)$$

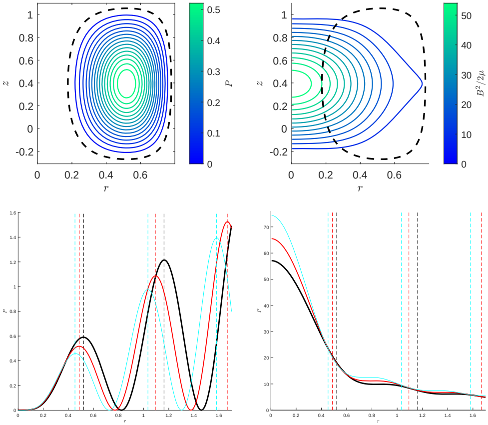
- A general linear combination: $\Psi(r,z) = \int_{-\infty}^{\infty} \psi_k(r,z)\, dk$

- **Example:** magnetic surfaces



The second family of axially symmetric solutions

- Configuration with lower external pressure and a current sheet



Outline

Helical symmetry

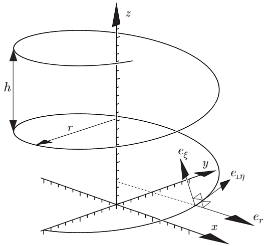
Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Rotational and translational symmetry $X_1 = \partial/\partial\varphi$, $X_2 = \partial/\partial z$
- Helical coordinates (r, η, ξ) in terms of cylindrical coordinates (r, φ, z):

$$r, \qquad \eta = \varphi + \gamma z/r^2, \qquad \xi = z - \gamma\varphi$$



Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Reduction to one PDE on one dependent variable (magnetic flux function):

$$\mathbf{B} = \frac{\psi_\xi}{r}\mathbf{e}_r + \frac{rI(\psi) + \gamma\psi_r}{r^2+\gamma^2}\mathbf{e}_\varphi + \frac{\gamma I(\psi) - r\psi_r}{r^2+\gamma^2}\mathbf{e}_z$$

- Johnson-Frieman-Kulsrud-Oberman (JFKO) equation

$$\frac{\psi_{\xi\xi}}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2+\gamma^2}\psi_r\right) + \frac{I(\psi)I'(\psi)}{r^2+\gamma^2} + \frac{2\gamma I(\psi)}{(r^2+\gamma^2)^2} = -\mu P'(\psi)$$

- $I(\psi)$, $P(\psi)$ "free"
- Consider again the linear homogeneous case: $P(\psi) = P_0 + \frac{1}{2}\sigma\psi^2$, $I(\psi) = \alpha\psi$

Helical symmetry

Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Linear JFKO:

$$\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2+\gamma^2}\frac{\partial \psi}{\partial r}\right) + \frac{\alpha^2 \psi}{r^2+\gamma^2} + \frac{2\gamma\alpha\psi}{(r^2+\gamma^2)^2} + 4\sigma\psi = 0$$

- Separated solutions $\psi(r,u) = R(r)\Xi(\xi)$

$$\Xi'' = \lambda \Xi, \quad \lambda = -\omega^2 < 0 \quad \to \quad \Xi(\xi) = C_3 \sin(\omega\xi) + C_4 \cos(\omega\xi)$$

$$r\left(\frac{r}{r^2+\gamma^2}R'\right)' + \left(\frac{\alpha^2 r^2}{r^2+\gamma^2} + \frac{2\gamma\alpha r^2}{(r^2+\gamma^2)^2} + 4ar^2\right)R = -\lambda R$$

- $\sigma = -\kappa^2 < 0$
- R-ODE:

$$r\left(\frac{r}{r^2+\gamma^2}R'\right)' + \left(\frac{\alpha^2 r^2}{r^2+\gamma^2} + \frac{2\gamma\alpha r^2}{(r^2+\gamma^2)^2} + 4\kappa^2 r^2\right)R = \omega^2 R.$$

- Solution:

$$R(r) = e^{-\kappa r^2}\left(C_1 r^b \mathcal{H}_C(a, b, -2, c, d, -r^2/\gamma^2) + C_2 r^{-b}\mathcal{H}_C(a, -b, -2, c, d, -r^2/\gamma^2)\right)$$

$$a = \kappa\gamma^2, \qquad b = \gamma\omega, \qquad c = \frac{\gamma^2(\gamma^2\kappa^2 - \alpha^2 + \omega^2)}{4},$$

$$d = 1 - \frac{\kappa^2\gamma^2}{4} + \frac{\alpha^2 - \omega^2}{4}\gamma^2 + \frac{\alpha\gamma}{2},$$

where $\mathcal{H}_C$ satisfies the confluent Heun ODE

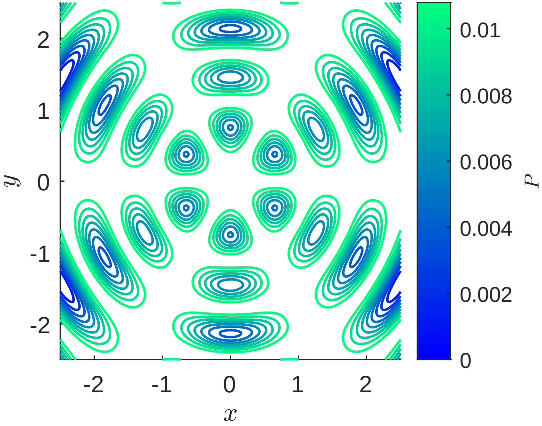
$$y'' - \frac{(-x^2 a + (-b+a)x + b + 1)}{x(x-1)}y' - \frac{((-ba-2c)x + (b+1)a + b - 2d + 2)}{2x(x-1)}y = 0$$

- Separated solutions of the JFKO:

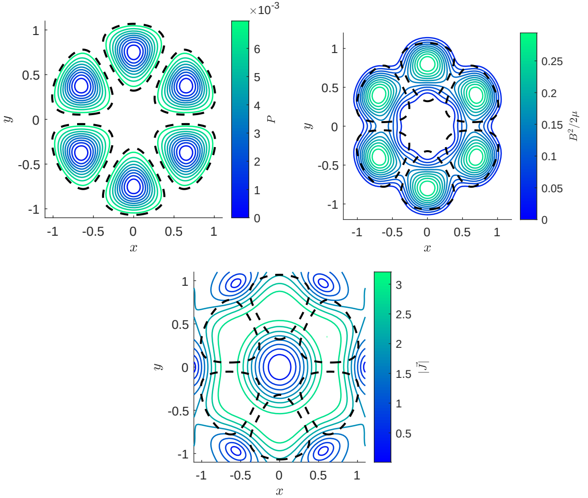
$$\psi_\omega(r,\xi) = e^{-\kappa r^2} r^b \mathcal{H}_C(a, b, -2, c, d, -r^2/\gamma^2)(C_3 \sin(\omega\xi) + C_4 \cos(\omega\xi)).$$

- A general linear combination: $\psi(r,\xi) = \int_{-\infty}^{\infty} \psi_\omega(r,\xi)\, d\omega$

- **First helical family: an example**
- Magnetic surfaces: global



- **First helical family: an example**
- Magnetic surfaces (local/bounded), magnetic energy, current density

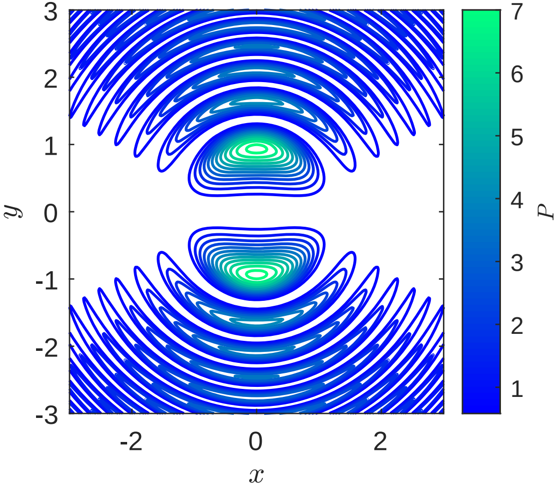


The second family of helically symmetric solutions

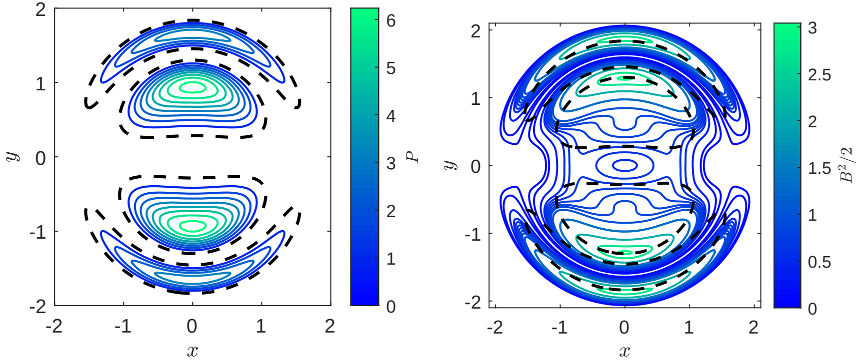
- $\sigma = \kappa^2 > 0$
- Real-valued separated solutions:

$$\psi_\omega(r,\xi) = e^{-i\kappa r^2} r^b \mathcal{H}_C(ia, b, -2, c, d, -r^2/\gamma^2)\,(C_1 \sin(\omega\xi) + C_2 \cos(\omega\xi))$$

- **Second helical family: an example**
- Magnetic surfaces: global



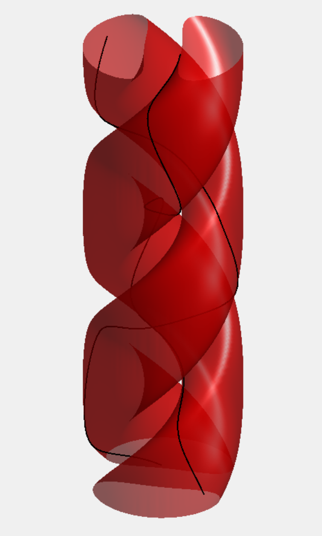
- **Second helical family: an example**
- Magnetic surfaces (bounded), magnetic energy



The second family of helically symmetric solutions

- **Second helical family: an example**

- A magnetic surface and tangent magnetic field lines:



Outline

Dynamic solutions

- Galas-Bogoyavlenskij transformations (potential symmetries of equilibrium MHD):

$$\mathbf{B}_1 = b(\psi)\mathbf{B} + c(\psi)\sqrt{\mu\rho}\mathbf{V},$$

$$\mathbf{V}_1 = \frac{c(\psi)}{a(\psi)\sqrt{\mu\rho}}\mathbf{B} + \frac{b(\psi)}{a(\psi)}\mathbf{V},$$

$$P_1 = CP + \frac{C\mathbf{B}^2 - \mathbf{B}_1^2}{2\mu},$$

$$\rho_1 = a^2(\psi)\rho$$

- $a = a(\psi)$, $b = \underline{(}\psi)$, $c = c(\psi)$ are constant on both magnetic fields lines and streamlines, and

$$b^2(\psi) - c^2(\psi) = C = \mathrm{const.}$$

Dynamic solutions

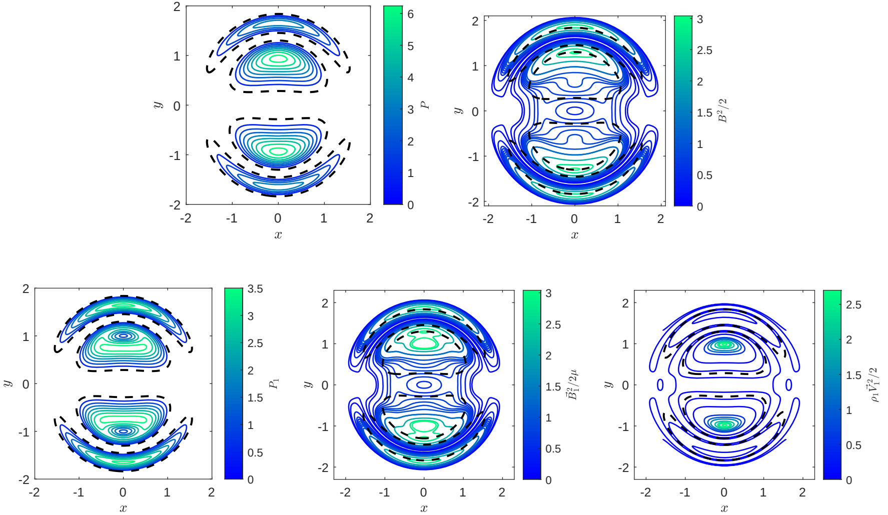
- Starting from a static equilibrium, get field-aligned solutions

$$
\begin{aligned}
\mathbf{B}_1 &= \sqrt{C + c^2(\psi)}\mathbf{B}, \\
\mathbf{V}_1 &= \frac{c(\psi)}{a(\psi)\sqrt{\mu\rho}}\mathbf{B}, \\
P_1 &= CP + -\frac{c^2(\psi)\mathbf{B}^2}{2\mu}, \\
\rho_1 &= a^2(\psi)\rho
\end{aligned}
$$

- Can apply to all of the above solutions!

Dynamic solutions

- **Second helical family: an example**
- Before and after: magnetic surfaces (bounded), magnetic energy, kinetic energy



Outline

Discussion

- New closed-form exact solutions in terms of special functions, in axial and helical symmetry, modeling prolonged static and dynamic plasma equilibria
- Field lines are not axially (or helically) symmetric
- Linear combinations of basic solutions $\rightarrow$ no z-periodicity
- Galas-Bogoyavlenskij symmetries: a rare example of useful nonlocal symmetries in multi-dimensions
- Linear and nonlinear stability? Numerical solutions?
- Heun functions are gaining popularity; implemented in numerical software
- Any physical invariant solutions from the time-dependent helical system [Dierkes and Oberlack (2017)]?

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