# Global and local conservation laws for physical models: Cases of static and moving domains


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# Some Collaborators

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# Goals of this talk

- Discuss a common framework for thinking about conservation laws (CL)
- Different CL types, different applications, examples
- Illustrate “what can happen”
- Discuss systematic CL computation
- The CL ideas are simple, general, and useful in various research areas

# Outline

# Notation

- Independent variables: $(x,t)$, or $(t,x,y,z)$, or $z=(z^1,...,z^n)$.
- Dependent variables: $u(x,t)$, or generally $v=(v^1(z),...,v^m(z))$.
- Derivatives:

$$\frac{d}{dt}w(t)=w'(t); \qquad \frac{\partial}{\partial x}u(x,t)=u_x; \qquad \frac{\partial}{\partial z^k}v^p(z)=v^p_k.$$

- All derivatives of order $p$: $\partial^p v$.
- A differential function:

$$H[v]=H(z,v,\partial v,\ldots,\partial^k v)$$

- A total derivative of a differential function: the chain rule

$$\mathrm{D}_i H[v]=\frac{\partial H}{\partial z^i}+\frac{\partial H}{\partial v^\alpha}\,v^\alpha_i+\frac{\partial H}{\partial v^\alpha_j}\,v^\alpha_{ij}+\ldots.$$

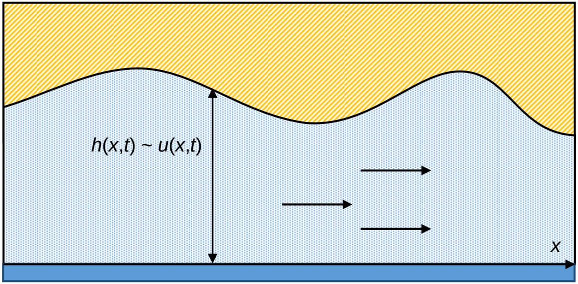
# Notation

- A PDE Example: the KdV (Korteweg-de Vries) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

for the dimensionless fluid depth $u = u(x,t)$ of long surface waves on shallow water:

$$G[u] = u_t + uu_x + u_{xxx} = 0.$$



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- $J^k(x,t|u)$: the $k$-th order jet space with coordinates $x, t$, $u$, $\partial u$, ..., $\partial^k u$.
- The solution manifold $\mathcal{E}$ in $J^k(x,t|u)$ is defined by the DE(s)+differential consequences to order $k$:

$$G[u] = 0, \quad \mathrm{D}_x G[u] = 0, \quad \mathrm{D}_t G[u] = 0, \ldots$$

- Statements are often formulated for differential functions defined in $J^k(x,t|u)$.

# Local and global conservation laws

- System of differential equations (PDE or ODE) $G[v] = 0$:

$$G^\sigma(z, v, \partial v, \ldots, \partial^{q_\sigma} v) = 0, \quad \sigma = 1, \ldots, M.$$

- The basic notion –

**A local (divergence-type) conservation law:**

A divergence expression

$$\boxed{\mathrm{D}_i \Phi^i[v] = 0}$$

vanishing on solutions of $G[v]$. Here $\mathbf{\Phi} = (\Phi^1[v], \ldots, \Phi^n[v])$ is the flux vector.

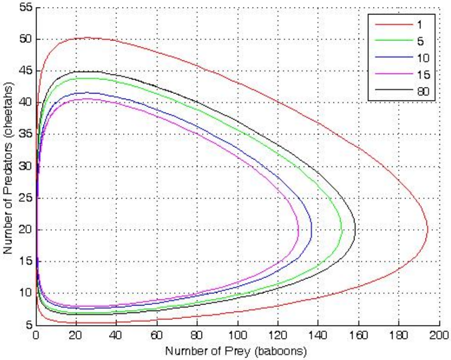
**ODE: A constant of motion (conserved quantity):**

$$v = v(t), \qquad \mathrm{D}_t T[v] = 0 \quad \Rightarrow \quad T[v] = \mathrm{const.}$$

# Local and global conservation laws – ODE examples

- Example 1: uniform rectilinear motion, $\boxed{m\ddot{x}(t) = 0.}$

$$\mathrm{D}_t\, P(t) = 0, \qquad P(t) = m\dot{x}(t) = \mathrm{const}.$$




- Example 2: the Lotka-Volterra model of a predator-prey interaction

$$x'(t) = \alpha x(t) - \beta x(t)y(t), \qquad y'(t) = \delta x(t)y(t) - \gamma y(t).$$

- Here $x(t)$=number of prey, (for example, baboons), $y(t)$=number of predator (e.g., cheetah), and $\alpha, \beta, \gamma, \delta = \text{const}$.
- A constant of motion: $\mathrm{D}_t\, V(t) = 0$,

$$V(t) = \delta x(t) - \gamma \ln x(t) + \beta y(t) - \alpha \ln y(t) = \text{const}.$$

Vol. 107, No. 957 The American Naturalist September–October 1973

# LETTER TO THE EDITORS

## DO HARES EAT LYNX?

To test a recently developed predator-prey model against reality, I chose the well-known Canadian hare-lynx system. A measure of the state of this system for the last 200-odd years is available in the fur catch records of the Hudson Bay Company (MacLulich 1937; Elton and Nicholson 1942). Although the accuracy of these data is questionable (see Elton and Nicholson 1942 for a full discussion), they represent the only long-term population record available to ecologists.

The model I tested is

$$dH/dt = H(r_H + C_{HL}L + S_H H + I_H H^2), \qquad (1a)$$

$$dL/dt = L(r_L + C_{LH}H + S_L L + I_L L^2), \qquad (1b)$$

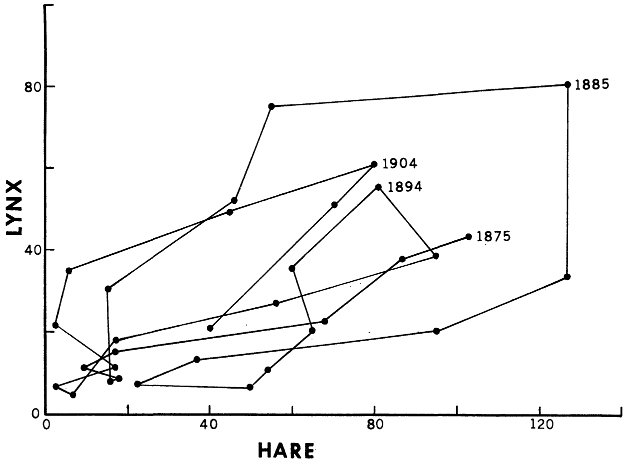



FIG. 1.—Yearly states of the Canadian lynx-hare system from 1875 to 1906. The numbers on the axes represent the numbers of the respective animals in thousands.

- For PDEs, the meaning of a local conservation law is different: the total amount of "density" is "conserved" in another sense.

- **(1+1)-dimensional PDEs:** $v = v(x,t)$, only one CL type.

  Local form:

$$\boxed{\mathrm{D}_t T[v] + \mathrm{D}_x \Psi[v] = 0.}$$

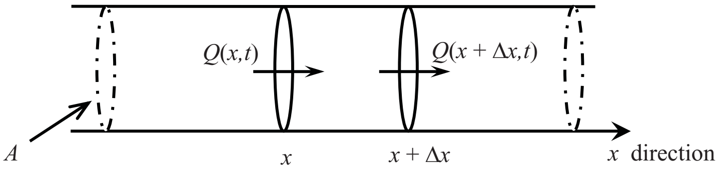
  Global form:

$$\boxed{\frac{d}{dt}\int_a^b T[v]\,dx = -\Psi[v]\Big|_a^b.}$$

## Conservation principles to derive model DEs.

- Continuity equation – gas/fluid flow:

$$\rho_t + (\rho v)_x = 0, \qquad \rho = \rho(x,t), \qquad v = v(x,t).$$

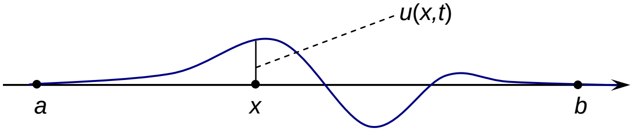



- Global form:

$$\frac{d}{dt} m = \frac{d}{dt} \int_x^{x+\Delta x} \rho \, dx = (\rho v)\Big|_x^{x+\Delta x}.$$

**(1+1)-dimensional linear wave equation:**

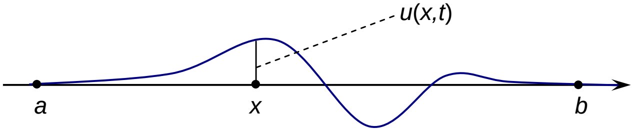
$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad c^2 = \tau/\rho, \quad a < x < b \text{ or } -\infty < x < \infty.$$



# Local and global conservation laws – PDE examples

**(1+1)-dimensional linear wave equation:**

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad c^2 = \tau/\rho, \quad a < x < b \text{ or } -\infty < x < \infty.$$



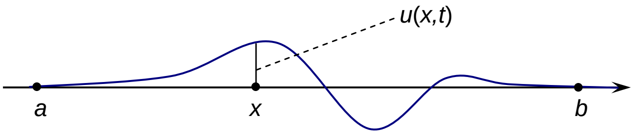

- A local CL – momentum conservation: $\mathrm{D}_t(\rho u_t) - \mathrm{D}_x(\tau u_x) = 0.$
- Global form:

$$\frac{d}{dt}M = \frac{d}{dt}\int_a^b \rho u_t \, dx = \tau u_x \Big|_a^b.$$

- $dM/dt = 0$ for zero Neumann BCs $\rightarrow$ the momentum is conserved, $M = \mathrm{const}$.
- (E.g., a finite perturbation of an infinite string.)

**(1+1)-dimensional linear wave equation:**

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t), \quad c^2 = \tau/\rho, \quad a < x < b \text{ or } -\infty < x < \infty.$$




- A local CL – energy conservation: $\mathrm{D}_t\left(\frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2}\right) - \mathrm{D}_x(\tau u_t u_x) = 0.$
- Global form:

$$\frac{d}{dt}E = \frac{d}{dt}\int\left(\frac{\rho u_t^2}{2} + \frac{\tau u_x^2}{2}\right)dx = \tau u_t u_x\Big|_a^b.$$

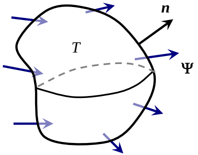
- For which BCs is $E = \mathrm{const}$?

- **(3+1)-dimensional PDEs:** $R[v] = 0, \quad v = v(t, x, y, z)$.

- Local form: $\boxed{D_t T[v] + \mathrm{Div}\, \mathbf{\Psi}[v] = 0} \quad \Leftrightarrow \quad \boxed{\mathrm{D}_i \Phi^i[v] = 0}$

- Global form: $\boxed{\dfrac{d}{dt} \displaystyle\int_{\mathcal{V}} T\, dV = -\oint_{\partial\mathcal{V}} \mathbf{\Psi} \cdot d\mathbf{S}}$

- Holds for all solutions $v(t, x, y, z) \in \mathcal{E}$, in some physical domain $\mathcal{V}$.




- In 3D, CLs of other types on static and moving domains can exist.

# Applications

# Applications of Conservation Laws

## Applications to ODEs

- Constants of motion:

$$\mathrm{D}_t T[v] = 0 \quad \Rightarrow \quad T[v] = \mathrm{const.}$$

- Reduction of order / integration.

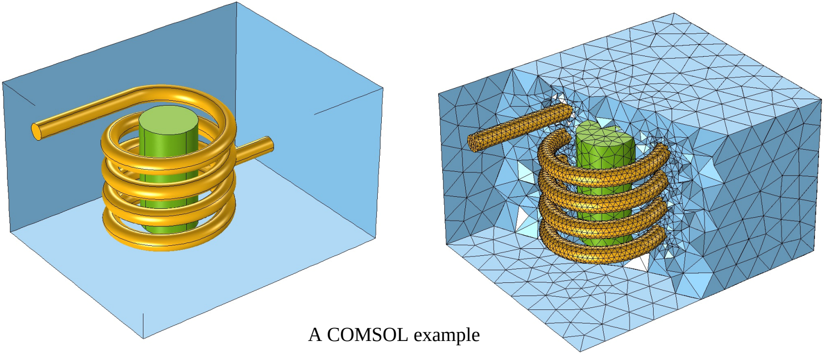
# Applications of Conservation Laws

## Applications to PDEs

$$D_t T[v] + \mathrm{Div}\, \boldsymbol{\Psi}[v] = 0$$

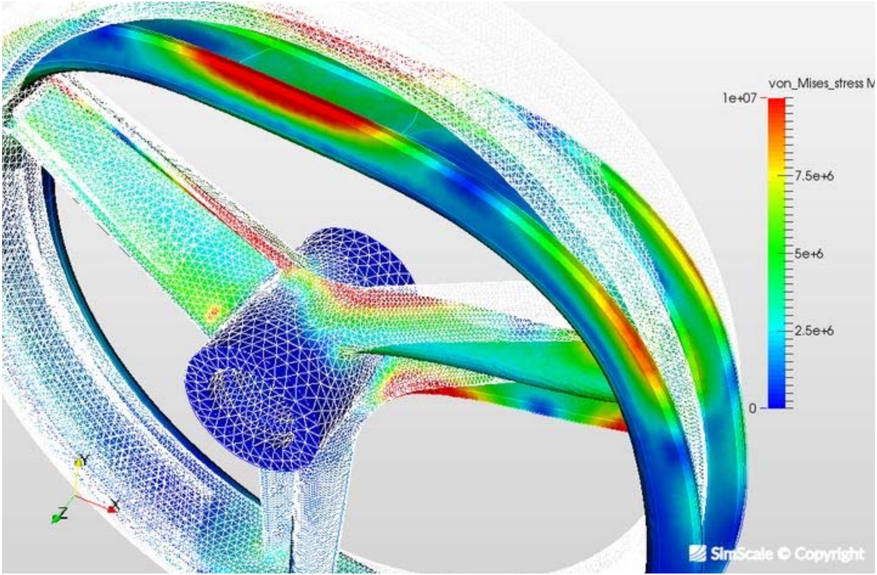
- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Analysis of solution behaviour: existence, uniqueness, stability.
- Potentials, stream functions, etc.
- An infinite number of CLs may indicate integrability/linearization.
- Conserved PDEs forms and constants of motion for numerical methods.

# Applications of Conservation Laws



A COMSOL example

# Applications of Conservation Laws



# CLs with no physical content?

# Trivial and equivalent local conservation laws

**Example: (1+1)-dimensional linear wave equation**

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t).$$

**Trivial conservation laws:**

- Density/flux vanishes on solutions (Type I, vanishing density/flux).
  For example,
  $$\mathrm{D}_t(u_{tt} - c^2 u_{xx}) + \mathrm{D}_x\left(2u\left[u_{ttx} - c^2 u_{xxx}\right]\right) = 0.$$
- Holds as an identity for any $u(x,t)$ (Type II, null divergence).
  For example,
  $$\mathrm{D}_t(x + u_x) + \mathrm{D}_x(2t - u_t) \equiv 0.$$
- A combination thereof.

# Trivial and equivalent local conservation laws

## Example: (1+1)-dimensional linear wave equation

$$u_{tt} = c^2 u_{xx}, \quad u = u(x,t).$$

## Equivalent conservation laws

- Differ by a trivial one. For example,

$$\mathrm{D}_t(u_t) - \mathrm{D}_x(c^2 u_x) = 0$$

  and

$$\mathrm{D}_t(u_t + x) - \mathrm{D}_x(c^2 u_x - 1) = 0$$

  describe the same physical quantity.

- Natural to seek all different equivalence classes of CLs.
- Same ideas for multi-dimensional models.

arXiv.org > math-ph > arXiv:1803.08859

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**Mathematical Physics**

# On the different types of global and local conservation laws for partial differential equations in three spatial dimensions

Stephen C. Anco, Alexei F. Cheviakov

*(Submitted on 23 Mar 2018)*

For systems of partial differential equations in three spatial dimensions, dynamical conservation laws holding on volumes, surfaces, and curves, as well as topological conservation laws holding on surfaces and curves, are studied in a unified framework. Both global and local formulations of these different conservation laws are discussed, including the forms of global constants of motion. The main results consist of providing an explicit characterization for when two conservation laws are locally or globally equivalent, and for when a conservation law is locally or globally trivial, as well as deriving relationships among the different types of conservation laws. In particular, the notion of a "trivial" conservation law is clarified for all of the types of conservation laws. Moreover, as further new results, conditions under which a trivial local conservation law on a domain can yield a non-trivial global conservation law on the domain boundary are determined and shown to be related to differential identities that hold for PDE systems containing both evolution equations and spatial constraint equations. Numerous physical examples from fluid flow, gas dynamics, electromagnetism, and magnetohydrodynamics are used as illustrations.

Comments: 55 pages
Subjects: **Mathematical Physics (math-ph)**; Fluid Dynamics (physics.flu-dyn)
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# Characteristic form of a CL

# Characteristic form of a CL

- What is an "algebraic handle" to compute divergence-type CLs

$$\mathrm{D}_i \Phi^i[v] = 0$$

of a DE system $G^\sigma[v] = 0$, $\sigma = 1, \ldots, M$?

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## Hadamard lemma for differential functions

A smooth differential function $Q[v]$ vanishes on solutions of a *totally nondegenerate* PDE system $G^\sigma[v] = 0$ if and only if it has the form, for all $v$,

$$Q[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v]\, \mathrm{D}_k G^\sigma[v] + \dots.$$

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- Off of solution set, for all $v$:

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- Off of solution set, for all $v$:

$$\mathrm{D}_i \Phi^i[v] = \Lambda_\sigma[v] G^\sigma[v] + \Lambda_\sigma^k[v] \, \mathrm{D}_k G^\sigma[v] + \ldots.$$

- An equivalent CL:

$$\mathrm{D}_i \tilde{\Phi}^i[v] = \tilde{\Lambda}_\sigma[v] G^\sigma[v].$$

# Characteristic form of a CL

## A characteristic form of a local CL:

$$\boxed{\mathrm{D}_i \Phi^i[v] = \Lambda_\sigma[v] G^\sigma[v].}$$

- $\Phi^i[v]$: fluxes.
- $\Lambda_\sigma[v]$: multipliers.
- There is "usually" a 1:1 correspondence between sets of (nontrivial) multipliers and the respective (nontrivial) local CLs.

# How many local CLs?

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- How many (linearly independent, nontrivial) local CLs does a given PDE system have?
- **Possibility I:** a finite number. For example:

**Theorem (Ibragimov, 1985)**

*For any* $(1+1)$*-dimensional even-order scalar evolution equation*

$$u_t = F(x, t, u, \partial_x u, \dots, \partial_x^{2k} u), \qquad u = u(x, t),$$

*the flux and the density of local CLs*

$$\mathrm{D}_t T[u] + \mathrm{D}_x \Psi[u] = 0$$

*(up to equivalence) depend only on* $x, t, u$ *and derivatives of* $u$ *with respect to* $x$*, and the maximal order of a derivative in the CL density* $T$ *is* $k$*.*

# How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility I:** a finite number. For example:

## A nonlinear diffusion equation

$$u_t = (u^2 u_x)_x, \qquad u = u(x, t).$$

Two local CLs only:

$$\mathrm{D}_t(u) - \mathrm{D}_x(u^2 u_x) = 0,$$

$$\mathrm{D}_t(xu) + \mathrm{D}_x\left(\frac{u^3}{3} - xu^2 u_x\right) = 0.$$

# How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility II:** an infinite countable set.
  E.g., CLs of an S-integrable equation.

## Example: the KdV

$$u_t + uu_x + u_{xxx} = 0, \qquad u = u(x,t).$$

A hierarchy of local CLs:

$$\Lambda(x,t) = 1, \qquad \mathrm{D}_t(u) + \mathrm{D}_x\left(\tfrac{1}{2}u^2 + u_{xx}\right) = 0,$$

$$\Lambda(x,t) = u, \qquad \mathrm{D}_t\left(\tfrac{1}{2}u^2\right) + \mathrm{D}_x\left(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2\right) = 0,$$

$$\Lambda(x,t) = \tfrac{1}{2}u^2, \quad \mathrm{D}_t\left(\tfrac{1}{6}u^3 - \tfrac{1}{2}u_x^2\right) + \mathrm{D}_x\left(\tfrac{1}{8}u^4 - uu_x^2 + \tfrac{1}{2}(u^2u_{xx} + u_{xx}^2) - u_x u_{xxx}\right) = 0,$$

$$\vdots$$

# How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility III:** an infinite CL family.
  E.g., CLs involving a free function.

## Constant-density Navier-Stokes equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p + \nu \, \Delta \mathbf{u}.$$

CLs [Gusyatnikova & Yumaguzhin, 1989]:

- Continuity (generalized): $\nabla \cdot (k(t)\,\mathbf{u}) = 0$.

- Momentum (generalized): $\mathrm{D}_t(f(t)u^1) + \mathrm{D}_x(\ldots) + \mathrm{D}_y(\ldots) + \mathrm{D}_z(\ldots) = 0$; same for $y, z$.

- Angular momentum: $\mathrm{D}_t(zu^2 - yu^3) + \mathrm{D}_x(\ldots) + \mathrm{D}_y(\ldots) + \mathrm{D}_z(\ldots) = 0$; same for $y, z$.

# How many local CLs?

- How many (linearly independent, nontrivial) local CLs does a given PDE system have?

- **Possibility III:** an infinite CL family.
  E.g., C-integrable equations, with CLs involving arbitrary solutions of linear PDEs.

**Example:**

- A linear heat equation $u_t = a^2 u_{xx}, \quad u = u(x,t)$.
- Local CLs: $\Lambda(x,t)(u_t - u_{xx}) = \mathrm{D}_t T + \mathrm{D}_x \Psi = 0$.
- The multiplier $\Lambda(x,t)$ is any solution of the adjoint linear PDE $\Lambda_t = -a^2 \Lambda_{xx}$.
- E.g., $\Lambda(x,t) = e^{a^2 t} \sin x$, then $\mathrm{D}_t \left(e^{a^2 t}\, u \sin x\right) + \mathrm{D}_x \left(a^2 e^{a^2 t} [u \cos x - u_x \sin x]\right) = 0$.

- Existence of a "large" CL family is a necessary condition of invertible linearization (e.g., Bluman, Anco & Wolf, 2008).

# How to compute CLs?

# The idea of the direct construction method

Independent and dependent variables of the problem:
$z = (z^1, ..., z^n), \quad v = v(z) = (v^1, ..., v^m).$

## Definition

The Euler operator with respect to an arbitrary function $v^j$:

$$\mathrm{E}_{v^j} = \frac{\partial}{\partial v^j} - \mathrm{D}_i \frac{\partial}{\partial v^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial v^j_{i_1 \ldots i_s}} + \cdots, \quad j = 1, \ldots, m.$$

## Theorem

*The equations*

$$\mathrm{E}_{v^j} F[v] \equiv 0, \quad j = 1, \ldots, m$$

*hold for arbitrary $v(z)$ if and only if*

$$F[v] \equiv \mathrm{D}_i \Phi^i$$

*for some functions $\Phi^i = \Phi^i[v]$.*

# The direct construction method

Given:

- A system of $M$ DEs $G^{\sigma}[v] = 0, \quad \sigma = 1, \ldots, M$.
- Variables: $z = (z^1, ..., z^n), \quad v = (v^1(z), ..., v^m(z))$.

Given:

- A system of $M$ DEs $G^\sigma[v] = 0, \quad \sigma = 1, \ldots, M$.
- Variables: $z = (z^1, ..., z^n), \quad v = (v^1(z), ..., v^m(z))$.

## The Direct CL Construction Method

1. Specify the dependence of multipliers: $\Lambda_\sigma = \Lambda_\sigma[z, v, \partial v, ...]$.
2. Solve the set of determining equations $\mathrm{E}_{v^j}(\Lambda_\sigma[v]G^\sigma[v]) \equiv 0, \quad j = 1, \ldots, m$, for arbitrary $v(z)$, to find all sets of multipliers.
3. Find the corresponding fluxes $\Phi^i[V]$ satisfying the identity
$$\Lambda_\sigma[v]G^\sigma[v] \equiv \mathrm{D}_i\Phi^i[v].$$
4. For each set of fluxes, on solutions, get a local conservation law
$$\boxed{\mathrm{D}_i\Phi^i[v] = 0.}$$

# Computational examples

# Example: CLs of Euler equations

## Constant-density Navier-Stokes equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p + \nu \, \Delta \mathbf{u}.$$

CLs [Gusyatnikova & Yumaguzhin, 1989]: CL order is bounded.

- Continuity (generalized): $\nabla \cdot (k(t)\, \mathbf{u}) = 0$.
- Momentum (generalized): $\mathrm{D}_t(f(t)u^1) + \mathrm{D}_x(\dots) + \mathrm{D}_y(\dots) + \mathrm{D}_z(\dots) = 0$; same for $y, z$.
- Angular momentum: $\mathrm{D}_t(zu^2 - yu^3) + \mathrm{D}_x(\dots) + \mathrm{D}_y(\dots) + \mathrm{D}_z(\dots) = 0$; same for $y, z$.

- No such result for Euler equations ($\nu = 0$).
- Also unknown for symmetry-reduced models (axial, helical...)

# Example: CLs of Euler equations

## Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

A. Cheviakov and M. Oberlack (2014)
Generalized Ertel's theorem and infinite hierarchies of conserved quantities for three-dimensional time-dependent Euler and NavierStokes equations. *JFM* 760: 368-386.

- seek CLs to second-order multipliers, depending on up to 45 variables,

$$t, x, y, z, \quad u^1, u^2, u^3, p, \quad u^1_y, u^1_z, \quad u^2_x, u^2_y, u^2_z, \quad u^3_x, u^3_y, u^3_z, \quad p_t, p_x, p_y, p_z,$$
$$u^1_{yy}, u^1_{yz}, u^1_{zz}, \quad u^2_{xx}, u^2_{xy}, u^2_{xz}, u^2_{yy}, u^2_{yz}, u^2_{zz}, \quad u^3_{xx}, u^3_{xy}, u^3_{xz}, u^3_{yy}, u^3_{yz}, u^3_{zz},$$
$$p_{tt}, p_{tx}, p_{ty}, p_{tz}, p_{xx}, p_{xy}, p_{xz}, p_{yy}, p_{yz}, p_{zz}.$$

# Example: CLs of Euler equations

## Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

## 1. Conservation of generalized momentum.

$$\Lambda_1 = f(t)u^1 - xf'(t), \quad \Lambda_2 = f(t), \quad \Lambda_3 = \Lambda_4 = 0;$$

$$\frac{\partial}{\partial t}(f(t)u^1) + \frac{\partial}{\partial x}\Big((u^1 f(t) - xf'(t))u^1 + f(t)p\Big)$$
$$+\frac{\partial}{\partial y}\Big((u^1 f(t) - xf'(t))u^2\Big) + \frac{\partial}{\partial z}\Big((u^1 f(t) - xf'(t))u^3\Big) = 0,$$

with analogous expressions holding for $y$- and the $z$-directions.

# Example: CLs of Euler equations

## Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

## 2. Conservation of the angular momentum.

$$\Lambda_1 = u^2_z - u^3_y, \quad \Lambda_2 = 0, \quad \Lambda_3 = z, \quad \Lambda_4 = -y;$$

$$\frac{\partial}{\partial t}(zu^2 - yu^3) + \frac{\partial}{\partial x}\Big((zu^2 - yu^3)u^1\Big)$$
$$+\frac{\partial}{\partial y}\Big((zu^2 - yu^3)u^2 + zp\Big) + \frac{\partial}{\partial z}\Big((zu^2 - yu^3)u^3 - yp\Big) = 0.$$

with cyclic permutations for $y$- and the $z$-directions.

### Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

### 3. Conservation of the kinetic energy.

$$\Lambda_1 = K + p, \quad [\Lambda_2, \Lambda_3, \Lambda_4] = \mathbf{u};$$

$$\frac{\partial}{\partial t} K + \nabla \cdot \Big( (K + p)\, \mathbf{u} \Big) = 0, \qquad K = \tfrac{1}{2} \left| \mathbf{u} \right|^2.$$

# Example: CLs of Euler equations

## Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

## 4. Generalized continuity equation.

$$\Lambda_1 = k(t), \quad \Lambda_2 = \Lambda_3 = \Lambda_4 = 0;$$

$$\nabla \cdot (k(t)\,\mathbf{u}) = 0.$$

# Example: CLs of Euler equations

## Constant-density Euler equations

$$\rho = \text{const}, \quad \operatorname{div} \mathbf{u} = 0, \quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\operatorname{grad} p.$$

## 5. Conservation of helicity.

$$\Lambda_1 = 0, \quad [\Lambda_2, \Lambda_3, \Lambda_4] = \boldsymbol{\omega} = \operatorname{curl} \mathbf{u};$$

$$h = \mathbf{u} \cdot \boldsymbol{\omega}; \quad E = K + p, \quad K = \frac{1}{2} |\mathbf{u}|^2;$$

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0.$$

# Example: CLs of NS and Euler equations under helical symmetry

Kelbin, O., Cheviakov, A.F., and Oberlack, M. (2013)
New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. *JFM* 721, 340-366.

## Helically-invariant equations

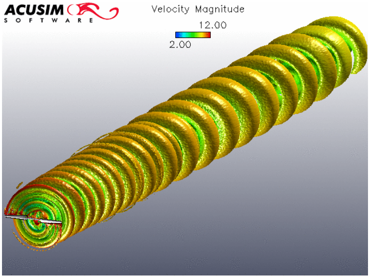
- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

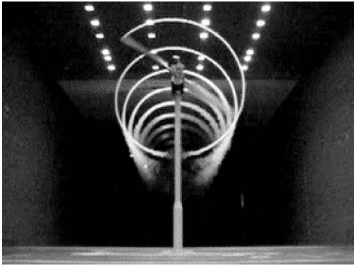
## Additional conservation laws – through direct construction

- Three-component Euler:
  - Generalized momenta. Generalized helicity. Additional vorticity CLs.
- Three-component Navier-Stokes:
  - Additional CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - Additional CLs in viscous and inviscid case, for plane and axisymmetric flows.

# Example: CLs of NS and Euler equations under helical symmetry

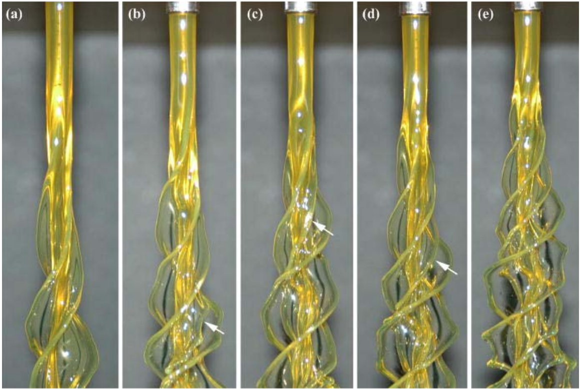
- Wind turbine wakes in aerodynamics [Vermeer, Sorensen & Crespo, 2003]



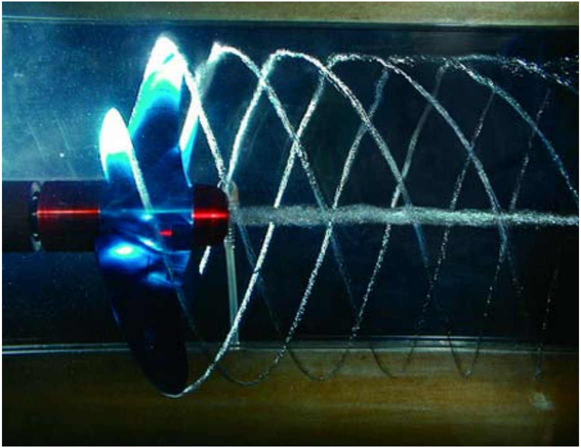
# Example: CLs of NS and Euler equations under helical symmetry

- Helical instability of rotating viscous jets [Kubitschek & Weidman, 2007]

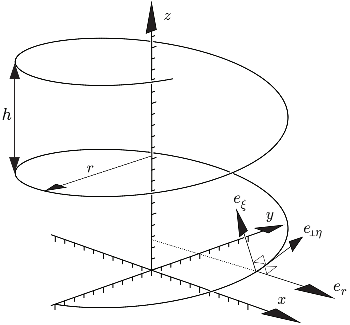


# Example: CLs of NS and Euler equations under helical symmetry

- Helical water flow past a propeller



# Example: CLs of NS and Euler equations under helical symmetry




## Helical Coordinates

- Helical coordinates: $(r, \eta, \xi)$;

$$\xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \qquad a, b = \text{const}, \quad a^2 + b^2 > 0.$$

- Helical invariance: $f = f(r, \xi), \quad a, b \neq 0.$
- Axial: $a = 1,\ b = 0.$ $\quad$ $z$-Translational: $a = 0,\ b = 1.$

# Divergence-type conservation laws – summary

# Divergence-type conservation laws – summary

For a DE system $G[v] = 0$:

- The solution manifold $\mathcal{E}$ is a geometric object.
- CLs reflect its properties, and are coordinate-independent. In particular,

$$\mathrm{D}_{(z^*)^i}(\Phi^*)^i[v^*] = J\,\mathrm{D}_i\Phi^i[v] = 0$$

  after a change of variables

$$\begin{aligned}(z^*)^i &= f^i(z,v), \qquad i = 1,\ldots,n,\\ (v^*)^k &= g^k(z,v), \qquad k = 1,\ldots,m.\end{aligned}$$

- CLs have a characteristic form: $\boxed{\mathrm{D}_i\Phi^i[v] = \Lambda_\sigma[v]G^\sigma[v].}$
- CLs can be systematically computed (the direct method and Maple/GeM implementations).
- The direct method is complete, within the chosen multiplier ansatz.

# Different types of CLs in 3D

# PDE models in three spatial dimensions

## General classical physical systems in 3D:

- Independent variables: coordinates $x = (x^1, x^2, x^3) \in \Omega$, and possibly time $t$.
- Dependent variables: $v = v(t, \mathbf{x})$ or $v(x)$; $m \geq 1$ scalars.
- PDEs: $G^\sigma[v] = 0$, $\sigma = 1, \ldots, M$.

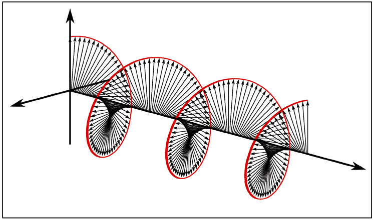
## Typical applications:

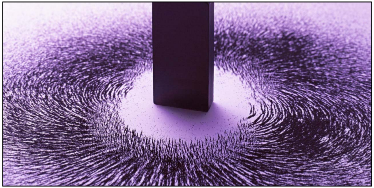
- Nonlinear mechanics, elasticity, viscoelasticity, plasticity
- Fluid mechanics
- Electromagnetism
- Wave propagation
- Thermodynamics, diffusion, ...

## Example: Microscopic Maxwell's equations in Gaussian units

$$\operatorname{div}\mathbf{B} = 0, \qquad \mathbf{B}_t + c\,\operatorname{curl}\mathbf{E} = 0,$$

$$\operatorname{div}\mathbf{E} = 4\pi\rho, \qquad \mathbf{E}_t - c\,\operatorname{curl}\mathbf{B} = -4\pi\mathbf{J}.$$



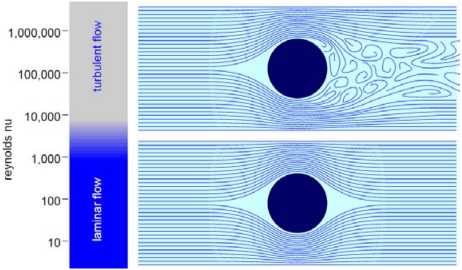


# PDE models in three spatial dimensions: examples

## Example: Navier-Stokes/Euler gas and fluid dynamics equations

$$\rho_t + \operatorname{div} \rho \mathbf{u} = 0,$$

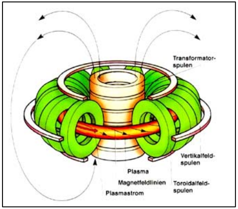
$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\operatorname{grad} p + \mu \Delta \mathbf{u}.$$

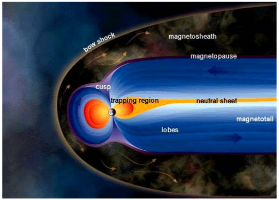


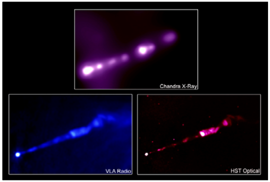
## Example: Ideal magnetohydrodynamics (MHD) equations

$$\rho_t + \operatorname{div} \rho \mathbf{u} = 0, \qquad \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\frac{1}{\mu}\mathbf{B}\times \operatorname{curl}\mathbf{B} - \operatorname{grad} p,$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{u}\times\mathbf{B}), \qquad \operatorname{div}\mathbf{B} = 0.$$



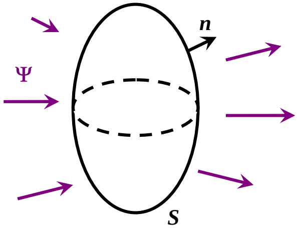

## Applications:

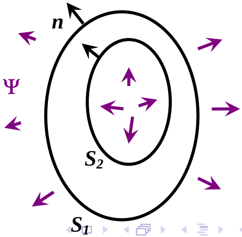
- Time-independent models.
- Differential constraints, e.g., $\operatorname{div} \mathbf{B} = 0, \ \operatorname{curl} \mathbf{u} = 0 \ldots$

## 1A. Spatial divergence/topological flux conservation laws

- Local form: $\boxed{\operatorname{Div} \mathbf{\Psi}[v] = 0.}$
- Global form in $\mathcal{V}$, $\partial\mathcal{V} = \mathcal{S}$: $\boxed{\oint_{\mathcal{S}} \mathbf{\Psi}[v] \cdot d\mathbf{S}\big|_{\mathcal{E}} = 0}$ (Gauss' theorem.)
- Global form when $\partial\mathcal{V} = \mathcal{S}_1 \cup \mathcal{S}_2$:

$$\boxed{\oint_{\mathcal{S}_1} \mathbf{\Psi}[v]|_{\mathcal{E}} \cdot d\mathbf{S} = \oint_{\mathcal{S}_2} \mathbf{\Psi}[v]|_{\mathcal{E}} \cdot d\mathbf{S}.}$$



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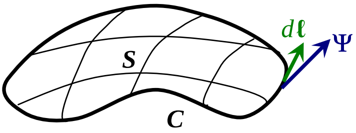
## Examples:

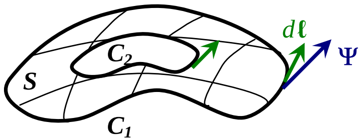
- Incompressible flow: $\mathrm{div}\, \mathbf{u} = 0.$
- Absence of magnetic sources: $\mathrm{div}\, \mathbf{B} = 0.$

## 1B. Spatial curl/topological circulation conservation laws

- Local form: $\boxed{\operatorname{Curl} \mathbf{\Psi}[v]|_{\mathcal{E}} = 0.}$
- Global form in $\mathcal{S}$, $\partial\mathcal{S} = \mathcal{C}$: $\boxed{\int_{\mathcal{C}} \mathbf{\Psi}[v] \cdot d\boldsymbol{\ell} = 0.}$
- Global form, $\partial\mathcal{S} = \mathcal{C}_1 \cup \mathcal{C}_2$:

$$\boxed{\oint_{\mathcal{C}_1} \mathbf{\Psi}[v]|_{\mathcal{E}} \cdot d\boldsymbol{\ell} = \oint_{\mathcal{C}_2} \mathbf{\Psi}[v]|_{\mathcal{E}} \cdot d\boldsymbol{\ell}.}$$



## 1B. Spatial curl/topological circulation conservation laws

- Local form: $\boxed{\operatorname{Curl} \mathbf{\Psi}[v]|_{\mathcal{E}} = 0.}$
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- Global form, $\partial \mathcal{S} = \mathcal{C}_1 \cup \mathcal{C}_2$:

$$\boxed{\oint_{\mathcal{C}_1} \mathbf{\Psi}[v]|_{\mathcal{E}} \cdot d\boldsymbol{\ell} = \oint_{\mathcal{C}_2} \mathbf{\Psi}[v]|_{\mathcal{E}} \cdot d\boldsymbol{\ell}.}$$

## Examples:

- Irrotational flow: $\operatorname{curl} \mathbf{u} = 0$.
- Equilibrium MHD–magnetic equation: $\operatorname{curl} (\mathbf{u} \times \mathbf{B}) = 0$
  $\Rightarrow$ circulation condition:

$$\forall \mathcal{S} \subset \Omega, \quad \int_{\partial \mathcal{S}} (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = 0.$$

## 2A. Volumetric conservation laws:

- A global volumetric conservation law of a given 3D PDE model, for $\mathcal{V} \subset \Omega$:

$$\boxed{\frac{d}{dt}\int_{\mathcal{V}} T\, dV = -\oint_{\partial\mathcal{V}} \mathbf{\Psi} \cdot d\mathbf{S},}$$

  holding for all solutions $v(t, \mathbf{x}) \in \mathcal{E}$.

- Local formulation: a continuity equation

$$\boxed{D_t T[v] + \mathrm{Div}\, \mathbf{\Psi}[v] = 0, \qquad v \in \mathcal{E}.}$$

- Scalar *conserved density*: $T = T[v]$, vector *spatial flux*: $\mathbf{\Psi} = \mathbf{\Psi}[v]$.

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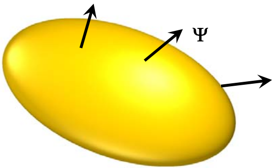
holding for all solutions $v(t, \mathbf{x}) \in \mathcal{E}$.

- Physical meaning: the rate of change of the volume quantity

$$\int_{\mathcal{V}} T[v]\, dV$$

is balanced by the surface flux

$$\oint_{\partial\mathcal{V}} \mathbf{\Psi}[v]\cdot d\mathbf{S}.$$



## Example: Microscopic Maxwell's equations in Gaussian units

$$\operatorname{div}\mathbf{B} = 0, \qquad \mathbf{B}_t + c\operatorname{curl}\mathbf{E} = 0,$$
$$\operatorname{div}\mathbf{E} = 4\pi\rho, \qquad \mathbf{E}_t - c\operatorname{curl}\mathbf{B} = -4\pi\mathbf{J}.$$

Conservation of electromagnetic energy:

$$\tfrac{1}{2}\partial_t\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) + c\operatorname{div}(\mathbf{E}\times\mathbf{B}) = 0.$$

## 2B. Surface-flux conservation laws:

- A global surface-flux conservation law of a given 3D PDE model:

$$\frac{d}{dt}\int_{\mathcal{S}} \mathbf{T}\cdot d\mathbf{S} = -\oint_{\partial\mathcal{S}} \mathbf{\Psi}\cdot d\boldsymbol{\ell}, \qquad v\in\mathcal{E}.$$

- Local formulation: a vector PDE

$$D_t\,\mathbf{T}[v] + \mathrm{Curl}\;\mathbf{\Psi}[v] = 0, \qquad v\in\mathcal{E}.$$

- $\mathcal{S}\subseteq\Omega$ is a fixed bounded surface.
- Vector *conserved flux density*: $\mathbf{T}=\mathbf{T}[v]$; vector *spatial circulation flux*: $\mathbf{\Psi}=\mathbf{\Psi}[v]$.
- Local form: three related scalar divergence-type CLs.

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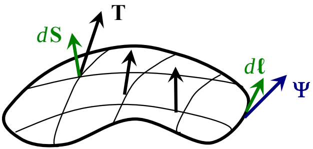


- Physical meaning: rate of change of the surface quantity

$$\int_{\mathcal{S}} \mathbf{T}[v]\cdot d\mathbf{S}$$

is balanced by the circulation

$$\oint_{\partial\mathcal{S}} \mathbf{\Psi}[v]\cdot d\boldsymbol{\ell}.$$

### Example: microscopic Maxwell's equations in Gaussian units

$$\operatorname{div}\mathbf{B} = 0, \qquad \boxed{\mathbf{B}_t + c\,\operatorname{curl}\mathbf{E} = 0,}$$

$$\operatorname{div}\mathbf{E} = 4\pi\rho, \qquad \mathbf{E}_t - c\,\operatorname{curl}\mathbf{B} = -4\pi\mathbf{J}.$$

Magnetic flux conservation: a global surface-flux conservation law (Faraday's law)

$$\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{S} = -c\oint_{\partial\mathcal{S}} \mathbf{E}\cdot d\boldsymbol{\ell}.$$

## Example: ideal magnetohydrodynamics (MHD) equations

$$\rho_t + \operatorname{div} \rho\mathbf{u} = 0,$$

$$\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\frac{1}{\mu}\mathbf{B}\times\operatorname{curl}\mathbf{B} - \operatorname{grad} p,$$

$$\operatorname{div}\mathbf{B} = 0,$$

$$\boxed{\mathbf{B}_t = \operatorname{curl}(\mathbf{u}\times\mathbf{B}).}$$

Conserved flux density, spatial circulation flux:

$$\mathbf{T} = \mathbf{B}, \qquad \mathbf{\Psi} = \mathbf{B}\times\mathbf{u}.$$

The global form of the surface-flux conservation law

$$\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{S} = -\oint_{\partial\mathcal{S}} (\mathbf{B}\times\mathbf{u})\cdot d\boldsymbol{\ell}$$

describes the time evolution of the total magnetic flux through a given fixed surface $\mathcal{S}$.

- A similar CL holds for non-ideal (resistive, viscous) plasmas.

## 2C. Circulatory conservation laws:

- A global circulatory conservation law of a given 3D PDE model:

$$\frac{d}{dt}\int_{\mathcal{C}} \mathbf{T}\cdot d\boldsymbol{\ell} = -\Psi\big|_{\partial\mathcal{C}}, \qquad v \in \mathcal{E}.$$

- Local local circulatory conservation law:

$$D_t\,\mathbf{T}[v] + \mathrm{Grad}\;\Psi[v] = 0, \qquad v \in \mathcal{E}.$$

- $\mathcal{C} \subseteq \Omega$ is a fixed simple curve.
- Vector *conserved circulation density*: $\mathbf{T} = \mathbf{T}[v]$; vector *spatial boundary flow*: $\Psi = \Psi[v]$.
- Local form: three related scalar divergence-type CLs.

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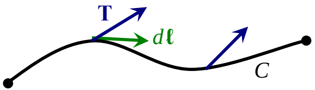



- Physical meaning: rate of change of the line integral quantity

$$\int_{\mathcal{C}} \mathbf{T}\cdot d\boldsymbol{\ell}$$

is balanced by the flow through the ends of the curve.

## Example: irrotational barotropic gas flow.

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0,$$

$$\mathbf{u}_t + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \operatorname{grad} f = 0, \qquad f = f_{\rm bar} = \frac{|\mathbf{u}|^2}{2} + \int \frac{p'(\rho)}{\rho}\, d\rho.$$

- Irrotational: $\operatorname{curl} \mathbf{u} = 0$.
- Barotropic: $p = p(\rho)$, $\Rightarrow$ $\boxed{\mathbf{u}_t + \operatorname{grad} f = 0.}$
- Circulatory conservation law over an arbitrary static curve $\mathcal{C}$:

$$\frac{d}{dt} \int_{\mathcal{C}} \mathbf{u} \cdot d\boldsymbol{\ell} = -f|_{\partial \mathcal{C}}.$$

- For closed curves, $\partial \mathcal{C} = \emptyset$:

$$\frac{d}{dt} \oint_{\mathcal{C}} \mathbf{u} \cdot d\boldsymbol{\ell} = 0,$$

conservation of a global velocity circulation around a static closed path.

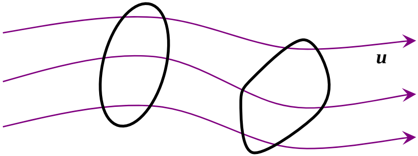
# Conservation laws on moving domains in 3D

- Suppose the model involves a velocity field $\mathbf{u}(t,\mathbf{x})$.
- $\mathbf{X}(t,\mathbf{x})$: material (Lagrangian) coordinates, macroscopic particle labels.
- Streamlines:

$$\frac{\mathrm{d}\mathbf{X}(t,\mathbf{x})}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \equiv \partial_t + \mathbf{u}\cdot\nabla.$$

- A moving material domain: consists of the same material points.

$$\mathbf{X}(t,\mathbf{x}(t)) = \mathrm{const}, \qquad \frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(t,\mathbf{x}(t)).$$



## Moving volume conservation laws:

- A moving volume conservation law of a given 3D PDE model:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} T[\mathbf{u},v]\,dV = -\oint_{\partial\mathcal{V}(t)} \mathbf{\Upsilon}[\mathbf{u},v]\cdot d\mathbf{S},$$

holding for all solutions $v = v(t,\mathbf{x}) \in \mathcal{E}$, for a volume $\mathcal{V}(t) \in \Omega$ transported by the flow.

## Local formulation:

- Leibniz's rule for moving domains:

$$\frac{d}{dt}\int_{\mathcal{V}(t)} T[\mathbf{u},v]\,dV = \int_{\mathcal{V}(t)} D_t T[\mathbf{u},v]\,dV + \oint_{\partial\mathcal{V}(t)} T[\mathbf{u},v]\,\mathbf{u}\cdot d\mathbf{S}$$

- Local form:

$$D_t\, T[\mathbf{u},v] + \mathrm{Div}\left(\mathbf{\Upsilon}[\mathbf{u},v] + T[\mathbf{u},v]\mathbf{u}\right) = 0.$$

# Moving volume CL example: helicity

## Conservation of helicity in a moving volume

- Constant-density fluid flow:

$$\operatorname{div} \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \operatorname{grad} f = 0, \qquad f = \frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho}.$$

- The fluid helicity: $h \equiv \mathbf{u} \cdot \boldsymbol{\omega}$.

- Helicity dynamics equation: $h_t + \operatorname{div} (\boldsymbol{\omega} \cdot \operatorname{grad} f + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0.$

- Moving volumetric CL, local form:

$$\boxed{D_t\, T[\mathbf{u}, v] + \operatorname{Div} (\boldsymbol{\Upsilon}[\mathbf{u}, v] + T[\mathbf{u}, v]\mathbf{u}) = 0, \qquad v \in \mathcal{E}.}$$

$$T = h = \mathbf{u} \cdot \boldsymbol{\omega}, \qquad \boldsymbol{\Upsilon} = (f - |\mathbf{u}|^2)\, \boldsymbol{\omega}.$$

- Global form:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} h\, dV = - \oint_{\partial \mathcal{V}(t)} (f - |\mathbf{u}|^2)\, \boldsymbol{\omega} \cdot d\mathbf{S}.$$

# Material conservation laws

## Important special case: a material CL

- A material conservation law: a moving volume CL with a vanishing spatial flux, $\boldsymbol{\Upsilon}[\mathbf{u}, v]|_{\mathcal{E}} = 0$. of a given 3D PDE model, for $\mathcal{V} \subset \Omega$:

$$\boxed{\frac{d}{dt} \int_{\mathcal{V}(t)} T[\mathbf{u}, v]\, dV = -\oint_{\partial \mathcal{V}(t)} \boldsymbol{\Upsilon}[\mathbf{u}, v] \cdot d\mathbf{S} = 0.}$$

- Local formulation:

$$\boxed{D_t T[\mathbf{u}, v] + \mathrm{Div}(T[\mathbf{u}, v]\mathbf{u}) = 0.}$$

- A well-known expression for incompressible flows $\mathrm{div}\,\mathbf{u} = 0$:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} T[\mathbf{u}, v] \;=\; 0,} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \;\equiv\; D_t + \mathbf{u} \cdot \mathrm{Grad}$$

## Material conservation laws: example

The continuity equation in gas/fluid dynamics:

$$\boxed{\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0,}$$

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = \mu\,\Delta\mathbf{u} + \rho\mathbf{g}.$$

Conservation of mass in a moving material domain :

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \rho\, dV = 0.$$

# B. Moving surface-flux conservation laws

## Moving surface-flux CLs:

- A moving surface-flux conservation law of a given 3D PDE model:

$$\boxed{\frac{d}{dt}\int_{\mathcal{S}(t)} \mathbf{T}[\mathbf{u}, v]\cdot d\mathbf{S} = -\oint_{\partial\mathcal{S}(t)} \boldsymbol{\Upsilon}[\mathbf{u}, v]\cdot d\boldsymbol{\ell},}$$

holding for all solutions $v = v(t, \mathbf{x}) \in \mathcal{E}$, for a surface $\mathcal{S}(t) \in \Omega$ transported by the flow.

## Physical meaning

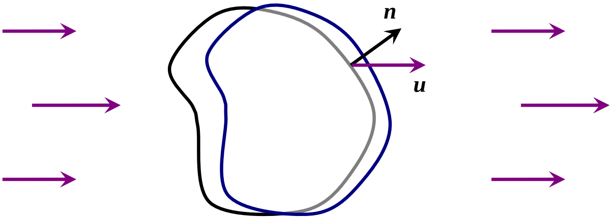
The rate of change of the total flux of the vector field $\mathbf{T}[\mathbf{u}, v]$ through the moving surface $\mathcal{S}(t)$ in terms of the net boundary circulation.

## Moving surface-flux CLs:

- A moving surface-flux conservation law of a given 3D PDE model:

$$\frac{d}{dt}\int_{\mathcal{S}(t)} \mathbf{T}[\mathbf{u}, v]\cdot d\mathbf{S} = -\oint_{\partial\mathcal{S}(t)} \boldsymbol{\Upsilon}[\mathbf{u}, v]\cdot d\boldsymbol{\ell},$$

holding for all solutions $v = v(t, \mathbf{x}) \in \mathcal{E}$, for a surface $\mathcal{S}(t) \in \Omega$ transported by the flow.



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holding for all solutions $v = v(t, \mathbf{x}) \in \mathcal{E}$, for a surface $\mathcal{S}(t) \in \Omega$ transported by the flow.

## Local formulation:

- Leibniz's rule for moving surfaces:

$$\frac{d}{dt}\int_{\mathcal{S}(t)} \mathbf{T}[\mathbf{u}, v]\cdot d\mathbf{S}$$
$$= \int_{\mathcal{S}(t)} D_t\,\mathbf{T}[\mathbf{u}, v]\cdot d\mathbf{S} + \int_{\mathcal{S}(t)} (\mathrm{Div}\,\mathbf{T}[\mathbf{u}, v])\,\mathbf{u}\cdot d\mathbf{S} + \oint_{\partial\mathcal{S}(t)} (\mathbf{T}[\mathbf{u}, v]\times\mathbf{u})\cdot d\boldsymbol{\ell}.$$

- Local form:

$$\boxed{D_t\,\mathbf{T}[\mathbf{u}, v] + (\mathrm{Div}\,\mathbf{T}[\mathbf{u}, v])\,\mathbf{u} + \mathrm{Curl}\,(\mathbf{T}[\mathbf{u}, v]\times\mathbf{u} + \boldsymbol{\Upsilon}[\mathbf{u}, v]) = 0, \qquad v \in \mathcal{E}.}$$

## Moving surface-flux CL example: MHD

- Non-ideal (finite conductivity) MHD:

$$\rho_t + \operatorname{div} \rho \mathbf{u} = 0, \qquad \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\frac{1}{\mu}\mathbf{B}\times\operatorname{curl}\mathbf{B} - \operatorname{grad} p,$$

$$\boxed{\mathbf{B}_t = \operatorname{curl}\left(\mathbf{u}\times\mathbf{B} + \frac{1}{\sigma}\mathbf{J}\right), \qquad \operatorname{div}\mathbf{B} = 0.}$$

- Plasma electric current density: $\mathbf{J} = (1/\mu)\operatorname{curl}\mathbf{B}$.
- Moving surface-flux conservation law on a material surface $\mathcal{S}(t)$:

$$\frac{d}{dt}\int_{\mathcal{S}(t)} \mathbf{B}\cdot d\mathbf{S} = -\frac{1}{\sigma}\oint_{\partial\mathcal{S}(t)} \mathbf{J}\cdot d\boldsymbol{\ell}.$$

- Describes yields a rate of change of the magnetic flux through $\mathcal{S}(t)$ in terms of the circulation of the electric current density.
- A material CL: for a closed $\mathcal{S}(t)$, or in the case of ideal plasma ($\sigma\to\infty$):

$$\frac{d}{dt}\int_{\mathcal{S}(t)} \mathbf{B}\cdot d\mathbf{S} = 0.$$

# C. Moving circulatory conservation laws

## Moving circulatory CLs

- A moving circulatory conservation law of a given 3D PDE model:

$$\boxed{\frac{d}{dt}\int_{\mathcal{C}(t)} \mathbf{T}[\mathbf{u}, v]\cdot d\boldsymbol{\ell} = -\Upsilon[\mathbf{u}, v]\big|_{\partial\mathcal{C}(t)},}$$

holding for all solutions $v = v(t,\mathbf{x}) \in \mathcal{E}$, for a curve $\mathcal{C}(t) \in \Omega$ transported by the flow.

## Physical meaning

The rate of change of the total flux of the moving line integral quantity in terms of the net flow out of the two ends.

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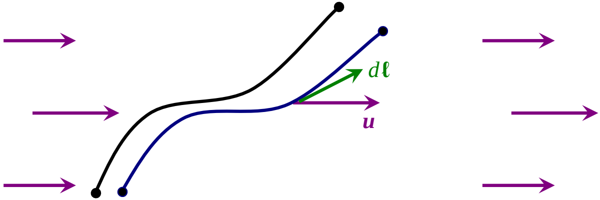

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holding for all solutions $v = v(t, \mathbf{x}) \in \mathcal{E}$, for a curve $\mathcal{C}(t) \in \Omega$ transported by the flow.

## Local formulation:

- Chain/Leibniz's rule for moving curves:

$$\frac{d}{dt}C[v; \mathcal{C}(t)] = \frac{d}{dt}\int_{\mathcal{C}(t)} \mathbf{T}[\mathbf{u}, v]\cdot d\boldsymbol{\ell}$$
$$= \int_{\mathcal{C}(t)} (D_t\,\mathbf{T}[\mathbf{u}, v])\cdot d\boldsymbol{\ell} + \int_{\mathcal{C}(t)} ((\mathrm{Curl}\ \mathbf{T}[\mathbf{u}, v])\times\mathbf{u})\cdot d\boldsymbol{\ell} + (\mathbf{T}[\mathbf{u}, v]\cdot\mathbf{u})\Big|_{\partial\mathcal{C}(t)}.$$

- Local form:

$$\boxed{D_t\,\mathbf{T}[\mathbf{u}, v] + (\mathrm{Curl}\ \mathbf{T}[\mathbf{u}, v])\times\mathbf{u} + \mathrm{Grad}\,(\Upsilon[\mathbf{u}, v] + \mathbf{T}[\mathbf{u}, v]\cdot\mathbf{u}) = 0.}$$

## Moving circulatory CL example: Euler model, velocity circulation

- Constant-density fluid flow:

$$\operatorname{div}\mathbf{u} = 0,$$

$$\mathbf{u}_t + (\operatorname{curl}\mathbf{u})\times\mathbf{u} + \operatorname{grad} f = 0, \qquad f = \frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho}.$$

- Local circulatory CL form:

$$\boxed{D_t\,\mathbf{T}[\mathbf{u},v] + (\operatorname{Curl}\,\mathbf{T}[\mathbf{u},v])\times\mathbf{u} + \operatorname{Grad}\left(\Upsilon[\mathbf{u},v] + \mathbf{T}[\mathbf{u},v]\cdot\mathbf{u}\right) = 0.}$$

- Velocity line integral: $\mathbf{T} = \mathbf{u}$, $\Upsilon = f - |\mathbf{u}|^2$.

- Global form:

$$\frac{d}{dt}\int_{\mathcal{C}(t)} \mathbf{u}\cdot d\boldsymbol{\ell} = -(f - |\mathbf{u}|^2)|_{\partial\mathcal{C}(t)}.$$

- Moving material closed curve – vanishing velocity circulation:

$$\frac{d}{dt}\oint_{\mathcal{C}(t)} \mathbf{u}\cdot d\boldsymbol{\ell} = 0.$$

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- Physical examples are readily available.

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- CLs are useful in physics, analysis, and numerical simulations.
- CLs have local and global forms.
- CLs are coordinate-independent.
- More than one kind of CLs exist, with different physical meaning. All are (locally) given in terms of divergence expressions.
- Theoretical methods and powerful symbolic software for systematic CL computations exists.

**Keywords related to what we did not discuss:**

- CL computational aspects: how to avoid trivial/equivalent CLs, singular multipliers, and yet retain completeness.
- Relationships with symmetries, Lagrangians, variational systems, 1st and 2nd Noether's theorems, integrability...
- Useful techniques to get CLs "cheaply".
- Nonlocal and approximate CLs.

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**Thank you for your attention!**