Symmetries of Differential Equations: Practical session

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Outline

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1. Point symmetries of ODEs

2. Local symmetries of PDEs

3. Nonlocal symmetries of PDEs

Applications of point symmetries of ODEs

- Every symmetry can be used to reduce order by 1 (by differential invariants, or canonical coordinates).

- Can find exact invariant solutions, or mappings: solutions $\to$ solutions.

Point symmetries of ODEs

- First-order ODEs

$$y'(x) = F(x, y(x))$$

  have infinitely many point symmetries

$$\mathrm{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

- Finding them is harder than to solve the ODE itself...

- Example 1: find point symmetries of the ODE

$$y'(x) = x^2 + y^2(x).$$

- Fact: nth order ODEs ($n > 2$) admit finitely many symmetries.

- One can show that a second order ODE admits at most an eight-parameter Lie group of transformations. An nth order ODE ($n > 2$) admits at most an ($n + 4$)-parameter Lie group of transformations [Lie (1893) Ovsiannikov (1982)].

- Example 2: find all (8) point symmetries of the second-order ODE

$$y''(x) = 0.$$

- Example 3: find all point symmetries of the second-order ODE

$$y''(x) = y(x)y'(x).$$

  Use the scaling symmetry and its canonical coordinates to reduce the ODE order to one.

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Applications of point symmetries of PDEs

- Every symmetry of a PDE/PDE system

$$\mathrm{X} = \xi^i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\mu}$$

  involving independent variables (some $\xi^i \neq 0$) can be used to seek invariant solutions.

- Invariant solutions are defined by DEs with a reduced number of independent variables (by 1); possibly ODEs.

- Symmetries can be used to construct new solutions from known ones.

- Symmetries are used to determine whether the PDE/system can be linearized by a point transformation.

- Other applications exist.

Point symmetries of PDEs

- Fact: Linear, or linearizable PDEs have infinitely many symmetries

$$\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^\mu},$$

  with components ξ^i, η^μ parameterized by solutions of linear PDEs.

- Nonlinear PDEs usually have a finite-dimensional Lie algebra of point symmetries, but sometimes admit infinitely many symmetries, with components ξ^i and/or η^μ involving some arbitrary functions.

- It is uncommon for nonlinear models to admit higher-order symmetries; their existence is often related to S-integrability.

- Example 4: find all point symmetries of the the nonlinear wave equation on $u(x, t)$ given by
$$u_{tt} = (1 + u_x^2) u_{xx}.$$

- Example 5: Find all local symmetries in the evolutionary form $\hat{X} = \zeta[u]\partial_u$ of the modified nonlinear heat equation

$$u_t = u_{xx} + u_x^2,$$

where $\zeta[u] = \zeta(x, t, u, u_x, u_{xx}, u_{xxx})$. Identify point and genuinely higher-order symmetries.

- Example 6: Compute the basis of the Lie algebra of point symmetries of the 2D Navier-Stokes equations of the constant-density fluid motion:

$$\begin{aligned} &u_x + v_y = 0,\\ &u_t + u u_x + v u_y = -P_x + \nu(u_{xx} + u_{yy}),\\ &v_t + u v_x + v v_y = -P_y + \nu(v_{xx} + v_{yy}), \end{aligned}$$

  where u, v, P are functions of t, x, y.

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Example 7: a potential symmetry

- Compute point symmetries of a nonlinear diffusion equation on $u(x,t)$:

$$U[u] = u_t - (L(u))_{xx} = 0, \qquad L'(u) = K(u) = u^{-2/3}.$$

- Compute point symmetries of the potential systems

$$UV[u,v]: \quad \begin{cases} v_x = u, \\ v_t = K(u)u_x \end{cases}$$

and

$$UA[u,a]: \quad \begin{cases} a_x = xu, \\ a_t = xK(u)u_x - L(u). \end{cases}$$

- Compute point symmetries of the couplet potential system

$$UVA[u,v,a]: \quad \begin{cases} v_x = u, \\ v_t = K(u)u_x, \\ a_x = xu, \\ a_t = xK(u)u_x - L(u). \end{cases}$$

- Find a potential symmetry of the nonlinear diffusion equation $U[u]$.