Symmetries of Differential Equations: Symbolic Computation using Maple

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Outline

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Definitions

Variables:

- Independent: $\mathbf{x} = (x^1, x^2, ..., x^n)$ or $(t, x^1, x^2, ...)$ or $(t, x, y, ...)$.
- Dependent: $\mathbf{u} = (u^1(\mathbf{x}), u^2(\mathbf{x}), ..., u^m(\mathbf{x}))$ or $(u(\mathbf{x}), v(\mathbf{x}), ...)$.

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Partial derivatives:

- Notation:
$$\frac{\partial u^k}{\partial x^i} = u^k_{x^i} = u^k_i = \partial_i u^k.$$
- E.g.,
$$\frac{\partial}{\partial t} u(x, y, t) = u_t = \partial_t u.$$
- All first-order partial derivatives of $\mathbf{u}$: $\partial \mathbf{u}$.
- E.g.,
$$\mathbf{u} = (u^1(x, t), u^2(x, t)), \qquad \partial \mathbf{u} = \{u^1_x,\ u^1_t,\ u^2_x,\ u^2_t\}.$$

Higher-order partial derivatives

- Notation: for example,

$$\frac{\partial^2}{\partial x^2}u(x,y,z) = u_{xx} = \partial_x^2 u.$$

- All $p^{\rm th}$-order partial derivatives: $\partial^p \mathbf{u}$.

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Jet spaces

- We wish to work with differential equations as with algebraic equations.
- Jet space of order p: linear space $J^p(\mathbf{x}|\mathbf{u})$ with coordinates $\mathbf{x}$, $\mathbf{u}$, $\partial \mathbf{u}$, ..., $\partial^p \mathbf{u}$.

Differential functions

- A differential function defined on a subset of $J^p(\mathbf{x}|\mathbf{u})$ is an expression that may involve independent and dependent variables, and derivatives of dependent variables to some order $\leq p$.

$$F[\mathbf{u}] = F(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^p\mathbf{u}).$$

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Differential equations

- A system of differential equations (PDE, ODE) of order k:

$$R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$$

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Example:

- The 1D diffusion equation for $u(x,t)$ can be written as
$$0 \; = \; u_t - u_{xx} \; = \; H(u, u_t, u_{xx}) \; = \; H[u],$$
that is, an algebraic equation in $J^2(x, t|u)$.

Definitions

The total derivative of a differential function:

- A basic chain rule for $u = u(x, y)$:

$$\frac{\partial}{\partial x} g(x, y, u, u_x, u_y) = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx} + \frac{\partial g}{\partial u_y} u_{xy}$$

- The total derivative does the same for differential functions on the jet space:

$$\mathrm{D}_x g[u] = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx} + \frac{\partial g}{\partial u_y} u_{xy},$$

where $x, y, u, u_x, u_y, u_{xx}, u_{xy}$ are coordinates in $J^2(x, y|u)$.

General case

- Independent variables: $\mathbf{x} = (x^1, x^2, ..., x^n)$; dependent: $\mathbf{u}(\mathbf{x}) = (u^1, ..., u^m)$.
- The total derivative operator with respect to x^i:

$$\mathrm{D}_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{ii_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots ,$$

Outline

Lie Groups of Point Transformations

- Variables $\mathbf{x} = (x^1, ..., x^n)$, $\mathbf{u} = (u^1, ..., u^m)$.
- A Lie group of point transformations:

$$(x^i)^* = f^i(\mathbf{x}, \mathbf{u}; \varepsilon),$$

$$(u^\mu)^* = g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon),$$

$$\varepsilon \in M \sim \mathbb{R}^n,$$

where ε is a vector parameter, and the transformations form a group with *some* composition law.

- Variables $\mathbf{x} = (x^1, ..., x^n), \quad \mathbf{u} = (u^1, ..., u^m)$.

- A one-parameter Lie group of point transformations:

$$\begin{aligned}(x^i)^* &= f^i(\mathbf{x}, \mathbf{u}; \varepsilon),\\ (u^\mu)^* &= g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon),\\ &\varepsilon \in I \subset \mathbb{R},\end{aligned} \tag{1}$$

where f, g are bijective and smooth in $\mathbf{x}, \mathbf{u}$, and analytic in ε.

- Lie's 1st theorem: WLOG (1) is Abelian, with an additive law of composition of the parameter:

$$(x^i)^{**} = f^i(\mathbf{x}^*, \mathbf{u}^*; \delta) = f^i(\mathbf{x}, \mathbf{u}; \varepsilon + \delta),$$

$$(u^\mu)^{**} = g^\mu(\mathbf{x}^*, \mathbf{u}^*; \delta) = g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon + \delta).$$

- A one-parameter Lie group of point transformations:

$$(x^i)^* = f^i(\mathbf{x}, \mathbf{u}; \varepsilon) = x^i + \varepsilon \xi^i(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2),$$
$$(u^\mu)^* = g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon) = u^\mu + \varepsilon \eta^\mu(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2).$$

- The corresponding infinitesimal generator (TVF):

$$\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\mu},$$

where

$$\xi^i = \left.\frac{\partial f^i}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \eta^\mu = \left.\frac{\partial g^\mu}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

- Fact: the global group, additive in parameter, is recovered from the solution of an ODE problem

$$\frac{d(x^i)^*(\varepsilon)}{d\varepsilon} = \xi^i(\mathbf{x}^*(\varepsilon), \mathbf{u}^*(\varepsilon)), \quad (x^i)^*(0) = x^i,$$
$$\frac{d(u^\mu)^*(\varepsilon)}{d\varepsilon} = \eta^\mu(\mathbf{x}^*(\varepsilon), \mathbf{u}^*(\varepsilon)), \quad (u^\mu)^*(0) = u^\mu.$$

- This integration can be automated using symbolic computations.

Invariance of algebraic equations:

A system of algebraic equations

$$R^{\sigma}(\mathbf{x},\mathbf{u}) = 0, \quad \sigma = 1,\ldots,N$$

is invariant under the transformation

$$(x^i)^* = f^i(\mathbf{x},\mathbf{u};\varepsilon), \quad (u^\mu)^* = g^\mu(\mathbf{x},\mathbf{u};\varepsilon), \quad \varepsilon \in \mathbb{R}$$

if and only if

$$\boxed{\mathrm{X} R^{\alpha}(\mathbf{x},\mathbf{u})\Big|_{R^\sigma=0,\ \sigma=1,\ldots,N} = 0, \quad \alpha = 1,\ldots,N.}$$

(That is, the curve (surface) in the $(\mathbf{x},\mathbf{u})$ - space is invariant.)

Invariance of differential equations:

A system of differential equations of order k

$$R^{\sigma}[\mathbf{u}] = 0, \quad \sigma = 1, \dots, N$$

is invariant under the transformation

$$(x^i)^* = f^i(\mathbf{x}, \mathbf{u}; \varepsilon), \quad (u^{\mu})^* = g^{\mu}(\mathbf{x}, \mathbf{u}; \varepsilon), \quad \varepsilon \in \mathbb{R}$$

if and only if

$$\boxed{\mathrm{X}^{(k)} R^{\alpha}[\mathbf{u}]\Big|_{R^{\sigma}[\mathbf{u}]=0,\ \sigma=1,\dots,N} = 0, \quad \alpha = 1, \dots, N.}$$

(That is, the curve (surface) in the jet space $J^q(\mathbf{x}|\mathbf{u})$ is invariant.)

- $\mathrm{X}^{(k)}$: the kth prolongation of X in $J^q(\mathbf{x}|\mathbf{u})$.
- **Note:** on solutions means using equations and their differential consequences as required.

Outline

Maple software and the GeM module

- GeM module: current version 32.12.
- Works with Maple versions 14...2017.
- Symmetries (point, local, approximate); equivalence transformations; conservation laws, more.
- Description, tutorial, examples: https://math.usask.ca/~shevyakov/gem/
- Implemented as text (MPL) file; all variables and functions are unprotected.
- The next version will use an object-oriented framework...

Step 1. Defining variables in Maple/GeM

A given DE system:

$$R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$$

- Independent: $\mathbf{x} = (x^1, x^2, ..., x^n)$ or $(t, x, y, ...)$.
- Dependent: $\mathbf{u}(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x}), ...)$.
- "Arbitrary" constant parameters: $c_1, \ldots, c_p$.
- "Arbitrary" constitutive functions: $F_1[\mathbf{u}], \ldots, F_q[\mathbf{u}]$.

Example – first step: read GeM package, define variables.

- Use the module: `read("d:/gem32_12.mpl"):`
- Declare variables and arbitrary functions/constants, for example,

```
gem_decl_vars(indeps=[x,t], deps=[U(x,t),V(x,t)],
               freeconst=[a,b], freefunc=[F(U(x,t), diff(U(x,t),x)]);
```

Step 2. Declare the DEs

- A given DE system:

$$R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$$

- Declare the equations:

```
gem_decl_eqs([...], solve_for=[...]);
```

- First list: a set of equations (ODE or PDE) involving only the pre-defined independent and dependent variables, arbitrary functions, and arbitrary constants.
- For symmetry computations, it is necessary that a given PDE system be in a solved form with respect to some leading derivatives, specified in the `\color{red}solve_for` parameter.

Extended Kovalevskaya form

A PDE system $\{R^\sigma[u]=0\}_{\sigma=1}^m$ consisting of m equations and m dependent variables $u=\{u^k(z)\}_{k=1}^m$ is represented in the *extended Kovalevskaya form* $\{\widehat{R}^\sigma[u]=0\}_{\sigma=1}^m$ with respect to an independent variable, say, z_n, if each equation has the form

$$\widehat{R}^\sigma[u]=\frac{\partial^{r_\sigma} u^\sigma}{\partial z_n^{r_\sigma}}-H^\sigma[u]=0, \quad \sigma=1,\ldots,m,$$

where $r_a \leqslant \rho$, and the functions $H^\sigma[u]$ may involve z, u and derivatives of the functions u with respect to z up to some maximal order ρ; moreover, each u^b is differentiated with respect to z_n at most r_b-1 times, $b=1,\ldots,m$.

In other words, a PDE system in extended Kovalevskaya form is solved for the *leading derivatives*

$$\frac{\partial^{r_\sigma} u^\sigma}{\partial z_n^{r_\sigma}}, \qquad \sigma=1,\ldots,m.$$

Step 3. Generate symmetry determining equations

- Generate symmetry determining equations

$$\boxed{\mathrm{X}^{(k)} R^{\alpha}[\mathbf{u}]\Big|_{R^{\sigma}[\mathbf{u}]=0,\ \sigma=1,\dots,N} = 0, \quad \alpha = 1, \dots, N.}$$

  and split them (setting coefficients at free derivatives to zero), using

```
det_eqs:=gem_symm_det_eqs([...]);
```

- The list `[...]` specifies the dependence of symmetry components. Can include independent and dependent variables (for point symmetries), and also derivatives of dependent variables (for higher-order symmetries).
- In principle, it is possible to specify different dependencies for different symmetry components (not explained now).
- The resulting Maple set of split symmetry determining equations will be placed in the variable `det_eqs` (can use another name).
- If one does not wish the determining equations to be split with respect to (higher) derivatives which do not participate in tangent vector field coordinates, one specifies an additional parameter

```
return_unsplit=true
```

Step 4. Store names of symmetry components

- Request names of symmetry components and place them in some user variable.

```
sym_components:=gem_symm_components();
```

Step 5 (optional). Simplify and reduce the determining equations

- DEtools[rifsimp] is a powerful Maple built-in routine for Gröbner basis-based elimination and simplification of systems of DEs.
- It is able to significantly reduce the number of symmetry determining equations.
- It provides the dimension of the solution set (option mindim=1) without solving the equations.
- rifsimp is able to do case splitting/clasifications (will discuss later).
- rifsimp **will not work** if any equations are not differential-polynomial (yet there are ways around it).
- Example:

```
simplified_eqs:=DEtools[rifsimp](det_eqs, sym_components, mindim=1);
```

Step 6. Solve the determining equations

- Use Maple pdsolve to solve the determining equations:

```
symm_sol:=pdsolve(simplified_eqs[Solved], sym_components);
```

- Could do directly without rifsimp (not always optimal):

```
symm_sol:=pdsolve(det_eqs, sym_components);
```

- Now the symmetry components are stored in symm_sol variable. Maple arbitrary constants _C1, _C2, etc., or free functions _F1, _F2, etc., correspond to linearly independent symmetry generators.

- pdsolve is a great solver but not a "universal black box". Its returned solution might not be complete; for example, in the case of linear equations.

Step 7. Output the symmetry generators

- Print all symmetry generators:

```
gem_output_symm(symm_sol);
```

- Can extract a single symmetry, e.g., one corresponding to `_C1`:

```
read("d:/gem_globgroup_and_equiv.txt");
X1:=gem_extract_symm(symm_sol,spec={_C1=1});
```

  and integrate to compute the global Lie group:

```
gem_global_group(X1, group_param_name='epsilon');
```

Outline

Interface recommendation: Tools/Options/Display/Input → Maple notation.

Computational examples

1. Symmetries of a second-order ODE: $\boxed{y''(x) = y(x)y'(x)}$

2. Symmetries of a PDE: Burgers equation $\boxed{u_t + uu_x = \nu u_{xx},}$ $u = u(x,t)$.
   - Only the case $\nu \neq 0$ (see code).

3. Symmetries of a PDE system – 2D constant-density Euler equations:

$$\boxed{\begin{aligned} & u_x + v_y = 0, \\ & \rho(u_t + uu_x + vu_y) = -p_x, \\ & \rho(v_t + uv_x + vv_y) = -p_y \end{aligned}}$$

   for the unknowns $u(t,x,y)$, $v(t,x,y)$, $p(t,x,y)$.
   - Equations solved with respect to the $x-$derivatives.
   - An infinite set of point symmetries (generalized Galilei group, pressure freedom).

Remarks

1. Linear and nonlinear ODEs of order $k > 1$ have a finite number of point symmetries.

2. A PDE or a PDE system can have an infinite number of point symmetries (involving arbitrary functions). [Examples: mechanical systems with generalized Galilei group; linear and linearizable PDEs.]

   In this case, `pdsolve` may or may not be helpful.

   - Example: linear heat equation $\boxed{u_t = u_{xx}.}$
   - Symmetries of linear equations and their computation will be discussed later.

3. A global multi-parameter Lie group can be computed by applying `gem_global_group` to a linear combination of symmetry generators.

Outline

Evolutionary Form of Point Transformation

- Variables $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{u} = (u^1, \dots, u^m)$.
- A one-parameter Lie group of point transformations:

$$(x^i)^* = f^i(\mathbf{x}, \mathbf{u}; \varepsilon) = x^i + \varepsilon \xi^i(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2),$$

$$(u^\mu)^* = g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon) = u^\mu + \varepsilon \eta^\mu(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2).$$

- The corresponding infinitesimal generator (TVF):

$$\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\mu}.$$

- The same local transformation in the evolutionary form:

$$(x^i)^\dagger = x^i,$$

$$(u^\mu)^\dagger = u^\mu + \varepsilon \zeta^\mu[\mathbf{u}] + O(\varepsilon^2).$$

- Evolutionary form of the symmetry generator:

$$\hat{\mathrm{X}} = \zeta^\mu[\mathbf{u}] \frac{\partial}{\partial u^\mu}, \qquad \boxed{\zeta^\mu[\mathbf{u}] = \eta^\mu[\mathbf{u}] - u^\mu_i \xi^i.}$$

Evolutionary Form of a Point Transformation: Example

- Consider an ODE $y' = -x/y \quad \Leftrightarrow \quad y^2 + x^2 = C = \text{const.}$
- A scaling symmetry:

$$x^* = e^\varepsilon x,$$
$$u^* = e^\varepsilon y.$$

- Point symmetry generator:

$$\mathrm{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \quad \xi = x, \quad \eta = y.$$

- Local form:

$$x^* = x + \varepsilon\xi(x,y) + O(\varepsilon^2),$$
$$y^* = y + \varepsilon\eta(x,y) + O(\varepsilon^2).$$

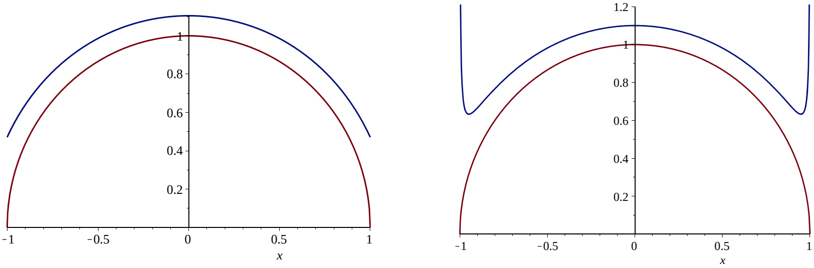
- Evolutionary form of the symmetry generator:

$$\hat{\mathrm{X}} = \zeta[y]\frac{\partial}{\partial y}, \quad \zeta[y] = \eta - y'(x)\,\xi = y + x^2/y.$$

- Local transformation in the evolutionary form:

$$x^\dagger = x,$$
$$u^\dagger = u + \varepsilon\left(y + \frac{x^2}{y}\right) + O(\varepsilon^2).$$

- $\varepsilon = 0.1$:



Evolutionary Form of a Point Transformation

- Is the evolutionary form of a point transformation *better*?

$$\mathrm{X} = \xi^i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\mu} \qquad \text{vs.} \qquad \hat{\mathrm{X}} = \zeta^\mu[\mathbf{u}]\frac{\partial}{\partial u^\mu}$$

- Evolutionary form generalizes to higher-order transformations (next).
- For a DE system

$$R^\sigma[\mathbf{u}] = 0, \quad \sigma = 1, \dots, N,$$

symmetry components in the evolutionary form $\zeta^\mu[\mathbf{u}] = v^\mu$ are solutions of the linearized equations (Fréchet derivative)

$$\mathrm{L}^\sigma_\rho[u]V^\rho = \left[\frac{\partial R^\sigma[u]}{\partial u^\rho} + \frac{\partial R^\sigma[u]}{\partial u^\rho_i}\mathrm{D}_i + \dots + \frac{\partial R^\sigma[u]}{\partial u^\rho_{i_1\dots i_k}}\mathrm{D}_{i_1}\dots\mathrm{D}_{i_k}\right] v^\rho = 0$$

which relates them to perturbation analysis, cf.

$$(x^i)^\dagger = x^i, \quad (u^\mu)^\dagger = u^\mu + \varepsilon\zeta^\mu[\mathbf{u}] + O(\varepsilon^2),$$

and the first Noether's theorem.

Higher-Order Local Transformations, Evolutionary Form

- Variables $\mathbf{x} = (x^1, ..., x^n)$, $\mathbf{u} = (u^1, ..., u^m)$.
- A one-parameter Lie group of higher-order transformations of order p:

$$(x^i)^\dagger = x^i,$$

$$(u^\mu)^\dagger = u^\mu + \varepsilon \zeta^\mu[\mathbf{u}] + O(\varepsilon^2),$$

where $\zeta^\mu[\mathbf{u}] = \zeta^\mu(\mathbf{x}, \mathbf{u}, \partial \mathbf{u}, \ldots, \partial^p \mathbf{u})$.

- If $p = 1$, and ζ^μ are linear in $\partial \mathbf{u}$, this corresponds to a point transformation.
- Otherwise, it is a higher-order (possibly contact) transformation.
- Generally higher-order transformations cannot be integrated to get a closed-form global expression for the group of transformations.
- Note that higher-order transformations can be sought in the non-evolutionary form

$$(x^i)^* = x^i + \varepsilon \xi^i[\mathbf{u}] + O(\varepsilon^2),$$

$$(u^\mu)^* = u^\mu + \varepsilon \eta^\mu[\mathbf{u}] + O(\varepsilon^2),$$

but then $\xi^i[\mathbf{u}], \eta^\mu[\mathbf{u}]$ are not unique: $\zeta^\mu[\mathbf{u}] = \eta^\mu[\mathbf{u}] - u^\mu_i \xi^i$.

Example: Higher-Order Symmetries of the KdV

- KdV: $\boxed{u_t + uu_x + u_{xxx} = 0.}$ S-integrable, has an infinite symmetry hierarchy.
- Seek symmetries up to order 5 (in x):

```
det_eqs:=gem_symm_det_eqs(
[x,t, U(x,t), diff(U(x,t),x),
diff(U(x,t),x,x), diff(U(x,t),x,x,x),
diff(U(x,t),x,x,x,x), diff(U(x,t),x,x,x,x,x)],
in_evolutionary_form=true );
```

- Which symmetries are higher-order?

```
> gem_output_symm(symm_sol);
```

Maximum number of free constants _C.. or free functions _F.. to be considered : 100

$$\_C5,\ X\_1 = Ux\, \mathrm{D}_U$$

$$\_C2,\ X\_2 = (Ux\, t - 1)\, \mathrm{D}_U$$

$$\_C4,\ X\_3 = (U\, Ux + Uxxx)\, \mathrm{D}_U$$

$$\_C1,\ X\_4 = \left(U\, Ux\, t - \frac{1}{3}\, Ux\, x - \frac{2}{3}\, U + Uxxx\, t\right) \mathrm{D}_U$$

$$\_C3,\ X\_5 = \left(\frac{1}{4}\, U^2\, Ux + Ux\, Uxx + \frac{1}{2}\, U\, Uxxx + \frac{3}{10}\, Uxxxxx\right) \mathrm{D}_U$$

Outline

Nonlocally related PDE systems:

- For a DE system $R^{\sigma}[\mathbf{u}] = 0$, a symmetry is any transformation that maps solutions to solutions.
- Symmetries are not exhausted by Lie groups of point & local symmetries.
- One extension: nonlocal symmetries that can arise as local symmetries of an equivalent nonlocally related PDE system.

Nonlocally related PDE systems:

- Solution set is equivalent to that of the given system.
- One-to-many solution set correspondence ("potential variables").
- Jet spaces are not isomorphic.
- Common examples: potential symmetries (following from a conservation law); subsystems (obtained by exclusion of dependent variables).

Nonlocally Related Systems: an Example

- A nonlinear diffusion equation on $u(x,t)$:

$$U[u] = \boxed{u_t - (L(u))_{xx} = 0}, \qquad L'(u) = K(u).$$

- Two conservation laws:

$$\mathrm{D}_t(u) - \mathrm{D}_x\Big((L(u))_x\Big) = 0, \qquad \mathrm{D}_t(xu) - \mathrm{D}_x\Big(x(L(u))_x - L(u)\Big) = 0.$$

- Potential system $UV[u,v]$: $\begin{cases} v_x = u, \\ v_t = K(u)u_x. \end{cases}$

- Potential system $UA[u,a]$: $\begin{cases} a_x = xu, \\ a_t = xK(u)u_x - L(u). \end{cases}$

- Potential system $UVA[u,v,a]$: $\begin{cases} v_x = u, \\ v_t = K(u)u_x, \\ a_x = xu, \\ a_t = xK(u)u_x - L(u). \end{cases}$

Nonlocally Related Systems: an Example

- Forms of point symmetries of $U[u]$:

$$\mathrm{X} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}.$$

- Point symmetries of $UV[u,v]$:

$$\mathrm{Y} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}.$$

- Point symmetries of $UA[u,a]$:

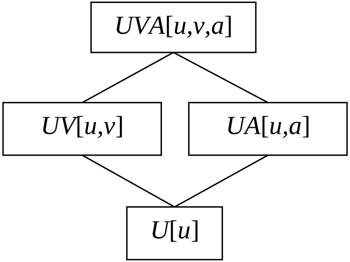
$$\mathrm{Z} = \xi(x,t,u,a)\frac{\partial}{\partial x} + \tau(x,t,u,a)\frac{\partial}{\partial t} + \eta(x,t,u,a)\frac{\partial}{\partial u} + \psi(x,t,u,a)\frac{\partial}{\partial a}.$$

- Point symmetries of $UVA[u,v,a]$:

$$\mathrm{W} = \xi(x,t,u,v,a)\frac{\partial}{\partial x} + \tau(x,t,u,v,a)\frac{\partial}{\partial t}$$
$$+\eta(x,t,u,v,a)\frac{\partial}{\partial u} + \phi(x,t,u,v,a)\frac{\partial}{\partial v} + \psi(x,t,u,v,a)\frac{\partial}{\partial a}.$$

Nonlocally Related Systems: an Example

- Point/local symmetry classifications of the given and potential systems may differ.
- A local symmetry of any PDE system can correspond to a nonlocal symmetry of a another, nonlocally related PDE system within the *tree*:



- There are further ways to obtain nonlocally related PDE systems.

Nonlocally Related Systems: an Example

- Symmetries of $U[u] = u_t - (L(u))_{xx} = 0$ for $L(u) = u^{n+1}/(n+1)$, for all n:

$$\mathrm{X}_1 = \frac{\partial}{\partial x}, \quad \mathrm{X}_2 = \frac{\partial}{\partial t},$$
$$\mathrm{X}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},$$
$$\mathrm{X}_4 = nt\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

- An extra point symmetry of $UVA[u, v, a]$, for $n = -2/3$:

$$\mathrm{W}_9 = (xv - a)\frac{\partial}{\partial x} - 3uv\frac{\partial}{\partial u} - v^2\frac{\partial}{\partial v} - va\frac{\partial}{\partial a}.$$

– a nonlocal symmetry of $U[u]$.

- Nonlocal symmetries arise in classifications for systems with essential arbitrary elements.

- Nonlocal symmetries of a given PDE system are computed as local symmetries of nonlocally related PDE systems.
- Various important examples exist, including linearizations through nonlocal symmetries, and infinite Galas-Bogoyavlenskij symmetries of MHD equations.
- More details, discussion, and examples: see the book *Applications of Symmetry Methods to Partial Differential Equations* (2010) by Bluman, Cheviakov, and Anco.

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