Equivalence Transformations and Their Symbolic Computation

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Outline

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Definitions

Variables:

- Independent: $\mathbf{x} = (x^1, x^2, ..., x^n)$ or $(t, x^1, x^2, ...)$ or $(t, x, y, ...)$.
- Dependent: $\mathbf{u} = (u^1(\mathbf{x}), u^2(\mathbf{x}), ..., u^m(\mathbf{x}))$ or $(u(\mathbf{x}), v(\mathbf{x}), ...)$.

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Partial derivatives:

- Notation:

$$\frac{\partial u^k}{\partial x^i} = u^k_{x^i} = u^k_i = \partial_i u^k.$$

- E.g.,

$$\frac{\partial}{\partial t} u(x, y, t) = u_t = \partial_t u.$$

- All first-order partial derivatives of $\mathbf{u}$: $\partial \mathbf{u}$.
- E.g.,

$$\mathbf{u} = (u^1(x, t), u^2(x, t)), \qquad \partial \mathbf{u} = \{u^1_x,\ u^1_t,\ u^2_x,\ u^2_t\}.$$

Definitions

Higher-order partial derivatives

- Notation: for example,

$$\frac{\partial^2}{\partial x^2} u(x,y,z) = u_{xx} = \partial_x^2 u.$$

- All p^{th}-order partial derivatives:

$$\begin{aligned} \partial^p u &= \left\{ u^\mu_{i_1 \ldots i_p} \;\middle|\; \mu = 1,\ldots,m; \quad i_1,\ldots,i_p = 1,\ldots,n \right\} \\ &= \left\{ \frac{\partial^p u^\mu(x)}{\partial x^{i_1} \ldots \partial x^{i_p}} \;\middle|\; \mu = 1,\ldots,m; i_1,\ldots,i_p = 1,\ldots,n \right\} \end{aligned}$$

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Jet spaces

- We wish to work with differential equations as with algebraic equations.
- Jet space of order p: linear space $J^p(\mathbf{x}|\mathbf{u})$ with coordinates $\mathbf{x}$, $\mathbf{u}$, $\partial\mathbf{u}$, ..., $\partial^p\mathbf{u}$.

Definitions

Differential functions

- A differential function defined on a subset of $J^p(\mathbf{x}|\mathbf{u})$ is an expression that may involve independent and dependent variables, and derivatives of dependent variables to some order $\leq p$.

$$F[\mathbf{u}] = F(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^p\mathbf{u}).$$

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Differential equations

- A system of differential equations (PDE, ODE)of order k:

$$R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$$

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Example:

- The 1D diffusion equation for $u(x,t)$ can be written as

$$0 \;=\; u_t - u_{xx} \;=\; H(u, u_t, u_{xx}) \;=\; H[u],$$

that is, an algebraic equation in $J^2(x, t|u)$.

Definitions

The total derivative of a differential function:

- A basic chain rule for $u = u(x, y)$:

$$\frac{\partial}{\partial x} g(x, y, u, u_x, u_y) = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx} + \frac{\partial g}{\partial u_y} u_{xy}$$

- The total derivative does the same for differential functions on the jet space:

$$\mathrm{D}_x g[u] = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx} + \frac{\partial g}{\partial u_y} u_{xy},$$

where $x, y, u, u_x, u_y, u_{xx}, u_{xy}$ are coordinates in $J^2(x, y|u)$.

General case

- Independent variables: $\mathbf{x} = (x^1, x^2, ..., x^n)$; dependent: $\mathbf{u}(\mathbf{x}) = (u^1, ... u^m)$.
- The total derivative operator with respect to x^i:

$$\mathrm{D}_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{ii_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots ,$$

Outline

1. Notation and Variables
2. Equivalence Transformations
3. Computation of Generalized Equivalence Transformations

Equivalence transformations – basic idea

Given:

- A family $\mathcal{F}_K$ of DEs/systems $\mathbf{R}\{x\,;u;K\}$

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N.$$

- arbitrary elements (constitutive functions and/or parameters):

$$K = (K^1, \ldots, K^L).$$

Equivalence transformations:

An *equivalence transformation* of a DE family $\mathcal{F}_K$ is a change of variables and arbitrary elements $(x, u, K) \to (x^*, u^*, K^*)$ which maps every DE system $\mathbf{R}\{x\,;u;K\} \in \mathcal{F}_K$ into a DE system $\mathbf{R}\{x^*\,;u^*;K^*\} \in \mathcal{F}_K$.

Equivalence transformations – basic idea

Example:

- A family of diffusion equations

$$u_t = c^2(u)u_{xx}.$$

- Arbitrary element: $c = c(u)$.
- Scaling and translation-type equivalence transformations

$$x^* = A_3 x + A_1, \quad t^* = A_4 t + A_2, \quad u^*(x^*, t^*) = A_5 u(x, t), \quad c^*(u^*) = \frac{A_3^2}{A_4} c(u).$$

- Then

$$u^*_{t^*} = (c^*(u^*))^2 u^*_{x^* x^*}.$$

- There are various kinds of equivalence transformations. Generally cannot be systematically computed.

A one-parameter Lie group of point equivalence transformations:

- equivalence transformations of the form

$$
\begin{aligned}
(x^*)^i &= f^i(x,u;\varepsilon) = x^i + \varepsilon\xi^i(x,u) + O(\varepsilon^2), \quad i = 1,\dots,n,\\
(u^*)^\mu &= g^\mu(x,u;\varepsilon) = u^\mu + \varepsilon\eta^\mu(x,u) + O(\varepsilon^2), \quad \mu = 1,\dots,m,\\
(K^*)^\ell &= G^\ell(Q^\ell;\varepsilon) = K^\ell + \varepsilon\kappa^\ell(Q^\ell) + O(\varepsilon^2), \quad \ell = 1,\dots,L,
\end{aligned}
$$

which form a Lie group.

Example:

- A family of diffusion equations $u_t = c^2(u)u_{xx}$.
- Arbitrary element: $c = c(u)$.
- Scaling and translation-type equivalence transformations

$$x^* = A_3 x + A_1, \quad t^* = A_4 t + A_2, \quad u^*(x^*, t^*) = A_5 u(x, t), \quad c^*(u^*) = \frac{A_3^2}{A_4} c(u).$$

- Then $u^*_{t^*} = (c^*(u^*))^2 u^*_{x^*x^*}$.
- A 5-dimensional Lie group form with parameters ε_i, $i = 1, \ldots, 5$:

$$x^* = e^{\varepsilon_3} x + \varepsilon_1, \quad t^* = e^{\varepsilon_4} t + \varepsilon_2, \quad u^*(x^*, t^*) = e^{\varepsilon_5} u(x, t), \quad c^*(u^*) = e^{\varepsilon_3 - \varepsilon_4/2} c(u),$$

- The corresponding infinitesimal generators:

$$\mathrm{X}_1 = \frac{\partial}{\partial x}, \quad \mathrm{X}_2 = \frac{\partial}{\partial t}, \quad \mathrm{X}_3 = x\frac{\partial}{\partial x} + c\frac{\partial}{\partial c}, \quad \mathrm{X}_4 = t\frac{\partial}{\partial t} - \frac{1}{2} c\frac{\partial}{\partial c}, \quad \mathrm{X}_5 = u\frac{\partial}{\partial u}.$$

- The KdV family:

$$u_t + au_x + buu_x + qu_{xxx} = 0,$$

- three constant parameters $a, b, q \in \mathbb{R}$, $b, q \neq 0$.
- A basic set of equivalence transformations of the PDE family (**??**) is given by infinitesimal generators

$$\mathrm{Y}_1 = \frac{\partial}{\partial x}, \quad \mathrm{Y}_2 = \frac{\partial}{\partial t}, \quad \mathrm{Y}_3 = \frac{\partial}{\partial u} - b\frac{\partial}{\partial a}, \quad \mathrm{Y}_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial a},$$

$$\mathrm{Y}_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + 2q\frac{\partial}{\partial q}, \quad \mathrm{Y}_6 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 2a\frac{\partial}{\partial a},$$

$$\mathrm{Y}_7 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2a\frac{\partial}{\partial a} - 2b\frac{\partial}{\partial b}.$$

Example – the KdV family

- The KdV family:

$$u_t + au_x + buu_x + qu_{xxx} = 0,$$

- three constant parameters $a, b, q \in \mathbb{R}$, $b, q \neq 0$.
- The corresponding point transformations:

$$x^* = \frac{A_5}{A_6 A_7}(x - A_4 t) + A_1, \qquad t^* = \frac{A_5}{A_6^3 A_7^3} t + A_2,$$

$$u^*(x^*, t^*) = A_6^2 \left(u(x,t) + A_3\right),$$

$$a^* = A_6^2 A_7^2 \left(a - A_3 b - A_4\right), \qquad b^* = A_7^2\, b, \qquad q^* = A_5^2\, q.$$

- Discrete, $u \to -u$, $b \to -b$: $A_5 = A_6 = i$, $A_7 = 1$.
- Another one, $b \to -b$, $q \to -q$: $A_5 = A_7 = i$, $A_6 = 1$.
- WLOG $b, q > 0$.

Example – the KdV family

- The KdV family:

$$u_t + au_x + buu_x + qu_{xxx} = 0,$$

- three constant parameters $a, b, q \in \mathbb{R}$, $b, q \neq 0$.
- Choices

$$A_1 = A_2 = A_3 = 0, \quad A_4 = a, \quad A_5 = A_6 = A_7 = 1,$$
$$x^* = x - at, \quad t^* = t, \quad u^*(x^*, t^*) = u(x, t)$$

and

$$A_1 = A_2 = A_4 = 0, \quad A_3 = a/b, \quad A_5 = A_6 = A_7 = 1,$$
$$x^* = x, \quad t^* = t, u^*(x^*, t^*) = u(x, t) + A_3$$

yield

$$a^* \propto a - A_3 b - A_4 = 0$$

and a reduced PDE class

$$u_t + buu_x + qu_{xxx} = 0.$$

Example – the KdV family

- The KdV family:

$$u_t + au_x + buu_x + qu_{xxx} = 0,$$

- three constant parameters $a, b, q \in \mathbb{R}$, $b, q \neq 0$.

- A further transformation with

$$A_1 = A_2 = A_3 = A_4 = 0, \quad A_5 = q^{-1/2}, \quad A_6 = 1, \quad A_7 = b^{-1/2}$$

maps

$$u_t + buu_x + qu_{xxx} = 0.$$

into the *standard KdV form* (no variable coefficients)

$$u_t + uu_x + u_{xxx} = 0.$$

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

A Lie group of point equivalence transformations:

$$\begin{aligned}
(x^*)^i &= f^i(x, u; \varepsilon) = x^i + \varepsilon \xi^i(x, u) + O(\varepsilon^2), \quad i = 1, \ldots, n, \\
(u^*)^\mu &= g^\mu(x, u; \varepsilon) = u^\mu + \varepsilon \eta^\mu(x, u) + O(\varepsilon^2), \quad \mu = 1, \ldots, m, \\
(K^*)^\ell &= G^\ell(Q^\ell; \varepsilon) = K^\ell + \varepsilon \kappa^\ell(Q^\ell) + O(\varepsilon^2), \quad \ell = 1, \ldots, L.
\end{aligned}$$

- A point symmetry of a DE system $\mathbf{R}\{x\,;u\} \in \mathcal{F}_K$ is an equivalence transformation of the family $\mathcal{F}_K$ if it is point symmetry for all systems in $\mathcal{F}_K$.

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- A point symmetry of a DE system $\mathbf{R}\{x\,;u\} \in \mathcal{F}_K$ is an equivalence transformation of the family $\mathcal{F}_K$ if it is point symmetry for all systems in $\mathcal{F}_K$.
- An equivalence transformation of the DE family $\mathcal{F}_K$ is point symmetry of its every member if and only if it does not involve components corresponding to the arbitrary elements of the family.

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \dots, \partial^k u, K) = 0, \quad \sigma = 1, \dots, N$$

with arbitrary elements $K = (K^1, \dots, K^L)$.

Generalized equivalence transformations

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

Non-Lie-type equivalence transformations:

$$\begin{aligned} (x^*)^i &= f^i[x, u, K], \quad i = 1, \ldots, n, \\ (u^*)^\mu &= g^\mu[x, u, K], \quad \mu = 1, \ldots, m, \\ (K^*)^\ell &= G^\ell[x, u, K], \quad \ell = 1, \ldots, L, \end{aligned}$$

Generalized equivalence transformations

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

"Generalized equivalence transformations":

Lie groups given by extended generators

$$\mathrm{X} = \xi^i(x, u, K)\frac{\partial}{\partial x^i} + \eta^\mu(x, u, K)\frac{\partial}{\partial u^\mu} + \theta^\nu(x, u, K)\frac{\partial}{\partial K^\nu}.$$

- Examples computed, e.g., in Popovych *et al* (2004) for a class of nonlinear (1+1)-dimensional Schrödinger equations with power nonlinearity

$$i\psi_t + \psi_{xx} + |\psi|^\gamma\psi + V(x,t)\psi = 0.$$

- Generalized equivalence transformations can often be computed as Lie point symmetries when arbitrary elements are treated as dependent variables.

A given DE family $\mathcal{F}_K$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N$$

with arbitrary elements $K = (K^1, \ldots, K^L)$.

- Further generalizations exist, including discrete and nonlocal equivalence transformations.
- For an overview of results and types of extended equivalence transformations the following sources and references therein.

Lisle, I. (1992).
Equivalence Transformations for Classes of Differential Equations. Ph.D. thesis, University of British Columbia.

Cheviakov, A. (2017).
Symbolic computation of equivalence transformations and parameter reduction for nonlinear physical models. *Computer Physics Communications*, 220, 56–73.

Outline

- Is it possible to compute all equivalence transformations of a given DE family?

Computation of equivalence transformations

- As usual, there is no single recipe... Every example must be understood in detail.
- Yet it is possible to systematically seek Lie groups of generalized equivalence transformations.
- Often can use Maple/GeM pair.

Given:

- A family $\mathcal{F}_K$ of DEs/systems $\mathbf{R}\{x\,; u; K\}$

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N.$$

- arbitrary elements (constitutive functions and/or parameters):

$$K = (K^1, \ldots, K^L).$$

Symbolic computation of equivalence transformations

1. Replace the constitutive functions and/or parameters $K = (K^1, \ldots, K^L)$ by new dependent variables $(K^1(x), \ldots, K^L(x))$. Thus consider a new DE system $\widetilde{\mathbf{R}}\{x\,; u, K\}$ with $m + L$ dependent variables and no arbitrary elements.

2. Seek point symmetries of $\widetilde{\mathbf{R}}\{x\,; u, K\}$, with infinitesimal generators

$$\mathrm{X} = \xi^i(x, u, K)\frac{\partial}{\partial x^i} + \eta^\mu(x, u, K)\frac{\partial}{\partial u^\mu} + \theta^\lambda(x, u, K)\frac{\partial}{\partial K^\lambda}.$$

3. Obtain the *split system of determining equations* for $\widetilde{\mathbf{R}}\{x\,; u, K\}$

$$\mathrm{X}^{(k)}\widetilde{R}^\alpha\Big|_{\widetilde{R}^\sigma=0,\ \sigma=1,\ldots,N} = 0.$$

4. If the arbitrary elements K of the original DE family contained arbitrary functions, introduce restrictions of the form

$$\frac{\partial \xi^i(x,u,K)}{\partial K^\gamma} = 0, \quad \frac{\partial \eta^\mu(x,u,K)}{\partial K^\delta} = 0, \quad \frac{\partial \theta^\lambda(x,u,K)}{\partial x^j} = 0,$$

as appropriate, to exclude the dependence of transformation components of the arbitrary elements on variables they do not depend on. For example, for the DE family

$$u_t = c^2(u) u_{xx},$$

the infinitesimal generator of the generalized equivalence transformations has the form

$$\mathrm{X} = \xi(x,t,u,c)\frac{\partial}{\partial x} + \tau(x,t,u,c)\frac{\partial}{\partial t} + \eta(x,t,u,c)\frac{\partial}{\partial u} + \theta(x,t,u,c)\frac{\partial}{\partial c},$$

and the transformation for $c(u)$ must not explicitly depend on the variables x, t. Therefore the restrictions on the component θ are given by

$$\frac{\partial \theta(x,t,u,c)}{\partial x} = \frac{\partial \theta(x,t,u,c)}{\partial t} = 0.$$

5. In order to simplify computations, additional restrictions may be introduced at this stage, for example,

$$\frac{\partial \xi^i(x,u,K)}{\partial K^\gamma} = 0, \quad i = 1,\ldots,n, \quad \gamma = 1,\ldots,L,$$

   if the transformations for the independent variables are assumed to be independent of the arbitrary elements.

6. Append all restrictions, as linear PDEs, to the split system of determining equations.

7. Simplify and solve the augmented split system of determining equations, to find the infinitesimal generators of the equivalence transformations.

8. Integrate to obtain the global group. For each infinitesimal generator, the corresponding one-parameter Lie group of equivalence transformations is found through the solution of the initial-value problem

$$\begin{aligned}
&\frac{d}{d\varepsilon}(x^*)^i = \xi^i(x^*, u^*, K^*), \quad i = 1, \ldots, n,\\
&\frac{d}{d\varepsilon}(u^*)^\mu = \eta^\mu(x^*, u^*, K^*), \quad \mu = 1, \ldots, m,\\
&\frac{d}{d\varepsilon}(K^*)^\lambda = \theta^\lambda(x^*, u^*, K^*), \quad \lambda = 1, \ldots, L,\\
&(x^*)^i|_{\varepsilon=0} = x^i, \quad (u^*)^\mu|_{\varepsilon=0} = u^\mu, \quad (K^*)^\lambda|_{\varepsilon=0} = K^\lambda,
\end{aligned}$$

where ε is the group parameter.

Restrictions for generalized equivalence transformations in Maple

- Generate restrictions in Maple/GeM:

```
restriction_eqs:=gem_generate_EquivTr_dependence([
     [[<variables1>],[<dep1>]],
     [[<variables2>],[<dep2>]],
     ...,
]);
```

- The KdV family:

$$u_t + au_x + buu_x + qu_{xxx} = 0.$$

- A basic set of equivalence transformations of the PDE family (**??**) is given by infinitesimal generators

$$\mathrm{Y}_1 = \frac{\partial}{\partial x}, \quad \mathrm{Y}_2 = \frac{\partial}{\partial t}, \quad \mathrm{Y}_3 = \frac{\partial}{\partial u} - b\frac{\partial}{\partial a}, \quad \mathrm{Y}_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial a},$$

$$\mathrm{Y}_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + 2q\frac{\partial}{\partial q}, \quad \mathrm{Y}_6 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 2a\frac{\partial}{\partial a},$$

$$\mathrm{Y}_7 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2a\frac{\partial}{\partial a} - 2b\frac{\partial}{\partial b}.$$

Potential equivalence transformations

- Nonlinear wave equations $u_{tt} = (c^2(u)u_x)_x$:

- One can show that the infinitesimal generators of the group of point equivalence transformations of the above family are given by

$$\mathrm{Z}_1 = \frac{\partial}{\partial t}, \quad \mathrm{Z}_2 = \frac{\partial}{\partial x}, \quad \mathrm{Z}_3 = \frac{\partial}{\partial u}, \quad \mathrm{Z}_4 = t\frac{\partial}{\partial u},$$

$$\mathrm{Z}_5 = u\frac{\partial}{\partial u}, \quad \mathrm{Z}_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad \mathrm{Z}_7 = t\frac{\partial}{\partial t} - c\frac{\partial}{\partial c},$$

  and the global group has the form

$$x^* = a_6 x + a_2, \quad t^* = a_6 a_7 t + a_1, \quad u^* = a_5 u + a_4 t + a_3, \quad c^*(u^*) = a_7^{-1} c(u).$$

  where $a_1, \dots, a_7$ are arbitrary constants with $a_5\, a_6\, a_7 \neq 0$.

- Nonlinear wave equations $u_{tt} = (c^2(u)u_x)_x$.
- A conservation law

$$\frac{\partial}{\partial t}(tu_t - u) - \frac{\partial}{\partial x}(tc^2(u)u_x) = 0,$$

- A potential system:

$$w_x = tu_t - u, \quad w_t = tc^2(u)u_x; \quad u = u(x,t), \quad w = w(x,t).$$

- Equivalence transformations:

$$\mathrm{W}_1 = \frac{\partial}{\partial w}, \quad \mathrm{W}_2 = \frac{\partial}{\partial x}, \quad \mathrm{W}_3 = \mathrm{Z}_3 - x\frac{\partial}{\partial w}, \quad \mathrm{W}_4 = t\frac{\partial}{\partial u},$$

$$\mathrm{W}_5 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x},$$

$$\mathrm{W}_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + w\frac{\partial}{\partial w}, \quad \mathrm{W}_7 = t\frac{\partial}{\partial t} - c\frac{\partial}{\partial c},$$

$$\mathrm{W}_8 = tu\frac{\partial}{\partial t} + w\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u} - 2uc\frac{\partial}{\partial x}.$$

- A potential system:

$$w_x = tu_t - u, \quad w_t = tc^2(u)u_x; \quad u = u(x,t), \quad w = w(x,t).$$

- Equivalence transformations – global group – part 1:

$$\begin{aligned} x^* &= A_5^{-1}A_6 x + A_2, \quad t^* = A_5^{-1}A_6 A_7 t, \quad u^* = A_5 u + A_4 t + A_3, \\ w^* &= A_6 w - A_3 A_5^{-1} A_6 x + A_1, \quad c^*(u^*) = A_7^{-1} c(u) \end{aligned}$$

- Equivalence transformations – global group – part 2:

$$x^* = x - Bw, \qquad t^* = \frac{t}{1+Bu}, \qquad u^* = \frac{u}{1+Bu},$$

$$w^* = w, \qquad c^*(u^*) = (1+Bu)^2\, c(u)$$

- These are nonlocal "projective-type" transformations of the nonlinear wave equation family.

Generalized equivalence transformations: Fiber-reinforced hyperelastic waves model

- A family of nonlinear wave equations

$$G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\,G_x + 2\sin^2\gamma\right)\right) G_{xx},$$

where $G(x,t)$ is the finite displacement amplitude of anti-plane shear motions, in the material z-direction, of a nonlinear incompressible hyperelasic medium, reinforced by fibers making a constant angle γ with the material direction x. The model involves three arbitrary elements, the material parameters $\alpha > 0$, $\beta > 0$, and the fiber angle $\gamma \in [0, \pi/2]$.

- The equivalence transformations look quite complicated. Denote:

$$k = \tan\gamma, \quad \sin\gamma = SG(k) = \frac{K}{\sqrt{K^2+1}}, \quad \cos\gamma = CG(k) = \frac{1}{\sqrt{K^2+1}}.$$

Generalized equivalence transformations: Fiber-reinforced hyperelastic waves model

- A family of nonlinear wave equations

$$G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.$$

- The equivalence transformation generators:

$$\mathrm{X}_1 = \frac{\partial}{\partial G}, \quad \mathrm{X}_2 = \frac{\partial}{\partial x}, \quad \mathrm{X}_3 = \frac{\partial}{\partial t}, \quad \mathrm{X}_4 = t\,\frac{\partial}{\partial G},$$
$$\mathrm{X}_5 = t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} + G\,\frac{\partial}{\partial G}, \quad \mathrm{X}_6 = -\frac{1}{2}\,t\,\frac{\partial}{\partial t} + a\,\frac{\partial}{\partial a} + b\,\frac{\partial}{\partial b},$$
$$\mathrm{X}_7 = -x\,\frac{\partial}{\partial x} - 2a\,\frac{\partial}{\partial a} - \frac{4b}{k^2+1}\,\frac{\partial}{\partial b} + k\,\frac{\partial}{\partial k},$$
$$\mathrm{X}_8 = -x\,\frac{\partial}{\partial G} + \frac{2bk}{(k^2+1)^2}\,\frac{\partial}{\partial a} + \frac{4bk}{k^2+1}\,\frac{\partial}{\partial b} + \frac{\partial}{\partial k}.$$

Generalized equivalence transformations: Fiber-reinforced hyperelastic waves model

- A family of nonlinear wave equations

$$G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.$$

- Equivalence transformation corresponding to X_8:

$$G^* = G - Sx, \qquad \tan\gamma^* = \tan\gamma + S,$$

$$\alpha^* = \alpha + 2\beta\cos^4\gamma\left(\frac{S^2}{2} + S\tan\gamma\right),$$

$$\beta^* = \beta\cos^4\gamma\left(\tan^2\gamma + 2S\tan\gamma + S^2 + 1\right)^2.$$

- The equations therefore can be mapped into the $\gamma = 0$ case:

$$G^*_{t^*t^*} = (\alpha^* + 3\beta^*(G^*_{x^*})^2)\, G^*_{x^*x^*}.$$

- Equilibrium fluid flow; stationary plasma equilibria in 3D:

$$\operatorname{div} \mathbf{B} = 0, \quad (\operatorname{curl} \mathbf{B}) \times \mathbf{B} = \operatorname{grad} P,$$

$$\mathbf{B} = B^1 \mathbf{e}_x + B^2 \mathbf{e}_y + B^3 \mathbf{e}_z.$$

- Axially symmetric case: four PDEs $\to$ one PDE:

$$\boxed{\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi).}$$

- The magnetic field and pressure are given by

$$\mathbf{B} = \frac{\psi_z}{r}\mathbf{e}_r + \frac{I(\psi)}{r}\mathbf{e}_\phi - \frac{\psi_r}{r}\mathbf{e}_z, \quad P = P(\psi).$$

- $I(\psi)$ and $P(\psi)$ are arbitrary constitutive functions.

- To compute in Maple: call $I(\psi)I'(\psi) = Q(\psi)$, $P'(\psi) = \tilde{P}(\psi)$.

Bragg-Hawthorne-Grad-Rubin-Shafranov model

- Bragg-Hawthorne-Grad-Rubin-Shafranov equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi).$$

- Equivalence transformations are given by

$$\tilde{r} = c_2 c_3^{-1} r, \quad \tilde{z} = c_2 c_3^{-1} z + c_1,$$

$$\tilde{\psi} = c_2^4 c_3^{-2} \psi,$$

$$\tilde{P}'(\tilde{\psi}) = c_3^2 P'(\psi),$$

$$\tilde{I}(\tilde{\psi})\tilde{I}'(\tilde{\psi}) = c_2^2 I(\psi)I'(\psi),$$

the pressure translation

$$\tilde{P}(\psi) = P(\psi) + c_4,$$

as well as the well-known transformation

$$\tilde{I}(\psi) = \pm\sqrt{I^2(\psi) + c_5},$$

where $c_1, \ldots, c_5$ are arbitrary constants, $c_2 c_3 \neq 0$.

Some references

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