An extended procedure for finding exact solutions of PDEs arising from potential symmetries

Alexei F. Cheviakov

University of Saskatchewan

Talk plan

- Local conservation laws of PDE systems
- Nonlocally related PDE systems
  - Potential systems, Subsystems
  - Trees of nonlocally related systems
- Example: Planar Gas Dynamics (PGD) equations
  - Conservation laws; Tree of nonlocally related systems
  - Nonlocal (potential) symmetries
- Construction of exact solutions from potential symmetries
  - Standard algorithm
  - Three refinements
  - New exact solutions for PGD equations

(I) Conservation laws and nonlocally related PDE systems

Local conservation laws

A PDE system: $$R^i[u] \equiv R^i(x, u, \partial u, ..., \partial^{(k)} u) = 0, \quad i = 1, ..., N;$$

$$x = (x^1, ..., x^n), \quad u = u(x) = (u^1, ..., u^m).$$

A conservation law: $$D_i \Phi^i[u] \equiv D_{x^1} \Phi^1[u] + ... + D_{x^n} \Phi^n[u] = 0.$$

Time-dependent systems: $$D_t \Psi[u] + D_{x^2} \Phi^2[u] ... + D_{x^n} \Phi^n[u] = 0.$$

For any physical PDE system (in the solved form), look for ***multipliers***

that yield conservation laws: $$\Lambda_\sigma[u] R^\sigma[u] \equiv \mathrm{D}_i \Phi^i[u] = 0.$$

Example: wave equation $\mathbf{U}\{x, t\,; u\}: \;\; u_{tt} = c^2(x)u_{xx}$

Conservation law: $\frac{\partial}{\partial t}(c^{-2}(x)u_t) - \frac{\partial}{\partial x}(u_x) = 0$

Potential equations:

$$\begin{cases} v_x = c^{-2}(x)u_t, \\ v_t = u_x. \end{cases}$$

Potential system: potential equations plus remaining equations

$$\mathbf{UV}\{x, t\,; u, v\}: \;\; \begin{cases} v_x = c^{-2}(x)u_t, \\ v_t = u_x. \end{cases}$$

Solution set: equivalent to that of the given system.

Nonlocally related *subsystems*:
exclude dependent variables using differential relations.

Given:

$$\mathbf{UV}\{x,t\,;u,v\}: \quad \begin{cases} v_x = c^{-2}(x)u_t, \\ v_t = u_x. \end{cases}$$

Nonlocally related subsystems:

$$\mathbf{U}\{x,t\,;u\}: \; u_{tt} = c^2(x)u_{xx}$$

$$\mathbf{V}\{x,t\,;v\}: \; v_{tt} = (c^2(x)v_x)_x$$

Construction of the tree of nonlocally related systems:

[*Bluman & Cheviakov, JMP 46 (2005);*
Bluman, Cheviakov & Ivanova, JMP 47 (2006)]

1. For a given PDE system, construct **local conservation laws**.
2. Construct **potential systems**.
   (include ones with pairs, triplets, quadruplets of potentials,…)
3. Construct **nonlocally related subsystems**.
4. Find further conservation laws.
5. Continue.

(II) Nonlocally related PDE systems of Planar Gas Dynamics

Lagrange PDE system of planar gas dynamics:

$$\mathbf{L}\{y, s\,; v, p, q\} : \quad \begin{cases} q_s - v_y = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q) v_y = 0. \end{cases}$$

Lagrangian coordinates (initial positions) of fluid particles: *y*

Time: *s*

Velocity: v

Density: $\rho = 1/q$

Local conservation laws: assume $\Lambda_i = \Lambda_i(y, s, V, P, Q)$

Conservation laws and potential systems

Multipliers $(\Lambda_1, \Lambda_2, \Lambda_3)$	Conservation law	Potential variable	Potential equations
$(1,0,0)$	$D_s(q) - D_y(v) = 0$	w^1	$w^1_y = q,\ w^1_s = v$
$(0,1,0)$	$D_s(v) + D_y(p) = 0$	w^2	$w^2_y = v,\ w^2_s = -p$
$(y,s,0)$	$D_s(sv + yq) + D_y(sp - yv) = 0$	w^3	$w^3_y = vs + qy,\ w^3_s = -sp + vy$
$(S_Q(P,Q), 0, S_P(P,Q))$	$D_s(S(p,q)) = 0$	w^4	$w^4_y = S(p,q),\ w^4_s = 0$
$(K_Q(P,Q), V, K_P(P,Q))$ $K_q(p,q) = B(p,q)K_p(p,q) - p$	$D_s\left(\frac{v^2}{2} + K(p,q)\right) + D_y(pv) = 0$	w^5	$w^5_y = v^2/2 + K(p,q),\ w^5_s = -pv$

$$\mathbf{L}\{y, s\,; v, p, q\} : \quad \begin{cases} q_s - v_y = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0. \end{cases}$$

Conservation laws and potential systems

Multipliers $(\Lambda_1, \Lambda_2, \Lambda_3)$	Conservation law	Potential variable	Potential equations
$(1, 0, 0)$	$D_s(q) - D_y(v) = 0$	w^1	$w^1_y = q,\ w^1_s = v$
$(0, 1, 0)$	$D_s(v) + D_y(p) = 0$	w^2	$w^2_y = v,\ w^2_s = -p$
$(y, s, 0)$	$D_s(sv + yq) + D_y(sp - yv) = 0$	w^3	$w^3_y = vs + qy,\ w^3_s = -sp + vy$
$(S_Q(P,Q), 0, S_P(P,Q))$	$D_s(S(p,q)) = 0$	w^4	$w^4_y = S(p,q),\ w^4_s = 0$
$(K_Q(P,Q), V, K_P(P,Q))$ $K_q(p,q) = B(p,q)K_p(p,q) - p$	$D_s\left(\frac{v^2}{2} + K(p,q)\right) + D_y(pv) = 0$	w^5	$w^5_y = v^2/2 + K(p,q),\ w^5_s = -pv$

$$\mathbf{LW}^1\{y, s\,; v, p, q, w^1\} : \quad \begin{cases} w^1_y = q, \\ w^1_s = v, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0; \end{cases}$$

Euler system

$$\mathbf{LW}^1\{y,s\,;v,p,q,w^1\}: \quad \begin{cases} w^1_y = q, \\ w^1_s = v, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0; \end{cases}$$

Change of variables.

- Dependent: $\alpha^1, v, p, \rho = 1/q$

- Independent: $x = w^1, t = s$

$$\Leftrightarrow \quad \mathbf{EA}^1\{x,t\,;v,p,\rho,\alpha^1\}: \quad \begin{cases} \alpha^1_x - \rho = 0, \\ \alpha^1_t + \rho v = 0, \\ \rho(v_t + v v_x) + p_x = 0, \\ \rho(p_t + v p_x) + B(p,1/\rho)v_x = 0. \end{cases}$$

Euler system

$$\mathbf{EA}^1\{x,t\,;v,p,\rho,\alpha^1\}: \quad \begin{cases} \alpha^1_x - \rho = 0, \\ \alpha^1_t + \rho v = 0, \\ \rho(v_t + v v_x) + p_x = 0, \\ \rho(p_t + v p_x) + B(p, 1/\rho) v_x = 0. \end{cases}$$

Exclude α^1 => nonlocally related subsystem **(Euler system)**

$$\mathbf{E}\{x,t\,;v,p,\rho\}: \quad \begin{cases} \rho_t + (\rho v)_x = 0, \\ \rho(v_t + v v_x) + p_x = 0, \\ \rho(p_t + v p_x) + B(p, 1/\rho) v_x = 0. \end{cases}$$

Conservation laws and potential systems

Multipliers $(\Lambda_1, \Lambda_2, \Lambda_3)$	Conservation law	Potential variable	Potential equations
$(1,0,0)$	$D_s(q) - D_y(v) = 0$	w^1	$w^1_y = q, \; w^1_s = v$
$(0,1,0)$	$D_s(v) + D_y(p) = 0$	w^2	$w^2_y = v, \; w^2_s = -p$
$(y,s,0)$	$D_s(sv + yq) + D_y(sp - yv) = 0$	w^3	$w^3_y = vs + qy, \; w^3_s = -sp + vy$
$(S_Q(P,Q), 0, S_P(P,Q))$	$D_s(S(p,q)) = 0$	w^4	$w^4_y = S(p,q), \; w^4_s = 0$
$(K_Q(P,Q), V, K_P(P,Q))$ $K_q(p,q) = B(p,q)K_p(p,q) - p$	$D_s\left(\frac{v^2}{2} + K(p,q)\right) + D_y(pv) = 0$	w^5	$w^5_y = v^2/2 + K(p,q), \; w^5_s = -pv$

$$\mathbf{LW}^2\{y, s\,; v, p, q, w^2\} : \quad \begin{cases} q_s - v_y = 0, \\ w^2_y = v, \\ w^2_s = -p, \\ p_s + B(p,q)v_y = 0; \end{cases}$$

Conservation laws and potential systems

Multipliers $(\Lambda_1, \Lambda_2, \Lambda_3)$	Conservation law	Potential variable	Potential equations
$(1,0,0)$	$D_s(q) - D_y(v) = 0$	w^1	$w^1_y = q,\ w^1_s = v$
$(0,1,0)$	$D_s(v) + D_y(p) = 0$	w^2	$w^2_y = v,\ w^2_s = -p$
$(y,s,0)$	$D_s(sv+yq) + D_y(sp-yv) = 0$	w^3	$w^3_y = vs+qy,\ w^3_s = -sp+vy$
$(S_Q(P,Q), 0, S_P(P,Q))$	$D_s(S(p,q)) = 0$	w^4	$w^4_y = S(p,q),\ w^4_s = 0$
$(K_Q(P,Q), V, K_P(P,Q))$ $K_q(p,q) = B(p,q)K_p(p,q) - p$	$D_s\left(\frac{v^2}{2} + K(p,q)\right) + D_y(pv) = 0$	w^5	$w^5_y = v^2/2 + K(p,q),\ w^5_s = -pv$

$$\mathbf{LW}^3\{y, s\,; v, p, q, w^3\} : \quad \begin{cases} w^3_y = sv + yq, \\ w^3_s = -sp + yv, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0; \end{cases}$$

Conservation laws and potential systems

Multipliers $(\Lambda_1, \Lambda_2, \Lambda_3)$	Conservation law	Potential variable	Potential equations
$(1,0,0)$	$D_s(q) - D_y(v) = 0$	w^1	$w^1_y = q,\ w^1_s = v$
$(0,1,0)$	$D_s(v) + D_y(p) = 0$	w^2	$w^2_y = v,\ w^2_s = -p$
$(y,s,0)$	$D_s(sv + yq) + D_y(sp - yv) = 0$	w^3	$w^3_y = vs + qy,\ w^3_s = -sp + vy$
$(S_Q(P,Q), 0, S_P(P,Q))$	$D_s(S(p,q)) = 0$	w^4	$w^4_y = S(p,q),\ w^4_s = 0$
$(K_Q(P,Q), V, K_P(P,Q))$ $K_q(p,q) = B(p,q)K_p(p,q) - p$	$D_s\left(\frac{v^2}{2} + K(p,q)\right) + D_y(pv) = 0$	w^5	$w^5_y = v^2/2 + K(p,q),\ w^5_s = -pv$

$$\mathbf{LW}^4\{y, s\,; v, p, q, w^4\} \quad \begin{cases} w^4_y = S(p,q), \\ w^4_s = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0; \end{cases}$$

Conservation laws and potential systems

Multipliers $(\Lambda_1, \Lambda_2, \Lambda_3)$	Conservation law	Potential variable	Potential equations
$(1,0,0)$	$D_s(q) - D_y(v) = 0$	w^1	$w^1_y = q,\ w^1_s = v$
$(0,1,0)$	$D_s(v) + D_y(p) = 0$	w^2	$w^2_y = v,\ w^2_s = -p$
$(y,s,0)$	$D_s(sv + yq) + D_y(sp - yv) = 0$	w^3	$w^3_y = vs + qy,\ w^3_s = -sp + vy$
$(S_Q(P,Q), 0, S_P(P,Q))$	$D_s(S(p,q)) = 0$	w^4	$w^4_y = S(p,q),\ w^4_s = 0$
$(K_Q(P,Q), V, K_P(P,Q))$ $K_q(p,q) = B(p,q)K_p(p,q) - p$	$D_s\left(\frac{v^2}{2} + K(p,q)\right) + D_y(pv) = 0$	w^5	$w^5_y = v^2/2 + K(p,q),\ w^5_s = -pv$

$$\mathbf{LW}^5\{y, s\,; v, p, q, w^5\} \quad \left\{ \begin{array}{l} w^5_y = \dfrac{v^2}{2} + K(p,q), \\ w^5_s = -pv, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0; \end{array} \right.$$

Other nonlocally related subsystems

$$\mathbf{L}\{y, s\,; v, p, q\} : \quad \begin{cases} q_s - v_y = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q) v_y = 0. \end{cases}$$

$\Downarrow$ Exclude v

$$\underline{\mathbf{L}}\{y, s\,; p, q\} : \quad \begin{cases} q_{ss} + p_{yy} = 0, \\ p_s + B(p,q) q_s = 0. \end{cases}$$

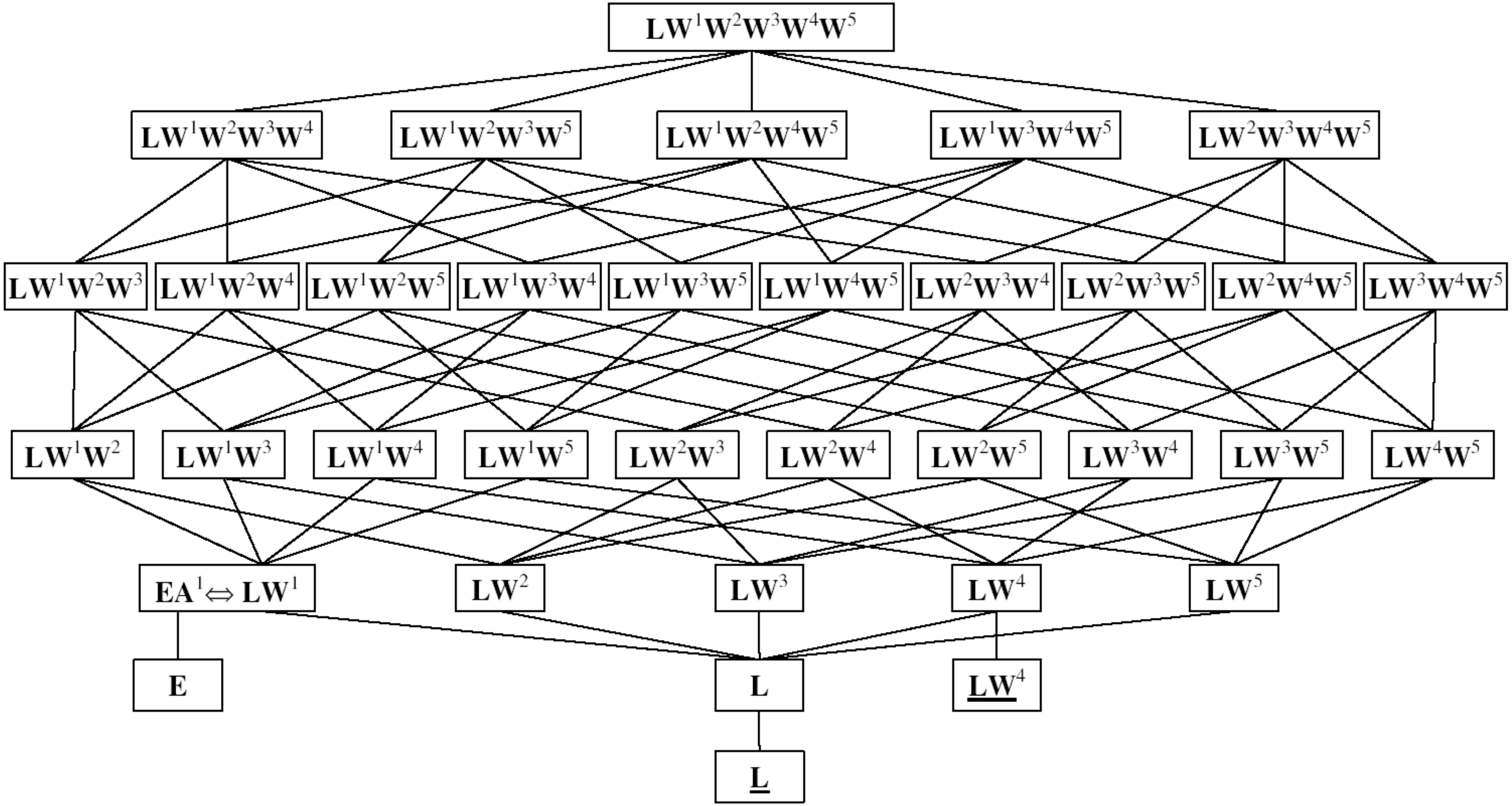
Other nonlocally related subsystems

$$\mathbf{LW}^4\{y,s\,;v,p,q,w^4\}\quad \begin{cases} w^4_y = S(p,q), \\ w^4_s = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0; \end{cases}$$

Exclude v

$$\underline{\mathbf{LW^4}}\{y,s\,;p,q,w^4\}:\quad \begin{cases} q_{ss} + p_{yy} = 0, \\ w^4_y = S(p,q), \\ w^4_s = 0, \\ p_s + B(p,q)q_s = 0, \\ S_q(p,q) = B(p,q)S_p(p,q). \end{cases}$$

Tree for the Lagrange PGD system



(III) Nonlocal symmetries for Planar Gas Dynamics

Nonlocal symmetries

Given system: $\mathbf{R}\{x,t\,;u\}$

Potential system: $\mathbf{RV}\{x,t\,;u,v\}$

A symmetry of $\mathbf{RV}\{x,t\,;u,v\}$

$$\mathrm{X}=\xi(x,t,u,v)\frac{\partial}{\partial x}+\tau(x,t,u,v)\frac{\partial}{\partial t}+\eta^{\sigma}(x,t,u,v)\frac{\partial}{\partial u^{\sigma}}+\zeta^{\mu}(x,t,u,v)\frac{\partial}{\partial v^{\mu}},$$

is a **nonlocal symmetry** of $\mathbf{R}\{x,t\,;u\}$,

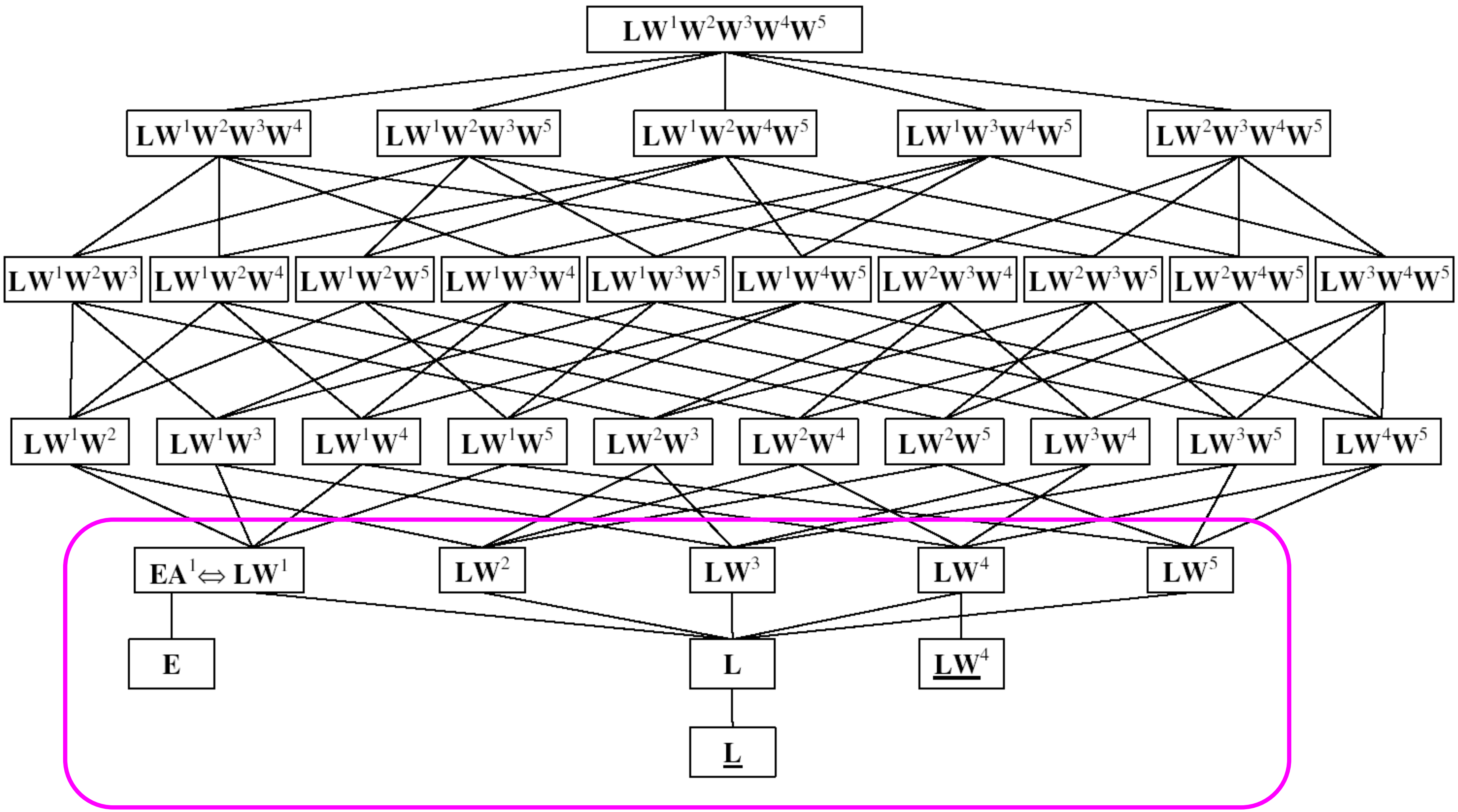
if one or more of $\xi(x,t,u,v),\tau(x,t,u,v),\eta^{\sigma}(x,t,u,v)$

depend on nonlocal variables.

Seek nonlocal symmetries of the Lagrange system $\mathbf{L}\{y,s\,;v,p,q\}$

in the polytropic case $B(p,q)=\gamma p/q,\quad \gamma=\text{const.}$

Tree for the Lagrange PGD system



Nonlocal symmetries of the Lagrange PGD system

γ	Admitted point symmetries		
	$\mathbf{E}\{x, t\,; v, p, \rho\}$	$\mathbf{L}\{y, s\,; v, p, q\}$	$\underline{\mathbf{L}}\{y, s\,; p, q\}$
Arbitrary	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$, $X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$, $X_5 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} - \rho\frac{\partial}{\partial \rho}$, $X_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}$.	$Z_1 = \frac{\partial}{\partial s}$, $Z_2 = y\frac{\partial}{\partial y} + s\frac{\partial}{\partial s}$, $Z_3 = \frac{\partial}{\partial v}$, $Z_4 = v\frac{\partial}{\partial v} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$, $Z_5 = y\frac{\partial}{\partial y} + p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}$, $Z_6 = \frac{\partial}{\partial y}$.	$\widehat{Z}_1 = Z_1$, $\widehat{Z}_2 = Z_2$, $\widehat{Z}_3 = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}$, $\widehat{Z}_4 = Z_5$, $\widehat{Z}_5 = Z_6$, $\widehat{Z}_6 = y^2\frac{\partial}{\partial y} + yp\frac{\partial}{\partial p} - 3yq\frac{\partial}{\partial q}$.
3	$X_1, X_2, X_3, X_4, X_5, X_6$, $X_7 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - vt)\frac{\partial}{\partial v}$ $- 3tp\frac{\partial}{\partial p} - t\rho\frac{\partial}{\partial \rho}$.	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$.	$\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3, \widehat{Z}_4, \widehat{Z}_5, \widehat{Z}_6$, $\widehat{Z}_7 = s^2\frac{\partial}{\partial s} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q}$.
-1	$X_1, X_2, X_3, X_4, X_5, X_6$.	$Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$, $Z_7 = \frac{\partial}{\partial p} + \frac{q}{p}\frac{\partial}{\partial q}$, $Z_8 = -s\frac{\partial}{\partial v} + y\frac{\partial}{\partial p} + \frac{yq}{p}\frac{\partial}{\partial q}$.	$\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3, \widehat{Z}_4, \widehat{Z}_5, \widehat{Z}_6$, $\widehat{Z}_8 = Z_7$, $\widehat{Z}_9 = y\frac{\partial}{\partial p} + \frac{yq}{p}\frac{\partial}{\partial q}$, $\widehat{Z}_{10} = s\frac{\partial}{\partial p} + \frac{sq}{p}\frac{\partial}{\partial q}$, $\widehat{Z}_{11} = sy\frac{\partial}{\partial p} + \frac{syq}{p}\frac{\partial}{\partial q}$.

Nonlocal symmetries of the Lagrange PGD system

γ	Admitted point symmetries		
	$\mathbf{LW}^1\{y, s\,; v, p, q, w^1\}$	$\mathbf{LW}^2\{y, s\,; v, p, q, w^2\}$	$\mathbf{LW}^3\{y, s\,; v, p, q, w^3\}$
Arbitrary	$\mathrm{I}_1 = \frac{\partial}{\partial w^1}$, $\mathrm{I}_2 = \mathrm{Z}_1$, $\mathrm{I}_3 = \mathrm{Z}_2 + w^1 \frac{\partial}{\partial w^1}$, $\mathrm{I}_4 = \mathrm{Z}_3 + s \frac{\partial}{\partial w^1}$, $\mathrm{I}_5 = \mathrm{Z}_4 + w^1 \frac{\partial}{\partial w^1}$, $\mathrm{I}_6 = \mathrm{Z}_5$, $\mathrm{I}_7 = \mathrm{Z}_6$.	$\mathrm{J}_1 = \frac{\partial}{\partial w^2}$, $\mathrm{J}_2 = \mathrm{Z}_1$, $\mathrm{J}_3 = \mathrm{Z}_2 + w^2 \frac{\partial}{\partial w^2}$, $\mathrm{J}_4 = \mathrm{Z}_3 + y \frac{\partial}{\partial w^2}$, $\mathrm{J}_5 = \mathrm{Z}_4 + w^2 \frac{\partial}{\partial w^2}$, $\mathrm{J}_6 = \mathrm{Z}_5 + w^2 \frac{\partial}{\partial w^2}$, $\mathrm{J}_7 = \mathrm{Z}_6$, $\mathrm{J}_8 = \widehat{\mathrm{Z}}_6 + (w^2 - yv)\frac{\partial}{\partial v} + yw^2 \frac{\partial}{\partial w^2}$.	$\mathrm{K}_1 = \frac{\partial}{\partial w^3}$, $\mathrm{K}_2 = \mathrm{Z}_2 + 2w^3 \frac{\partial}{\partial w^3}$, $\mathrm{K}_3 = \mathrm{Z}_3 + ys \frac{\partial}{\partial w^3}$, $\mathrm{K}_4 = \mathrm{Z}_4 + w^3 \frac{\partial}{\partial w^3}$, $\mathrm{K}_5 = \mathrm{Z}_5 + w^3 \frac{\partial}{\partial w^3}$.
3	$\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4, \mathrm{I}_5, \mathrm{I}_6, \mathrm{I}_7$, $\mathrm{I}_8 = s^2 \frac{\partial}{\partial s} + (w^1 - sv)\frac{\partial}{\partial v} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q} + sw^1 \frac{\partial}{\partial w^1}$.	$\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3, \mathrm{J}_4, \mathrm{J}_5, \mathrm{J}_6, \mathrm{J}_7, \mathrm{J}_8$.	$\mathrm{K}_1, \mathrm{K}_2, \mathrm{K}_3, \mathrm{K}_4, \mathrm{K}_5$.
-1	$\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4, \mathrm{I}_5, \mathrm{I}_6, \mathrm{I}_7$.	$\mathrm{J}_1, \mathrm{J}_2, \mathrm{J}_3, \mathrm{J}_4, \mathrm{J}_5, \mathrm{J}_6, \mathrm{J}_7, \mathrm{J}_8$, $\mathrm{J}_9 = \mathrm{Z}_7 - s\frac{\partial}{\partial w^2}$, $\mathrm{J}_{10} = \mathrm{Z}_8 - sy\frac{\partial}{\partial w^2}$	$\mathrm{K}_1, \mathrm{K}_2, \mathrm{K}_3, \mathrm{K}_4, \mathrm{K}_5$.

Nonlocal symmetries of the Lagrange PGD system

γ	Admitted point symmetries		
	$\mathbf{LW}^4\{y, s\,; p, q, w^4\}$	$\mathbf{LW}^4\{y, s\,; v, p, q, w^4\}$	$\mathbf{LW}^5\{y, s\,; v, p, q, w^5\}$
Arbitrary	$\widehat{\mathrm{L}}_1 = \frac{\partial}{\partial w^4}$, $\widehat{\mathrm{L}}_2 = \mathrm{Z}_1$, $\widehat{\mathrm{L}}_3 = \mathrm{Z}_2 + w^4 \frac{\partial}{\partial w^4}$, $\widehat{\mathrm{L}}_4 = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + (\gamma+1)w^4 \frac{\partial}{\partial w^4}$ $\widehat{\mathrm{L}}_5 = \mathrm{Z}_5 + (2-\gamma)w^4 \frac{\partial}{\partial w^4}$, $\widehat{\mathrm{L}}_6 = \mathrm{Z}_6$.	$\mathrm{L}_1 = \widehat{\mathrm{L}}_1$, $\mathrm{L}_2 = \mathrm{Z}_1$, $\mathrm{L}_3 = \widehat{\mathrm{L}}_3$, $\mathrm{L}_4 = \mathrm{Z}_3$, $\mathrm{L}_5 = v\frac{\partial}{\partial v} + \widehat{\mathrm{L}}_4$, $\mathrm{L}_6 = \widehat{\mathrm{L}}_5$, $\mathrm{L}_7 = \mathrm{Z}_6$.	$\mathrm{M}_1 = \frac{\partial}{\partial w^5}$, $\mathrm{M}_2 = \mathrm{Z}_1$, $\mathrm{M}_3 = \mathrm{Z}_2 + w^5 \frac{\partial}{\partial w^5}$, $\mathrm{M}_4 = \mathrm{Z}_4 + 2w^5 \frac{\partial}{\partial w^5}$, $\mathrm{M}_5 = \mathrm{Z}_5 + w^5 \frac{\partial}{\partial w^5}$, $\mathrm{M}_6 = \mathrm{Z}_6$.
3	$\widehat{\mathrm{L}}_1, \widehat{\mathrm{L}}_2, \widehat{\mathrm{L}}_3, \widehat{\mathrm{L}}_4, \widehat{\mathrm{L}}_5, \widehat{\mathrm{L}}_6$, $\widehat{\mathrm{L}}_7 = s^2\frac{\partial}{\partial s} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q}$.	$\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3, \mathrm{L}_4, \mathrm{L}_5, \mathrm{L}_6, \mathrm{L}_7$.	$\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3, \mathrm{M}_4, \mathrm{M}_5, \mathrm{M}_6$.
-1	$\widehat{\mathrm{L}}_1, \widehat{\mathrm{L}}_2, \widehat{\mathrm{L}}_3, \widehat{\mathrm{L}}_4, \widehat{\mathrm{L}}_5, \widehat{\mathrm{L}}_6$, $\widehat{\mathrm{L}}_7 = \mathrm{Z}_7$, $\widehat{\mathrm{L}}_8 = \mathrm{Z}_8$, $\widehat{\mathrm{L}}_9 = \widehat{\mathrm{Z}}_{10}$, $\widehat{\mathrm{L}}_{10} = \widehat{\mathrm{Z}}_{11}$.	$\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3, \mathrm{L}_4, \mathrm{L}_5, \mathrm{L}_6, \mathrm{L}_7$, $\mathrm{L}_8 = \mathrm{Z}_7$, $\mathrm{L}_9 = \mathrm{Z}_8$.	$\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3, \mathrm{M}_4, \mathrm{M}_5, \mathrm{M}_6$.
1	$\widehat{\mathrm{L}}_1, \widehat{\mathrm{L}}_2, \widehat{\mathrm{L}}_3, \widehat{\mathrm{L}}_4, \widehat{\mathrm{L}}_5, \widehat{\mathrm{L}}_6$, $\widehat{\mathrm{L}}_{11} = \widehat{\mathrm{Z}}_6$.	$\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3, \mathrm{L}_4, \mathrm{L}_5, \mathrm{L}_6, \mathrm{L}_7$.	$\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3, \mathrm{M}_6$, $\mathrm{M}_7 = \mathrm{Z}_4 - \mathrm{Z}_5 + w^5 \frac{\partial}{\partial w^5}$.

(IV) Exact solutions arising from nonlocal symmetries

The standard algorithm for invariant solutions

Given system: $\mathbf{R}\{x,t\,;u\}$, **Potential system:** $\mathbf{RV}\{x,t\,;u,v\}$

(For simplicity: consider scalar u, v.)

Potential symmetry of $\mathbf{R}\{x,t\,;u\}$:

$$\mathrm{X} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \zeta(x,t,u,v)\frac{\partial}{\partial v}.$$

The standard algorithm for invariant solutions

(1) Characteristic equations: $\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta} = \frac{dv}{\zeta}$

(2) Solutions (invariants):

$$z = Z(x,t,u,v), \;\; h_1 = H_1(x,t,u,v), \;\; h_2 = H_2(x,t,u,v)$$

(3) Translation coordinate: $\hat{z} = \hat{Z}(x,t,u,v) : \;\; \mathrm{X}\hat{Z}(x,t,u,v) = 1$

(4) Change variables in the potential system $\mathbf{RV}\{x,t\,;u,v\}$:

$$(x,t,u,v) \to (z,\hat{z},h_1,h_2)$$

(5) Drop dependence on $\hat{z}$: $h_1 = h_1(z), h_2 = h_2(z).$

(6) Solve ODEs to get $h_1 = h_1(z), h_2 = h_2(z).$

(7) Express $u, v.$

The first refinement [Pucci & Saccomandi (1993)]

...

---

(4) Change variables in the **<u>given system</u>** $\mathbf{R}\{x, t\,; u\}$,

$$(x, t, u, v) \to (z, \hat{z}, h_1, h_2)$$

---

(5) Drop dependence on $\hat{z}:\quad h_1 = h_1(z), h_2 = h_2(z).$

---

(6) Solve ODEs to get $h_1 = h_1(z), h_2 = h_2(z).$

---

(7) Express $u, v.$

The <u>potential variable</u> is sought in the ***invariant form*,**
but is ***not a solution*** of the potential equations.

The second refinement [Sjoberg & Mahomed (2004)]

...

---

(4) Change variables in the potential system $\mathbf{RV}\{x, t\,; u, v\}$:

$$(x, t, u, v) \to (z, \hat{z}, h_1, h_2)$$

---

(5) In the expression for u,
drop dependence on $\hat{z}$: $h_1 = h_1(z), h_2 = h_2(z).$

---

(6) Solve ODEs to get $h_1 = h_1(z), h_2 = h_2(z).$

---

(7) Express $u, v.$

The potential variable is ***not*** sought in the invariant form.

The combined approach

Do both:

The potential variable is ***not*** sought in the invariant form,

and ***is not*** a solution of the potential equations

(i.e., ansatz is substituted into the **given system**).

Example: Exact solutions of the Lagrange system

Lagrange polytropic system:

$$\mathbf{L}\{y, s\,; v, p, q\}: \quad \begin{cases} q_s - v_y = 0, \\ v_s + p_y = 0, \\ p_s + \gamma \dfrac{p}{q} v_y = 0. \end{cases}$$

Potential system:

$$\mathbf{LW}^2\{y, s\,; v, p, q, w^2\}: \quad \begin{cases} q_s - v_y = 0, \\ w^2_y = v, \\ w^2_s = -p, \\ p_s + \gamma \dfrac{p}{q} v_y = 0; \end{cases}$$

Nonlocal symmetry:

$$\mathrm{J}_8 = y^2 \frac{\partial}{\partial y} + yp \frac{\partial}{\partial p} - 3yq \frac{\partial}{\partial q} + (w^2 - yv) \frac{\partial}{\partial v} + yw^2 \frac{\partial}{\partial w^2}$$

Nonlocal symmetry:

$$\mathrm{J}_8 = y^2\frac{\partial}{\partial y} + yp\frac{\partial}{\partial p} - 3yq\frac{\partial}{\partial q} + (w^2 - yv)\frac{\partial}{\partial v} + yw^2\frac{\partial}{\partial w^2} = \frac{\partial}{\partial \hat{z}}$$

Invariants:

$$z = s, \quad h_1 = \frac{p}{y}, \quad h_2 = y^3 q, \quad h_3 = \frac{w^2}{y}, \quad h_4 = yv - w^2.$$

Translation coordinate: $\hat{z} = 1/y.$

Invariant form:

$$p(y,s) = yh_1(s), \quad q(y,s) = \frac{h_2(s)}{y^3},$$

$$v(y,s) = \frac{h_4(s)}{y} + h_3(s), \quad w^2(y,s) = yh_3(s).$$

Standard invariant solution:

$$v(y,s) = -C_1 s + C_3, \quad p(y,s) = C_1 y, \quad q(y,s) = \frac{C_2}{y^3}.$$

Exact solutions: Extended (combined) approach

Translation coordinate: $\hat{z} = 1/y.$

Invariant form:

$$p(y,s) = y h_1(s), \quad q(y,s) = \frac{h_2(s)}{y^3},$$

$$v(y,s) = \frac{h_4(s)}{y} + h_3(s), \quad \boxed{w^2(y,s) = y h_3(y,s).}$$

Substitute into $\mathbf{L}\{y,s\,;v,p,q\}$ **not** $\mathbf{LW}^2\{y,s\,;v,p,q,w^2\}.$

Solutions:

$$\mathcal{F}_1: \quad v(y,s) = -a_1 s + a_3, \quad p(y,s) = a_1 y, \quad q(y,s) = \frac{a_2}{y^3}.$$

$$\mathcal{F}_2: \quad v(y,s) = \frac{b_1}{y^2} + b_2, \quad p(y,s) = 0, \quad q(y,s) = \frac{-2b_1 s + b_3}{y^3}.$$

$$\mathcal{F}_3: \quad \begin{cases} v(y,s) = \dfrac{c_1 n^n (-1)^{n-1}}{n-1}(s+c_2)^{1-n} + c_3 - \dfrac{c_4}{y^2}, \\ p(y,s) = c_1 n^n (-1)^{n-1} (s+c_2)^{-n} y, \\ q(y,s) = \dfrac{2c_4(s+c_2)}{y^3}. \qquad \text{(Integer } \gamma = n \neq 1.) \end{cases}$$

Standard invariant solution:

$$v(y,s) = -C_1 s + C_3, \quad p(y,s) = C_1 y, \quad q(y,s) = \frac{C_2}{y^3}.$$

Exact solutions: Extended (combined) approach

Theorem:

- Families $\mathcal{F}_2$ and $\mathcal{F}_3$ do not arise as invariant solutions of the Lagrange or potential system with respect to any of their point symmetries.

- Families $\mathcal{F}_2$ and $\mathcal{F}_3$ only arise from the extended (combined) algorithm and not from first or second refinement.

Conclusions

- One can systematically seek nonlocal symmetries of PDE systems;

- If a nonlocal (potential) symmetry of a PDE system is found, an *extended* procedure (presented in this talk) can yield additional solutions compared to the classical method.

Some references

• G. Bluman, A. Cheviakov, N. Ivanova, *Framework for nonlocally related PDE systems and nonlocal symmetries: extension, simplification, and examples,* J. Math. Phys. 47 (2006), 113505.

• A. Sjoberg and F.M. Mahomed, *Non-local symmetries and conservation laws for one-dimensional gas dynamics equations.* App. Math. Comp. 150 (2004), 379–397

• E. Pucci and G. Saccomandi, *Potential symmetries and solutions by reduction of partial differential equations.* J. Phys. A: Math. Gen. 26 (1993), 681-690.

• A. Cheviakov, *An extended procedure for finding exact solutions of partial differential equations arising from potential symmetries*, J. Math. Phys. 49 (2008), 083502.

Thank you for your attention!