Two approximate symmetry frameworks for nonlinear DEs with a small parameter. Comparisons, relations, approximate solutions

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Symmetry, Invariants, and their Applications

A celebration of Peter Olver's 70^{th} birthday

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Outline

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Perturbed DEs

Regular perturbation of an ODE

- Original (unperturbed) ODE:

$$y^{(n)}(x) = f_0[y] \quad \equiv \quad f_0(x, y(x), \ldots, y^{(n-1)}(x))$$

- Small parameter: ϵ of an ODE: the leading derivative(s) does not change.

- Perturbed ODE:

$$y^{(n)}(x) = f_0[y] + \epsilon f_1[y] + o(\epsilon).$$

ODE systems, PDEs, PDE systems

- Same idea: a regular perturbation where $O(\epsilon)$ terms do not break the structure of the solved form; DEs still have the same leading derivatives.

Difficulty:

- Analytical structure of the unperturbed model may be lost under perturbation.

Consider a nonlinear wave-type equation on $u = u(x, t)$:

$$\boxed{u_{tt} = u_x u_{xx}} \tag{1}$$

Exact point symmetry generator:

$$X^0 = \xi_0^1(x, t, u)\frac{\partial}{\partial x} + \xi_0^2(x, t, u)\frac{\partial}{\partial t} + \eta_0(x, t, u)\frac{\partial}{\partial u}$$

Final result: the PDE (1) admits six point symmetries given by

$$\begin{gathered} X_1^0 = \frac{\partial}{\partial u}, \quad X_2^0 = t\frac{\partial}{\partial u}, \quad X_3^0 = \frac{\partial}{\partial t}, \quad X_4^0 = \frac{\partial}{\partial x}, \\ X_5^0 = u\frac{\partial}{\partial u} - \frac{t}{2}\frac{\partial}{\partial t}, \quad X_6^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \end{gathered} \tag{2}$$

corresponding to three translations (X_1^0, X_3^0 and X_4^0), the Galilei group (X_2^0), and two scalings (X_5^0 and X_6^0).

A simple example: lost point symmetries

A perturbed PDE:

$$\boxed{u_{tt} + \epsilon u u_t = u_x u_{xx}} \tag{3}$$

- The perturbation term distorts symmetry structure. Exact point symmetries of (3) are generated by

$$\begin{aligned} Y_1 &= X_3^0 = \frac{\partial}{\partial t}, \\ Y_2 &= X_4^0 = \frac{\partial}{\partial x}, \\ Y_3 &= \frac{4}{3}X_5^0 - \frac{1}{3}X_6^0 = -t\frac{\partial}{\partial t} - \frac{x}{3}\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \end{aligned} \tag{4}$$

  a three-dimensional subalgebra of the six-dimensional Lie algebra of point symmetries (2).

- Where did the other three go? Stable vs. unstable symmetries.

- Can unstable symmetries re-appear as, in some sense, "approximate" symmetries of the perturbed PDE (3)?

Outline

BGI approximate symmetries

- For simplicity, consider a single PDE on $u = u(x) = (x_1, \ldots, x_n)$.
- Original (unperturbed) PDE:

$$F_0[u] = 0 \tag{5}$$

- Perturbed DE (regular perturbation):

$$F_0[u] + \epsilon F_1[u] = 0 \tag{6}$$

- BGI approximate point symmetry generator:

$$X = X^0 + \epsilon X^1 = \left(\xi_0^i(x, u) + \epsilon \xi_1^i(x, u)\right) \frac{\partial}{\partial x^i} + (\eta_0(x, u) + \epsilon \eta_1(x, u)) \frac{\partial}{\partial u}$$

- Easier to work with characteristic forms which generalize to local symmetries:

$$\hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = (\zeta_0[u] + \epsilon \zeta_1[u]) \frac{\partial}{\partial u}$$

- Determining equations:

$$(\hat{X}^{0\,\infty} + \epsilon \hat{X}^{1\,\infty})(F_0[u] + \epsilon F_1[u])\Big|_{F_0[u] + \epsilon F_1[u] = o(\epsilon)} = o(\epsilon)$$

BGI approximate symmetries

- $O(\epsilon^0)$ part of the determining equations:

$$\hat{X}^{0\,\infty} F_0[u]\Big|_{F_0[u]=0} = 0$$

  - Every BGI approximate point symmetry of the perturbed equation corresponds to an exact point symmetry $\hat{X}^{0\,\infty}$ of the unperturbed equation.
  - Opposite is not true (previous example).

- $O(\epsilon)$ part of the determining equations:

$$\hat{X}^{1\,\infty} F_0[u]\Big|_{F_0[u]=0} = H[u],$$

where $H[u]$ is the $O(\epsilon)$ part of the expression

$$-\hat{X}^{0\,\infty}(F_0[u] + \epsilon F_1[u])\Big|_{F_0[u]+\epsilon F_1[u]=o(\epsilon)}\,.$$

  - These are extra conditions on $\hat{X}^0$ that can make some symmetries unstable.

Outline

FS approximate symmetries

- Original (unperturbed) DE:

$$F_0[u] = 0 \tag{7}$$

- Perturbed DE (regular perturbation):

$$F_0[u] + \epsilon F_1[u] = 0 \tag{8}$$

- Seek solution as a regular perturbation

$$u(x) = v(x) + \epsilon w(x) + o(\epsilon).$$

  Split (8) into $O(1)$ and $O(\epsilon)$ parts (FS system):

$$\begin{aligned} &G_1[v, w] \equiv F_0[v] = 0, \\ &G_2[v, w] \equiv F_{0\,v} w + F_{0\,v_i} w_i + F_{0\,v_{ij}} w_{ij} + \ldots + F_{0\,v_{i_1 i_2 \ldots i_k}} w_{i_1 i_2 \ldots i_k} + F_1[v] = 0 \end{aligned} \tag{9}$$

- Find usual point/local symmetries of (9): infinitesimal generators

$$\hat{Z} = \zeta_0[v, w]\frac{\partial}{\partial v} + \zeta_1[u, w]\frac{\partial}{\partial w}$$

- Determining equations are again restrictive on ζ_0; can lead to stable or unstable local symmetries of (7), now in the FS sense.

BGI vs. FS approximate symmetry forms

- Point:

$$X = \left(\xi_0^i(x,u) + \epsilon\xi_1^i(x,u)\right)\frac{\partial}{\partial x^i} + (\eta_0(x,u) + \epsilon\eta_1(x,u))\frac{\partial}{\partial u}$$

$$Z = \lambda^i(x,v,w)\frac{\partial}{\partial x^i} + \phi_1(x,v,w)\frac{\partial}{\partial v} + \phi_2(x,v,w)\frac{\partial}{\partial w}$$

- General local, characteristic form:

$$\hat{X} = \hat{X}^0 + \epsilon\hat{X}^1 = (\zeta_0[u] + \epsilon\zeta_1[u])\frac{\partial}{\partial u}$$

$$\hat{Z} = \zeta_0[v,w]\frac{\partial}{\partial v} + \zeta_1[u,w]\frac{\partial}{\partial w}$$

Which is more general?

In both BGI and FS approximate symmetries, the following kinds arise.

- Directly inherited from the original PDE: $\zeta_0 = \zeta_0[u]$, $\zeta_1 = 0$
- Genuine approximate: $\zeta_0, \zeta_1 \neq 0$
- "Trivial" approximate: $\zeta_0 = 0$ ("trivial" = "'always appear")

Outline

BGI vs. FS: a computational comparison

- Unperturbed and perturbed PDEs, $u = u(x,t)$:

$$\boxed{u_{tt} = u_x u_{xx}} \qquad \boxed{u_{tt} + \epsilon F_1(u, u_t) = u_x u_{xx}}$$

- Six point symmetries of the unperturbed PDE:

$$X_1^0 = \frac{\partial}{\partial u}, \quad X_2^0 = t\frac{\partial}{\partial u}, \quad X_3^0 = \frac{\partial}{\partial t}, \quad X_4^0 = \frac{\partial}{\partial x},$$
$$X_5^0 = u\frac{\partial}{\partial u} - \frac{t}{2}\frac{\partial}{\partial t}, \quad X_6^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},$$

- Which of these are stable (as point symmetries) when $F_1 \neq 0$, in BGI and/or FS frameworks?

- Classify with respect to inequivalent forms of $F_1(u, u_t)$.

BGI vs. FS: a computational comparison

$\hat{X}_1^0 = \dfrac{\partial}{\partial u}$	$F_1 = Q_1(u_t) + a_1 u u_t + a_2 u,$ $\hat{X}_1 = \left(1 - \epsilon\left(\dfrac{a_1}{10} t(t u_t + 4u) + \dfrac{a_2}{2} t^2\right)\right)\dfrac{\partial}{\partial u}$	• $F_1 = e^{a_3 v} Q_4(v_t) + a_2 v_t + a_1,$ $\hat{Z}_1 = \dfrac{\partial}{\partial v} + a_3\left(\dfrac{a_2}{10} t(t v_t + 4v) + w + \dfrac{a_1}{2} t^2\right)\dfrac{\partial}{\partial w}$ • $F_1 = Q_4(v_t) + a_1 v v_t + a_2 v,$ $\hat{Z}_1 = \dfrac{\partial}{\partial v} - \left(\dfrac{a_1}{10} t(t v_t + 4v) + \dfrac{a_2}{2} t^2\right)\dfrac{\partial}{\partial w}$
$\hat{X}_2^0 = t\dfrac{\partial}{\partial u}$	$F_1 = a_1 u_t^2 + a_2 u_t + a_3 u + a_4,$ $\hat{X}_2 = \left(t - \epsilon\left(\dfrac{a_1}{5} t(t u_t + 4u) + \dfrac{1}{6} t^2 (a_3 t + 3a_2)\right)\right)\dfrac{\partial}{\partial u}$	• $F_1 = a_1 v_t^2 + a_2 v_t + a_3 v + a_4,$ $\hat{Z}_2 = t\dfrac{\partial}{\partial v} - \left(\dfrac{a_1}{5} t(t v_t + 4v) + \dfrac{1}{6} t^2 (a_3 t + 3a_2)\right)\dfrac{\partial}{\partial w}$ • $F_1 = a_3 e^{a_4 v_t} + a_2 v_t + a_1,$ $\hat{Z}_2 = t\dfrac{\partial}{\partial v} + \left(\dfrac{a_2 a_4}{10} t(t v_t + 4v) + \dfrac{a_1 a_4 - a_2}{2} t^2 + a_4 w\right)\dfrac{\partial}{\partial w}$
$\hat{X}_3^0 = u_t\dfrac{\partial}{\partial u}$	$F_1 = F_1(u, u_t), \quad \hat{X}_3 = u_t \dfrac{\partial}{\partial u}$	$F_1 = F_1(v, v_t), \quad \hat{Z}_3 = v_t\dfrac{\partial}{\partial v} + w_t\dfrac{\partial}{\partial w}$

BGI vs. FS: a computational comparison

$\hat{X}_4^0 = u_x \frac{\partial}{\partial u}$	$F_1 = F_1(u, u_t), \quad \hat{X}_4 = u_x \frac{\partial}{\partial u}$	$F_1 = F_1(v, v_t), \quad \hat{Z}_4 = v_x \frac{\partial}{\partial v} + w_x \frac{\partial}{\partial w}$
$\hat{X}_5^0 = \left(u + \frac{tu_t}{2}\right) \frac{\partial}{\partial u}$	$F_1 = u^2 Q_2 \left(u_t/u^{3/2}\right) + a_2 u_t + a_1$ $\hat{X}_5 = \left(u + \frac{tu_t}{2} + \epsilon\left(a_1 t^2 + \frac{a_2}{20} t(tu_t + 4u)\right)\right) \frac{\partial}{\partial u}$	$F_1 = v^{a_3} Q_5 \left(v_t/v^{3/2}\right) + a_2 v_t + a_1,$ $\hat{Z}_5 = \left(v + \frac{tv_t}{2}\right) \frac{\partial}{\partial v} + \left((a_3 - 1)w + \frac{tw_t}{2} + \frac{a_2}{20}(2a_3 - 3)t(tv_t + 4v) + \frac{a_1 a_3}{2} t^2\right) \frac{\partial}{\partial w}$
$\hat{X}_6^0 = (u - xu_x - tu_t) \frac{\partial}{\partial u}$	$F_1 = u^{-1} Q_3(u_t) + a_2 u_t + a_1,$ $\hat{X}_6 = \left(u - xu_x - tu_t - \epsilon\left(\frac{a_2}{10} t(tu_t + 4u) + \frac{a_1}{2} t^2\right)\right) \frac{\partial}{\partial u}$	$F_1 = v^{a_3} Q_6(v_t) + a_2 v_t + a_1,$ $\hat{Z}_6 = (v - xv_x - tv_t) \frac{\partial}{\partial v} + \left((a_3 + 2)w - xw_x - tw_t + \frac{a_2 a_3}{10} t(tv_t + 4v) + \frac{a_1 a_3}{2} t^2\right) \frac{\partial}{\partial w}$

Outline

A nonlinear wave equation

- A class of nonlinear wave equations with a small parameter:

$$\boxed{u_{tt} = (1 + \epsilon Q(u_x))u_{xx}}$$

- FS system: $v_{tt} = v_{xx}$, $w_{tt} = w_{xx} + Q(v_x)v_{xx}$.
- Power nonlinearity $Q(v_x) = v_x^s$ $(s \neq -1)$: genuine approximate FS symmetry

$$Z = v\frac{\partial}{\partial v} + (s+1)w\frac{\partial}{\partial w}$$

- "Large" and "small" solution components are scaled differently

- An approximately invariant solution:

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dv}{v} = \frac{dw}{(s+1)w}$$

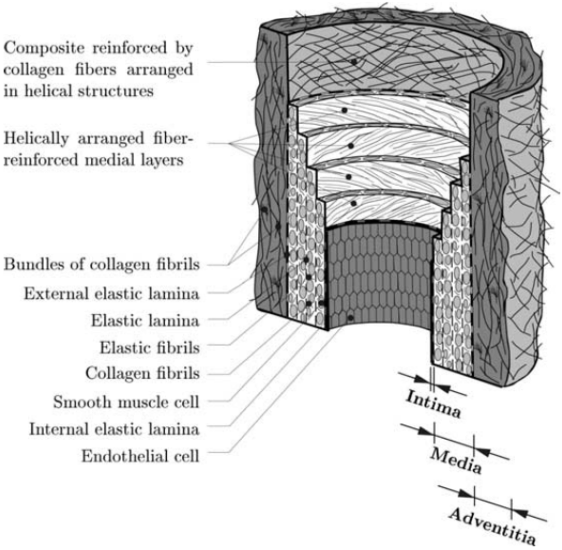
Take $v = g(x \pm t)$. Then $w = g^{s+1}\phi$, and

$$g^{s+1}(\phi_{tt} - \phi_{xx}) + 2(s+1)g^s g'(\pm\phi_t - \phi_x) - (g')^s g'' = 0$$

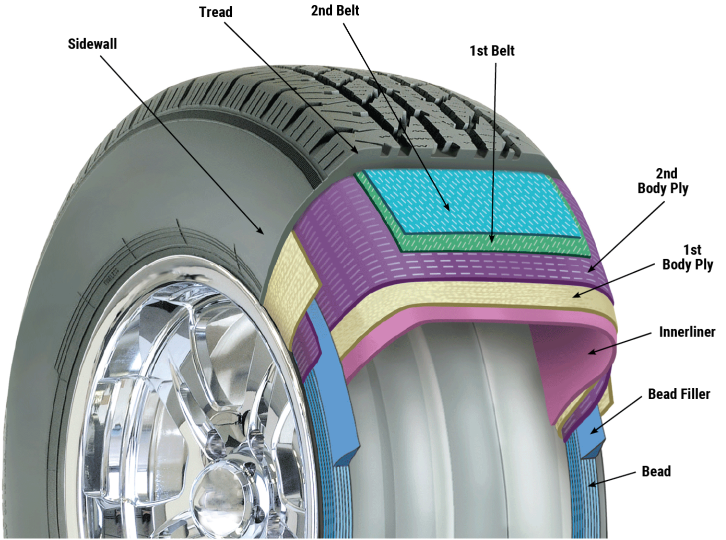
- $\phi = \phi(x,t)$, and so the approximate solution $u = v + \epsilon w$, can be found explicitly.

- **Motivation:** biological and artificial elastic materials with families of aligned fibers
- Anisotropic elastodynamics

1D nonlinear waves in fiber-reinforced solids



1D nonlinear waves in fiber-reinforced solids



1D nonlinear waves in fiber-reinforced solids

- Fully nonlinear Eulerian shear displacements $u(x,t)$ (not small) in terms of material coordinate X

- Viscoelastic dynamics [A.S. and Ganghoffer (2015)]

$$\begin{aligned} u_{tt} = \; & \left(\alpha + 3\beta u_x^2\right) u_{xx} \\ & + \eta\, u_x \left(2\,(4u_x^2+1) u_{xx} u_{tx} + (2u_x^2+1) u_x u_{txx}\right) \\ & + \zeta\, u_x^3 \left(4\,(6u_x^2+1) u_{xx} u_{tx} + (4u_x^2+1) u_x u_{txx}\right) \end{aligned}$$

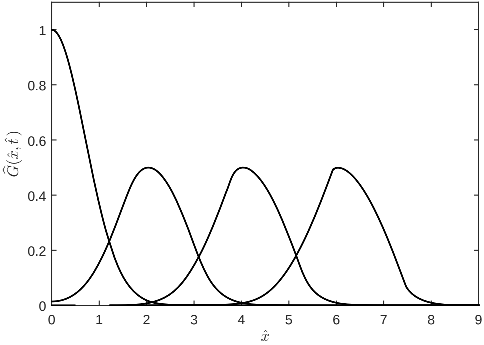
- Possible small parameters: β, η, ζ

- Hyperelastic simplification:

$$u_{tt} = \left(\alpha + 3\beta u_x^2\right) u_{xx} \quad \to \quad \boxed{u_{tt} = (1+\epsilon u_x^2) u_{xx}}$$

  - Member of the previous wave equation family with power nonlinearity u^s, $s=2$
  - Produces "breaking waves:" finite-time singularity formation

- Numerical solution, $\epsilon = 0.5$:

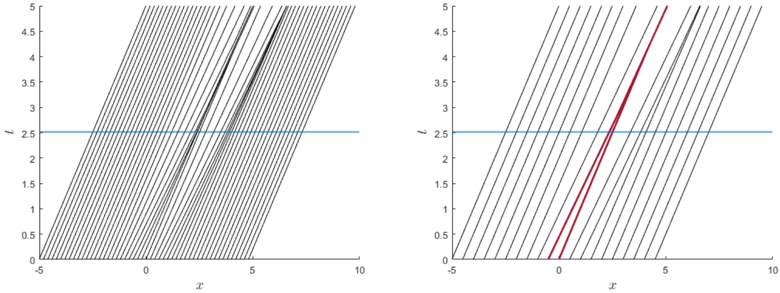


- One can show that the PDE $u_{tt} = (1 + \epsilon u_x^2) u_{xx}$ can be reduced to the first-order characteristic form

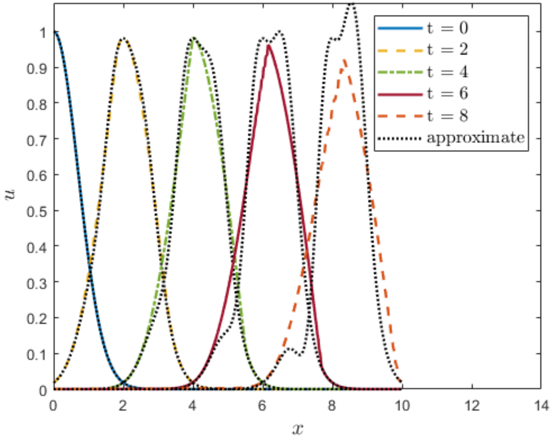
$$u_t = \pm \frac{1}{2\sqrt{\epsilon}} \left(\sqrt{\epsilon}\, u_x \sqrt{1 + \epsilon\, (u_x)^2} + \ln \left(\sqrt{\epsilon}\, u_x + \sqrt{1 + \epsilon\, (u_x)^2} \right) \right)$$

on the characteristic curves

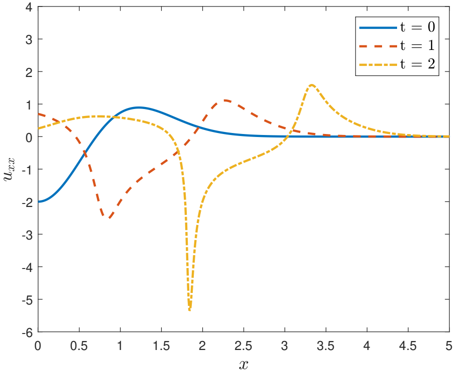
$$\frac{dx}{dt} = \pm \sqrt{1 + \epsilon\, (u_x)^2}\,.$$

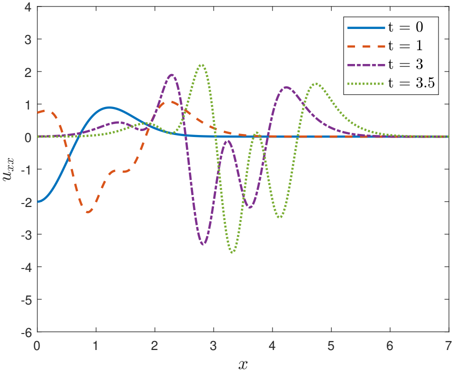


- Compare behaviour with the approximate solution

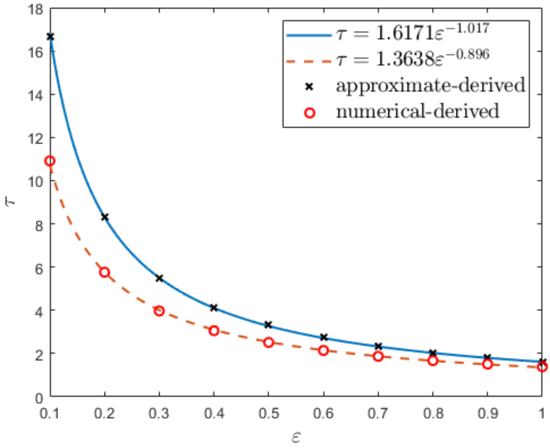


- Wave breaking ∼ formation of an extra inflection point on the approximate solution.
- Left: u_{xx} for the numerical solution; right: same for the approximate solution





- Estimate wave breaking times [Tarayrah, Pitzel, A.S.]



Outline

Discussion

- Approximate symmetries can be found using BGI and FS frameworks ("somewhat" related), and are useful.
- Approximate conservation laws can arise in a similar manner.
- PDE models with multiple small parameters?

Anisotropic dynamic vicoelasticity, shear waves

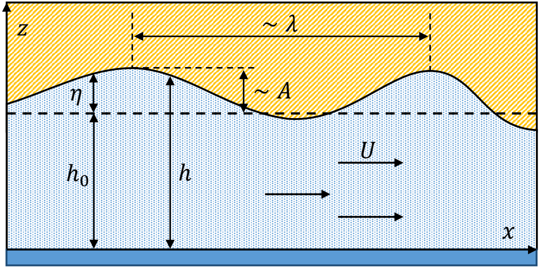
$$\begin{aligned} u_{tt} = \quad & \left(\alpha + 3\beta u_x^2\right) u_{xx} \\ & + \eta\, u_x \Big(2\,(4u_x^2 + 1) u_{xx} u_{tx} + (2u_x^2 + 1) u_x G_{txx}\Big) \\ & + \zeta\, u_x^3 \Big(4\,(6u_x^2 + 1) u_{xx} u_{tx} + (4u_x^2 + 1) u_x u_{txx}\Big) \end{aligned}$$

Serre-Su-Gardner-Green-Naghdi shallow water equations

$$u_t + \epsilon u u_x^* + \eta_x = \frac{\delta^2}{3h}\left(h^3\left(u_{xt} + \epsilon u u_{xx} - \epsilon u_x^2\right)\right)_x ,$$

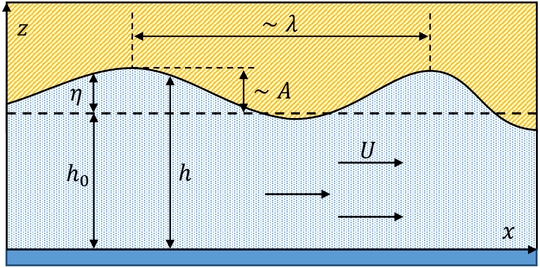
$$h_t + \epsilon (hu)_x = 0$$

Outline



- $\delta = \dfrac{h_0}{\lambda}$: dispersion parameter; $\varepsilon = \dfrac{A}{h_0}$: amplitude parameter
- Weakly nonlinear dispersionless equations, characterized by $\delta^2 \ll \varepsilon \ll 1$.
  Example: shallow water equations

$$u^*_{t^*} + \eta^*_{x^*} + \varepsilon u^* u^*_{x^*} = 0,$$
$$h^*_t + (h^* u^*)_{x^*} = 0$$

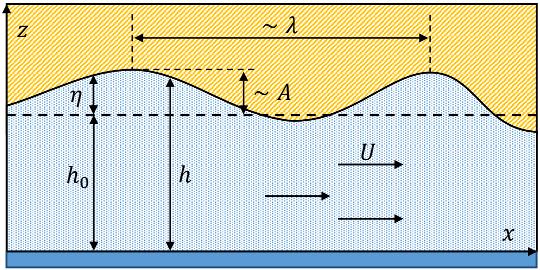


- $\delta = \dfrac{h_0}{\lambda}$: dispersion parameter; $\varepsilon = \dfrac{A}{h_0}$: amplitude parameter
- Weakly nonlinear and weakly dispersive equations: Boussinesq regime $\delta^2 \sim \varepsilon \ll 1$.
  Examples: the Boussinesq equation

$$\eta^*_{t^*t^*} = \eta^*_{x^*x^*} + \frac{\varepsilon}{2}\left(((\eta^*)^2)_{x^*x^*} + 2((u_0^*)^2)_{x^*x^*}\right) + \frac{\delta^2}{3}\eta^*_{x^*x^*x^*x^*}$$

  The KdV equation

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\,\eta^*\eta^*_{x^*} + \frac{\delta^2}{6}\,\eta^*_{x^*x^*x^*} = 0\,.$$



- $\delta = \dfrac{h_0}{\lambda}$: dispersion parameter; $\varepsilon = \dfrac{A}{h_0}$: amplitude parameter
- Strongly nonlinear weakly dispersive models: $\delta^2 \ll 1$, $\varepsilon = O(1)$.
  Example: the Su-Gardner equations

$$u^*_{t^*} + \varepsilon u^* u^*_{x^*} + \eta^*_{x^*} = \frac{\delta^2}{3h^*}\left((h^*)^3 \left(u^*_{x^*t^*} + \varepsilon^* u^*_{x^*x^*} - \varepsilon(u^*_{x^*})^2\right)\right)_{x^*},$$
$$h^*_{t^*} + \varepsilon(h^* u^*)_{x^*} = 0$$

Physical dimensionless vs. canonical forms

- KdV canonical: $u_t + 6uu_x + u_{xxx} = 0$
- BBM canonical: $u_t + u_x + uu_x - u_{xxt} = 0$
- KdV physical dimensionless:

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\,\eta^*\eta^*_{x^*} + \frac{\delta^2}{6}\,\eta^*_{x^*x^*x^*} = 0$$

- BBM physical dimensionless:

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\eta^*\eta^*_{x^*} - \frac{\delta^2}{6}\eta^*_{x^*x^*t^*} = 0$$

- Both in the Boussinesq regime $\delta^2 \sim \varepsilon$
- Same order of approximation $o(\varepsilon^2)$, very different analytical properties.

Arnold's principle

If a model bears a name, it is not the name of the person who discovered it.

Examples:

- Korteweg-de Vries $\rightarrow$ Boussinesq (25 years earlier)
- Su-Gardner (Green-Naghdi) $\rightarrow$ Serre (13 and 21 years earlier)
- Camassa-Holm $\rightarrow$ Fokas and Fuchssteiner (12 years earlier)
- ...

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