Conservation Laws For Viscous and Inviscid Flows in Helical, Plane and Rotational Symmetry

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March 25, 2013

Outline

Collaborators

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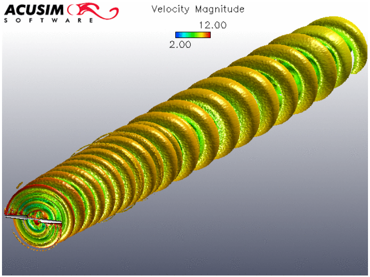
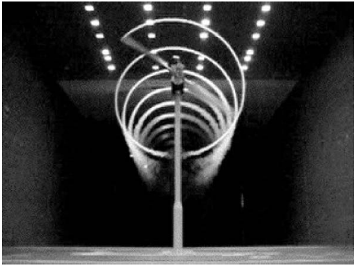
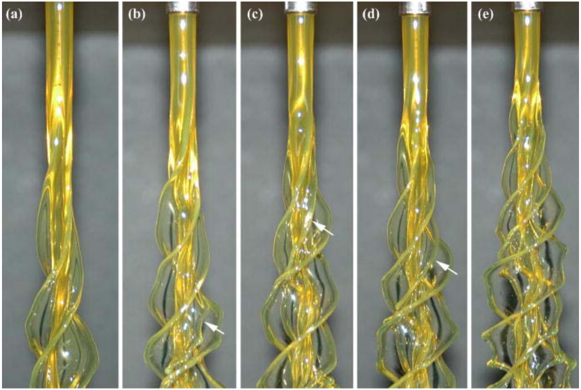
- Wind turbine wakes in aerodynamics [*Vermeer, Sorensen & Crespo, 2003*]



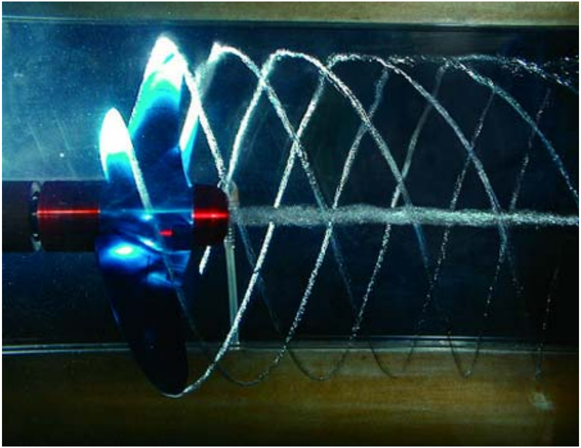


Examples of Helical Flows in Nature

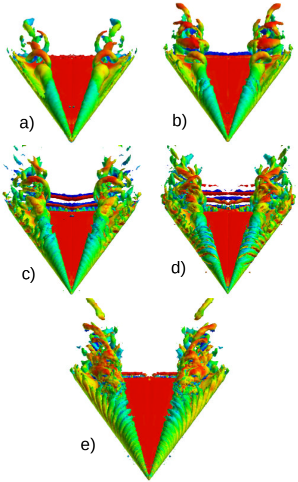
- Helical instability of rotating viscous jets [*Kubitschek & Weidman, 2007*]



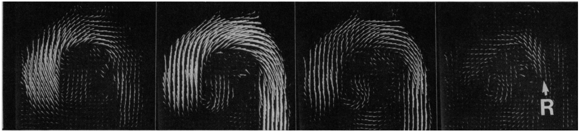
- Helical water flow past a propeller

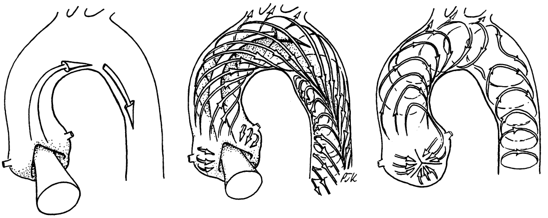


- Wing tip vortices, in particular, on delta wings [*Mitchell, Morton & Forsythe, 1997*]



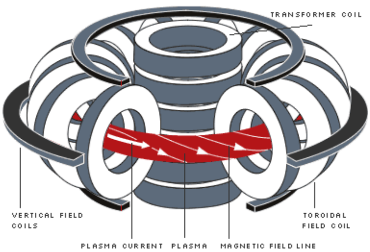
- Helical blood flow patterns in the aortic arch [*Kilner et al, 1993*]

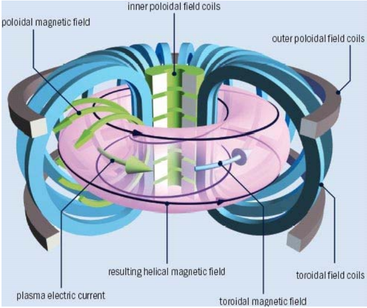




Examples of Helical Flows in Nature

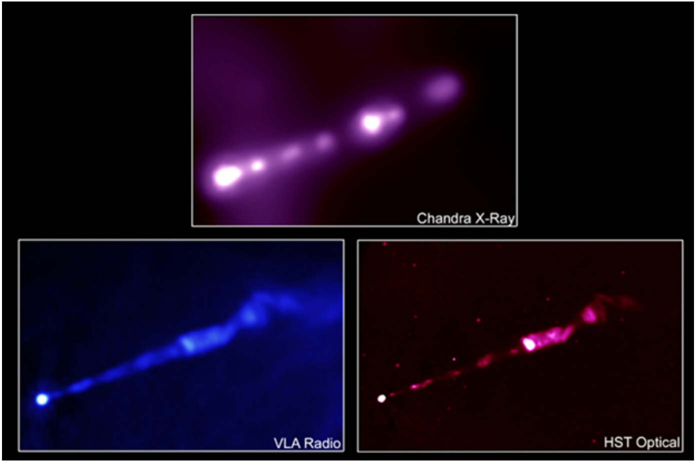
- Helical plasma flows in tokamaks





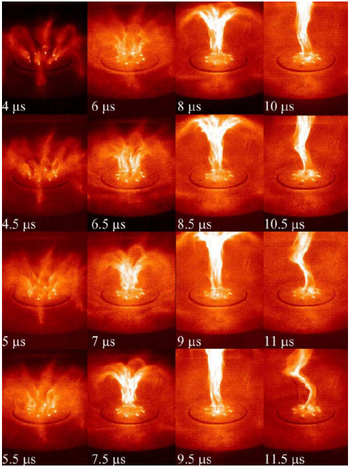
Examples of Helical Flows in Nature

- Helical plasma structures in astrophysics



Examples of Helical Flows in Nature

- Collimated helical plasma jet formation in a plasma discharge



Incompressible Fluid Flow Equations

Navier-Stokes Equations

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

- Euler/inviscid: $\nu = 0$.
- Constant-density (WLOG $\rho = 1$).

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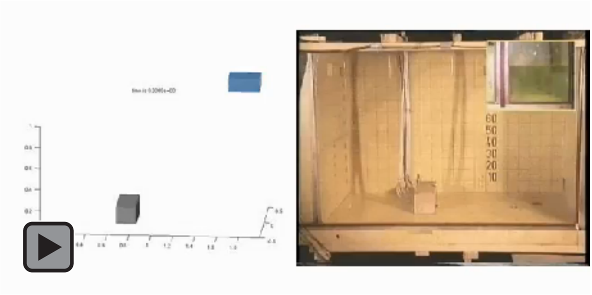
[*K.M.T. Kleefsman, MARIN, U. Groningen*]



Local Divergence-type Conservation Laws

Conservation laws

Independent variables: $\mathbf{x} = (t, x, y, ...)$; **dependent variables:** $\mathbf{q} = (q^1, q^2, ...)$.

Local conservation law:

$$\boxed{\mathrm{D}_t \Theta + \operatorname*{div}_{x,y,\dots} \mathbf{\Phi} = 0.}$$

Density: $\Theta(\mathbf{x}, \mathbf{q}, ...)$. Spatial fluxes: $\mathbf{\Phi} = (\Phi^1(\mathbf{x}, \mathbf{q}, ...), \Phi^2(\mathbf{x}, \mathbf{q}, ...), \cdots)$.

Conserved quantities

$$\mathrm{D}_t \int_V \Theta \, dV = 0.$$

Material conservation laws

For incompressible flows with velocity field $\mathbf{u}$, $\operatorname{div} \mathbf{u} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta \equiv \mathrm{D}_t \Theta + \mathbf{u} \cdot \nabla \Theta = \mathrm{D}_t \Theta + \operatorname*{div}_{x,y,\dots} \left(\Theta \mathbf{u} \right) = 0.$$

Euler equations in $3+1$ dimensions

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

Basic conservation laws:

- Kinetic energy: $\Theta = \frac{1}{2}\mathbf{u}^2$.
- Momentum / generalized momentum: $\Theta = f(t)u^i, \quad i = 1, 2, 3.$
- Angular momentum: $\Theta = (\mathbf{r} \times \mathbf{u})^i, \quad i = 1, 2, 3.$

Euler Equations in vorticity formulation:

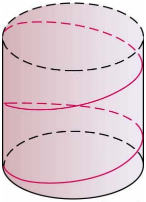
$$\nabla \cdot \mathbf{u} = 0, \quad \boldsymbol{\omega} = \nabla \times \mathbf{u},$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = 0.$$

- Vorticity is conserved: $\Theta = \omega^i, \quad i = 1, 2, 3.$

Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega}.$$



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Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega}.$$

Conservation:

$$\mathrm{D}_t\,(h) + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0,$$

where total energy density is

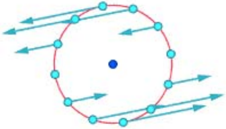
$$E = \frac{1}{2}|\mathbf{u}|^2 + p = \frac{1}{2}\left((u^r)^2 + (u^\eta)^2 + \left(u^\xi\right)^2\right) + p.$$

Euler classical two-component plane flow:

$$u^z = \omega^x = \omega^y = 0;$$

$$\begin{cases} (u^x)_x + (u^y)_y = 0, \\ (u^x)_t + u^x(u^x)_x + u^y(u^x)_y = -p_x, \\ (u^y)_t + u^x(u^y)_x + u^y(u^y)_y = -p_y; \end{cases}$$

$$\begin{cases} \omega^z + (u^x)_y - (u^y)_x = 0, \\ (\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y = 0. \end{cases}$$



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Enstrophy Conservation

- Enstrophy: $\mathcal{E} = |\boldsymbol{\omega}|^2 = (\omega^z)^2$.
- Material conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathrm{D}_t\, \mathcal{E} + \mathrm{D}_x\, (u^x \mathcal{E}) + \mathrm{D}_y\, (u^y \mathcal{E}) = 0.$$

- Was only known to hold for plane flows, $(2+1)$-dimensions.

Conservation Laws of Navier-Stokes Equations

Navier-Stokes Equations equations in $3 + 1$ dimensions

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

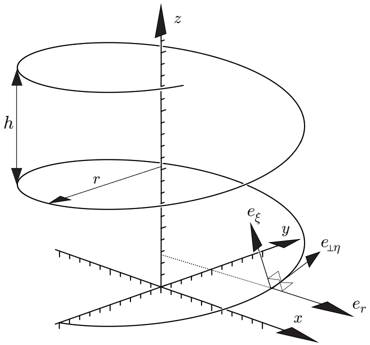
Vorticity formulation:

$$\nabla \cdot \mathbf{u} = 0, \quad \boldsymbol{\omega} = \nabla \times \mathbf{u},$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

Basic conservation laws:

- Momentum / generalized momentum: $\Theta = f(t) u^i, \quad i = 1, 2, 3.$
- Angular momentum: $\Theta = (\mathbf{r} \times \mathbf{u})^i, \quad i = 1, 2, 3.$
- Vorticity: $\Theta = \omega^i, \quad i = 1, 2, 3.$

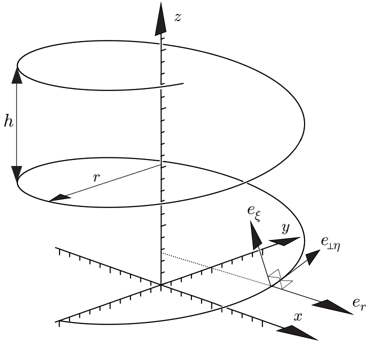


Helical Coordinates

- Cylindrical coordinates: (r, φ, z). **Helical coordinates:** (r, η, ξ)

$$\xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \qquad a, b = \text{const}, \quad a^2 + b^2 > 0.$$

Helical Coordinates

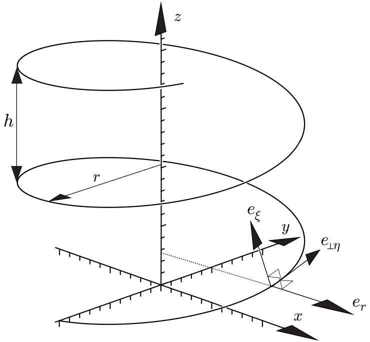


Orthogonal Basis

$$\mathbf{e}_r = \frac{\nabla r}{|\nabla r|}, \quad \mathbf{e}_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad \mathbf{e}_{\perp\eta} = \frac{\nabla_\perp \eta}{|\nabla_\perp \eta|} = \mathbf{e}_\xi \times \mathbf{e}_r.$$

- Scaling factors: $H_r = 1, H_\eta = r, H_\xi = B(r), \qquad B(r) = \dfrac{r}{\sqrt{a^2 r^2 + b^2}}.$

Helical Coordinates

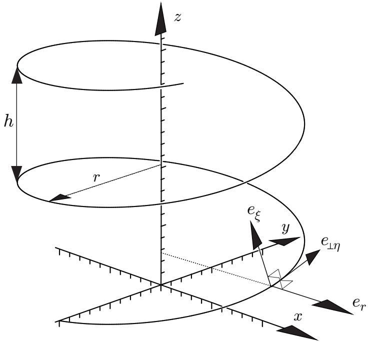


Vector expansion

$$\mathbf{u} \;=\; u^r\mathbf{e}_r + u^\varphi\mathbf{e}_\varphi + u^z\mathbf{e}_z \;=\; u^r\mathbf{e}_r + u^\eta\mathbf{e}_{\perp\eta} + u^\xi\mathbf{e}_\xi .$$

$$u^\eta = \mathbf{u}\cdot\mathbf{e}_{\perp\eta} = B\left(au^\varphi - \frac{b}{r}u^z\right), \qquad u^\xi = \mathbf{u}\cdot\mathbf{e}_\xi = B\left(\frac{b}{r}u^\varphi + au^z\right).$$

Helical Coordinates



Helical invariance: generalizes axal and translational invariance

- Helical coordinates: r, $\xi = az + b\varphi$, $\eta = a\varphi - bz/r^2$.
- **General helical symmetry:** $f = f(r, \xi)$, $a, b \neq 0$.
- **Axial:** $a = 1$, $b = 0$. **z-Translational:** $a = 0$, $b = 1$.

Helically Invariant Navier-Stokes Equations

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

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$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

Continuity:

$$\frac{1}{r} u^r + (u^r)_r + \frac{1}{B}(u^\xi)_\xi = 0$$

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

r-momentum:

$$(u^r)_t + u^r (u^r)_r + \frac{1}{B} u^\xi (u^r)_\xi - \frac{B^2}{r} \left(\frac{b}{r} u^\xi + a u^\eta \right)^2 = -p_r$$

$$+ \nu \left[\frac{1}{r} (r (u^r)_r)_r + \frac{1}{B^2} (u^r)_{\xi\xi} - \frac{1}{r^2} u^r - \frac{2bB}{r^2} \left(a (u^\eta)_\xi + \frac{b}{r} (u^\xi)_\xi \right) \right]$$

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

η-momentum:

$$(u^\eta)_t + u^r (u^\eta)_r + \frac{1}{B} u^\xi (u^\eta)_\xi + \frac{a^2 B^2}{r} u^r u^\eta$$

$$= \nu \left[\frac{1}{r}(r(u^\eta)_r)_r + \frac{1}{B^2}(u^\eta)_{\xi\xi} + \frac{a^2 B^2 (a^2 B^2 - 2)}{r^2} u^\eta + \frac{2abB}{r^2} \left((u^r)_\xi - \left(B u^\xi \right)_r \right) \right]$$

Helically Invariant Navier-Stokes Equations

Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

ξ-momentum:

$$(u^\xi)_t + u^r (u^\xi)_r + \frac{1}{B} u^\xi (u^\xi)_\xi + \frac{2abB^2}{r^2} u^r u^\eta + \frac{b^2 B^2}{r^3} u^r u^\xi = -\frac{1}{B} p_\xi$$
$$+ \nu \left[\frac{1}{r}(r(u^\xi)_r)_r + \frac{1}{B^2}(u^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2} u^\xi + \frac{2bB}{r} \left(\frac{b}{r^2}(u^r)_\xi + \left(\frac{aB}{r} u^\eta \right)_r \right) \right]$$

Helically Invariant Vorticity Formulation

Navier-Stokes Equations, Vorticity Formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

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$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

Vorticity definition:

$$\omega^r = -\frac{1}{B}(u^\eta)_\xi,$$

$$\omega^\eta = \frac{1}{B}(u^r)_\xi - \frac{1}{r}\left(r u^\xi\right)_r - \frac{2abB^2}{r^2}u^\eta + \frac{a^2 B^2}{r}u^\xi,$$

$$\omega^\xi = (u^\eta)_r + \frac{a^2 B^2}{r}u^\eta$$

Navier-Stokes Equations, Vorticity Formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

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r-Momentum:

$$(\omega^r)_t + u_r(\omega^r)_r + \frac{1}{B} u^\xi (\omega^r)_\xi = \omega^r (u^r)_r + \frac{1}{B}\omega^\xi (u^r)_\xi$$
$$+ \nu \left[\frac{1}{r}(r(\omega^r)_r)_r + \frac{1}{B^2}(\omega^r)_{\xi\xi} - \frac{1}{r^2}\omega^r - \frac{2bB}{r^2}\left(a(\omega^\eta)_\xi + \frac{b}{r}(\omega^\xi)_\xi \right) \right]$$

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$$\nabla \cdot \mathbf{u} = 0,$$

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$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

η-Momentum:

$$(\omega^\eta)_t + u^r (\omega^\eta)_r + \frac{1}{B} u^\xi (\omega^\eta)_\xi$$
$$- \frac{a^2 B^2}{r}(u^r \omega^\eta - u^\eta \omega^r) + \frac{2abB^2}{r^2}(u^\xi \omega^r - u^r \omega^\xi) = \omega^r (u^\eta)_r + \frac{1}{B} \omega^\xi (u^\eta)_\xi$$
$$+ \nu \left[\frac{1}{r}(r(\omega^\eta)_r)_r + \frac{1}{B^2}(\omega^\eta)_{\xi\xi} + \frac{a^2 B^2 (a^2 B^2 - 2)}{r^2} \omega^\eta + \frac{2abB}{r^2} \left((\omega^r)_\xi - \left(B \omega^\xi \right)_r \right) \right]$$

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ξ-Momentum:

$$(\omega^\xi)_t + u^r(\omega^\xi)_r + \frac{1}{B} u^\xi (\omega^\xi)_\xi$$
$$+ \frac{1 - a^2 B^2}{r}(u^\xi \omega^r - u^r \omega^\xi) = \omega^r (u^\xi)_r + \frac{1}{B}\omega^\xi (u^\xi)_\xi$$
$$+ \nu \left[\frac{1}{r}(r(\omega^\xi)_r)_r + \frac{1}{B^2}(\omega^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2}\omega^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}(\omega^r)_\xi + \left(\frac{aB}{r}\omega^\eta \right)_r \right) \right]$$

Conservation laws

Independent variables: $\mathbf{x} = (t, x, y, \ldots)$; **dependent variables:** $\mathbf{q} = (q^1, q^2, \ldots)$.

Local conservation law:

$$\boxed{\mathrm{D}_t\Theta + \operatorname*{div}_{x,y,\ldots} \mathbf{\Phi} = 0.}$$

Density: $\Theta(\mathbf{x}, \mathbf{q}, \ldots)$. Spatial fluxes: $\mathbf{\Phi} = (\Phi^1(\mathbf{x}, \mathbf{q}, \ldots), \Phi^2(\mathbf{x}, \mathbf{q}, \ldots), \cdots)$.

Conserved quantities

$$\mathrm{D}_t \int_V \Theta \, dV = 0.$$

Material conservation laws

For incompressible flows with velocity field $\mathbf{u}$, $\operatorname{div}\mathbf{u} = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta \equiv \mathrm{D}_t\Theta + \mathbf{u}\cdot\nabla\Theta = \mathrm{D}_t\Theta + \operatorname*{div}_{x,y,\ldots}\left(\Theta\mathbf{u}\right) = 0.$$

Applications to PDEs

- Direct physical meaning. Constants of motion.
- Analysis: existence, uniqueness, stability.
- Nonlocally related PDE systems, exact solutions. Potentials, stream functions, etc.
- An infinite number of conservation laws can indicate integrability / linearization.
- Fully conserved form of equations is required by modern numerical methods, e.g., Discontinuous Galerkin.

Direct Construction Method [Anco, Bluman (1997,2002)]

- Given: a PDE system $R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N$.
- Specify dependence of multipliers: $\Lambda_\sigma = \Lambda_\sigma(\mathbf{x}, \mathbf{U}, ...), \quad \sigma = 1, ..., N$.
- Solve the determining equations for arbitrary $\mathbf{U}(\mathbf{x})$ (off of solutions)
$$\mathrm{E}_{U^j}(\Lambda_\sigma[\mathbf{U}]R^\sigma[\mathbf{U}]) \equiv 0, \quad j = 1, \ldots, m.$$
- Find the corresponding fluxes $\Phi^i(\mathbf{x}, \mathbf{U}, ...)$ satisfying $\Lambda_\sigma R^\sigma \equiv \mathrm{D}_i\Phi^i$.
- Each set multipliers yields a local conservation law holding on solutions $\mathbf{u}(\mathbf{x})$:
$$\mathrm{D}_i\Phi^i(\mathbf{x}, \mathbf{u}, ...) = 0.$$
- The Direct Method is **complete** for PDE systems that can be written in a **solved form**.

For helically symmetric flows:

- Seek local conservation laws

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{\Phi} \equiv \frac{\partial \Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Phi^r\right) + \frac{1}{B}\frac{\partial \Phi^\xi}{\partial \xi} = 0$$

using divergence expressions

$$\frac{\partial \Gamma^1}{\partial t} + \frac{\partial \Gamma^2}{\partial r} + \frac{\partial \Gamma^3}{\partial \xi} = r\left[\frac{\partial}{\partial t}\left(\frac{\Gamma^1}{r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Gamma^2}{r}\right) + \frac{1}{B}\frac{\partial}{\partial \xi}\left(\frac{B}{r}\Gamma^3\right)\right] = 0,$$

i.e.,

$$\Theta \equiv \frac{\Gamma^1}{r}, \quad \Phi^r \equiv \frac{\Gamma^2}{r}, \quad \Phi^\xi \equiv \frac{B}{r}\Gamma^3.$$

- 1st-order multipliers in primitive variables.
- 0th-order multipliers in vorticity formulation.

Conservation Laws for Helically Symmetric Inviscid Flows: $\nu = 0$

Primitive variables - EP1 - Kinetic energy

$$\Theta = K, \quad \Phi^r = u^r(K+p), \quad \Phi^\xi = u^\xi(K+p), \qquad K = \frac{1}{2}|\mathbf{u}|^2.$$

Primitive variables - EP2 - z-momentum

$$\Theta = B\left(-\frac{b}{r}u^\eta + au^\xi\right) = u^z, \quad \Phi^r = u^r u^z, \quad \Phi^\xi = u^\xi u^z + aBp.$$

Primitive variables - EP3 - z-angular momentum

$$\Theta = rB\left(au^\eta + \frac{b}{r}u^\xi\right) = ru^\varphi, \quad \Phi^r = ru^r u^\varphi, \quad \Phi^\xi = ru^\xi u^\varphi + bBp.$$

Primitive variables - EP4 - Generalized momenta/angular momenta (NEW)

$$\Theta = F\left(\frac{r}{B}u^\eta\right), \quad \Phi^r = u^r F\left(\frac{r}{B}u^\eta\right), \quad \Phi^\xi = u^\xi F\left(\frac{r}{B}u^\eta\right),$$

where $F(\cdot)$ is an arbitrary function.

Vorticity formulation - EV1 - Conservation of helicity

Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi.$$

The conservation law:

$$\begin{aligned} \Theta &= h, \\ \Phi^r &= \omega^r \left(E - (u^\eta)^2 - \left(u^\xi\right)^2 \right) + u^r \left(h - u^r \omega^r \right), \\ \Phi^\xi &= \omega^\xi \left(E - (u^r)^2 - (u^\eta)^2 \right) + u^\xi \left(h - u^\xi \omega^\xi \right), \end{aligned}$$

where

$$E = \frac{1}{2}|\mathbf{u}|^2 + p = \frac{1}{2}\left((u^r)^2 + (u^\eta)^2 + \left(u^\xi\right)^2 \right) + p$$

is the total energy density. In vector notation:

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0.$$

Vorticity formulation - EV2 - Generalized helicity (NEW)

Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi.$$

$$\frac{\partial}{\partial t}\left(h\,H\left(\frac{r}{B}u^\eta\right)\right) + \nabla \cdot \left[H\left(\frac{r}{B}u^\eta\right)\left[\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}\right] + E u^\eta \mathbf{e}_{\perp\eta} \times \nabla H\left(\frac{r}{B}u^\eta\right)\right] = 0$$

for an arbitrary function $H = H(\cdot)$.

Conservation Laws for Helically Symmetric Inviscid Flows: $\nu = 0$

Vorticity formulation - EV3 - Vorticity conservation laws (NEW)

$$\Theta = \frac{Q(t)}{r}\omega^{\varphi},$$

$$\Phi^r = \frac{1}{r}\left(Q(t)[u^r\omega^{\varphi} - \omega^r u^{\varphi}] + Q'(t)u^z\right),$$

$$\Phi^{\xi} = -\frac{aB}{r}\left(Q(t)\left[u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}\right] + Q'(t)u^r\right),$$

where $Q(t)$ is an arbitrary function.

Vorticity formulation - EV4 - Vorticity conservation law (NEW)

$$\Theta = -rB\left(a^3\omega^{\eta} - \frac{b^3}{r^3}\omega^{\xi}\right),$$

$$\Phi^r = -2a^2u^r u^z - a^3Br\left(u^r\omega^{\eta} - u^{\eta}\omega^r\right) + \frac{Bb^3}{r^2}\left(u^r\omega^{\xi} - u^{\xi}\omega^r\right),$$

$$\Phi^{\xi} = a^3B\left[(u^r)^2 + (u^{\eta})^2 - (u^{\xi})^2 + r\left(u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}\right)\right] + \frac{2a^2bB}{r}u^{\eta}u^{\xi}.$$

Conservation Laws for Helically Symmetric Inviscid Flows: $\nu = 0$

Vorticity formulation - EV5 - Vorticity conservation law (NEW)

$$\Theta = -\frac{B}{r^2}\left(\frac{b^2 r^2}{B^2}\omega^\xi + a^3 r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2 r^2}{B^2}\omega^\xi + \frac{a^3 r^4}{B}\omega^z\right),$$

$$\begin{aligned}\Phi^r =& a^3 r B\left(2u^r\left(a u^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right)\\ &- \frac{a^4 r^4 + a^2 r^2 b^2 + b^4}{r\sqrt{a^2 r^2 + b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right),\end{aligned}$$

$$\Phi^\xi = -a^3 b B\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4 r B u^\eta u^\xi.$$

Vorticity formulation - EV6 - Vorticity conservation law (NEW)

$$\nabla\cdot\mathbf{\Phi} = 0, \quad \Phi^r = N\omega^r - \frac{1}{B}N_\xi u^\eta, \quad \Phi^\xi = N\omega^\xi,$$

for an arbitrary $N(t,\xi)$.

- Generalization of the obvious divergence expression $\nabla\cdot(G(t)\boldsymbol{\omega}) = 0$.

Primitive variables - NSP1 - z-momentum.

$$\Theta = u^z, \quad \Phi^r = u^r u^z - \nu (u^z)_r, \quad \Phi^\xi = u^\xi u^z + aBp - \frac{\nu}{B}(u^z)_\xi.$$

Primitive variables - NSP2 - generalized momentum (NEW)

$$\Theta = \frac{r}{B} u^\eta,$$

$$\begin{aligned}\Phi^r &= \frac{r}{B} u^r u^\eta - \nu \left[-2aB \left(a u^\eta + 2\frac{b}{r} u^\xi \right) + \left(\frac{r}{B} u^\eta \right)_r \right] \\ &= \frac{r}{B} u^r u^\eta - \nu \left[-2a u^\varphi + \left(\frac{r}{B} u^\eta \right)_r \right],\end{aligned}$$

$$\Phi^\xi = \frac{r}{B} u^\eta u^\xi - \nu \frac{1}{B} \left[\frac{2abB^2}{r} u^r + \left(\frac{r}{B} u^\eta \right)_\xi \right].$$

Vorticity formulation - NSV1 - Family of vorticity conservation laws (NEW)

$$\Theta = \frac{Q(t)}{r}B\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) = \frac{Q(t)}{r}\omega^\varphi,$$

$$\Phi^r = \frac{1}{r}\Big\{Q(t)\left[u^r B\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) - \omega^r B\left(au^\eta + \frac{b}{r}u^\xi\right)\right] + Q'(t)B\left(-\frac{b}{r}u^\eta + au^\xi\right)$$
$$-Q(t)\nu\left[\frac{aB}{r}\omega^\eta + \frac{b^2 B}{r\left(a^2r^2 + b^2\right)}\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) + B\left(a\omega^\eta_r + \frac{b}{r}\omega^\xi_r\right)\right]\Big\},$$

$$\Phi^\xi = -\frac{B}{r}\Big\{aQ(t)\left[u^\eta\omega^\xi - u^\xi\omega^\eta\right] + aQ'(t)u^r$$
$$+\frac{Q(t)}{r^3}\nu\left[\frac{r^3}{B}\left(a\omega^\eta_\xi + \frac{b}{r}\omega^\xi_\xi\right) + 2br\omega^r\right]\Big\},$$

for an arbitrary function where $Q(t)$.

Vorticity formulation - NSV2 - Vorticity conservation law (NEW)

$$\Theta = \quad -rB\left(a^3\omega^\eta - \frac{b^3}{r^3}\omega^\xi\right),$$

$$\Phi^r = \quad -\frac{B}{r^2}\left(a^3r^3\left(u^r\omega^\eta - u^\eta\omega^r\right) - b^3\left(u^r\omega^\xi - u^\xi\omega^r\right)\right) - 2a^2Bu^r\left(-\frac{b}{r}u^\eta + au^\xi\right)$$
$$-\frac{B}{r^2}\nu\left[\frac{r^2}{B^2}\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) - r^3\left(a^3\omega^\eta_r - \frac{b^3}{r^3}\omega^\xi_r\right) + abB^2r\left(\frac{b^3}{r^3}\omega^\eta + a^3\omega^\xi\right)\right],$$

$$\Phi^\xi = \quad a^3B\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + \frac{2a^2bB}{r}u^\eta u^\xi$$
$$+\frac{2a^2bB}{r}\nu\left[\left(1 - \frac{b^2}{a^2r^2}\right)\omega^r + \frac{r^2}{2a^2bB}\left(a^3\omega^\eta_\xi - \frac{b^3}{r^3}\omega^\xi_\xi\right)\right].$$

Vorticity formulation - NSV3 - Vorticity conservation law (NEW)

$$\Theta = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + a^3r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + \frac{a^3r^4}{B}\omega^z\right),$$

$$\begin{aligned}\Phi^r = \; & a^3rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right) \\ & -\frac{a^4r^4 + a^2r^2b^2 + b^4}{r\sqrt{a^2r^2+b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right) \\ & +\nu\left[4a^3B\left(au^\eta + \frac{b}{r}u^\xi\right) - a^3brB(\omega^\eta)_r + \frac{B}{r^3}\left(b^4 - a^4r^4 - \frac{a^6r^6}{a^2r^2+b^2}\right)\omega^\xi\right. \\ & \left. + \frac{B}{r^2}\left(a^4r^4 + a^2r^2b^2 + b^4\right)(\omega^\xi)_r + \frac{ab}{B}\left(2 + \frac{a^4r^4}{(a^2r^2+b^2)^2}\right)\omega^\eta\right],\end{aligned}$$

$$\begin{aligned}\Phi^\xi = \; & -a^3bB\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4rBu^\eta u^\xi \\ & +\nu\left[\frac{1}{r^2}\left(a^4r^4 + a^2r^2b^2 + b^4\right)(\omega^\xi)_\xi - a^3br(\omega^\eta)_\xi - \frac{4a^3bB}{r}u^r + \frac{2b^4B}{r^3}\omega^r\right].\end{aligned}$$

Generalized enstrophy for inviscid plane flow (known)

$$\Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

Some Conservation Laws for Two-Component Flows

Generalized enstrophy for inviscid plane flow (known)

$$\Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

Generalized enstrophy for inviscid axisymmetric flow (NEW)

$$\Theta = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right)$$

for arbitrary $S(\cdot)$.

Some Conservation Laws for Two-Component Flows

Generalized enstrophy for inviscid plane flow (known)

$$\Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

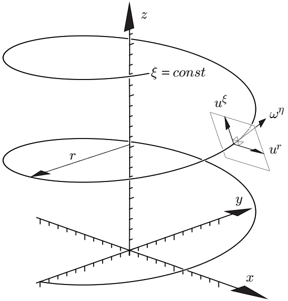
$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

Generalized enstrophy for inviscid axisymmetric flow (NEW)

$$\Theta = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right)$$

for arbitrary $S(\cdot)$.

- Several additional new conservation laws for **plane** and **axisymmetric**, inviscid and viscous flows (details in paper).



Generalized enstrophy for general inviscid helical 2-component flow (NEW)

$$\Theta = T\left(\frac{B}{r}\omega^\eta\right), \quad \Phi^r = u^r T\left(\frac{B}{r}\omega^\eta\right), \quad \Phi^\xi = u^\xi T\left(\frac{B}{r}\omega^\eta\right),$$

for an arbitrary $T(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\, T\left(\frac{B}{r}\omega^\eta\right) = 0.$$

Results and Open Problems

Helically-Invariant Equations

- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

New Conservation Laws

- Three-component Euler:
  - Generalized momenta. Generalized helicity. Additional vorticity CLs.
- Three-component Navier-Stokes:
  - New CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - New CLs in viscous and inviscid case, for plane and axisymmetric flows.

Open problems

- Understand the nature of the new CLs.
- Explore the usefulness of the new CLs for numerical simulation and analysis (e.g., computing stability conditions for equilibria).

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Thank you for your attention!