On Conservation Laws and Potential Systems of Vorticity-Type Equations

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Introduction


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Notation

Derivatives

- Independent variables: $x = (x^1, \dots, x^n) = (x, y, z, \dots)$.
- Dependent variables: $u = (u^1(x), \dots, u^m(x)) = (u, v, w, \dots)$.
- Partial derivatives:

$$\frac{\partial u^j}{\partial x} = u^j_x = u_1; \quad \frac{\partial^2 u^j}{\partial x \partial y} = u^j_{xy} = u^j_{12}.$$

- All 1st-order and kth-order partial derivatives:

$$\partial u = \underset{1}{u}, \qquad \partial^k u = \underset{k}{u}.$$

Total derivative (chain rule)

- Let $F = F(x, u, \partial u, \dots, \partial^q u)$.
- Total derivative: $D_i F = \dfrac{\partial}{\partial x^i} + u^\mu_i \dfrac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \dfrac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{ii_1 i_2} \dfrac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots$
- Summation assumed here and in many other places.

Outline

Conservation Laws and Potential Systems in 2D

- Given PDE system:

$$R^\sigma[u] = R^\sigma(x, t, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

- A conservation law:

$$\mathrm{div}_{(t,x)}(\Theta[u], \Phi[u]) \equiv \boxed{D_t(\Theta[u]) + D_x(\Phi[u]) = 0.}$$

- The corresponding potential system:

$$v_x = \Theta[u], \quad v_t = -\Phi[u].$$

Example:

- $u = u(x, t)$,
- the nonlinear wave equation $u_{tt} = (c^2(u)u_x)_x$.

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- A couple of conservation laws:

$$D_t(u_t) - D_x(c^2(u)u_x) = 0, \qquad D_t(tu_t - u) - D_x(tc^2(u)u_x) = 0.$$

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- Potential systems:

$$v^1_x = u_t, \qquad v^2_x = tu_t - u,$$
$$v^1_t = c^2(u). \qquad v^2_t = tc^2(u)u_x.$$

Conservation Laws and Potential Systems in 2D – Differential-Geometric Notation

- Given PDE system: $u = u(x,t)$;

$$R^\sigma[u] = R^\sigma(x, t, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

Definition

A conservation law is a differential form $\omega^{(r)}[U]$ whose exterior derivative vanishes on solutions $U = u$ of $\{R^\sigma[u] = 0\}$.

$$\omega^{(1)}[U] = \Theta[U]\,\mathrm{d}x - \Phi[U]\,\mathrm{d}t \quad \to \quad \Omega^{(2)} = \mathrm{d}\omega^{(1)} = (D_t(\Theta[U]) + D_x(\Phi[U]))\,\mathrm{d}t \wedge \mathrm{d}x.$$

- On solutions, $\Omega^{(2)}[u] = \mathrm{d}\omega^{(1)}[u] = 0$, hence locally, $\omega^{(1)}[u] = \mathrm{d}\widetilde{\omega}^{(0)}$:

$$\widetilde{\omega}^{(0)} = v(x,t); \quad \mathrm{d}\widetilde{\omega}^{(0)} = v_x \mathrm{d}x + v_t \mathrm{d}t = \omega^{(1)}[u] = \Theta[u]\mathrm{d}x + (-\Phi[u])\mathrm{d}t.$$

- The potential system: $v_x = \Theta[u], \quad v_t = -\Phi[u]$.

Conservation Laws and Potential Systems in 2D – Differential-Geometric Notation

- Given PDE system: $u = u(x, t)$;

$$R^\sigma[u] = R^\sigma(x, t, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

- $\widetilde{\omega}^{(0)} = v(x, t)$: potential(s).
- $\omega^{(1)}[u] = \Theta[u]\,\mathrm{d}x - \Phi[u]\,\mathrm{d}t$: density/flux(es).
- $\Omega^{(2)}[u] = (D_t\Theta + D_x\Phi)\,\mathrm{d}t \wedge \mathrm{d}x$: conserved form.

$$\widetilde{\omega}^{(0)} \stackrel{\mathrm{d}}{\to} \omega^{(1)}[u] \stackrel{\mathrm{d}}{\to} \Omega^{(2)}[u] \stackrel{\mathrm{d}}{\to} 0.$$

- Given PDE system: $u = u(x, y, z)$,

$$R^{\sigma}[u] = R^{\sigma}(x, y, z,\ u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

(a) Divergence-type conservation laws:

$$\boxed{\operatorname{div} \mathbf{\Phi}[u] = \Phi^1_x[u] + \Phi^2_y[u] + \Phi^3_z[u] = 0.}$$

- Potential equations:

$$\operatorname{curl} \mathbf{\Gamma}(x, y, z) = \mathbf{\Phi}[u].$$

- Gauge freedom:

$$\mathbf{\Gamma} \to \mathbf{\Gamma} + \operatorname{grad}\ \phi(x, y, z)$$

$\Rightarrow$ under-determined potential system.

Theorem (Anco, Bluman (1997))

Every local symmetry of an under-determined potential system projects onto a local symmetry of the given PDE system.

- Some common gauge constraints in $n = 3$ dimensions:
  - Divergence (Coulomb) gauge: $\operatorname{div} \mathbf{\Gamma} \equiv \Gamma^1_x + \Gamma^2_y + \Gamma^3_z = 0$.
  - Spatial gauge: $\Gamma^k = 0$, $k = 1$ or 2 or 3.
  - Poincaré gauge: $x\Gamma^1 + y\Gamma^2 + z\Gamma^3 = 0$.
- For time-dependent systems in 2+1 dimensions, $x = (t, x, y)$:
  - Lorentz gauge: $\Gamma^1_t - \Gamma^2_x - \Gamma^3_y = 0$.
  - Cronstrom gauge: $t\Gamma^1 - x\Gamma^2 - y\Gamma^3 = 0$.

- Given PDE system: $u = u(x, y, z)$,

$$R^\sigma[u] = R^\sigma(x, y, z,\ u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

(a) Divergence-type conservation laws:

$$\boxed{\operatorname{div} \mathbf{\Phi}[u] = \Phi^1_x[u] + \Phi^2_y[u] + \Phi^3_z[u] = 0.}$$

- $\widetilde{\omega}^{(1)} = \Gamma^1\, \mathrm{d}x + \Gamma^2\, \mathrm{d}y + \Gamma^3\, \mathrm{d}z$: potential.
- $\omega^{(2)}[u] = \Phi^1\, \mathrm{d}y \wedge \mathrm{d}z + \Phi^2\, \mathrm{d}z \wedge \mathrm{d}x + \Phi^3\, \mathrm{d}x \wedge \mathrm{d}y$: fluxes.
- $\Omega^{(3)}[u] = (\Phi^1_x + \Phi^2_y + \Phi^3_z)\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$: conserved form.

$$\widetilde{\omega}^{(1)} \xrightarrow{\mathrm{d}} \omega^{(2)}[u] \xrightarrow{\mathrm{d}} \Omega^{(3)}[u] \xrightarrow{\mathrm{d}} 0.$$

- Given PDE system: $u = u(x, y, z)$,

$$R^\sigma[u] = R^\sigma(x, y, z,\ u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

(a) Divergence-type conservation laws:

$$\boxed{\operatorname{div} \boldsymbol{\Phi}[u] = \Phi^1_x[u] + \Phi^2_y[u] + \Phi^3_z[u] = 0.}$$

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- $\omega^{(2)}[u] = \Phi^1\, \mathrm{d}y \wedge \mathrm{d}z + \Phi^2\, \mathrm{d}z \wedge \mathrm{d}x + \Phi^3\, \mathrm{d}x \wedge \mathrm{d}y$: fluxes.
- $\Omega^{(3)}[u] = (\Phi^1_x + \Phi^2_y + \Phi^3_z)\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$: conserved form.

- Potential equations: $\mathrm{d}\widetilde{\omega}^{(1)} = \omega^{(2)}[u] \quad \Rightarrow \quad \operatorname{curl} \boldsymbol{\Gamma} = \boldsymbol{\Phi}[u].$

$$\Gamma^3_y - \Gamma^2_z = \Phi^1, \quad \Gamma^1_z - \Gamma^3_x = \Phi^2, \quad \Gamma^2_x - \Gamma^1_y = \Phi^3.$$

- Given PDE system: $u = u(x, y, z)$,

$$R^\sigma[u] = R^\sigma(x, y, z,\; u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

(b) Curl-type conservation laws:

$$\boxed{\mathrm{curl}\,\mathbf{B}[u] = 0.}$$

- Potential equations:

$$\mathrm{grad}\;\phi(x, y, z) = \mathbf{B}[u].$$

- No gauge freedom: scalar potential.

$$\phi \to \phi + \mathrm{const}.$$

$\Rightarrow$ determined potential system.

- Given PDE system: $u = u(x, y, z)$,

$$R^\sigma[u] = R^\sigma(x, y, z, \; u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

(b) Curl-type conservation laws:

$$\boxed{\operatorname{curl} \mathbf{B}[u] = 0.}$$

- $\widetilde{\omega}^{(0)} = \phi$: potential.
- $\omega^{(1)}[u] = B^1 \,\mathrm{d}x + B^2 \,\mathrm{d}y + B^3 \,\mathrm{d}z$: fluxes.
- $\Omega^{(2)}[u] = (\mathrm{D}_y B^3 - \mathrm{D}_z B^2)\mathrm{d}y \wedge \mathrm{d}z + (\mathrm{D}_z B^1 - \mathrm{D}_x B^3)\mathrm{d}z \wedge \mathrm{d}x + (\mathrm{D}_x B^2 - \mathrm{D}_y B^1)\mathrm{d}x \wedge \mathrm{d}y$: conserved form.

$$\widetilde{\omega}^{(0)} \;\xrightarrow{\mathrm{d}}\; \omega^{(1)}[u] \;\xrightarrow{\mathrm{d}}\; \Omega^{(2)}[u].$$

- Given PDE system: $u = u(x, y, z)$,

$$R^\sigma[u] = R^\sigma(x, y, z,\ u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N.$$

(b) Curl-type conservation laws:

$$\boxed{\operatorname{curl} \mathbf{B}[u] = 0.}$$

- Curl-type conservation laws in applications:
  - Irrotational flows: $\operatorname{curl} \mathbf{V} = 0.$
  - Ideal MHD equilibrium equations: $\operatorname{curl} (\mathbf{V} \times \mathbf{B}) = 0.$

- $x = (x^1, \ldots, x^n) \in \mathbb{R}^n, \quad u = u(x).$
- Conservation laws of degree $k = 1, \ldots, n-1$, given by $\omega^{(k)}[u]$, may exist.
- $\widetilde{\omega}^{(k-1)}$: potential(s).
- $\Omega^{(k+1)}[u]$: conserved form.

CL degree							
1	$\widetilde{\omega}^{(0)} \to$	$\omega^{(1)} \to$	$\Omega^{(2)}$				
2		$\widetilde{\omega}^{(1)} \to$	$\omega^{(2)} \to$	$\Omega^{(3)}$			
...		...	...	...			
n − 1					... $\widetilde{\omega}^{(n-2)} \to$	$\omega^{(n-1)} \to$	$\Omega^{(n)}$

Outline

"Vorticity-type Equations":

- Independent variables: t, x, y, z.
- Vector fields: $\mathbf{N}, \mathbf{M} \in \mathbb{R}^3$.

$$\boxed{\operatorname{div} \mathbf{N} = 0, \quad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0.}$$

Some applications:

- Maxwell equations;
- Hydrodynamics/vorticity equations;
- Plasma dynamics / Magnetohydrodynamics (MHD).

- Denote the four scalar PDEs $\boxed{\operatorname{div}\mathbf{N}=0, \quad \mathbf{N}_t+\operatorname{curl}\mathbf{M}=0}$ by

$$E^1 = N^1_x + N^2_y + N^3_z, \qquad E^2 = N^1_t + M^3_y - M^2_z,$$
$$E^3 = N^2_t + M^1_z - M^3_x, \qquad E^4 = N^3_t + M^2_x - M^1_y.$$

- Consider a differential two-form

$$\begin{aligned}\omega^{(2)} = \quad & -M^1\,\mathrm{d}t\wedge\mathrm{d}x - M^2\,\mathrm{d}t\wedge\mathrm{d}y - M^3\,\mathrm{d}t\wedge\mathrm{d}z\\ & +N^3\,\mathrm{d}x\wedge\mathrm{d}y + N^2\,\mathrm{d}z\wedge\mathrm{d}x + N^1\,\mathrm{d}y\wedge\mathrm{d}z.\end{aligned}$$

- Then the exterior derivative $\Omega^{(3)}=\mathrm{d}\omega^{(2)}$ (off solutions) is given by

$$\begin{aligned}\Omega^{(3)} = \quad & E^1\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + E^2\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}t\\ & -E^3\,\mathrm{d}z\wedge\mathrm{d}t\wedge\mathrm{d}x + E^4\,\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y,\end{aligned}$$

- On solutions, $\Omega^{(3)}=\mathrm{d}\omega^{(2)}=0$, hence one has a conservation law of degree two.

- Denote the four scalar PDEs $\boxed{\operatorname{div} \mathbf{N} = 0, \quad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0}$ by

$$E^1 = N^1_x + N^2_y + N^3_z, \qquad E^2 = N^1_t + M^3_y - M^2_z,$$
$$E^3 = N^2_t + M^1_z - M^3_x, \qquad E^4 = N^3_t + M^2_x - M^1_y.$$

- Consider a differential two-form

$$\omega^{(2)} = -M^1\,dt \wedge dx - M^2\,dt \wedge dy - M^3\,dt \wedge dz$$
$$+N^3\,dx \wedge dy + N^2\,dz \wedge dx + N^1\,dy \wedge dz.$$

- The "vorticity tensor" (parallel to the "electromagnetic tensor"):

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -M^1 & -M^2 & -M^3 \\ M^1 & 0 & N^3 & -N^2 \\ M^2 & -B^3 & 0 & N^1 \\ M^3 & N^2 & -N^1 & 0 \end{pmatrix}.$$

The Vorticity System as a Lower-Degree Conservation Law

- Denote the four scalar PDEs $\boxed{\operatorname{div}\mathbf{N}=0, \quad \mathbf{N}_t+\operatorname{curl}\mathbf{M}=0}$ by

$$E^1 = N^1_x + N^2_y + N^3_z, \qquad E^2 = N^1_t + M^3_y - M^2_z,$$
$$E^3 = N^2_t + M^1_z - M^3_x, \qquad E^4 = N^3_t + M^2_x - M^1_y.$$

- Consider a differential two-form

$$\begin{aligned}\omega^{(2)} = \quad & -M^1\,\mathrm{d}t\wedge\mathrm{d}x - M^2\,\mathrm{d}t\wedge\mathrm{d}y - M^3\,\mathrm{d}t\wedge\mathrm{d}z\\ & + N^3\,\mathrm{d}x\wedge\mathrm{d}y + N^2\,\mathrm{d}z\wedge\mathrm{d}x + N^1\,\mathrm{d}y\wedge\mathrm{d}z.\end{aligned}$$

- On solutions, $\Omega^{(3)}=\mathrm{d}\omega^{(2)}=0$, hence $\omega^{(2)}=\mathrm{d}\widetilde{\omega}^{(1)}$ for the potential 1-form

$$\widetilde{\omega}^{(1)} = \theta^t(t,x,y,z)\,\mathrm{d}t + \theta^x(t,x,y,z)\,\mathrm{d}x + \theta^y(t,x,y,z)\,\mathrm{d}y + \theta^z(t,x,y,z)\,\mathrm{d}z.$$

- Potential equations:

$$\begin{array}{lll} -M^1 = \theta^x_t - \theta^t_x, & -M^2 = \theta^y_t - \theta^t_y, & -M^3 = \theta^z_t - \theta^t_z,\\ N^1 = \theta^z_y - \theta^y_z, & N^2 = \theta^x_z - \theta^z_x, & N^3 = \theta^y_x - \theta^x_y. \end{array}$$

The Potentials for the Vorticity System

Result:

The vorticity PDEs $\boxed{\operatorname{div}\mathbf{N} = 0, \quad \mathbf{N}_t + \operatorname{curl}\mathbf{M} = 0}$ form a conservation law of degree two.

The potential equations are given by

$$\begin{array}{lll} -M^1 = \theta^x_t - \theta^t_x, & -M^2 = \theta^y_t - \theta^t_y, & -M^3 = \theta^z_t - \theta^t_z, \\ N^1 = \theta^z_y - \theta^y_z, & N^2 = \theta^x_z - \theta^z_x, & N^3 = \theta^y_x - \theta^x_y. \end{array}$$

Note

The potential equations are under-determined. Gauge symmetry:

$$\theta \to \theta + \mathrm{d}f$$

for an arbitrary scalar function $f(t, x, y, z)$.

Outline

Example A: Maxwell's Equations

The dimensionless PDE system of Maxwell's equations:

$$\operatorname{div}\mathbf{B} = 0, \quad \mathbf{B}_t = -\operatorname{curl}\mathbf{E},$$

$$\mathbf{E}_t = \operatorname{curl}\mathbf{B} - \mathbf{J}, \quad \operatorname{div}\mathbf{E} = \rho,$$

the charge density ρ, the magnetic field, the electric field and the current density $\mathbf{B}, \mathbf{E}, \mathbf{J} \in \mathbb{R}^3$ are functions of t, x, y, z.

- In the required form $\boxed{\operatorname{div}\mathbf{N} = 0, \quad \mathbf{N}_t + \operatorname{curl}\mathbf{M} = 0,}$
- $\mathbf{N} = \mathbf{B}, \quad \mathbf{M} = \mathbf{E}.$

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the charge density ρ, the magnetic field, the electric field and the current density $\mathbf{B}, \mathbf{E}, \mathbf{J} \in \mathbb{R}^3$ are functions of t, x, y, z.

- Electromagnetic field tensor F in the 4D Minkowski spacetime $(x^0, x^1, x^2, x^3) = (t, x, y, z)$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}.$$

- The blue equations $\Leftrightarrow$ $\mathrm{d}F = 0$.

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the charge density ρ, the magnetic field, the electric field and the current density $\mathbf{B}, \mathbf{E}, \mathbf{J} \in \mathbb{R}^3$ are functions of t, x, y, z.

- Potential equations $F = \mathrm{d}\,\widetilde{\omega}^{(1)}$:

$$\widetilde{\omega}^{(1)} = \theta^t\,\mathrm{d}t + \theta^x\,\mathrm{d}x + \theta^y\,\mathrm{d}y + \theta^z\,\mathrm{d}z.$$

$$(\theta^t, \theta^x, \theta^y, \theta^z) = (\Theta, \mathbf{A})$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \operatorname{grad} \Theta(t, x, y, z) = \mathbf{A}_t + \mathbf{E}.$$

The dimensionless PDE system of Vacuum Maxwell's equations:

$$\operatorname{div} \mathbf{B} = 0, \quad \mathbf{B}_t = -\operatorname{curl} \mathbf{E},$$

$$\mathbf{E}_t = \operatorname{curl} \mathbf{B}, \quad \operatorname{div} \mathbf{E} = 0.$$

- In an alternative required form $\boxed{\operatorname{div} \mathbf{N} = 0, \quad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0,}$
- $\mathbf{N} = \mathbf{E}, \quad \mathbf{M} = -\mathbf{B}.$

The dimensionless PDE system of Vacuum Maxwell's equations:

$$\operatorname{div}\mathbf{B} = 0, \quad \mathbf{B}_t = -\operatorname{curl}\mathbf{E},$$

$$\mathbf{E}_t = \operatorname{curl}\mathbf{B}, \quad \operatorname{div}\mathbf{E} = 0.$$

- An additional lower-degree conservation law $\mathrm{d} *F = 0$.
- Dual electromagnetic field tensor:

$$*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta},$$

$$\eta^{\mu\nu} = \operatorname{diag}(-1,1,1,1).$$

- Vacuum Maxwell's equations are symmetrically written as two conservation laws of degree two

$$\mathrm{d}F = 0, \quad \mathrm{d} *F = 0.$$

Example B: Vorticity Equations of Fluid Dynamics

Euler and Navier-Stokes Equations in 3D:

- Incompressible constant-density viscous fluid flow, no external forcing:

$$\mathrm{div}\ \mathbf{V} = 0,$$

$$\mathbf{V}_t + \mathrm{curl}\,\mathbf{V} \times \mathbf{V} + \mathrm{grad}\,\left(p + \frac{|\mathbf{V}|^2}{2}\right) = \nu\,\nabla^2\mathbf{V}.$$

- Vorticity formulation: $\mathbf{w} = \mathrm{curl}\ \mathbf{V}$.

$$\mathrm{div}\ \mathbf{w} = 0, \quad \mathbf{w}_t + \mathrm{curl}\,(\mathbf{w} \times \mathbf{V} - \nu\,\nabla^2\mathbf{V}) = 0.$$

- In the required form $\boxed{\mathrm{div}\,\mathbf{N} = 0, \quad \mathbf{N}_t + \mathrm{curl}\,\mathbf{M} = 0,}$
- $\mathbf{N}_w = \mathbf{w}, \quad \mathbf{M}_w = \mathbf{w} \times \mathbf{V} - \nu\,\nabla^2\mathbf{V}.$

Euler and Navier-Stokes Equations in 3D:

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- Vorticity formulation: $\mathbf{w} = \mathrm{curl}\ \mathbf{V}$.

$$\mathrm{div}\ \mathbf{w} = 0, \quad \mathbf{w}_t + \mathrm{curl}\,(\mathbf{w} \times \mathbf{V} - \nu\,\nabla^2\mathbf{V}) = 0.$$

- Conservation law of degree two: $\mathrm{d}\,\omega^{(2)}_{fluid} = 0$,

$$(\omega_{fluid})_{\mu\nu} = \begin{pmatrix} 0 & -M^1_w & -M^2_w & -M^3_w \\ M^1_w & 0 & N^3_w & -N^2_w \\ M^2_w & -N^3_w & 0 & N^1_w \\ M^3_w & N^2_w & -N^1_w & 0 \end{pmatrix}.$$

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- Incompressible constant-density viscous fluid flow, no external forcing:

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- Vorticity formulation: $\mathbf{w} = \mathrm{curl}\ \mathbf{V}$.

$$\mathrm{div}\ \mathbf{w} = 0, \quad \mathbf{w}_t + \mathrm{curl}\,(\mathbf{w} \times \mathbf{V} - \nu\,\nabla^2\mathbf{V}) = 0.$$

- Potential equations $\omega^{(2)}_{fluid} = \mathrm{d}\,\widetilde{\omega}^{(1)} = \mathrm{d}\,(\theta^t\,\mathrm{d}t + \theta^x\,\mathrm{d}x + \theta^y\,\mathrm{d}y + \theta^z\,\mathrm{d}z.)$
  - $\mathbf{q} := (\theta^x, \theta^y, \theta^z)$;
  - $\mathrm{curl}\ \mathbf{q} = \mathbf{w}, \quad \Rightarrow \quad \mathbf{q} = \mathbf{V} + \mathrm{grad}\,\chi$;
  - $\theta^t = -p - |\mathbf{V}|^2/2 + \chi_t \quad \Rightarrow \quad \mathbf{V}_t + \mathrm{grad}\,(p + |\mathbf{V}|^2/2) = -(\mathbf{w} \times \mathbf{V} - \nu\nabla^2\mathbf{V})$.

- Potentialization $\Leftrightarrow$ inversion of the spatial curl operator.

Example C: Magnetohydrodynamics Equations

MHD Equations in 3D:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

$$\rho \mathbf{V}_t + \rho \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{|\mathbf{V}|^2}{2} + \mu_1 \nabla^2 \mathbf{V},$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

- $\mu, \mu_1, \eta, \sigma = \text{const.}$
- Plasma parameters depend on t, x, y, z.
  - ρ: plasma density. $\mathbf{V} = (V^1, V^2, V^3)$: velocity.
  - $\mathbf{B} = (B^1, B^2, B^3)$: magnetic field. P: pressure.

Example C: Magnetohydrodynamics Equations

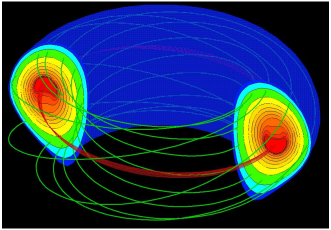
MHD Equations in 3D:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

$$\rho \mathbf{V}_t + \rho \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{|\mathbf{V}|^2}{2} + \mu_1 \nabla^2 \mathbf{V},$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

- Adequate description of industrial/laboratory plasmas...



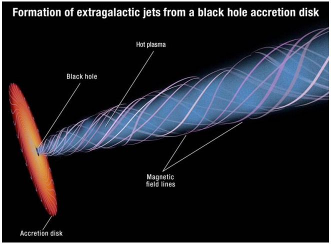
MHD Equations in 3D:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

$$\rho \mathbf{V}_t + \rho \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{|\mathbf{V}|^2}{2} + \mu_1 \nabla^2 \mathbf{V},$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

- ... as well as astrophysical ones.



Ideal MHD equilibrium equations:

$$\operatorname{div} \mathbf{V} = 0, \quad \operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0,$$

$$\rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{|\mathbf{V}^2|}{2} = 0,$$

- Magnetic flux function: $\mathbf{V} \times \mathbf{B} = \operatorname{grad} \Psi(x, y, z)$.
- Magnetic surfaces: $\Psi = \text{const}, \quad \mathbf{B}, \mathbf{V} \perp \operatorname{grad} \Psi(x, y, z)$.

Ideal MHD equilibrium equations:

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- Assume ρ is constant on both magnetic field lines and streamlines ($\operatorname{grad} \rho \perp \mathbf{V}, \mathbf{B}$).
- Galas-Bogoyavlenskij symmetries:

$$\mathbf{B}_1 = b\mathbf{B} + c\sqrt{\mu\rho}\mathbf{V}, \qquad \mathbf{V}_1 = \frac{c}{a\sqrt{\mu\rho}}\mathbf{B} + \frac{b}{a}\mathbf{V},$$

$$\rho_1 = a^2(\mathbf{r})\rho, \qquad P_1 = CP + (C\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu).$$

- Here a, b, c are functions of (x, y, z) constant on both magnetic field lines and streamlines,

$$b^2 - c^2 = C = \text{const}.$$

- An infinite set of nonlocal symmetries.
- Led to the construction of important classes of physical solutions.

MHD Equations in 3D:

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(ctd.) Example C: Magnetohydrodynamics Equations

MHD Equations in 3D:

$$\rho_t + \mathrm{div}\,\rho\mathbf{V} = 0, \qquad \mathrm{div}\,\mathbf{B} = 0,$$

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$$\mathbf{B}_t = \mathrm{curl}(\mathbf{V}\times\mathbf{B}) + \eta\,\nabla^2\mathbf{B}.$$

- $\mathbf{N}_m = \mathbf{B}, \qquad \mathbf{M}_m = \mathbf{B}\times\mathbf{V} - \eta\,\nabla^2\mathbf{A} \qquad (\mathbf{B} = \mathrm{curl}\,\mathbf{A}).$
- Conservation law of degree two: $\mathrm{d}\,\omega^{(2)}_{MHD} = 0$. The MHD tensor:

$$(\omega_{MHD})_{\mu\nu} = \begin{pmatrix} 0 & -M_m^1 & -M_m^2 & -M_m^3 \\ M_m^1 & 0 & N_m^3 & -N_m^2 \\ M_m^2 & -N_m^3 & 0 & N_m^1 \\ M_m^3 & N_m^2 & -N_m^1 & 0 \end{pmatrix}.$$

MHD Equations in 3D:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

$$\rho \mathbf{V}_t + \rho \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{|\mathbf{V}|^2}{2} + \mu_1 \nabla^2 \mathbf{V},$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

- Potential equations: $\omega_{MHD} = \mathrm{d}\, \widetilde{\omega}^{(1)}$:

$$\widetilde{\omega}^{(1)} = \theta^t \, \mathrm{d}t + \theta^x \, \mathrm{d}x + \theta^y \, \mathrm{d}y + \theta^z \, \mathrm{d}z.$$

$$(\theta^x, \theta^y, \theta^z) = \mathbf{A}, \quad \theta^t = -\Psi.$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \operatorname{grad} \Psi = \mathbf{V} \times \mathbf{B} - \mathbf{A}_t - \eta \operatorname{curl} \mathbf{B}.$$

(ctd.) Example C: Magnetohydrodynamics Equations

MHD Equations in 3D:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

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$$(\theta^x, \theta^y, \theta^z) = \mathbf{A}, \quad \theta^t = -\Psi.$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \underbrace{\operatorname{grad} \Psi = \mathbf{V} \times \mathbf{B}}_{\text{Galas-Bogoyavlenskij}} - \mathbf{A}_t - \eta \operatorname{curl} \mathbf{B}.$$

Outline

Another result (by Direct CL Construction):

- The vorticity PDEs $\boxed{\operatorname{div}\mathbf{N} = 0, \quad \mathbf{N}_t + \operatorname{curl}\mathbf{M} = 0}$.
- Admitted multipliers: $\Lambda^1 = -F_t, \quad \Lambda^2 = F_x, \quad \Lambda^3 = F_y, \quad \Lambda^4 = F_z.$
- $F(t,x,y,z)$: arbitrary function.
- An infinite family of divergence-type conservation laws:

$$\boxed{(\mathbf{N}\cdot\nabla F)_t + \operatorname{div}(\mathbf{M}\times\nabla F - F_t\,\mathbf{N}) = 0.}$$

An Infinite Set of Divergence-type Conservation Laws

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Discussion

- Parallel to the 2nd Noether's theorem, but for a non-variational system.
- Formally trivial, related to the abnormality of the given equations.
- Physical triviality?

Euler and Navier-Stokes Equations in 3D:

- Incompressible constant-density viscous fluid flow, no external forcing:

$$\mathrm{div}\ \mathbf{V} = 0,$$

$$\mathbf{V}_t + \mathrm{curl}\,\mathbf{V} \times \mathbf{V} + \mathrm{grad}\left(p + \frac{|\mathbf{V}|^2}{2}\right) = \nu\,\nabla^2\mathbf{V}.$$

- Vorticity formulation: $\mathbf{w} = \mathrm{curl}\ \mathbf{V}$.

$$\mathrm{div}\ \mathbf{w} = 0, \quad \mathbf{w}_t + \mathrm{curl}\,(\mathbf{w} \times \mathbf{V} - \nu\,\nabla^2\mathbf{V}) = 0.$$

The conservation laws:

$$(\mathbf{w} \cdot \nabla F)_t + \mathrm{div}\left([\mathbf{w} \times \mathbf{V} - \nu\,\nabla^2\mathbf{V}] \times \nabla F - F_t\,\mathbf{w}\right) = 0,$$

holding for an arbitrary $F = F(t, x, y, z)$.

The MHD equations:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

$$\rho \mathbf{V}_t + \rho \operatorname{curl} \mathbf{V} \times \mathbf{V} = -\frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} \, P - \rho \operatorname{grad} \frac{|\mathbf{V}|^2}{2} + \mu_1 \nabla^2 \mathbf{V},$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

The conservation laws:

$$(\mathbf{B} \cdot \nabla F)_t + \operatorname{div} \left(\left[\mathbf{B} \times \mathbf{V} + \frac{1}{\sigma} \mathbf{J} \right] \times \nabla F - F_t \, \mathbf{B} \right) = 0,$$

holding for an arbitrary $F = F(t, x, y, z)$.

- $\mathbf{J} = \dfrac{1}{\mu} \operatorname{curl} \mathbf{B}, \qquad \sigma = \dfrac{1}{\mu \, \eta}.$
- For ideal plasmas where $\sigma \to +\infty$, the conservation laws do not involve the current density.

Outline

Conclusions and Open Problems

Conclusions

- The vorticity equations in 3+1 dimensions, $\boxed{\operatorname{div}\mathbf{N}=0, \quad \mathbf{N}_t+\operatorname{curl}\mathbf{M}=0}$:
  - Are a part of important physical systems (Maxwell, Navier-Stokes, MHD...).
  - Yield a conservation law of degree two.
  - Lead to under-determined potential equations with four potential variables.
  - Admit an infinite set of local conservation laws given by

$$(\mathbf{N}\cdot\nabla F)_t+\operatorname{div}(\mathbf{M}\times\nabla F-F_t\,\mathbf{N})=0, \quad F=F(t,x,y,z)$$

- Galas-Bogoyavlenskij potential generalized to non-ideal, time-dependent MHD flows.

Future work:

- Study properties and applications of the potential system, in particular, in the MHD context.
- Study meaning and usefulness of the infinite set of the divergence-type conservation laws.

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Thank you for your attention!