Constitutive Relations, Natural States, and Travelling Waves in Two-Dimensional Nonlinear Elastodynamics

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Outline

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Notation

Notation

- $\dfrac{\partial u}{\partial x} \equiv u_x.$

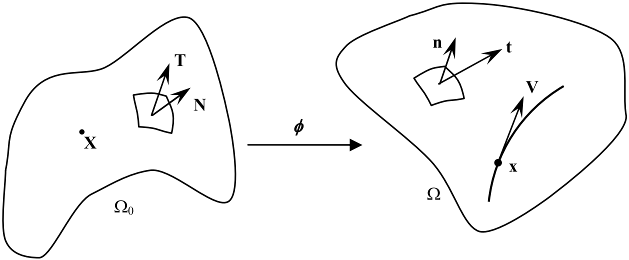

Fig. 1. Material and Eulerian coordinates.

Material picture

- A solid body occupies the reference (Lagrangian) volume $\Omega_0 \subset \mathbb{R}^3$.
- Actual (Eulerian) configuration: $\Omega \subset \mathbb{R}^3$.
- Material points are labelled by $\mathbf{X} \in \Omega_0$.
- The actual position of a material point: $\mathbf{x} = \phi(\mathbf{X}, t) \in \Omega$.

Notation; Material Picture

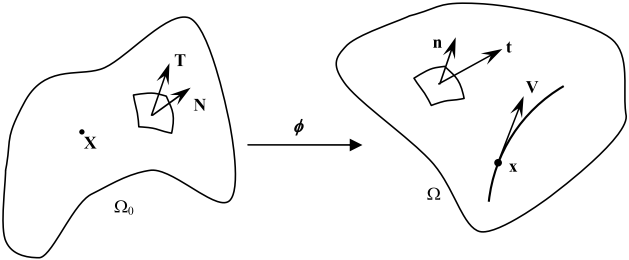

Fig. 1. Material and Eulerian coordinates.

Material picture

- Velocity of a material point $\mathbf{X}$: $\mathbf{v}(\mathbf{X}, t) = \dfrac{d\mathbf{x}}{dt}$.
- Jacobian matrix (deformation gradient):

$$\mathbf{F}(\mathbf{X}, t) = \nabla\phi; \quad J = \det \mathbf{F} > 0;$$

$$\mathbf{F} = \{F_{ij}\} = \{F^{i}{}_{j}\}.$$

Notation; Material Picture

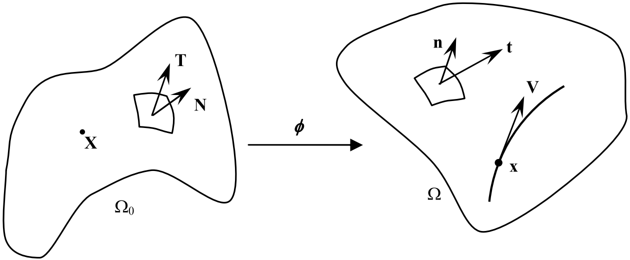

Fig. 1. Material and Eulerian coordinates.

Material picture

- Boundary force (per unit area) in Eulerian configuration: $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$.
- Boundary force (per unit area) in Lagrangian configuration: $\mathbf{T} = \mathbf{P}\mathbf{N}$.
- $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ is the Cauchy stress tensor.
- $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$ is the first Piola-Kirchhoff tensor.

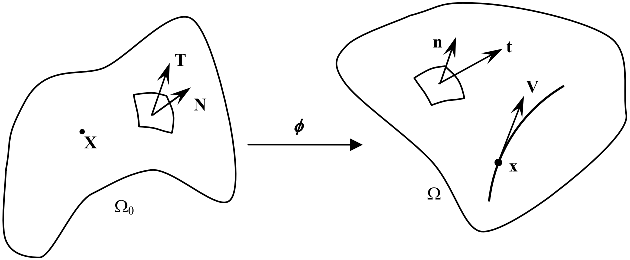

Fig. 1. Material and Eulerian coordinates.

Material picture

- Density in reference configuration: $\rho_0 = \rho_0(\mathbf{X})$ (time-independent).
- Density in actual configuration:

$$\rho = \rho(\mathbf{X}, t) = \rho_0/J.$$

Equations of motion (no dissipation, purely elastic setting):

$$\rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(X)}\mathbf{P} + \rho_0 \mathbf{R}, \tag{1}$$

- $\mathbf{R} = \mathbf{R}(\mathbf{X}, t)$: total body force per unit mass.
- $(\mathrm{div}_{(X)}\mathbf{P})^i = \dfrac{\partial P^{ij}}{\partial X^j}$.

Cauchy stress tensor symmetry (conservation of angular momentum):

$$\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T \quad \Leftrightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{2}$$

The first Piola-Kirchhoff stress tensor:

$$\mathbf{P} = \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \qquad P^{ij} = \rho_0 \frac{\partial W}{\partial F_{ij}}. \tag{3}$$

- $W = W(\mathbf{X}, \mathbf{F})$: a scalar strain energy density function.

Strain Energy Density for Isotropic Homogeneous Hyperelastic Materials

Isotropic Homogeneous Hyperelastic Materials

- Strain energy density W depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3).$$

- For the left Cauchy-Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$,

$$\begin{aligned} I_1 &= \mathrm{Tr}\,\mathbf{B} = F^i{}_k F^i{}_k, \\ I_2 &= \tfrac{1}{2}[(\mathrm{Tr}\,\mathbf{B})^2 - \mathrm{Tr}(\mathbf{B}^2)] = \tfrac{1}{2}(I_1^2 - B^{ik}B^{ki}), \\ I_3 &= \det \mathbf{B} = J^2. \end{aligned} \tag{4}$$

Strain Energy Density for Isotropic Homogeneous Hyperelastic Materials

Isotropic Homogeneous Hyperelastic Materials

- Strain energy density W depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3).$$

Table 1: Neo-Hookean and Mooney-Rivlin constitutive models

Type	Neo-Hookean	Mooney-Rivlin
Standard	$W = aI_1$, $a > 0$.	$W = aI_1 + bI_2$, $a, b > 0$
Generalized	$W = a\bar{I}_1 + c(J-1)^2$, $a, c > 0$.	$W = a\bar{I}_1 + b\bar{I}_2 + c(J-1)^2$ $a, b, c > 0$
Generalized (Ciarlet) “compressible”	$W = aI_1 + \Gamma(J)$, $\Gamma(q) = cq^2 - d\log q, \quad a, c, d > 0$	$W = aI_1 + bI_2 + \Gamma(J)$ $\Gamma(q) = cq^2 - d\log q, \quad a, b, c, d > 0$

Strain Energy Density for Isotropic Homogeneous Hyperelastic Materials

Isotropic Homogeneous Hyperelastic Materials

- Strain energy density W depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3).$$

Example: the Neo-Hookean Case

- Strain energy density: $W = a\, I_1, \quad a = \mathrm{const.}$
- Equations of motion are linear and decoupled:

$$(x^k)_{tt} = a\left(\frac{\partial^2}{\partial (X^1)^2} + \frac{\partial^2}{\partial (X^2)^2} + \frac{\partial^2}{\partial (X^3)^2}\right) x^k\,,$$
$$k = 1, 2, 3.$$

Outline

Ciarlet-Mooney-Rivlin solids in 2D: Governing equations

General equations:

$$\rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(X)} \mathbf{P} + \rho_0 \mathbf{R},$$

$$\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T,$$

$$\mathbf{P} = \rho_0 \, \partial W / \partial \mathbf{F}.$$

Assumptions:

- Two-dimensional: $x^{1,2} = x^{1,2}(X^1, X^2, t)$; the third coordinate $x^3 = X^3$ is fixed.
- Ciarlet-Mooney-Rivlin constitutive relation (4 parameters):

$$W = aI_1 + bI_2 - cI_3 - \frac{1}{2} d \log I_3, \quad a > 0, \quad b, c, d \geq 0, \tag{4}$$

$$\mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\rho_0 (x^1)_{tt} - \frac{\partial P^{11}}{\partial X^1} - \frac{\partial P^{12}}{\partial X^2} - \rho_0 R^1 = 0,$$

$$\rho_0 (x^2)_{tt} - \frac{\partial P^{21}}{\partial X^1} - \frac{\partial P^{22}}{\partial X^2} - \rho_0 R^2 = 0.$$

Ciarlet-Mooney-Rivlin constitutive relation:

$$W = aI_1 + bI_2 - cI_3 - \frac{1}{2} d \log I_3, \quad a > 0, \quad b, c, d \geq 0.$$

Equivalence Transformations:

$$\widetilde{t} = e^{\varepsilon_2} t + \varepsilon_1, \qquad \widetilde{x}^1 = e^{2\varepsilon_2} x^1, \qquad \widetilde{x}^2 = e^{2\varepsilon_2} x^2,$$

$$\widetilde{X}^1 = e^{\varepsilon_3} \left(X^1 \cos \varepsilon_7 - X^2 \sin \varepsilon_7\right) + \varepsilon_4, \qquad \widetilde{X}^2 = e^{\varepsilon_3} \left(X^1 \sin \varepsilon_7 + X^2 \sin \varepsilon_7\right) + \varepsilon_5,$$

$$\widetilde{\rho}_0 = e^{\varepsilon_6} \rho_0, \qquad \widetilde{R}^1 = R^1, \qquad \widetilde{R}^2 = R^2,$$

$$\widetilde{a} = -b + e^{2\varepsilon_3 - 2\varepsilon_2}(a + b), \qquad \widetilde{b} = b, \qquad \widetilde{c} = -b + e^{4\varepsilon_3 - 6\varepsilon_2}(b + c), \qquad \widetilde{d} = e^{2\varepsilon_2} d.$$

Ciarlet-Mooney-Rivlin constitutive relation:

$$W = aI_1 + bI_2 - cI_3 - \frac{1}{2} d \log I_3, \quad a > 0, \quad b, c, d \geq 0.$$

Principal Result 1:

- The model essentially depends on three constitutive parameters:

$$A = 2(a+b) \geq 0, \quad B = 2(b+c) \geq 0, \quad d.$$

- the two-dimensional first Piola-Kirchhoff stress tensor is given by

$$\mathbf{P_2} = \rho_0 \left[A\,\mathbf{F_2} + B\,J\,\mathbf{C_2} - \frac{d}{J}\,\mathbf{C_2} \right],$$

where

$$\mathbf{F_2} = \begin{bmatrix} F^1{}_1 & F^1{}_2 \\ F^2{}_1 & F^2{}_2 \end{bmatrix}, \quad \mathbf{C_2} = \begin{bmatrix} F^2{}_2 & -F^2{}_1 \\ -F^1{}_2 & F^1{}_1 \end{bmatrix},$$

$$\mathbf{F_2} = \nabla_{(X)}\,\mathbf{x}, \quad \mathbf{x} = [x^1(\mathbf{X}, t), x^2(\mathbf{X}, t)]^T.$$

Outline

General 2D Case, No Forcing

Governing equations:

- No forcing: $R^1 = R^2 = 0$.
- Dynamic equations:

$$\rho_0 (x^1)_{tt} - \frac{\partial P^{11}}{\partial X^1} - \frac{\partial P^{12}}{\partial X^2} = 0,$$
$$\rho_0 (x^2)_{tt} - \frac{\partial P^{21}}{\partial X^1} - \frac{\partial P^{22}}{\partial X^2} = 0.$$

- C-M-R constitutive relation:

$$\mathbf{P}_2 = \rho_0 \left[A\, \mathbf{F}_2 + B\, J\, \mathbf{C}_2 - \frac{d}{J}\, \mathbf{C}_2 \right],$$

$$\mathbf{F}_2 = \begin{bmatrix} F^1{}_1 & F^1{}_2 \\ F^2{}_1 & F^2{}_2 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} F^2{}_2 & -F^2{}_1 \\ -F^1{}_2 & F^1{}_1 \end{bmatrix},$$

$$\mathbf{F}_2 = \nabla_{(X)}\, \mathbf{x}, \quad \mathbf{x} = [x^1(\mathbf{X}, t), x^2(\mathbf{X}, t)]^T.$$

Specific Forms of Equations

$$
\begin{aligned}
x1_{t,\,t} = {} & \frac{1}{\left(x1_{X1}\, x2_{X2} - x1_{X2}\, x2_{X1}\right)^2} \Big(x2_{X1,\,X2}\, x1_{X2}{}^3\, x2_{X1}{}^3\, B \\
& + x1_{X2,\,X2}\, x1_{X2}{}^2\, x2_{X1}{}^4\, B + x1_{X1,\,X1}\, x1_{X1}{}^2\, x2_{X2}{}^2\, A \\
& + x1_{X2,\,X2}\, x1_{X1}{}^2\, x2_{X2}{}^2\, A + x1_{X1,\,X1}\, x1_{X2}{}^2\, x2_{X1}{}^2\, A \\
& + x1_{X2,\,X2}\, x1_{X2}{}^2\, x2_{X1}{}^2\, A + x2_{X1,\,X2}\, x1_{X1}\, x2_{X2}\, d + x2_{X1,\,X2}\, x1_{X2}\, x2_{X1}\, d \\
& - 2\, x1_{X1,\,X2}\, x2_{X2}\, x2_{X1}\, d - x2_{X1,\,X1}\, x2_{X2}\, x1_{X2}\, d + x1_{X1,\,X1}\, x2_{X2}{}^2\, d \\
& + x1_{X2,\,X2}\, x2_{X1}{}^2\, d - x2_{X2,\,X2}\, x1_{X1}{}^3\, x2_{X2}{}^2\, x2_{X1}\, B \\
& - x2_{X1,\,X1}\, x1_{X1}{}^2\, x2_{X2}{}^3\, x1_{X2}\, B - 2\, x1_{X1,\,X2}\, x1_{X1}{}^2\, x2_{X2}{}^3\, x2_{X1}\, B \\
& + x1_{X2,\,X2}\, x1_{X1}{}^2\, x2_{X2}{}^2\, x2_{X1}{}^2\, B - x2_{X2,\,X2}\, x1_{X1}\, x1_{X2}{}^2\, x2_{X1}{}^3\, B \\
& + x1_{X1,\,X1}\, x2_{X2}{}^2\, x1_{X2}{}^2\, x2_{X1}{}^2\, B - x2_{X1,\,X1}\, x2_{X2}\, x1_{X2}{}^3\, x2_{X1}{}^2\, B \\
& - 2\, x1_{X1,\,X2}\, x2_{X2}\, x1_{X2}{}^2\, x2_{X1}{}^3\, B - x2_{X2,\,X2}\, x1_{X1}\, x2_{X1}\, d \\
& - 2\, x1_{X2,\,X2}\, x1_{X1}\, x2_{X2}\, x1_{X2}\, x2_{X1}\, A - x2_{X1,\,X2}\, x1_{X1}{}^2\, x2_{X2}{}^2\, x1_{X2}\, x2_{X1}\, B \\
& + 2\, x2_{X2,\,X2}\, x1_{X1}{}^2\, x2_{X2}\, x1_{X2}\, x2_{X1}{}^2\, B \\
& - 2\, x1_{X1,\,X1}\, x1_{X1}\, x2_{X2}{}^3\, x1_{X2}\, x2_{X1}\, B \\
& + 2\, x2_{X1,\,X1}\, x1_{X1}\, x2_{X2}{}^2\, x1_{X2}{}^2\, x2_{X1}\, B \\
& + 4\, x1_{X1,\,X2}\, x1_{X1}\, x2_{X2}{}^2\, x1_{X2}\, x2_{X1}{}^2\, B - x2_{X1,\,X2}\, x1_{X1}\, x2_{X2}\, x1_{X2}{}^2\, x2_{X1}{}^2\, B \\
& - 2\, x1_{X2,\,X2}\, x1_{X1}\, x2_{X2}\, x1_{X2}\, x2_{X1}{}^3\, B - 2\, x1_{X1,\,X1}\, x1_{X1}\, x2_{X2}\, x1_{X2}\, x2_{X1}\, A \\
& + x2_{X1,\,X2}\, x1_{X1}{}^3\, x2_{X2}{}^3\, B + x1_{X1,\,X1}\, x1_{X1}{}^2\, x2_{X2}{}^4\, B \Big)
\end{aligned}
$$

$$x2_{t,t} = \frac{1}{\left(x1_{X2}\, x2_{X1} - x1_{X1}\, x2_{X2}\right)^2} \Big(x2_{X1,X1}\, {x1_{X2}}^2\, d + x2_{X2,X2}\, {x1_{X1}}^2\, d$$

$$- 2\, x2_{X1,X1}\, x1_{X1}\, x2_{X2}\, {x1_{X2}}^3\, x2_{X1}\, B - x1_{X1,X2}\, x1_{X1}\, x2_{X2}\, {x1_{X2}}^2\, {x2_{X1}}^2\, B$$

$$+ 4\, x2_{X1,X2}\, {x1_{X1}}^2\, x2_{X2}\, {x1_{X2}}^2\, x2_{X1}\, B$$

$$+ 2\, x1_{X1,X1}\, x1_{X1}\, {x2_{X2}}^2\, {x1_{X2}}^2\, x2_{X1}\, B$$

$$+ 2\, x1_{X2,X2}\, {x1_{X1}}^2\, x2_{X2}\, x1_{X2}\, {x2_{X1}}^2\, B$$

$$- 2\, x2_{X2,X2}\, {x1_{X1}}^3\, x2_{X2}\, x1_{X2}\, x2_{X1}\, B - x1_{X1,X2}\, {x1_{X1}}^2\, {x2_{X2}}^2\, x1_{X2}\, x2_{X1}\, B$$

$$- 2\, x2_{X1,X1}\, x1_{X1}\, x2_{X2}\, x1_{X2}\, x2_{X1}\, A - 2\, x2_{X2,X2}\, x1_{X1}\, x2_{X2}\, x1_{X2}\, x2_{X1}\, A$$

$$+ x2_{X1,X1}\, {x1_{X1}}^2\, {x2_{X2}}^2\, {x1_{X2}}^2\, B - 2\, x2_{X1,X2}\, x1_{X1}\, {x1_{X2}}^3\, {x2_{X1}}^2\, B$$

$$- x1_{X1,X1}\, x2_{X2}\, {x1_{X2}}^3\, {x2_{X1}}^2\, B - x1_{X2,X2}\, x1_{X1}\, {x1_{X2}}^2\, {x2_{X1}}^3\, B$$

$$+ x2_{X2,X2}\, {x1_{X1}}^2\, {x1_{X2}}^2\, {x2_{X1}}^2\, B - 2\, x2_{X1,X2}\, {x1_{X1}}^3\, {x2_{X2}}^2\, x1_{X2}\, B$$

$$- x1_{X1,X1}\, {x1_{X1}}^2\, {x2_{X2}}^3\, x1_{X2}\, B - x1_{X2,X2}\, {x1_{X1}}^3\, {x2_{X2}}^2\, x2_{X1}\, B$$

$$- x1_{X1,X1}\, x2_{X2}\, x1_{X2}\, d - 2\, x2_{X1,X2}\, x1_{X1}\, x1_{X2}\, d + x1_{X1,X2}\, x1_{X1}\, x2_{X2}\, d$$

$$+ x1_{X1,X2}\, x1_{X2}\, x2_{X1}\, d - x1_{X2,X2}\, x1_{X1}\, x2_{X1}\, d + x2_{X1,X1}\, {x1_{X2}}^4\, {x2_{X1}}^2\, B$$

$$+ x1_{X1,X2}\, {x1_{X2}}^3\, {x2_{X1}}^3\, B + x2_{X2,X2}\, {x1_{X1}}^4\, {x2_{X2}}^2\, B$$

$$+ x1_{X1,X2}\, {x1_{X1}}^3\, {x2_{X2}}^3\, B + x2_{X1,X1}\, {x1_{X2}}^2\, {x2_{X1}}^2\, A$$

$$+ x2_{X1,X1}\, {x1_{X1}}^2\, {x2_{X2}}^2\, A + x2_{X2,X2}\, {x1_{X2}}^2\, {x2_{X1}}^2\, A$$

$$+ x2_{X2,X2}\, {x1_{X1}}^2\, {x2_{X2}}^2\, A \Big)$$

Outline

Consider a general DE system

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N$$

with variables $x = (x^1, ..., x^n), \quad u = u(x) = (u^1, ..., u^m)$.

Definition

A transformation

$$x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2),$$
$$u^* = g(x, u; a) = u + a\eta(x, u) + O(a^2).$$

depending on a parameter a is a *point symmetry* of $R^\sigma[u]$ if the equations are the same in new variables x^*, u^*.

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Example 1: translations

The translation

$$x^* = x + C, \quad t^* = t, \quad u^* = u$$

leaves KdV invariant:

$$u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^*x^*x^*}.$$

Consider a general DE system

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N$$

with variables $x = (x^1, ..., x^n)$, $u = u(x) = (u^1, ..., u^m)$.

Definition

A transformation

$$x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2),$$
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depending on a parameter a is a *point symmetry* of $R^\sigma[u]$ if the equations are the same in new variables x^*, u^*.

Example 2: scaling

Same for the scaling:

$$x^* = \alpha x, \quad t^* = \alpha^3 t, \quad u^* = \alpha u.$$

One has

$$u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^* x^* x^*}.$$

Nonlinear DEs

- Numerical solutions: resource/time consuming; lack generality.
- Solution methods for linear DEs do not work.
- Symmetry analysis: a general systematic framework leading to useful results.

Symmetries for ODEs

- Reduction of order / complete integration.
- All known methods of solution of specific classes of ODEs follow from symmetries!

Symmetries for PDEs

- Exact symmetry-invariant (e.g., self-similar) solutions.
- Transformations: solutions $\Rightarrow$ new solutions.
- Mappings relating classes of equations; linearizations.
- Symmetry-preserving numerical methods.

Computation of Symmetries

- Lie point symmetries and other types are computed <u>systematically</u> for any DE.
- Literature widely available.
- Symbolic software packages available.
- A popular approach to analyze complicated DEs arising in applied science:
  - fluid and solid mechanics,
  - rocket science,
  - meteorology,
  - biological applications, ...

Table 1: Point symmetry classification for the 2D Ciarlet-Mooney-Rivlin models with zero forcing and $\rho_0 = \mathrm{const} > 0$.

Case	Point symmetries
General	$\mathrm{Y}_1 = \frac{\partial}{\partial t}, \mathrm{Y}_2 = \frac{\partial}{\partial X^1}, \mathrm{Y}_3 = \frac{\partial}{\partial X^2}, \mathrm{Y}_4 = \frac{\partial}{\partial x^1}, \mathrm{Y}_5 = \frac{\partial}{\partial x^2}, \mathrm{Y}_6 = t\frac{\partial}{\partial x^1}, \mathrm{Y}_7 = t\frac{\partial}{\partial x^2},$ $\mathrm{Y}_8 = X^2\frac{\partial}{\partial X^1} - X^1\frac{\partial}{\partial X^2}, \mathrm{Y}_9 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2},$ $\mathrm{Y}_{10} = t\frac{\partial}{\partial t} + X^1\frac{\partial}{\partial X^1} + X^2\frac{\partial}{\partial X^2} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$
$A = 0$, B, d arbitrary	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5, \mathrm{Y}_6, \mathrm{Y}_7, \mathrm{Y}_8, \mathrm{Y}_9, \mathrm{Y}_{10},$ $\mathrm{Y}_{11} = f_1(X^2)\frac{\partial}{\partial X^1}, \mathrm{Y}_{12} = \left(\frac{\partial}{\partial X_2} f_2(X^1, X^2)\right)\frac{\partial}{\partial X^1} - \left(\frac{\partial}{\partial X_1} f_2(X^1, X^2)\right)\frac{\partial}{\partial X^2},$ $f_1(X^2)$, $f_2(X^1, X^2)$ are arbitrary functions
$A = d = 0$ B arbitrary	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5, \mathrm{Y}_6, \mathrm{Y}_7, \mathrm{Y}_8, \mathrm{Y}_9, \mathrm{Y}_{10}, \mathrm{Y}_{11}, \mathrm{Y}_{12},$ $\mathrm{Y}_{13} = t\frac{\partial}{\partial t} + X^1\frac{\partial}{\partial X^1}$
$A = B = 0$ d arbitrary	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5, \mathrm{Y}_6, \mathrm{Y}_7, \mathrm{Y}_8, \mathrm{Y}_9, \mathrm{Y}_{10}, \mathrm{Y}_{11}, \mathrm{Y}_{12},$ $\mathrm{Y}_{14} = X^1\frac{\partial}{\partial X^1}$

Traveling Wave Ansatz Along X^1

Traveling Wave Ansatz

- No forcing: $R^1 = R^2 = 0$.
- Ansatz:
$$x^i(X^1, X^2, t) = w^i(z, X^2), \quad z = X^1 - st, \quad i = 1, 2;$$
$$\rho_0 = \rho_0(X^2).$$
- s is the constant wave speed.

Symmetry Classification in Traveling Wave Coordinates

#	Case	Point symmetries
1	General	$Y_1 = \frac{\partial}{\partial z}$, $Y_2 = \frac{\partial}{\partial w^1}$, $Y_3 = \frac{\partial}{\partial w^2}$, $Y_4 = w^2 \frac{\partial}{\partial w^1} - w^1 \frac{\partial}{\partial w^2}$
2	$\rho_0(X^2) = (X^2 + q_1)^{q_2}$, $q_1, q_2 = \mathrm{const}$, $q_2 \neq 0$, A, B, d, s arbitrary	Y_1, Y_2, Y_3, Y_4, $Y_5 = z\frac{\partial}{\partial z} + (X^2 + q_1)\frac{\partial}{\partial X^2} + w^1 \frac{\partial}{\partial w^1} + w^2 \frac{\partial}{\partial w^2}$
3a	$\rho_0(X^2) = \exp(q_1 X^2)$, $q_1 = \mathrm{const} \neq 0$, A, B, d, s arbitrary	Y_1, Y_2, Y_3, Y_4, $Y_6 = \frac{\partial}{\partial X^2}$
3b	$\rho_0(X^2) = \exp(q_1 X^2)$, $q_1 = \mathrm{const} \neq 0$, A, d arbitrary, $B = 0$, $s^2 = A$	Y_1, Y_2, Y_3, Y_4, $Y^{\infty}_{(1)} = -\left(\frac{1}{q_1}\frac{d}{dz} f_1(z)\right)\frac{\partial}{\partial X^2} + f_1(z)\frac{\partial}{\partial z}$
4a	$\rho_0(X^2) > 0$ arbitrary, A, B arbitrary, $d = 0$, $s^2 = A$	Y_1, Y_2, Y_3, Y_4, $Y_7 = z\frac{\partial}{\partial z} + w^1 \frac{\partial}{\partial w^1} + w^2 \frac{\partial}{\partial w^2}$, $Y_8 = \left(\rho_0 \int \frac{1}{\rho_0} dX^2\right)\frac{\partial}{\partial X^2}$, $Y^{\infty}_{(2)} = f_2(z)\rho_0 \frac{\partial}{\partial X^2}$, $f_2(z)$ is an arbitrary function
4b	$\rho_0(X^2) > 0$ arbitrary, A, d arbitrary, $B = 0$, $s^2 = A$	Y_1, Y_2, Y_3, Y_4, $Y_9 = z\frac{\partial}{\partial z}$
5a	$\rho_0 = \mathrm{const}$ A, B, d, s arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_6$, $Y_{10} = X^2 \frac{\partial}{\partial z} - \frac{Az}{A - s^2}\frac{\partial}{\partial X^2}$
5b	$\rho_0 = \mathrm{const}$, $s^2 = A$, A, B, d arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0)$, $Y^{\infty}_{(3)} = f_3(z)\frac{\partial}{\partial X^2}$, $f_3(z)$ is an arbitrary function
5c	$\rho_0 = \mathrm{const}$, $s^2 = A$, A, d arbitrary, $B = 0$	$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_9, Y^{\infty}_{(3)}$,

Examples of Exact Solutions

Example:

- Case: $\rho_0 = \text{const}, \quad R^1 = R^2 = 0, \quad A = s^2, \quad d = s^2 + B.$
- A basic solution:

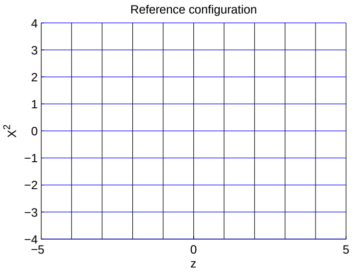
$$w^1 = z \quad \Leftrightarrow \quad x^1(X^1, X^2, t) = X^1 - st, \qquad w^2 = x^2(X^1, X^2, t) = X^2,$$

- A symmetry-transformed solution:

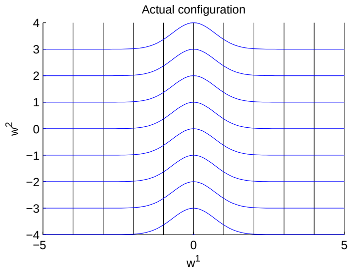
$$\begin{array}{lcl} w^1 = z & \Leftrightarrow & x^1(X^1, X^2, t) = X^1 - st, \\ w^2 = X^2 - f(z) & \Leftrightarrow & x^2(X^1, X^2, t) = X^2 - f(X^1 - st) \end{array}$$

- Figure:
  - (a) A rectangular grid in the reference configuration.
  - (b) The propagating deformation, $f(z) = -\exp(-z^2)$.
  - (c) The propagating deformation, $f(z) = -(1 + \tanh z)/2$.

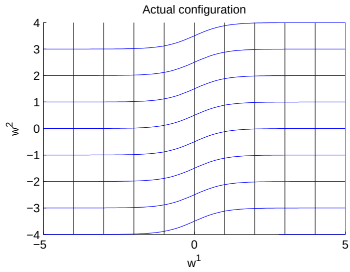
Examples of Exact Solutions



(a)



(b)



(c)

Outline

Conclusions and Open Problems

Conclusions

- Symmetry properties of dynamic equations for 2D planar Ciarlet-Mooney-Rivlin materials were studied:
  - in a general setting;
  - in traveling wave coordinates.
- The number of essential constitutive parameters in the model were reduced through equivalence transformations.
- New traveling-wave type exact solutions were obtained for the nonlinear model.

Future/ongoing work

- Use computed symmetries to derive new exact solutions in 2D.
- Consider important *non-planar* two-dimensional reductions (including axial symmetry), and 3D.
- Generalize to other constitutive models, in particular, models of anisotropic materials.

Some references

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Thank you for attention!