Nonlinear Elastodynamic Models of Wave Propagation and Conservation Laws for Fiber-Reinforced Materials

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Outline

1. Local Conservation Laws
2. Fiber-Reinforced Materials; Governing Equations
3. Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves
4. Single Fiber Family, Ansatz 2 – 2D Shear Waves
5. Two Fiber Families, Planar Case
6. A Viscoelastic Model, Single Fiber Family, 1D Shear Waves
7. Discussion

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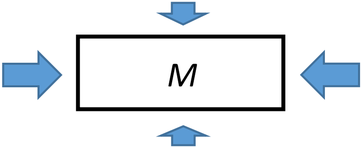
Motivation

- Interesting mathematics!
- Study of fundamental properties of nonlinear elastodynamics equations arising in applications.

Notation

- $\mathrm{D}_t = \dfrac{\partial u}{\partial t} \equiv u_t.$

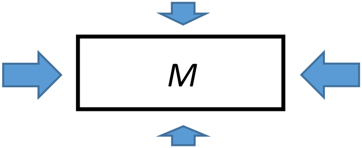


Global form

- Global quantity $M \in \mathcal{D}$ changes only due to boundary fluxes.

$$M = \int_{\mathcal{D}} \Theta \, dV; \qquad \frac{d}{dt} M = \oint_{\partial\mathcal{D}} \mathbf{\Psi} \cdot d\mathbf{S}.$$

- $\Theta[\mathbf{u}]$: conserved density; $\mathbf{\Psi}$: flux vector.



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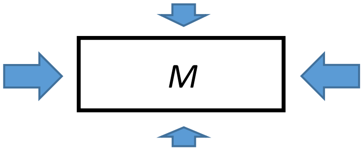
Local form

- A local conservation law: a divergence expression equal to zero, e.g.,

$$\mathrm{D}_t \, \Theta[\mathbf{u}] + \mathrm{D}_i \, \Psi^i[\mathbf{u}] = 0.$$



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Global conserved quantity:

$$\frac{d}{dt} M = \mathrm{D}_t \int_V \Theta \, dV = 0 \qquad \text{when} \qquad \oint_{\partial V} \mathbf{\Psi} \cdot d\mathbf{S} = 0.$$

Applications of Conservation Laws

ODEs

- Constants of motion.
- Integration.

PDEs

- Rates of change of physical variables; constants of motion.
- Differential constraints.
- Analysis: existence, uniqueness, stability, integrability, linearization.
- Potentials, stream functions, etc.
- Conserved forms for numerical methods (finite volume, etc.).
- Numerical method testing.

For equations following from a variational principle:

- Can use Noether's theorem.
- Conservation laws are connected with variational symmetries.
- Technically difficult.
- For the majority of DE systems, classical variational formulation is not available.

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For generic models: Direct conservation law construction method

- Conservation laws are be sought in the characteristic form $\boxed{\Lambda_\sigma R^\sigma \equiv \mathrm{D}_i \Phi^i.}$
- Systematically find the multipliers Λ_σ.
- Direct method is complete for a wide class of systems.
- Implemented in Maple/GeM: symbolic computations.

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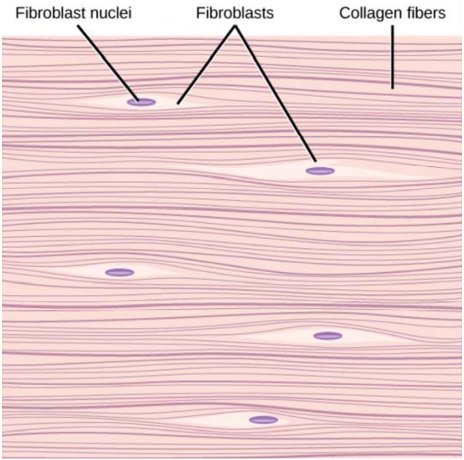
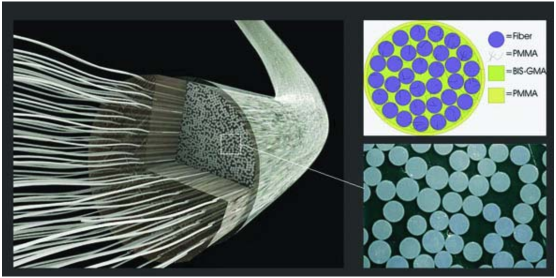


Collagen fiber in tendons.

Single fiber family.

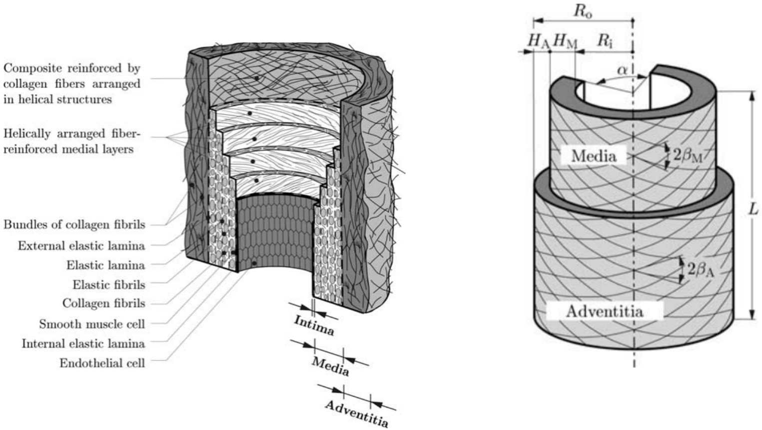
Examples



A fiber-reinforced composite in dentistry.

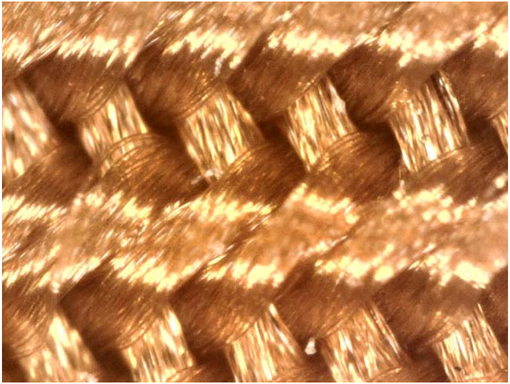
Single fiber family.

Examples



Arterial tissue (Holzapfel, Gasser, and Ogden, 2000).

Two helically arranged fiber families.



Fabric – two fiber families.

Examples

- **Appropriate framework:** incompressible hyperelasticity / viscoelasticity.

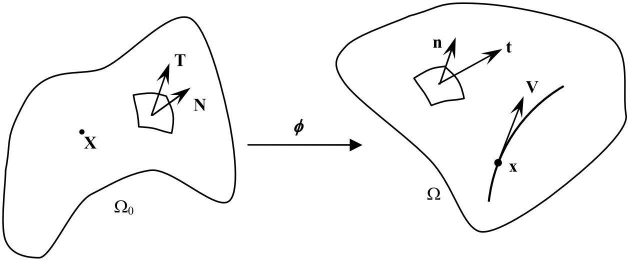


Fig. 1. Material and Eulerian coordinates.

Material picture

- Material points $\mathbf{X} \in \Omega_0$.
- Actual position of a material point: $\mathbf{x} = \phi\,(\mathbf{X}, t) \in \Omega$.
- Deformation gradient: $\mathbf{F}(\mathbf{X}, t) = \nabla\phi, \qquad F^i{}_j = \dfrac{\partial x^i}{\partial X^j}.$

Incompressibility:

$$J = \det \mathbf{F} = \left| \frac{\partial x^i}{\partial X^j} \right| = 1, \qquad \rho = \rho_0 / J = \rho_0(\mathbf{X}).$$

Governing Equations

Incompressibility:

$$J = \det \mathbf{F} = \left| \frac{\partial x^i}{\partial X^j} \right| = 1, \qquad \rho = \rho_0 / J = \rho_0(\mathbf{X}).$$

Equations of motion:

$$\rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(X)} \mathbf{P} + \rho_0 \mathbf{R}, \qquad J = 1.$$

- $\mathbf{R} = \mathbf{R}(\mathbf{X}, t)$: total body force per unit mass; $\rho_0(\mathbf{X})$: density.
- Assumptions: $\mathbf{R} = 0$, $\rho_0 = \mathrm{const}$.

Stress tensor (incompressible):

$$P^{ij} = -p\,(F^{-1})^{ji} + \rho_0 \frac{\partial W}{\partial F_{ij}}, \tag{1}$$

- W: scalar strain energy density; p: hydrostatic pressure.

Strain Energy Density

$$W = W_{iso} + W_{aniso}.$$

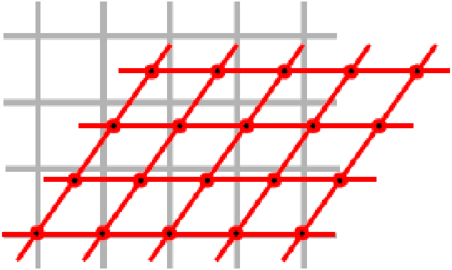
Isotropic Strain Energy Density

- Right Cauchy-Green strain tensor: $\mathbf{C} = \mathbf{F}^T\mathbf{F}$,

$$I_1 = \mathrm{Tr}\,\mathbf{C}, \qquad I_2 = \tfrac{1}{2}[(\mathrm{Tr}\,\mathbf{C})^2 - \mathrm{Tr}(\mathbf{C}^2)]. \tag{2}$$

- Mooney-Rivlin materials:

$$W_{iso} = a(I_1 - 3) + b(I_2 - 3), \quad a, b > 0.$$



Fiber directions

- Reference configuration: fibers along $\mathbf{A}$ $(|\mathbf{A}| = 1)$.
- Actual configuration: fibers along $\mathbf{a}$ $(|\mathbf{a}| = 1)$.
- Fiber stretch factor:

$$\lambda \mathbf{a} = \mathbf{F}\mathbf{A} \quad \Rightarrow \quad \lambda^2 = \mathbf{A}^T\,\mathbf{C}\,\mathbf{A}.$$

Anisotropic Strain Energy Density

- Fiber invariants:

$$I_4 = \mathbf{A}^T \mathbf{C} \mathbf{A}, \quad I_5 = \mathbf{A}^T \mathbf{C}^2 \mathbf{A}.$$

- General constitutive model:

$$W_{aniso} = f\left(I_4 - 1,\, I_5 - 1\right), \quad f(0,0) = 0.$$

- Standard reinforcement model: $W_{aniso} = q\left(I_4 - 1\right)^2$.

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$$P^{ij} = -p\,(F^{-1})^{ji} + \rho_0 \frac{\partial W}{\partial F_{ij}}.$$

- Strain energy density, single fiber family:

$$W = W_{iso} + W_{aniso} = a(I_1 - 3) + b(I_2 - 3) + q\,(I_4 - 1)^2; \quad a, b, q > 0.$$

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Ansatz 1 Compatible with Incompressibility

Equilibrium and Displacements

- Equilibrium/no displacement: $\mathbf{x} = \mathbf{X}$, natural state.
- Time-dependent, with displacement: $\mathbf{x} = \mathbf{X} + \mathbf{G}, \quad \mathbf{G} = \mathbf{G}(\mathbf{X}, t)$.
- No linearization, or assumption of smallness of $\mathbf{G}$, etc.

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Motions Transverse to a Plane

$$\mathbf{x} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

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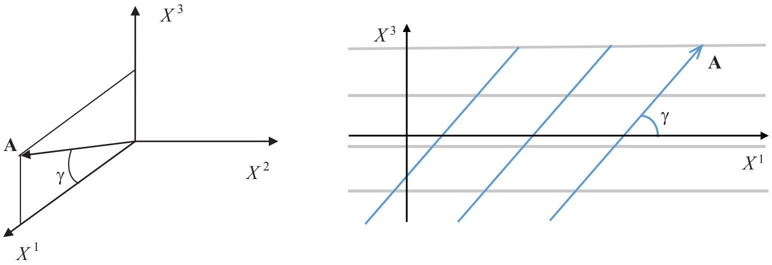
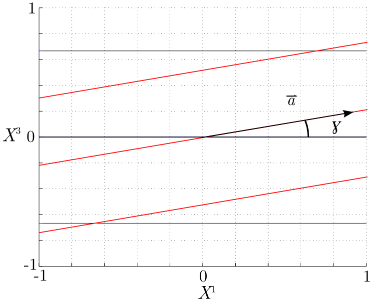
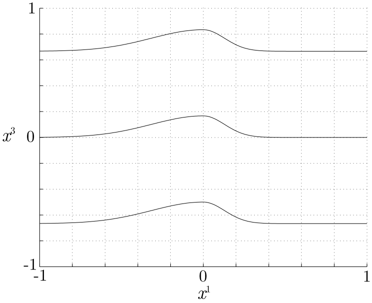
Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial G/\partial X_1 & 0 & 1 \end{bmatrix}, \qquad J = |\mathbf{F}| \equiv 1.$$


X3
A
γ
X2
X1
X3
A
γ
X1



Reference Configuration



Actual Configuration

Equation of motion for one-dimensional displacements:

- Denote

$$X^1 = x, \quad G = G(x,t), \quad \alpha = 2(a+b) > 0, \quad \beta = 4q > 0.$$

- Single nonlinear PDE:

$$\boxed{G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.}$$

- Pressure is found explicitly:

$$\boxed{p = \beta\rho_0 \cos^3\gamma\,(\cos\gamma\, G_x + 2\sin\gamma)\, G_x + f(t).}$$

1D wave model in the case of a single fiber family

- Wave equation:

$$G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma \,(G_x)^2 + 6\sin\gamma\cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.$$

- General PDE class: $\boxed{G_{tt} = \left(A\,(G_x)^2 + BG_x + C\right) G_{xx},}$

$$\begin{aligned} A &= 3\beta\cos^4\gamma > 0, \\ B &= 6\beta\sin\gamma\,\cos^3\gamma, \qquad\qquad 0 \le \gamma < \pi/2. \\ C &= \alpha + \tfrac{1}{2}\beta\sin^2(2\gamma) > 0, \end{aligned}$$

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Loss of hyperbolicity:

- May occur when $B^2 - 4AC \geq 0$, i.e., $\sin^2(2\gamma) \geq \dfrac{4\alpha}{\beta}$.
- Can only happen for "strong" fiber contribution: $\beta \geq \dfrac{4\alpha}{\sin^2(2\gamma)}$.

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Variational structure

- Any nonlinear PDE of the above class follows from a variational principle, with the Lagrangian density (*up to equivalence*)

$$\mathcal{L} = \frac{1}{2}G_t^2 + \frac{A}{4}GG_x^2G_{xx} + \frac{B}{3}GG_xG_{xx} - \frac{C}{2}G_x^2.$$

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Simplification

- Depending on the sign of $B^2 - 4AC$, equivalence transformations can be used to map the wave equation into

$$u_{tt} = \left((u_x)^2 + K\right) u_{xx}, \qquad K = 0, \pm 1.$$

Ansatz 1: Nonlinear Wave Equation and Its Properties

1D wave model in the case of a single fiber family

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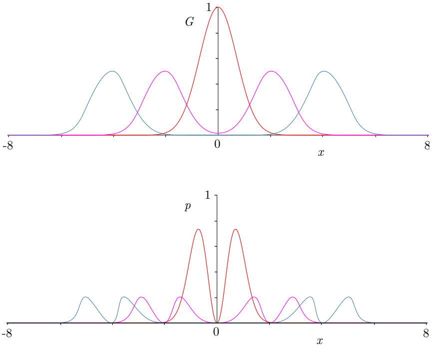
$$C = \alpha + \tfrac{1}{2}\beta\sin^2(2\gamma) > 0,$$

Reduction to the case $\gamma = 0$: X^1-aligned fibers

- In particular (for the hyperbolic case $B^2 - 4AC < 0$), the wave PDE is equivalent to the one with $\gamma = 0$:

$$\boxed{u_{tt} = \left((u_x)^2 + 1\right) u_{xx}.}$$

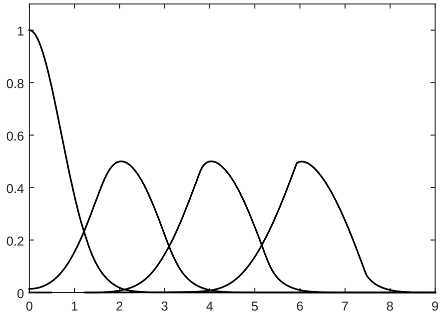
- Numerical d'Alembert-type solution of $u_{tt} = \left((u_x)^2 + 1\right) u_{xx}$: Gaussian bell IC.



- Wave speed dependent on u_x.
- Numerical instabilities.
- Wave breaking.

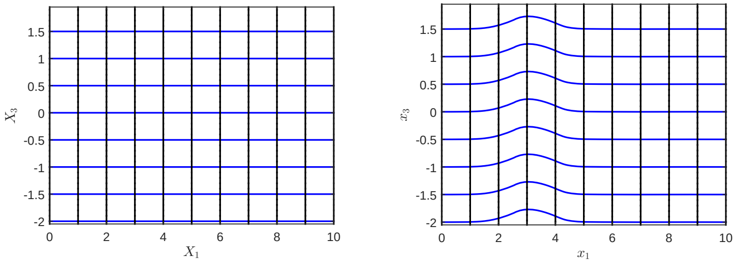
- Right-traveling wave profiles for the dimensionless wave PDE for a stationary Gaussian initial displacement

$$u(x,0) = \exp(-x^2), \qquad u_t(x,0) = 0.$$

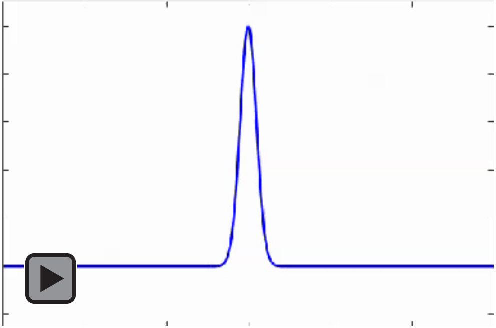


- Dimensional plot for the fiber angle $\gamma = 0$ and the parameter values

$$\rho_0 = 1.1 \cdot 10^3 \text{ kg/m}^3, \quad a = 1.5 \cdot 10^3 \text{ Pa}, \quad b = 0, \quad q = 1.18 \cdot 10^3 \text{ Pa} :$$



- Material lines $X^1 = \text{const}$ (vertical) and the fiber lines $X^3 = \text{const}$ (blue, horizontal), for the Gaussian initial condition, and the time $t = 1.82 \cdot 10^{-3}$ s (the corresponding dimensionless time is $\hat{t} = 3$). The material configuration (left) and the actual configuration (right). Spatial coordinates are dimensional, given in millimeters.



Direct Construction of Conservation Laws for Ansatz 1

Find local CLs for the nonlinear wave equation

- Model: $u_{tt} = \left(u_x^2 + 1\right) u_{xx}$.
- Conserved form: $\Lambda[u] \left(u_{tt} - \left(u_x^2 + 1\right) u_{xx}\right) = \mathrm{D}_t\, \Theta + \mathrm{D}_x\, \Psi = 0$.
- Basic CLs:

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Eulerian momentum:

- $\Lambda = 1$,

$$\mathrm{D}_t(u_t) - \mathrm{D}_x \left[u_x \left(\frac{1}{3} u_x^2 + 1 \right) \right] = 0.$$

Lagrangian momentum:

- $\Lambda = u_x$,

$$\mathrm{D}_t(u_x u_t) - \mathrm{D}_x \left(\frac{1}{2}(u_t^2 + u_x^2) + \frac{1}{4} u_x^4 \right) = 0.$$

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Energy:

- $\Lambda = u_t$,

$$\mathrm{D}_t \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{1}{12} u_x^4\right) - \mathrm{D}_x \left[u_t u_x \left(\frac{1}{3} u_x^2 + 1\right)\right] = 0.$$

Center of mass theorem:

- $\Lambda = t$,

$$\mathrm{D}_t(t u_t - u) - \mathrm{D}_x \left[t u_x \left(\frac{1}{3} u_x^2 + 1\right)\right] = 0.$$

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An infinite family of conservation laws

- Multiplier: any function $\Lambda(u_t, u_x)$ satisfying

$$\Lambda_{u_x,u_x} = \left(u_x^2 + 1\right)\Lambda_{u_t,u_t}.$$

Linearization by a Legendre contact transformation:

$$y = u_x, \quad z = u_t, \quad w(y,z) = u(x,t) - xu_x - tu_t;$$

$$w_{yy} = \left(y^2+1\right) w_{zz}.$$

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A more exotic, 2nd-order CL:

- For Λ depending on 3rd derivatives, can have, e.g.,

$$\mathrm{D}_t \frac{u_{xx}}{u_{tx} - (u_x^2 + 1)u_{xx}} + \mathrm{D}_x \frac{u_{tx}}{u_{tx}^2 - (u_x^2 + 1)u_{xx}^2} = 0.$$

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Displacements transverse to an axis:

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Ansatz 2 Compatible with Incompressibility

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Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \partial H/\partial X_1 & 1 & 0 \\ \partial G/\partial X_1 & 0 & 1 \end{bmatrix}, \qquad J = |\mathbf{F}| \equiv 1.$$

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \partial H/\partial X_1 & 1 & 0 \\ \partial G/\partial X_1 & 0 & 1 \end{bmatrix}, \qquad J = |\mathbf{F}| \equiv 1.$$

Governing PDEs:

- Denote $X^1 = x$, $G = G(x,t)$, $H = H(x,t)$.

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Coupled nonlinear wave equations:

$$0 = p_x - 2\beta\rho_0 \cos^3\gamma \left[(\cos\gamma\, G_x + \sin\gamma)\, G_{xx} + \cos\gamma\, H_x H_{xx}\right],$$

$$H_{tt} = \quad \alpha H_{xx} + \beta\cos^3\gamma \left[\cos\gamma \left(\left[G_x^2 + H_x^2\right] H_{xx} + 2 G_x H_x G_{xx}\right) + 2\sin\gamma \frac{\partial}{\partial x}\left(G_x H_x\right)\right],$$

$$G_{tt} = \quad \alpha G_{xx} + \beta\cos^2\gamma \left[2\sin^2\gamma\ G_{xx} + \cos^2\gamma \left(2 G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2\right) G_{xx}\right)\right.$$
$$\left. + \sin 2\gamma \left(3 G_x G_{xx} + H_x H_{xx}\right)\right].$$

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

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$$\left. + \sin 2\gamma \left(3 G_x G_{xx} + H_x H_{xx} \right) \right].$$

Subcase 1: $\gamma = \pi/2$

$$H_{tt} = \alpha H_{xx}, \qquad G_{tt} = \alpha G_{xx}.$$

Ansatz 2 Compatible with Incompressibility

Subcase 2: $\gamma = 0$

$$H_{tt} = \quad \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2 \right] H_{xx} + 2G_x H_x G_{xx} \right) \right],$$

$$G_{tt} = \quad \alpha G_{xx} + \beta \left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2 \right) G_{xx} \right) \right].$$

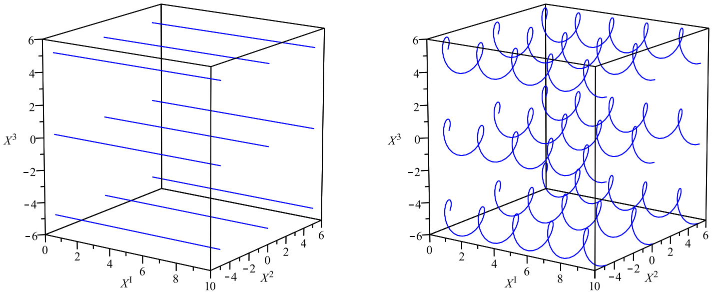
Ansatz 2 Compatible with Incompressibility

Subcase 2: $\gamma = 0$

$$H_{tt} = \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2\right] H_{xx} + 2G_x H_x G_{xx}\right)\right],$$

$$G_{tt} = \alpha G_{xx} + \beta \left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2\right) G_{xx}\right)\right].$$

- Exact traveling wave solutions can be derived [*A. C., J.-F. G., S. St.Jean (2015)*].
- e.g. Carrol-type nonlinear rotational shear waves



Compute local CLs for the coupled model

$$
\begin{aligned}
H_{tt} = &\quad \alpha H_{xx} + \beta\left[\left(\left[3H_x^2 + G_x^2\right] H_{xx} + 2G_x H_x G_{xx}\right)\right],\\
G_{tt} = &\quad \alpha G_{xx} + \beta\left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2\right) G_{xx}\right)\right].
\end{aligned}
$$

Compute local CLs for the coupled model

$$H_{tt} = \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2 \right] H_{xx} + 2G_x H_x G_{xx} \right) \right],$$

$$G_{tt} = \alpha G_{xx} + \beta \left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2 \right) G_{xx} \right) \right].$$

Linear momenta:

$$\Theta_1 = H_t, \qquad \Theta_2 = G_t,$$

x-components of the Lagrangian and the Angular momentum:

$$\Theta_3 = G_x G_t + G_x G_t, \qquad \Theta_4 = -GH_t + HG_t,$$

Compute local CLs for the coupled model

$$\begin{aligned} H_{tt} &= \alpha H_{xx} + \beta\left[\left(\left[3H_x^2 + G_x^2\right] H_{xx} + 2G_x H_x G_{xx}\right)\right], \\ G_{tt} &= \alpha G_{xx} + \beta\left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2\right) G_{xx}\right)\right]. \end{aligned}$$

Energy:

$$\Theta_5 = \frac{1}{2}(G_t^2 + H_t^2) + \frac{\alpha}{2}(G_x^2 + H_x^2) + \frac{\beta}{4}(G_x^2 + H_x^2)^2.$$

Center of mass theorem:

$$\Theta_6 = tG_t - G, \qquad \Theta_7 = tH_t - H.$$

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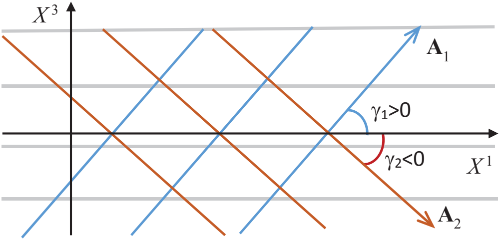
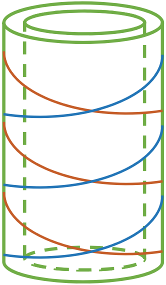
A Two-Fiber Planar Model – "Flat Artery"





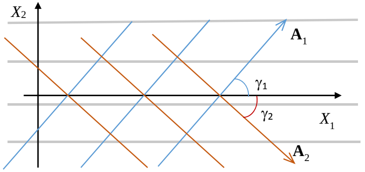
Fiber invariants:

$$I_4 = \lambda_1^2 = \mathbf{A}_1^T \mathbf{C} \mathbf{A}_1, \qquad I_6 = \lambda_2^2 = \mathbf{A}_2^T \mathbf{C} \mathbf{A}_2, \qquad I_8 = (\mathbf{A}_1^T \mathbf{A}_2)(\mathbf{A}_1^T \mathbf{C} \mathbf{A}_2).$$

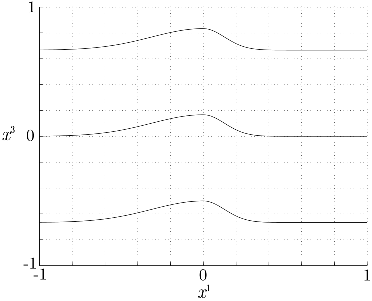
Strain energy density:

$$W = a(I_1 - 3) + b(I_2 - 3) + q_1 (I_4 - 1)^2 + q_2 (I_6 - 1)^2 + K_1 I_8^2 + K_2 I_8.$$

One-Dimensional Shear Waves



Reference Configuration



Actual Configuration

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + G\left(X^1, t\right) \end{bmatrix}, \qquad p = p(X^1, t).$$

A Two-Fiber Planar Model, 1D Shear Waves

Displacements transverse to an axis:

$$\mathbf{X} = \left[\begin{array}{c} X^1 \\ X^2 + G\left(X^1, t\right) \end{array} \right], \qquad p = p(X^1, t).$$

Equations:

- Denote $X^1 = x$.
- Incompressibility condition is again identically satisfied.
- $p(x, t)$ found explicitly.
- Displacement $G(x, t)$ satisfies a PDE from the same general class

$$\boxed{G_{tt} = \left(A\left(G_x\right)^2 + BG_x + C\right) G_{xx},}$$

where the constants A, B, C are rather complicated functions of material parameters:

$$A = A(K_1, q_{1,2}, \gamma_{1,2}), \quad B = B(K_1, q_{1,2}, \gamma_{1,2}), \quad C = C(K_{1,2}, q_{1,2}, \gamma_{1,2}),$$

A Two-Fiber Planar Model, 1D Shear Waves

Nonlinear wave equation

$$\boxed{G_{tt} = \left(A\left(G_x\right)^2 + BG_x + C\right) G_{xx},}$$

- Same conservation laws as found before!

A Two-Fiber Planar Model, 1D Shear Waves

Nonlinear wave equation

$$\boxed{G_{tt} = \left(A\,(G_x)^2 + BG_x + C\right)\,G_{xx},}$$

- Same conservation laws as found before!

Variational structure

- Any nonlinear PDE of the above class follows from a variational principle, with the Lagrangian density (*up to equivalence*)

$$\mathcal{L} = \frac{1}{2}G_t^2 + \frac{A}{4}GG_x^2G_{xx} + \frac{B}{3}GG_xG_{xx} - \frac{C}{2}G_x^2$$

A Two-Fiber Planar Model, 1D Shear Waves

Nonlinear wave equation

$$\boxed{G_{tt} = \left(A\left(G_x\right)^2 + BG_x + C\right) G_{xx},}$$

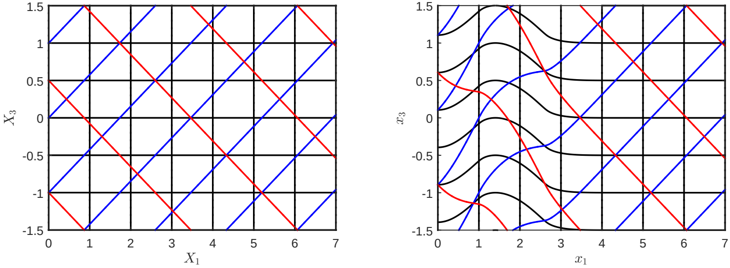
- Same conservation laws as found before!

Simplification

- Depending on the sign of $B^2 - 4AC$, PDE can be transformed to

$$u_{tt} = \left(\left(u_x\right)^2 \pm K\right) u_{xx}, \qquad K = 0, \pm 1.$$

- Dimensional plot for the fiber angles $\gamma_1 = -\gamma_2 \simeq \pi/6$ and same parameters as before:



- Material lines $X^1 = \text{const}$, $X^3 = \text{const}$ (black, vertical and horizontal) and the fiber lines $X^3 - X^1 \tan\gamma_i = \text{const}$, $i = 1, 2$ (blue, red) for the one-dimensional hyperelastic two-fiber anti-plane shear model. The material configuration (left) and the actual configuration (right) at the time $t = 5 \cdot 10^{-4}$ s. Spatial coordinates are given in millimeters.

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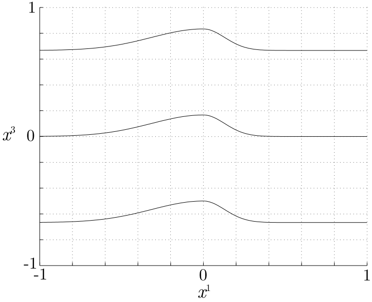
A Viscoelastic Planar Model

A hyper-viscoelastic model:

- An extra "invariant": $J_2 = \mathrm{Tr}(\dot{\mathbf{C}}^2)$.

Total potential, one fiber family:

$$W = a(I_1 - 3) + b(I_2 - 3) + q_1\,(I_4 - 1)^2 + \frac{\eta}{4} J_2 (I_1 - 3).$$

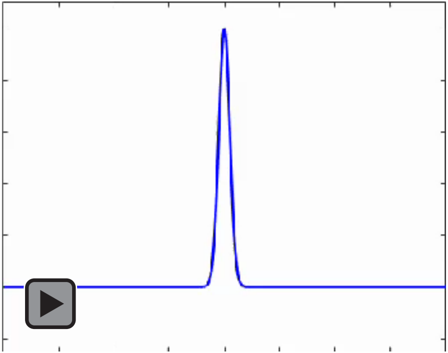


Equation of motion:

- Case: shear wave propagating along the fibers, X^1.
- Single nonlinear PDE:

$$G_{tt} = (\alpha + 3\beta G_x^2) G_{xx} + \eta \left[2(1 + 4G_x^2) G_x G_{tx} G_{xx} + (1 + 2G_x^2) G_x^2 G_{txx} \right].$$

- D'Alembert-type example: no wave breaking...



Compute local CLs for the coupled model

$$G_{tt} = (\alpha + 3\beta G_x^2) G_{xx} + \eta \left[2(1 + 4G_x^2) G_x G_{tx} G_{xx} + (1 + 2G_x^2) G_x^2 G_{txx} \right].$$

- $\alpha = \eta = 1.$

Compute local CLs for the coupled model

$$G_{tt} = (\alpha + 3\beta G_x^2) G_{xx} + \eta \left[2(1 + 4G_x^2) G_x G_{tx} G_{xx} + (1 + 2G_x^2) G_x^2 G_{txx} \right].$$

- $\alpha = \eta = 1$.

CL 1:

$$\mathrm{D}_t(u_t - (1 + 2u_x^2) u_x^2 u_{xx}) - \mathrm{D}_x((1 + \beta u_x^2) u_x) = 0.$$

Potential system:

$$v_x = u_t - (1 + 2u_x^2) u_x^2 u_{xx}, \qquad v_t = (1 + \beta u_x^2) u_x.$$

Evolution equations:

$$u_t = v_x + (1 + 2u_x^2) u_x^2 u_{xx},$$

$$v_t = (1 + \beta u_x^2) u_x.$$

Conservation Laws for the Viscoelastic Shear Waves

Compute local CLs for the coupled model

$$\boxed{G_{tt} = (\alpha + 3\beta G_x^2)G_{xx} + \eta\left[2(1+4G_x^2)G_x G_{tx} G_{xx} + (1+2G_x^2)G_x^2 G_{txx}\right].}$$

- $\alpha = \eta = 1$.

CL 2:

$$\mathrm{D}_t(tu_t - u - t(1+2u_x^2)u_x^2 u_{xx}) - \mathrm{D}_x\left[\left(t - \left(\frac{1}{3} - \beta t\right) + \frac{2}{5}u_x^4\right)u_x\right] = 0.$$

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Summary

Incompressible hyperelastic models

- Fundamental nonlinear equations for finite-amplitude waves are systematically obtained.
- Anti-plane shear wave equations derived for one- and two-fiber-family cases, as well as wave equations for motions transverse to an axis.
- Variational structure is inherited in all hyperelastic models.
- Wave breaking in the one-dimensional case.
- Local conservation laws are computed.

Viscoelastic models

- A one-dimensional finite-amplitude nonlinear anti-plane shear wave model is derived, for the two-fiber-family case.
- No wave breaking.
- No of variational formulation.
- Local conservation laws are considered; potential system used for numerical simulations.

Future work

Further research

- Consider different geometries/curviinear coordinates/symmetric settings of interest for applications.
- Use local conservation laws for further optimization and testing of numerical methods.

Merodio, J. and Ogden, R.W. (2005)
Mechanical response of fiber-reinforced incompressible non-linearly elastic solids. *Int. J. Nonlin. Mech.* **40** (2), 213–227.

Bluman, G.W., Cheviakov, A.F., and Anco, S.C. (2010)
Applications of Symmetry Methods to Partial Differential Equations.
Springer: Applied Mathematical Sciences, Vol. 168.

Cheviakov, A. F. (2007)
GeM software package for computation of symmetries and conservation laws of differential equations. *Comput. Phys. Comm.* **176**, 48–61.

Cheviakov, A. F., Ganghoffer, J.-F., and St. Jean, S. (2015)
Fully nonlinear wave models in fiber-reinforced anisotropic incompressible hyperelastic solids. *Int. J. Nonlin. Mech.* **71**, 8–21.

Cheviakov, A. F. and Ganghoffer, J.-F. (2016)
One-dimensional nonlinear elastodynamic models and their local conservation laws with applications to biological membranes. *J. Mech. Behav. Biomed. Mater.* **58**, 105–121.