Vorticity-Type Equations: Conservation Laws and Applications

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June 25, 2016

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Outline

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Vorticity-type equations:

$$\boxed{\operatorname{div} \mathbf{N} = 0, \qquad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0,}$$

$$\mathbf{N} = \mathbf{N}(t, x, y, z) \in \mathbb{R}^3, \qquad \mathbf{M} = \mathbf{M}(t, x, y, z) \in \mathbb{R}^3.$$

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Euler and Navier-Stokes equations of fluid flow:

$$\operatorname{div} \mathbf{V} = 0, \qquad \mathbf{V}_t + (\mathbf{V} \cdot \nabla)\mathbf{V} + \operatorname{grad} p = \nu \, \Delta \mathbf{V}$$

- Vorticity dynamics equations: $\boldsymbol{\omega} = \operatorname{curl} \mathbf{V}$,

$$\operatorname{div} \boldsymbol{\omega} = 0, \qquad \boldsymbol{\omega}_t + \operatorname{curl} (\boldsymbol{\omega} \times \mathbf{V} - \nu \, \Delta \mathbf{V}) = 0.$$

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Magnetohydrodynamic (MHD) equations:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \boxed{\operatorname{div} \mathbf{B} = 0,}$$

$$\rho \mathbf{V}_t + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\mu}\mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P + \mu_1 \Delta \mathbf{V},$$

$$\boxed{\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \Delta \mathbf{B}.}$$

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$$\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta\, \Delta \mathbf{B}.$$

- $\Delta \mathbf{B} = -\operatorname{curl}(\operatorname{curl} \mathbf{B})$;
- Plasma electric current density and conductivity:

$$\mathbf{J} = (1/\mu) \operatorname{curl} \mathbf{B}, \qquad \sigma = 1/(\mu\, \eta).$$

- Obtain:

$$\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{B}_t + \operatorname{curl}\, (\mathbf{B} \times \mathbf{V} + (1/\sigma)\mathbf{J}) = 0.$$

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Dimensionless Maxwell's equations:

$$\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{B}_t + \operatorname{curl} \mathbf{E} = 0,$$

$$\operatorname{div} \mathbf{E} = \rho, \qquad \mathbf{E}_t - \operatorname{curl} \mathbf{B} = -\mathbf{J}.$$

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Vacuum Maxwell's equations:

$$\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{B}_t + \operatorname{curl} \mathbf{E} = 0,$$

- Second "vorticity-type" subsystem:

$$\operatorname{div} \mathbf{E} = \rho, \qquad \mathbf{E}_t - \operatorname{curl} \mathbf{B} = 0.$$

Outline

Notation

Variables:

- Independent: $\mathbf{x} = (x^1, x^2, ..., x^n)$ or $(t, x^1, x^2, ...)$ or $(t, x, y, ...)$.
- Dependent: $\mathbf{u} = (u^1(\mathbf{x}), u^2(\mathbf{x}), ..., u^m(\mathbf{x}))$ or $(u(\mathbf{x}), v(\mathbf{x}), ...)$.

Partial derivatives:

- Notation:
$$\frac{\partial u^k}{\partial x^m} = u^k_{x^m} = \partial_{x^m} u^k.$$
- E.g.,
$$\frac{\partial}{\partial t} u(x, y, t) = u_t = \partial_t u.$$

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Total derivative operators:

$$\mathrm{D}_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ii_1} \frac{\partial}{\partial u^\mu_{i_1}} + u^\mu_{ii_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \cdots .$$

Local Conservation Laws

Conservation laws

- A local conservation law: a divergence expression equal to zero,

$$\mathrm{D}_i \Psi^i[\mathbf{u}] \equiv \operatorname{div} \mathbf{\Psi}^{\mathbf{i}}[\mathbf{u}] = 0.$$

- For models involving time:

$$\mathrm{D}_t\, \Theta[\mathbf{u}] + \operatorname{div}_{\mathbf{x}}\, \mathbf{\Psi}[\mathbf{u}] = 0.$$

- $\Theta[\mathbf{u}]$: conserved density; $\mathbf{\Psi}[\mathbf{u}]$: flux vector.

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Globally Conserved Quantities

- When the total flux vanishes, $\oint_{\partial V} \mathbf{\Psi}[\mathbf{u}] \cdot d\mathbf{S} = 0$, one has $\dfrac{d}{dt} \displaystyle\int_V \Theta[\mathbf{u}]\, dV = 0.$

- Other applications.

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Direct CL construction

- For a PDE system $R^\sigma[\mathbf{u}] \equiv R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N$, one may seek local CLs as follows:

$$\mathrm{D}_i \Phi^i[\mathbf{u}] = \Lambda_\sigma[\mathbf{u}] R^\sigma[\mathbf{u}] = 0.$$

- Unknown multipliers / characteristics: $\{\Lambda_\sigma[\mathbf{u}]\}$.

The Infinite Set of Conservation Laws

Vorticity-type equations:

$$\operatorname{div} \mathbf{N} = 0, \qquad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0.$$

Theorem (Principal Result 1)

The set of vorticity-type equations admits an infinite family of local conservation laws given by

$$\boxed{(\mathbf{N} \cdot \nabla F)_t + \operatorname{div}(\mathbf{M} \times \nabla F - F_t\, \mathbf{N}) = 0,}$$

depending on an arbitrary function $F = F(t, x, y, z)$. *In particular,*

$$\Lambda^1 = -F_t, \quad \Lambda^2 = F_x, \quad \Lambda^3 = F_y, \quad \Lambda^4 = F_z.$$

- *F may be a differential function of the dependent variables.*

Physical Examples

Euler and Navier-Stokes equations of fluid flow:

- PDEs:

$$\mathrm{div}\ \mathbf{V} = 0, \qquad \mathbf{V}_t + (\mathbf{V}\cdot\nabla)\mathbf{V} + \mathrm{grad}\ p = \nu\,\Delta\mathbf{V};$$

$$\mathrm{div}\,\boldsymbol{\omega} = 0, \qquad \boldsymbol{\omega}_t + \mathrm{curl}\ (\boldsymbol{\omega}\times\mathbf{V} - \nu\,\Delta\mathbf{V}) = 0.$$

- An infinite CL set:

$$(\boldsymbol{\omega}\cdot\nabla F)_t + \mathrm{div}\left([\boldsymbol{\omega}\times\mathbf{V} - \nu\,\nabla^2\mathbf{V}]\times\nabla F - F_t\,\boldsymbol{\omega}\right) = 0.$$

- $F = F(t, x, y, z)$.
- All well-known CLs of fluid dynamics that are linear in $\boldsymbol{\omega}$ arise from this family.

Magnetohydrodynamic equations:

- PDEs:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

$$\rho \mathbf{V}_t + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\mu}\mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad}\, P + \mu_1 \Delta \mathbf{V},$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \Delta \mathbf{B}.$$

- An infinite CL set:

$$(\mathbf{B} \cdot \nabla F)_t + \operatorname{div} \left(\left[\mathbf{B} \times \mathbf{V} + \frac{1}{\sigma}\mathbf{J} \right] \times \nabla F - F_t \mathbf{B} \right) = 0.$$

- $F = F(t, x, y, z)$.

Dimensionless Maxwell's equations:

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$$\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{B}_t + \operatorname{curl} \mathbf{E} = 0,$$

$$\operatorname{div} \mathbf{E} = \rho, \qquad \mathbf{E}_t - \operatorname{curl} \mathbf{B} = -\mathbf{J}.$$

- An infinite "magnetic CL" set:

$$(\mathbf{B} \cdot \nabla F)_t + \operatorname{div} (\mathbf{E} \times \nabla F - F_t \, \mathbf{B}) = 0.$$

- $F = F(t, x, y, z)$.

Vacuum Maxwell's equations:

- For $\mathbf{J}, \rho = 0$, additionally, an "electric CL" with $G = G(t, x, y, z)$:

$$(\mathbf{E} \cdot \nabla G)_t - \operatorname{div} (\mathbf{B} \times \nabla G - G_t \, \mathbf{E}) = 0.$$

Outline

Divergence-type conservation laws in $\mathbb{R}^3$:

- $\mathbf{x} = (x, y, z), \quad \mathbf{\Psi} = (\Psi^1, \Psi^2, \Psi^3);$
- CL: $\quad \mathrm{div}\ \mathbf{\Psi}[\mathbf{u}] = 0.$
- Potential equations: $\quad \mathbf{\Psi}[\mathbf{u}] = \mathrm{curl}\ \mathbf{A}[\mathbf{u}].$

An Example in $\mathbb{R}^3$

Divergence-type conservation laws in $\mathbb{R}^3$:

- $\mathbf{x} = (x, y, z), \quad \boldsymbol{\Psi} = (\Psi^1, \Psi^2, \Psi^3);$
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- Potential equations: $\quad \boldsymbol{\Psi}[\mathbf{u}] = \mathrm{curl}\ \mathbf{A}[\mathbf{u}].$

Curl-type (lower-degree) conservation laws in $\mathbb{R}^3$:

- $\boldsymbol{\Phi} = (\Phi^1, \Phi^2, \Phi^3);$
- CL: $\quad \mathrm{curl}\ \boldsymbol{\Phi}[\mathbf{u}] = 0.$
- Potential equations: $\quad \boldsymbol{\Phi}[\mathbf{u}] = \mathrm{grad}\ \phi[\mathbf{u}].$

- A differential r-form:

$$\omega^{(r)} = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}.$$

Definition

A conservation law of degree r $(1 \le r \le n-1)$ of a PDE system is an r-form $\omega^{(r)}[\mathbf{U}]$, such that its exterior derivative

$$\Omega^{(r+1)}[\mathbf{u}] = \mathrm{d}\omega^{(r)}[\mathbf{u}] = 0$$

on all solutions $\mathbf{U} = \mathbf{u}(\mathbf{x})$ of the PDE system.

- A conservation law of degree $n-1$: divergence-type, $\mathrm{D}_i\Psi^i[\mathbf{u}] = 0$.
- One may consider $\binom{n}{r}$ potential equations

$$\omega_{\mu_1\ldots\mu_r}[\mathbf{u}] = \sum_{i=1}^{r}(-1)^{i-1}\frac{\partial}{\partial x^{\mu_i}}\widetilde{\omega}_{\mu_1\ldots\overline{\mu_i}\ldots\mu_r}[\mathbf{u}]$$

for $\binom{n}{r-1}$ potential variables given by the independent components of $\widetilde{\omega}^{(r-1)}[\mathbf{u}]$.

The Lower-Degree CL Structure of Vorticity-type Equations

Vorticity-type equations:

$$\operatorname{div} \mathbf{N} = 0, \qquad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0.$$

Theorem

The vorticity-type PDEs are equivalent to a lower-degree (degree two) conservation law in the four-dimensional space of variables t, x, y, z.

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The vorticity-type PDEs are equivalent to a lower-degree (degree two) conservation law in the four-dimensional space of variables t, x, y, z.

- Denote these four scalar PDEs by

$$E^1 = N^1_x + N^2_y + N^3_z, \qquad E^2 = N^1_t + M^3_y - M^2_z,$$
$$E^3 = N^2_t + M^1_z - M^3_x, \qquad E^4 = N^3_t + M^2_x - M^1_y.$$

Vorticity-type equations:

$$\operatorname{div}\mathbf{N} = 0, \qquad \mathbf{N}_t + \operatorname{curl}\mathbf{M} = 0.$$

Theorem

The vorticity-type PDEs are equivalent to a lower-degree (degree two) conservation law in the four-dimensional space of variables t, x, y, z.

- Let

$$\begin{aligned}\omega = \;& -M^1\,\mathrm{d}t\wedge\mathrm{d}x - M^2\,\mathrm{d}t\wedge\mathrm{d}y - M^3[\mathbf{U}]\,\mathrm{d}t\wedge\mathrm{d}z\\ & +N^3\,\mathrm{d}x\wedge\mathrm{d}y + N^2\,\mathrm{d}z\wedge\mathrm{d}x + N^1\,\mathrm{d}y\wedge\mathrm{d}z,\end{aligned}$$

$$\Omega[\mathbf{U}] = E^1[\mathbf{U}]\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + E^2[\mathbf{U}]\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}t - E^3[\mathbf{U}]\,\mathrm{d}z\wedge\mathrm{d}t\wedge\mathrm{d}x + E^4[\mathbf{U}]\,\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y.$$

- Then $\Omega[\mathbf{U}] = \mathrm{d}\omega[\mathbf{U}]$.
- On solutions, $\mathrm{d}\omega[\mathbf{u}] = \Omega[\mathbf{u}] = 0$.

Outline

Vorticity-type equations:

$$\operatorname{div} \mathbf{N} = 0, \qquad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0.$$

- Degree two differential form:

$$\begin{aligned} \omega = \quad & -M^1\, \mathrm{d}t \wedge \mathrm{d}x - M^2\, \mathrm{d}t \wedge \mathrm{d}y - M^3[\mathbf{U}]\, \mathrm{d}t \wedge \mathrm{d}z \\ & + N^3\, \mathrm{d}x \wedge \mathrm{d}y + N^2\, \mathrm{d}z \wedge \mathrm{d}x + N^1\, \mathrm{d}y \wedge \mathrm{d}z, \end{aligned}$$

- Equations: $\mathrm{d}\omega[\mathbf{u}] = \Omega[\mathbf{u}] = 0$.
- Potential equations: $\omega[\mathbf{u}] = \mathrm{d}\theta[\mathbf{u}]$,

$$\theta = \theta^t(t,x,y,z)\,\mathrm{d}t + \theta^x(t,x,y,z)\,\mathrm{d}x + \theta^y(t,x,y,z)\,\mathrm{d}y + \theta^z(t,x,y,z)\,\mathrm{d}z.$$

- In components:

$$\begin{array}{lll} -M^1[\mathbf{u}] = \theta^x_t - \theta^t_x, & -M^2[\mathbf{u}] = \theta^y_t - \theta^t_y, & -M^3[\mathbf{u}] = \theta^z_t - \theta^t_z, \\ N^1[\mathbf{u}] = \theta^z_y - \theta^y_z, & N^2[\mathbf{u}] = \theta^x_z - \theta^z_x, & N^3[\mathbf{u}] = \theta^y_x - \theta^x_y. \end{array}$$

Potential Systems for the Physical Examples

Euler and Navier-Stokes equations of fluid flow:

- PDEs:

$$\mathrm{div}\ \mathbf{V} = 0, \qquad \mathbf{V}_t + (\mathbf{V}\cdot\nabla)\mathbf{V} + \mathrm{grad}\ p = \nu\,\Delta\mathbf{V};$$

$$\mathrm{div}\,\boldsymbol{\omega} = 0, \qquad \boldsymbol{\omega}_t + \mathrm{curl}\ (\boldsymbol{\omega}\times\mathbf{V} - \nu\,\Delta\mathbf{V}) = 0.$$

- Denote

$$(\theta^x, \theta^y, \theta^z) = \mathbf{V}, \qquad \theta^t = -p.$$

- Potential equations: recover the momentum PDEs

$$\mathbf{V}_t + \mathrm{grad}\ p = -(\boldsymbol{\omega}\times\mathbf{V} - \nu\Delta\mathbf{V}).$$

Potential Systems for the Physical Examples

Magnetohydrodynamic equations:

- PDEs:

$$\rho_t + \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \mathbf{B} = 0,$$

$$\rho \mathbf{V}_t + \rho(\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\mu}\mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P + \mu_1 \Delta \mathbf{V},$$

$$\mathbf{B}_t = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \Delta \mathbf{B}.$$

- Denote

$$(\theta^x, \theta^y, \theta^z) = \mathbf{A}, \qquad \theta^t = -\Psi.$$

- Potential equations:

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \operatorname{grad} \Psi = \mathbf{V} \times \mathbf{B} - \mathbf{A}_t - \eta \operatorname{curl} \mathbf{B}.$$

- $\Psi(t, x, y, z)$: generalization of the famous Galas-Bogoyavlenskij potential in plasma physics.

Outline

Discussion

Vorticity-type system:

$$\operatorname{div} \mathbf{N} = 0, \qquad \mathbf{N}_t + \operatorname{curl} \mathbf{M} = 0.$$

- By itself, is underdetermined.
- Is a part of important physical models.
- Has a special geometric structure of a lower-degree conservation law, $\mathrm{d}\omega[\mathbf{u}] = 0$:

$$\begin{aligned}\omega = \quad & -M^1 \,\mathrm{d}t \wedge \mathrm{d}x - M^2 \,\mathrm{d}t \wedge \mathrm{d}y - M^3[\mathbf{U}] \,\mathrm{d}t \wedge \mathrm{d}z \\ & + N^3 \,\mathrm{d}x \wedge \mathrm{d}y + N^2 \,\mathrm{d}z \wedge \mathrm{d}x + N^1 \,\mathrm{d}y \wedge \mathrm{d}z,\end{aligned}$$

- Has a corresponding differential identity $\mathrm{d}^2\omega[\mathbf{u}] = 0$.
- Admits an infinite family of local divergence-type conservation laws

$$(\mathbf{N} \cdot \nabla F)_t + \operatorname{div}(\mathbf{M} \times \nabla F - F_t \,\mathbf{N}) = 0,$$

corresponding to that identity (cf. Noether's second theorem).

Some references

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Thank you for your attention!