Exact Solutions of a Fully Nonlinear Two-Fluid Model

Alexey Shevyakov

(*alternative spelling: Alexei Cheviakov*)

Department of Mathematics and Statistics,

University of Saskatchewan, Saskatoon, Canada

June 28, 2018

Outline

Outline

Euler and Navier-Stokes equations

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0,$$

$$\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \operatorname{grad} p = \mathbf{f} \;\; + \mu\Delta\mathbf{v}.$$

- ... add an *equation of state*.
- 1757 & 1822
- Velocity $\mathbf{v}(t,\mathbf{x}) \;=\; (u, v, w)$
- Pressure $p(t,\mathbf{x})$
- Density $\rho(t,\mathbf{x})$
- Viscosity μ

Euler and Navier-Stokes equations

$$\rho_t + \mathrm{div}(\rho \mathbf{v}) = 0,$$

$$\rho(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) + \mathrm{grad}\, p = \mathbf{f} \ + \mu \Delta \mathbf{v}.$$

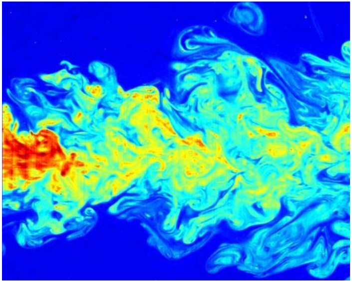
- Appropriate for the description of a wide range of physical phenomena...

Euler and Navier-Stokes equations

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0,$$

$$\rho(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) + \operatorname{grad} p = \mathbf{f} \; + \mu \Delta \mathbf{v}.$$

- ... including turbulence.



Classical PDEs of Gas & Fluid Dynamics

Euler and Navier-Stokes equations

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0,$$

$$\rho(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) + \operatorname{grad} p = \mathbf{f} \;\; + \mu \Delta \mathbf{v}.$$

- Multiple open questions, of physical and mathematical nature (e.g., solution existence, regularity, stability...).
- Direct numerical simulations: high cost, low precision.
- Knowledge of analytical properties and any exact or approximate solutions is of importance.
- Geometric reductions & various simplified models are common.

Outline

Two Stratified Non-Mixing Fluids in a Horizontal Channel

- W. Choi & R. Camassa, *"Fully nonlinear internal waves in a two-fluid system."* [JFM, 1999]

Two Stratified Non-Mixing Fluids in a Horizontal Channel

- W. Choi & R. Camassa, *"Fully nonlinear internal waves in a two-fluid system."* [JFM, 1999]
- Models a stratified system of two non-mixing fluids of different densities.

Two Stratified Non-Mixing Fluids in a Horizontal Channel

- W. Choi & R. Camassa, *"Fully nonlinear internal waves in a two-fluid system."* [JFM, 1999]
- Models a stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.

Two Stratified Non-Mixing Fluids in a Horizontal Channel

- W. Choi & R. Camassa, *"Fully nonlinear internal waves in a two-fluid system."* [JFM, 1999]
- Models a stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.
- Describes nonlinear internal/interfacial waves, propagating in both directions.

Two Stratified Non-Mixing Fluids in a Horizontal Channel

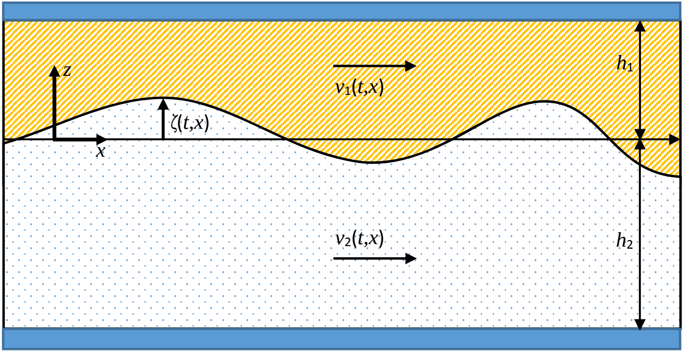
- W. Choi & R. Camassa, *"Fully nonlinear internal waves in a two-fluid system."* [JFM, 1999]
- Models a stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.
- Describes nonlinear internal/interfacial waves, propagating in both directions.
- Provides good agreement with experiment and Euler-based DNS.

Two Stratified Non-Mixing Fluids in a Horizontal Channel

- W. Choi & R. Camassa, *"Fully nonlinear internal waves in a two-fluid system."* [JFM, 1999]
- Models a stratified system of two non-mixing fluids of different densities.
- A (1+1)-dimensional asymptotic model based on incompressible Euler equations.
- Describes nonlinear internal/interfacial waves, propagating in both directions.
- Provides good agreement with experiment and Euler-based DNS.
- Reduces to shallow-water and KdV models in limiting cases.

Outline

The Governing Equations



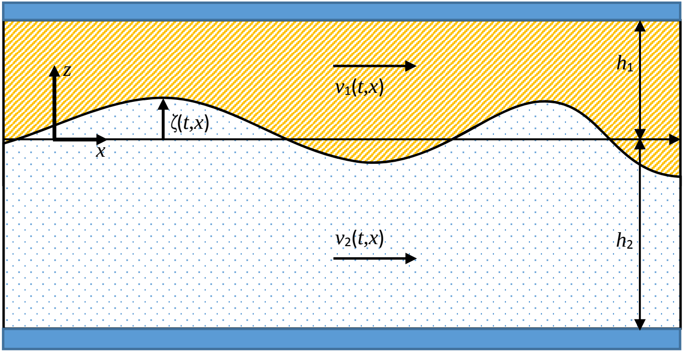
Euler equations of incompressible constant-density flow in gravity field, 3D

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\,\mathrm{grad}\, p - \mathbf{g},$$

$$\mathrm{div}\ \mathbf{v} = 0, \qquad \mathbf{g} = -g\mathbf{k}.$$

- Here $\mathbf{v} = (u(t,\mathbf{x}), 0, w(t,\mathbf{x}))$; $p = p(t,\mathbf{x})$; $\rho = \mathrm{const}$.

The Governing Equations

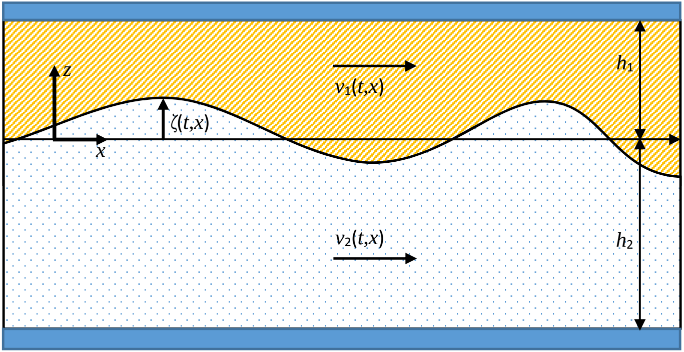


Two-dimensional Euler equations in the (x, z)-plane

$$u_x + w_z = 0,$$

$$u_t + uu_x + wu_z = -p_x/\rho,$$

$$w_t + uw_x + ww_z = -p_z/\rho - g.$$

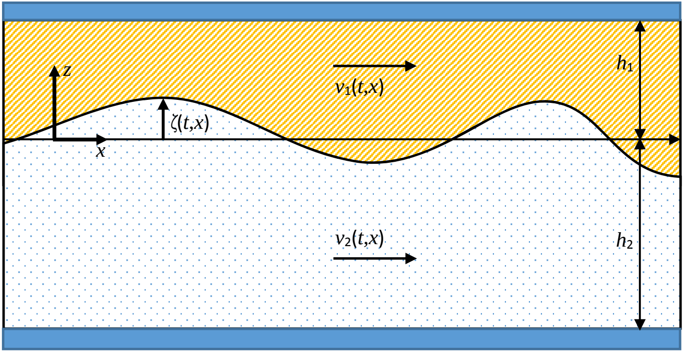


Boundary conditions

- No-leak: $w_1(t,x,h_1) = w_2(t,x,-h_2) = 0.$
- At the interface $z = \zeta(t,x)$:

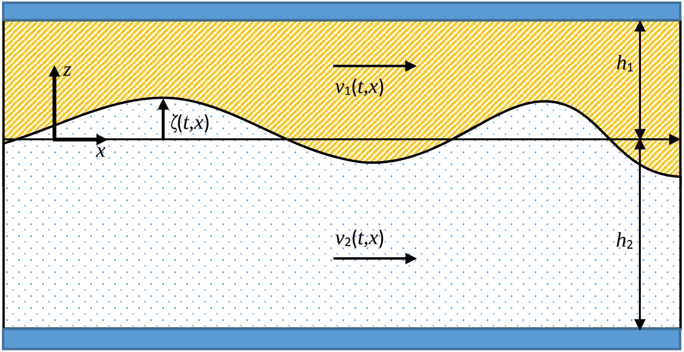
$$\zeta_t + u_1\zeta_x = w_1, \quad \zeta_t + u_2\zeta_x = w_2, \quad p_1 = p_2.$$

Asymptotic Assumptions and the CC Model



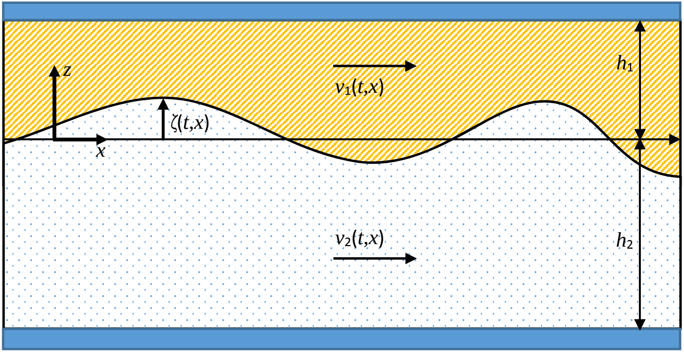
- Fluid depth $\ll$ characteristic length: $h_i/L = \epsilon \ll 1.$
- Continuity equation $\to$ $w_i/u_i = O(h_i/L) = O(\epsilon) \ll 1.$
- Finite-amplitude waves: $\zeta \lesssim h_i$.

$$u_i/U_0 = O(\zeta/h_i) = O(1), \quad U_0 = (gH)^{1/2}, \quad H = h_1 + h_2.$$



- Actual fluid layer thicknesses: $\eta_1 = h_1 - \zeta, \;\; \eta_2 = h_2 + \zeta.$
- Layer-average (depth-mean) horizontal velocities:

$$v_1 = \frac{1}{\eta_1} \int_{\zeta}^{h_1} u_1(t,x,z)\, dz, \qquad v_2 = \frac{1}{\eta_2} \int_{-h_2}^{\zeta} u_2(t,x,z)\, dz.$$



The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x + O(\epsilon^4), \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

Outline

Some Properties of the CC Model

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

Variables, unknowns, order

- (1+1) – dimensional.
- Independent: x, t.
- Dependent: v_1, v_2, P, ζ.
- $\eta_1 = h_1 - \zeta$, $\eta_2 = h_2 + \zeta$.
- 4 PDEs, two third-order, mixed space-time derivatives.

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

Asymptotic horizontal velocity estimates

One can show that in terms of the mean velocity of each fluid layer, the corresponding horizontal velocities $u_i(t, x, z)$ are given by

$$u_i(t, x, z) = v_i + \left(\tfrac{1}{6}\eta_i^2 - \tfrac{1}{2}(z \mp h_i)^2\right) v_{i\,xx} + O(\epsilon^4).$$

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

An average velocity relationship

From the first two PDEs,

$$\frac{\partial}{\partial x}(\eta_1 v_1 + \eta_2 v_2) = 0, \quad \Rightarrow \quad \eta_1 v_1 + \eta_2 v_2 = (\eta_1 v_1 + \eta_2 v_2)|_{\pm\infty}\ .$$

- In the case of no velocity shear boundary condition, $v_1|_{\pm\infty} = v_2|_{\pm\infty} = 0$, one has

$$\frac{v_2}{v_1} = -\frac{\eta_1}{\eta_2}.$$

- We don't assume this is the case.

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

Symmetry properties

- Translations and the Galilei group:

$$x^* = x + x_0 + Ct, \qquad t^* = t + t_0, \qquad (v_i)^* = v_i + C,$$

$$P^* = P + P_0(t, \eta_1 v_1 + \eta_2 v_2),$$

$$x_0, t_0, C = \text{const}.$$

- Time inversion:

$$x^* = x, \qquad t^* = -t, \qquad (v_i)^* = -v_i, \qquad P^* = P.$$

The Quality of Approximation

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

Two-dimensional Euler equations in the (x, z)-plane

$$u_x + w_z = 0,$$

$$u_t + uu_x + wu_z = -p_x/\rho,$$

$$w_t + uw_x + ww_z = -p_z/\rho - g.$$

- $\rho = \rho_i, \quad i = 1, 2.$

- Choi and Camassa (1999): semi-numerical solitary wave solutions.

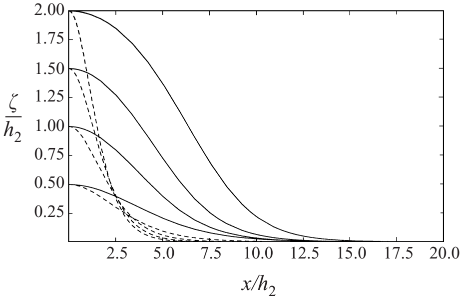


FIGURE 4. Solitary wave solutions (——) of (3.51) for $\rho_1/\rho_2 = 0.63$, $h_1/h_2 = 5.09$ and $a/h_2 =$(0.5, 1, 1.5, 2) compared with KdV solitary waves (– – –) of the same amplitude given by (3.39). Here and in the following figure of wave profiles, the waves are symmetric with respect to reflections $X \to -X$ and only half of the wave profile is shown.

- Wave amplitude vs. effective wavelength: CC model solutions provide a “better” agreement with Euler dynamics than, e.g., Korteweg - de Vries (KdV) solitons,

$$u_t + uu_x + u_{xxx} = 0.$$

The Quality of Approximation

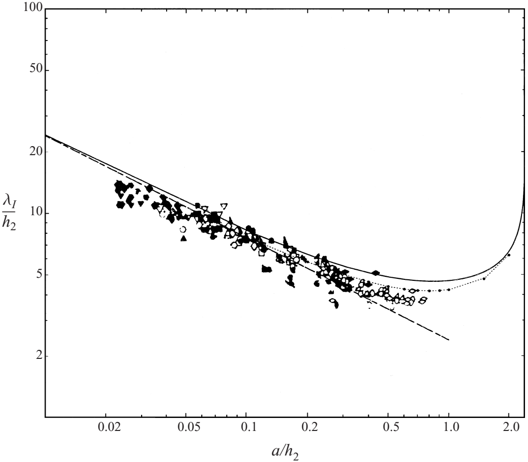


FIGURE 5. Effective wavelength λ_I versus wave amplitude a curves compared with experimental data (symbols, reproduced with permission from Cambridge University Press) by Koop & Butler (1981) for $\rho_1/\rho_2 = 0.63$ and $h_1/h_2 = 5.09$: ——, fully nonlinear theory given by (3.62); – – –, weakly nonlinear (KdV) theory given by (3.67); $\cdots\bullet\cdots$, numerical solutions of the full Euler equations by Grue *et al.* (1997).

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

Old and new variables

- Original form: five constant physical parameters: $g,\ \rho_1,\ \rho_2,\ h_1,\ h_2$.
- Total channel depth: $H = h_1 + h_2$.
- Density ratio: $S = \rho_1/\rho_2$, $0 < S < 1$.
- Relative depth of the top fluid level (dimensionless):

$$\hat{Z} = \frac{h_1 - \zeta}{H} \equiv \frac{\eta_1}{H}, \qquad 0 < \hat{Z} < 1.$$

The Dimensionless Form and Parameter Reduction

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

Dimensionless forms of other variables

$$t = Q_t\,\hat{t}, \qquad x = Q_h\,\hat{x}, \qquad P(t,x) = Q_P\,\hat{P}(\hat{t},\hat{x}), \qquad v_i(t,x) = Q_i\,\hat{v}_i(\hat{t},\hat{x}),$$

$$i = 1, 2,$$

where the scaling factors are chosen to remove most of the constant coefficients:

$$Q_h = H, \quad Q_t = \sqrt{\frac{H}{g}}, \quad Q_1 = Q_2 = \sqrt{gH}, \quad Q_P = \rho_1 g H.$$

The Dimensionless Form and Parameter Reduction

The Choi-Camassa (CC) model:

$$\eta_{i\,t} + (\eta_i v_i)_x = 0, \qquad i = 1, 2,$$

$$v_{i\,t} + v_i v_{i\,x} + g\zeta_x = -\frac{P_x}{\rho_i} + \frac{1}{3\eta_i}\left(\eta_i^3 G_i\right)_x, \qquad G_i \equiv v_{i\,tx} + v_i v_{i\,xx} - (v_{i\,x})^2.$$

The dimensionless Miyata-Choi-Camassa system

$$\hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_1)_{\hat{x}} = 0, \qquad \hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_2)_{\hat{x}} - (\hat{v}_2)_{\hat{x}} = 0,$$

$$\hat{v}_{1\hat{t}} + \hat{v}_1\hat{v}_{1\,\hat{x}} - \hat{Z}_{\hat{x}} + \hat{P}_{\hat{x}} - \hat{Z}\hat{Z}_{\hat{x}}\hat{G}_1 - \tfrac{1}{3}\hat{Z}^2\hat{G}_{1\,\hat{x}} = 0,$$

$$\hat{v}_{2\hat{t}} + \hat{v}_2\hat{v}_{2\,\hat{x}} - \hat{Z}_{\hat{x}} + S\hat{P}_{\hat{x}} - \tfrac{1}{3}(1-\hat{Z})^2\hat{G}_{2\,\hat{x}} + (1-\hat{Z})\hat{Z}_{\hat{x}}\hat{G}_2 = 0,$$

$$\hat{G}_i \equiv \hat{v}_{i\,tx} + \hat{v}_i\hat{v}_{i\,xx} - (\hat{v}_{i\,\hat{x}})^2, \qquad i = 1, 2.$$

- Loss of "symmetry" between layers (though there was *no actual symmetry*!)
- A single constitutive parameter: S.

Outline

The ODE Governing Traveling Wave Solutions

Dimensionless Choi-Camassa PDEs:

$$\hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_1)_{\hat{x}} = 0, \qquad \hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_2)_{\hat{x}} - (\hat{v}_2)_{\hat{x}} = 0,$$

$$\hat{v}_{1\hat{t}} + \hat{v}_1\hat{v}_{1\,\hat{x}} - \hat{Z}_{\hat{x}} + \hat{P}_{\hat{x}} - \hat{Z}\hat{Z}_{\hat{x}}\hat{G}_1 - \tfrac{1}{3}\hat{Z}^2\hat{G}_{1\,\hat{x}} = 0,$$

$$\hat{v}_{2\hat{t}} + \hat{v}_2\hat{v}_{2\,\hat{x}} - \hat{Z}_{\hat{x}} + S\hat{P}_{\hat{x}} - \tfrac{1}{3}(1-\hat{Z})^2\hat{G}_{2\,\hat{x}} + (1-\hat{Z})\hat{Z}_{\hat{x}}\hat{G}_2 = 0,$$

$$\hat{G}_i \equiv \hat{v}_{i\,tx} + \hat{v}_i\hat{v}_{i\,xx} - (\hat{v}_{i\,\hat{x}})^2, \qquad i = 1, 2.$$

Traveling wave coordinate

- Point symmetry generator:

$$\mathrm{X} = \hat{c}\,\frac{\partial}{\partial\hat{x}} + \frac{\partial}{\partial\hat{t}}\,.$$

- Dimensionless traveling wave coordinate and the ansatz:

$$\hat{r} = \hat{r}(t,x) = \hat{x} - \hat{c}\hat{t} + \hat{x}_0 = \frac{1}{H}(x - ct + x_0);$$

$$\hat{Z}, \hat{v}_1, \hat{v}_2, \hat{P} = \hat{Z}, \hat{v}_1, \hat{v}_2, \hat{P}\ (\hat{r}).$$

The ODE Governing Traveling Wave Solutions

Dimensionless Choi-Camassa PDEs:

$$\hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_1)_{\hat{x}} = 0, \qquad \hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_2)_{\hat{x}} - (\hat{v}_2)_{\hat{x}} = 0,$$

$$\hat{v}_{1\hat{t}} + \hat{v}_1\hat{v}_{1\,\hat{x}} - \hat{Z}_{\hat{x}} + \hat{P}_{\hat{x}} - \hat{Z}\hat{Z}_{\hat{x}}\hat{G}_1 - \tfrac{1}{3}\hat{Z}^2\hat{G}_{1\,\hat{x}} = 0,$$

$$\hat{v}_{2\hat{t}} + \hat{v}_2\hat{v}_{2\,\hat{x}} - \hat{Z}_{\hat{x}} + S\hat{P}_{\hat{x}} - \tfrac{1}{3}(1-\hat{Z})^2\hat{G}_{2\,\hat{x}} + (1-\hat{Z})\hat{Z}_{\hat{x}}\hat{G}_2 = 0,$$

$$\hat{G}_i \equiv \hat{v}_{i\,tx} + \hat{v}_i\hat{v}_{i\,xx} - (\hat{v}_{i\,\hat{x}})^2, \qquad i = 1, 2.$$

First two equations; velocity expressions

$$\hat{c}\hat{Z}' = (\hat{Z}\hat{v}_1)' = (\hat{Z}\hat{v}_2)' - \hat{v}_2', \quad \Rightarrow$$

$$\hat{v}_1 = \hat{c} + \frac{C_1}{\hat{Z}}, \qquad \hat{v}_2 = \hat{c} + \frac{C_2}{1-\hat{Z}}, \qquad C_1, C_2 = \text{const}.$$

- Galilei invariance, WLOG $\hat{c} = 0$.

Dimensionless Choi-Camassa PDEs:

$$\hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_1)_{\hat{x}} = 0, \qquad \hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_2)_{\hat{x}} - (\hat{v}_2)_{\hat{x}} = 0,$$

$$\hat{v}_{1\hat{t}} + \hat{v}_1\hat{v}_{1\,\hat{x}} - \hat{Z}_{\hat{x}} + \hat{P}_{\hat{x}} - \hat{Z}\hat{Z}_{\hat{x}}\hat{G}_1 - \tfrac{1}{3}\hat{Z}^2\hat{G}_{1\,\hat{x}} = 0,$$

$$\hat{v}_{2\hat{t}} + \hat{v}_2\hat{v}_{2\,\hat{x}} - \hat{Z}_{\hat{x}} + S\hat{P}_{\hat{x}} - \tfrac{1}{3}(1-\hat{Z})^2\hat{G}_{2\,\hat{x}} + (1-\hat{Z})\hat{Z}_{\hat{x}}\hat{G}_2 = 0,$$

$$\hat{G}_i \equiv \hat{v}_{i\,tx} + \hat{v}_i\hat{v}_{i\,xx} - (\hat{v}_{i\,\hat{x}})^2, \qquad i = 1, 2.$$

Third equation; pressure

$$\hat{P} = \hat{P}_0 + \hat{Z} - \frac{C_1^2}{6\hat{Z}^2}\left(2\hat{Z}\hat{Z}'' - (\hat{Z}')^2 + 3\right), \qquad \hat{P}_0 = \mathrm{const}.$$

The ODE Governing Traveling Wave Solutions

Dimensionless Choi-Camassa PDEs:

$$\hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_1)_{\hat{x}} = 0, \qquad \hat{Z}_{\hat{t}} + (\hat{Z}\hat{v}_2)_{\hat{x}} - (\hat{v}_2)_{\hat{x}} = 0,$$

$$\hat{v}_{1\hat{t}} + \hat{v}_1\hat{v}_{1\,\hat{x}} - \hat{Z}_{\hat{x}} + \hat{P}_{\hat{x}} - \hat{Z}\hat{Z}_{\hat{x}}\hat{G}_1 - \tfrac{1}{3}\hat{Z}^2\hat{G}_{1\,\hat{x}} = 0,$$

$$\hat{v}_{2\hat{t}} + \hat{v}_2\hat{v}_{2\,\hat{x}} - \hat{Z}_{\hat{x}} + S\hat{P}_{\hat{x}} - \tfrac{1}{3}(1-\hat{Z})^2\hat{G}_{2\,\hat{x}} + (1-\hat{Z})\hat{Z}_{\hat{x}}\hat{G}_2 = 0,$$

$$\hat{G}_i \equiv \hat{v}_{i\,tx} + \hat{v}_i\hat{v}_{i\,xx} - (\hat{v}_{i\,\hat{x}})^2, \qquad i = 1, 2.$$

Fourth equation: $\hat{v}_{2\hat{t}} + \ldots = 0$

- 3rd-order, complicated-looking ODE for $\hat{Z}(\hat{r})$: $E_4[\hat{Z}] = 0$.
- Seek integrating factors (conservation law multipliers): $\Lambda_k[\hat{Z}]E_4[\hat{Z}] = \dfrac{d}{d\hat{r}}\Phi_k[\hat{Z}]$.
- Find two factors assuming $\Lambda_k = \Lambda_k(\hat{r}, \hat{Z})$ (GeM symbolic software):

$$\Lambda_1 = \hat{Z}^{-3}(1-\hat{Z})^{-3}, \qquad \Lambda_2 = \hat{Z}^{-2}(1-\hat{Z})^{-3}.$$

The ODE Governing Traveling Wave Solutions (ctd.)

- Two respective constants of motion (first integrals):

$$\begin{aligned}\Phi_1[\hat{Z}] &= -\frac{1}{2\hat{Z}^2(1-\hat{Z})^2}\Big[2\hat{Z}(1-\hat{Z})(\alpha_1\hat{Z}+\alpha_0)\hat{Z}'' \\ &\quad +\left(\alpha_0(1-2\hat{Z})-\alpha_1\hat{Z}^2\right)\left(3-(\hat{Z}')^2\right)+6(1-S)\hat{Z}^3(1-\hat{Z})^2\Big] \\ &= K_1 = \text{const},\end{aligned}$$

$$\begin{aligned}\Phi_2[\hat{Z}] &= -\frac{1}{2\hat{Z}(1-\hat{Z})^2}\Big[2\hat{Z}(1-\hat{Z})(\alpha_1\hat{Z}+\alpha_0)\hat{Z}'' \\ &\quad +\left(\alpha_1\hat{Z}(1-2\hat{Z})+\alpha_0(2-3\hat{Z})\right)\left(3-(\hat{Z}')^2\right)+3(1-S)\hat{Z}^3(1-\hat{Z})^2\Big] \\ &= K_2 = \text{const}.\end{aligned}$$

- Solve for $\hat{Z}''$, $\hat{Z}'$ in terms of $\hat{Z}$; obtain a 1st-order autonomous ODE on $\hat{Z}$.
- Here we denoted $\alpha_0 = C_1^2 S$, $\alpha_1 = C_2^2 - \alpha_0$.

The ODE Governing Traveling Wave Solutions (ctd.)

The final ODE:

$$(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0} =: Q(\hat{Z})$$

- Relationships between parameters:

$$A_4 = 3(1 - S), \quad A_3 = 2K_1 - A_4,$$

$$A_2 = -2(K_1 + K_2), \quad A_1 = 2K_2 + 3\alpha_1, \quad A_0 = 3\alpha_0.$$

- Four independent constant parameters. For example, one may choose

$$\alpha_0 \geq 0, \quad \alpha_1 \geq -\alpha_0, A_2, A_3 \in \mathbb{R}$$

as arbitrary constants. Then

$$A_1 = 3\alpha_1 - (A_2 + A_3 + A_4), \quad \alpha_0 + \alpha_1 \geq 0, \quad A_4 > 0.$$

The ODE Governing Traveling Wave Solutions (ctd.)

The final ODE:

$$\boxed{(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0} =: \; Q(\hat{Z})}$$

- The above ODE has not been generally studied.
- Implicit solution – not so practical:

$$\pm\int^{\hat{Z}} Q(s)^{-1/2}\, ds = r - r_0,$$

- Transformation $Y(\hat{r}) = (\hat{Z} - \alpha_0/\alpha_1)^{-1}$ maps the above ODE to an ODE with the 5th-degree polynomial right-hand side

$$(Y')^2 = \frac{A_0}{\alpha_1}Y^5 + \frac{A_1}{\alpha_1}Y^4 + \frac{A_2}{\alpha_1}Y^3 + \frac{A_3}{\alpha_1}Y^2 + \frac{A_4}{\alpha_1}Y.$$

- If $\alpha_1 = 0$, 4th-degree polynomial right-hand side.

The ODE Governing Traveling Wave Solutions (ctd.)

The final ODE:

$$\boxed{(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0} =: \ Q(\hat{Z})}$$

Classical ODEs with polynomial right-hand side:

- Weierstrass ODE (cubic RHS) $\rightarrow$ Weierstrass function $\wp()$;
- KdV reduction (cubic RHS) $\rightarrow$ $\mathrm{sech}^2()$;
- Jacobi elliptic ODEs (4th degree polynomial RHS)

  $\rightarrow$ Jacobi elliptic functions $\mathrm{cn}(), \mathrm{sn}(), \mathrm{dn}()$.

Outline

Exact Traveling Wave Solutions: (A) Cnoidal and Solitary Waves

The ODE family:

$$(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0}$$

- The following theorem is proven by a direct substitution.

The ODE family:

$$(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0}$$

Theorem

The above family of ODEs admits exact solutions in the form

$$\hat{Z}(\hat{r}) = B_1\,\mathrm{sn}^2(\gamma\,\hat{r}, k) + B_2,$$

for arbitrary constants k, B_1, B_2. The remaining constants γ and $\alpha_{1,2}$ are given by one of the following relationships.

- **Case 1:**

$$\alpha_0 = -\alpha_1 = \frac{A_4 B_2}{3k^2}(B_1 + B_2)(B_1 + B_2 k^2), \qquad \gamma^2 = \frac{A_4 B_1}{4k^2\alpha_1};$$

The ODE family:

$$(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0}$$

Theorem

The above family of ODEs admits exact solutions in the form

$$\hat{Z}(\hat{r}) = B_1\,\mathrm{sn}^2(\gamma\,\hat{r}, k) + B_2,$$

for arbitrary constants k, B_1, B_2. The remaining constants γ and $\alpha_{1,2}$ are given by one of the following relationships.

- **Case 2:**

$$\alpha_0 = 0, \qquad \alpha_1 = -\frac{A_4}{3k^2}(B_2-1)(B_1+B_2-1)\big(B_1+k^2(B_2-1)\big), \qquad \gamma^2 = \frac{A_4B_1}{4k^2\alpha_1}.$$

The ODE family:

$$(\hat{Z}')^2 = \frac{A_4 \hat{Z}^4 + A_3 \hat{Z}^3 + A_2 \hat{Z}^2 + A_1 \hat{Z} + A_0}{\alpha_1 \hat{Z} + \alpha_0}$$

Theorem

The above family of ODEs admits exact solutions in the form

$$\hat{Z}(\hat{r}) = B_1 \,\mathrm{sn}^2(\gamma\, \hat{r}, k) + B_2,$$

for arbitrary constants k, B_1, B_2. The remaining constants γ and $\alpha_{1,2}$ are given by one of the following relationships.

- Natural choice: $B_2 = \dfrac{h_1}{H} - B_1$. Then the dimensional interface displacement is

$$\zeta(x,t) = H B_1 \,\mathrm{cn}^2(\gamma\, \hat{r}(x,t), k).$$

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = B_1 \operatorname{sn}^2(\gamma\, \hat{r}, k) + B_2$.

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = B_1 \,\mathrm{sn}^2(\gamma\,\hat{r}, k) + B_2$.
- Layer-average velocities, **Case 1:**

$$v_1(x,t) = \sqrt{gH}\left(\hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1\,\mathrm{sn}^2(\gamma\,\hat{r}(x,t), k) + B_2}\right), \qquad v_2(x,t) = \hat{c}\sqrt{gH} = \mathrm{const},$$

Exact Traveling Wave Solutions: (A) Cnoidal and Solitary Waves

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = B_1 \,\mathrm{sn}^2(\gamma\, \hat{r}, k) + B_2$.
- Layer-average velocities, **Case 1:**

$$v_1(x,t) = \sqrt{gH}\left(\hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1 \,\mathrm{sn}^2(\gamma\, \hat{r}(x,t), k) + B_2}\right), \qquad v_2(x,t) = \hat{c}\sqrt{gH} = \mathrm{const},$$

- Layer-average velocities, **Case 2:**

$$v_1(x,t) = \hat{c}\sqrt{gH} = \mathrm{const}, \qquad v_2(x,t) = \sqrt{gH}\left(\hat{c} \pm \frac{\sqrt{\alpha_1}}{1 - B_1 \,\mathrm{sn}^2(\gamma\, \hat{r}(x,t), k) - B_2}\right).$$

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = B_1 \,\mathrm{sn}^2(\gamma\,\hat{r}, k) + B_2$.
- Layer-average velocities, **Case 1:**

$$v_1(x,t) = \sqrt{gH}\left(\hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1 \,\mathrm{sn}^2(\gamma\,\hat{r}(x,t), k) + B_2}\right), \qquad v_2(x,t) = \hat{c}\sqrt{gH} = \mathrm{const},$$

- Layer-average velocities, **Case 2:**

$$v_1(x,t) = \hat{c}\sqrt{gH} = \mathrm{const}, \qquad v_2(x,t) = \sqrt{gH}\left(\hat{c} \pm \frac{\sqrt{\alpha_1}}{1 - B_1 \,\mathrm{sn}^2(\gamma\,\hat{r}(x,t), k) - B_2}\right).$$

- Signs can be chosen independently.

Exact Traveling Wave Solutions: (A) Cnoidal and Solitary Waves

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = B_1 \operatorname{sn}^2(\gamma\,\hat{r}, k) + B_2$.
- Layer-average velocities, **Case 1:**

$$v_1(x,t) = \sqrt{gH}\left(\hat{c} \pm \frac{\sqrt{\alpha_0/S}}{B_1 \operatorname{sn}^2(\gamma\,\hat{r}(x,t),k) + B_2}\right), \qquad v_2(x,t) = \hat{c}\sqrt{gH} = \text{const},$$

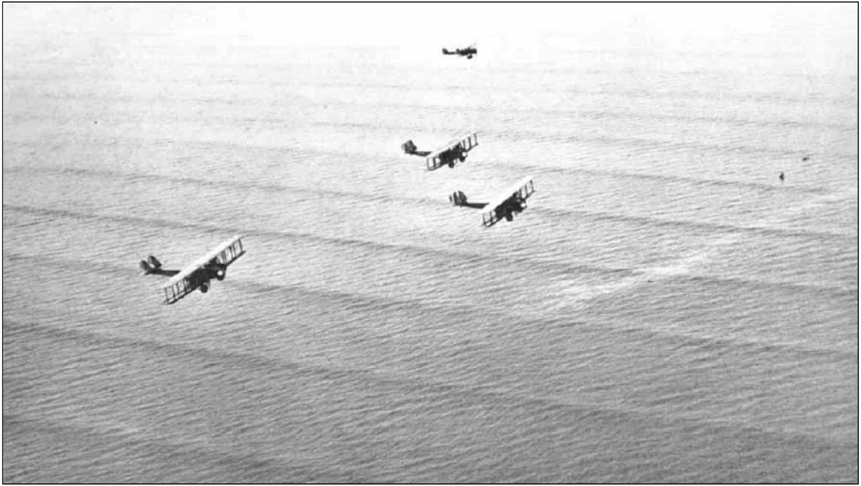
- Layer-average velocities, **Case 2:**

$$v_1(x,t) = \hat{c}\sqrt{gH} = \text{const}, \qquad v_2(x,t) = \sqrt{gH}\left(\hat{c} \pm \frac{\sqrt{\alpha_1}}{1 - B_1 \operatorname{sn}^2(\gamma\,\hat{r}(x,t),k) - B_2}\right).$$

- Signs can be chosen independently.
- Pressure: from appropriate formula that uses $\hat{Z}(\hat{r})$.

Cnoidal Waves

- Cnoidal waves in nature:



Jacobi Elliptic Functions

- $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$; $0 \le k \le 1$.
- Doubly periodic meromorphic functions on the complex plane.
- Related to elliptic integrals, elliptic curves.
- Can be defined as solutions of special ODEs.
  E.g., $y = \mathrm{sn}(x+c,k)$ is a general solution of

$$\left(\frac{dy}{dx}\right)^2 = (1-y^2)(1-k^2y^2).$$

- Identities, e.g.,

$$\mathrm{sn}^2(x,k)+\mathrm{cn}^2(x,k)=1; \qquad \frac{d}{dx}\,\mathrm{sn}(x,k)=\mathrm{cn}(x,k)\,\mathrm{dn}(x,k).$$

- Limits:

$$\lim_{k\to 0^+}\mathrm{sn}(x,k)=\sin x; \qquad \lim_{k\to 1^-}\mathrm{sn}(x,k)=\tanh x;$$

$$\lim_{k\to 0^+}\mathrm{cn}(x,k)=\cos x; \qquad \lim_{k\to 1^-}\mathrm{cn}(x,k)=\frac{1}{\cosh x}.$$

Periods of Cnoidal Wave Solutions

- Spatial period (wavelength) of the elliptic sine $\mathrm{sn}(x, k)$:

$$\tau = \frac{2\pi}{\mathrm{AGM}(1, \sqrt{1-k^2})},$$

$AGM(a, b)$ denoting the Gauss' algebraic-geometric mean of a, b.

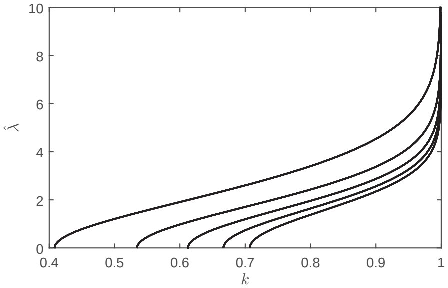
- $a_1 = \sqrt{ab}, \quad b_1 = (a+b)/2,$
- $a_2 = \sqrt{a_1 b_1}, \quad b_2 = (a_1 + b_1)/2,$
- ...
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = AGM(a, b).$

Periods of Cnoidal Wave Solutions

- Wavelength of the cnoidal traveling wave $\zeta(x,t) = HB_1 \operatorname{cn}^2\left(\gamma \dfrac{x-ct}{H},\ k\right)$:

$$\hat{\lambda} = \frac{\pi}{\gamma\,\mathrm{AGM}(1, \sqrt{1-k^2})}, \qquad \lambda = H\hat{\lambda}.$$

- γ, k are related.
- $\lim_{k\to 1^-} \hat{\lambda} = +\infty$.
- Dimensionless wavelength $\hat{\lambda}$ as a function of k, for different B_1:



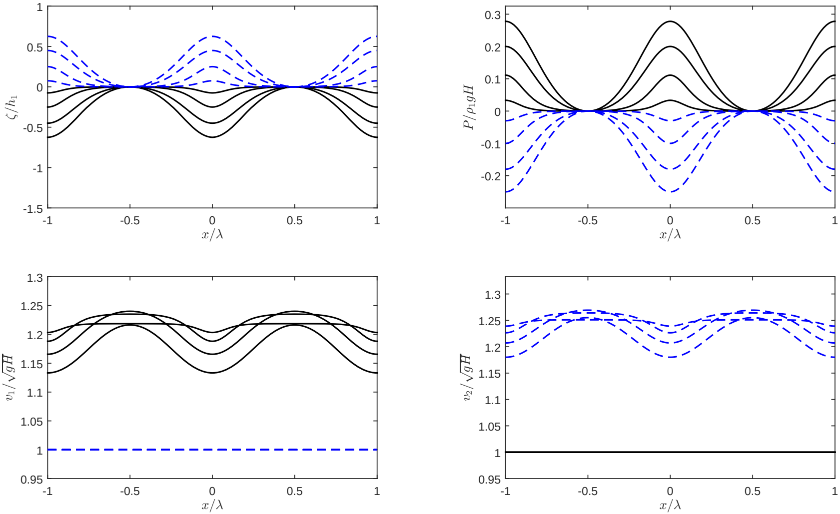
Cnoidal Waves – Sample Plots

- Sample exact solution parameters and wavelengths for the exact periodic cnoidal wave solutions:

$$\hat{c} = 1, \quad S = 0.9, \quad x_0 = t = 0,$$
$$h_1 = 0.4 \text{ m}, \quad h_2 = 0.6 \text{ m}, \quad H = 1 \text{ m}, \quad g = 9.8 \text{ m/s}^2.$$

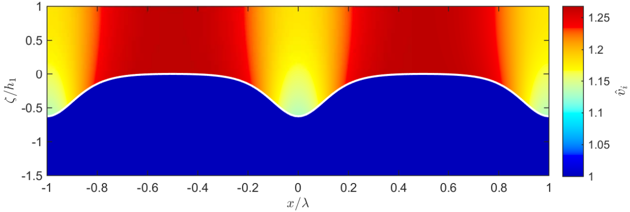
Case	k	B_1	λ, m	$\epsilon = H/\lambda$
1	0.9990	-0.0300	15.7055	0.0637
2	0.9990	0.0300	90.4410	0.0111
1	0.9900	-0.1000	6.8466	0.1461
2	0.9900	0.1000	22.0327	0.0454
1	0.9000	-0.1800	3.2188	0.3107
2	0.9000	0.1800	7.1438	0.1400
1	0.8000	-0.2500	1.9146	0.5223
2	0.8000	0.2500	3.1912	0.3134
1	0.9900	-0.2500	5.2898	0.1890

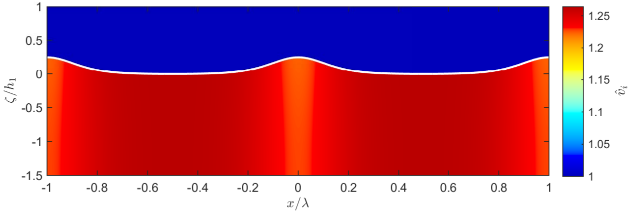
- **Case 1:** solid black, negative amplitude. **Case 2:** dashed blue, positive amplitude.



Flood diagrams for the right-propagating cnoidal wave solutions:

- **Case 1:** $k = 0.99$, $B_1 = -0.25$.
- **Case 2:** $k = 0.99$, $B_1 = 0.1$.





- For $k = 1$, obtain solitary wave solutions (different in Cases 1,2):

$$\hat{Z}(\hat{r}) = (B_1 + B_2) - B_1 \cosh^{-2}(\gamma \hat{r}).$$

In particular, under the natural choice $B_2 = \dfrac{h_1}{H} - B_1$, one has

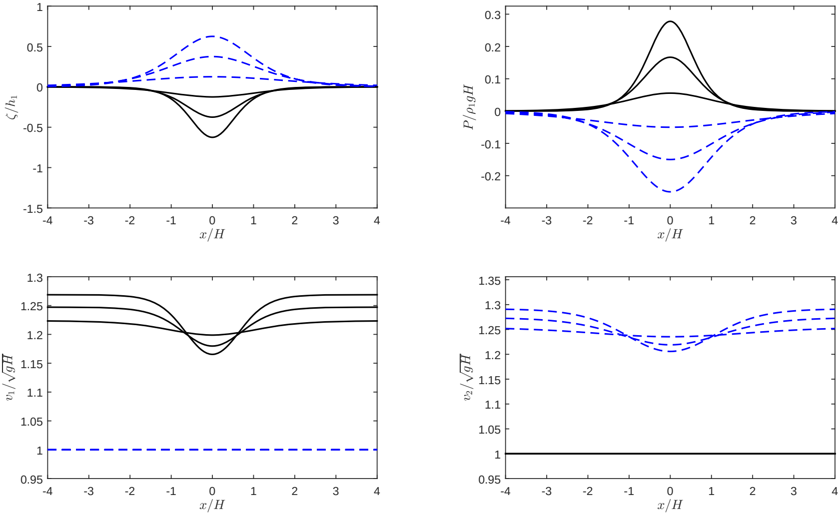
$$\zeta(x, t) = H B_1 \cosh^{-2}\left(\gamma \frac{x - ct}{H}\right).$$

- Characteristic spike width:

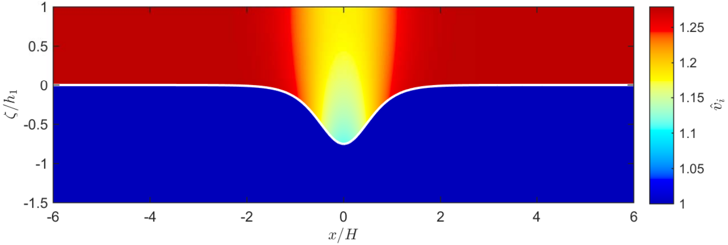
$$\lambda_s = \frac{H}{\gamma(B_1, B_2)}.$$

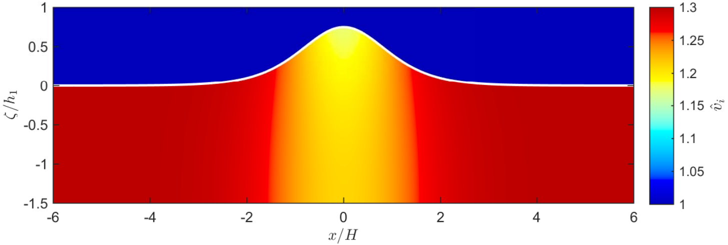
- Depression-type waves: Case 1, $B_1 < 0$.
- Elevation-type waves: Case 2, $B_1 > 0$.

- **Case 1:** solid black, depression-type. **Case 2:** dashed blue, elevation-type.



- Flood diagrams for the right-propagating solitary waves: $B_1 = \pm 0.3$.





Solitary Waves – Velocity Shear?

- Interface displacement: $\zeta(x,t) = HB_1 \cosh^{-2}\left(\gamma \frac{x-ct}{H}\right)$.
- Physical conditions: $-h_2 < HB_1 < h_1$.
- The dimensionless velocity shear values at infinity $|\Delta\hat{v}|_\infty = |\hat{v}_1 - \hat{v}_2|_{x=\pm\infty}$:

$$|\Delta\hat{v}|_\infty^{(1)} = \sqrt{\frac{1-S}{S}\left(\frac{h_1}{H} - B_1\right)} \neq 0,$$

$$|\Delta\hat{v}|_\infty^{(2)} = \sqrt{(1-S)\left(\frac{h_2}{H} + B_1\right)} \neq 0.$$

Outline

The ODE family:

$$(\hat{Z}')^2 = \frac{A_4\hat{Z}^4 + A_3\hat{Z}^3 + A_2\hat{Z}^2 + A_1\hat{Z} + A_0}{\alpha_1\hat{Z} + \alpha_0}$$

- The following theorem is also proven by a direct substitution.

Theorem

The above family of ODEs admits exact solutions in the form

$$\hat{Z}(\hat{r}) = \frac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$$

for arbitrary constants B_1, B_2, S. The remaining constants γ, k and $\alpha_{1,2}$ are given by one of the following relationships.

- **Case 1:**

$$\alpha_0 = -\alpha_1 = -\frac{A_4 B_1^3}{6B_2(1-B_2^2)},$$

$$\gamma^2 = \frac{3B_2^2}{B_1^2}, \qquad k^2 = \frac{(1-(B_1-B_2)^2)}{B_2(2B_1-B_2)(B_1^2+(B_1-B_2)^2)+(B_1-B_2)^2}.$$

- Here $\alpha_0 + \alpha_1 = C_2 = 0$, hence the mean velocity of the bottom layer $v_2(t,x) = \mathrm{const}$.

Theorem

The above family of ODEs admits exact solutions in the form

$$\hat{Z}(\hat{r}) = \frac{B_1}{\mathrm{sn}(\gamma\, \hat{r}, k) + B_2}$$

for arbitrary constants B_1, B_2, S. *The remaining constants* γ, k *and* $\alpha_{1,2}$ *are given by one of the following relationships.*

- **Case 2:**

$$\alpha_0 = 0, \qquad \alpha_1 = \frac{A_4(2B_2 - B_1)(1 - (B_1 - B_2)^2)}{6B_2(1 - B_2^2)},$$

$$\gamma^2 = \frac{3B_1 B_2^2}{(2B_2 - B_1)(1 - (B_1 - B_2)^2)}, \qquad k^2 = B_2^{-2}.$$

- Here $\alpha_0 = C_1 = 0$, which yields a constant mean velocity of the top layer, $v_1(t, x) = \mathrm{const}$.

Exact Traveling Wave Solutions: (B) Further Cnoidal and Kink Waves

Theorem

The above family of ODEs admits exact solutions in the form

$$\hat{Z}(\hat{r}) = \frac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$$

for arbitrary constants B_1, B_2, S. The remaining constants γ, k and $\alpha_{1,2}$ are given by one of the following relationships.

- **Case 3:**

$$\alpha_0 = \frac{A_4 B_1^3}{3(1-B_2^2)} \, \frac{1-(B_1-B_2)^2}{B_2(4B_1^2 - 5B_1B_2 + 2B_2^2) - 2B_2 + B_1}, \qquad \alpha_1 = 0,$$

$$\gamma^2 = \frac{3}{B_1^2} \, \frac{B_2(2B_1 - B_2)(B_1^2 + (B_1-B_2)^2) + (B_1-B_2)^2}{1-(B_1-B_2)^2}, \qquad k^2 = \gamma^{-2}.$$

- For this case, both mean horizontal velocities are non-constant.

Exact Traveling Wave Solutions: Second Cnoidal Family

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$.

Exact Traveling Wave Solutions: Second Cnoidal Family

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$.
- Layer-average velocities and pressure: same formulas as before, through $\hat{Z}(\hat{r})$.

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$.
- Layer-average velocities and pressure: same formulas as before, through $\hat{Z}(\hat{r})$.
- Dimensionless and dimensional wavelength:

$$\hat{\lambda} = \frac{2\pi}{\gamma\,\mathrm{AGM}(1, \sqrt{1-k^2})}, \qquad \lambda = H\hat{\lambda}.$$

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}.$

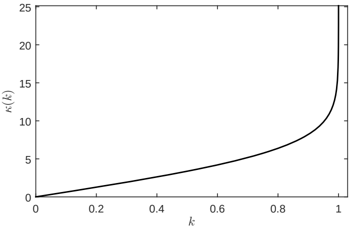
- Layer-average velocities and pressure: same formulas as before, through $\hat{Z}(\hat{r})$.

- Dimensionless and dimensional wavelength:

$$\hat{\lambda} = \frac{2\pi}{\gamma\,\mathrm{AGM}(1, \sqrt{1-k^2})}, \qquad \lambda = H\hat{\lambda}.$$

- In **Case 3**, $k = 1/\gamma$, and

$$\hat{\lambda}(k) = 2\pi k/\mathrm{AGM}(1, \sqrt{1-k^2}), \qquad \lim_{k\to 1^-} \hat{\lambda} = +\infty.$$



Second Cnoidal Family – Sample Plots

- Case 3, sample parameters and wavelengths for the second cnoidal solution family:

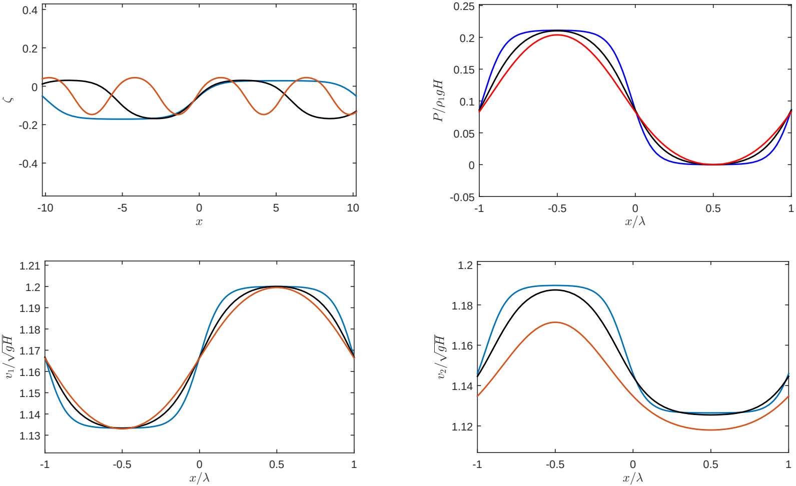
$$\hat{c} = 1, \quad x_0 = t = 0, \quad S = 0.9,$$

$$h_1 = 3/7 \text{ m}, \quad h_2 = 4/7 \text{ m}, \quad H = 1 \text{ m}, \quad g = 9.8 \text{ m/s}^2.$$

B_1	B_2	k	λ, m	$\epsilon = H/\lambda$
2.3995	5	0.9950	20.4057	0.0980
2.3881	5	0.8996	11.3073	0.1769
2.3037	5	0.6000	5.5882	0.3579

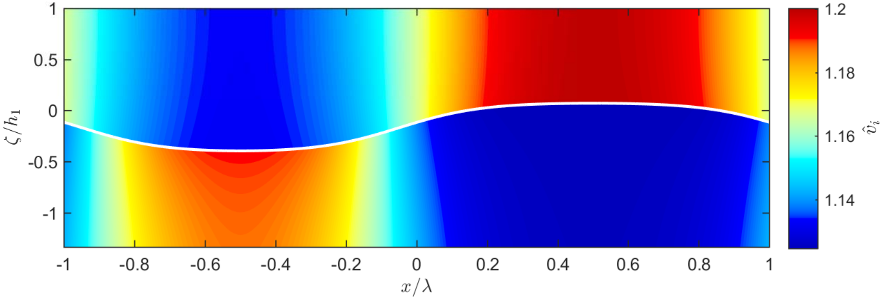
Second Cnoidal Family – Sample Plots

- Solution plots: curve colors blue, black, and red correspond to the tree rows of the above table.



Second Cnoidal Family – Sample Plots

Sample flood diagram, for the solution parameters in the second row of the table:



- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$.

Exact Traveling Wave Solutions: Kink/Anti-Kink Solutions

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\operatorname{sn}(\gamma\,\hat{r}, k) + B_2}$.
- In the limit $k \to 1^-$: $\operatorname{sn}(y, 1) = \tanh y$.

Exact Traveling Wave Solutions: Kink/Anti-Kink Solutions

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$.
- In the limit $k \to 1^-$: $\mathrm{sn}(y, 1) = \tanh y$.
- Resulting exact solution: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\tanh(\gamma\,\hat{r}) + B_2}$.

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$.
- In the limit $k \to 1^-$: $\mathrm{sn}(y, 1) = \tanh y$.
- Resulting exact solution: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\tanh(\gamma\,\hat{r}) + B_2}$.
- Dimensional interface displacement: $\zeta(x, t) = h_1 - \dfrac{HB_1}{\tanh\left(\gamma\,\dfrac{x - ct}{H}\right) + B_2}$.

- Cnoidal traveling wave: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\mathrm{sn}(\gamma\,\hat{r}, k) + B_2}$.
- In the limit $k \to 1^-$: $\mathrm{sn}(y, 1) = \tanh y$.
- Resulting exact solution: $\hat{Z}(\hat{r}) = \dfrac{B_1}{\tanh(\gamma\,\hat{r}) + B_2}$.
- Dimensional interface displacement: $\zeta(x, t) = h_1 - \dfrac{HB_1}{\tanh\left(\gamma\,\dfrac{x - ct}{H}\right) + B_2}$.
- Case 3: the dimensional amplitude and the characteristic wavelength:

$$a = H|B_2|^{-1}, \qquad \lambda = \frac{H}{\gamma} = \frac{H|B_1|}{\sqrt{3}}.$$

Kink/Anti-Kink Solutions – Sample Plots

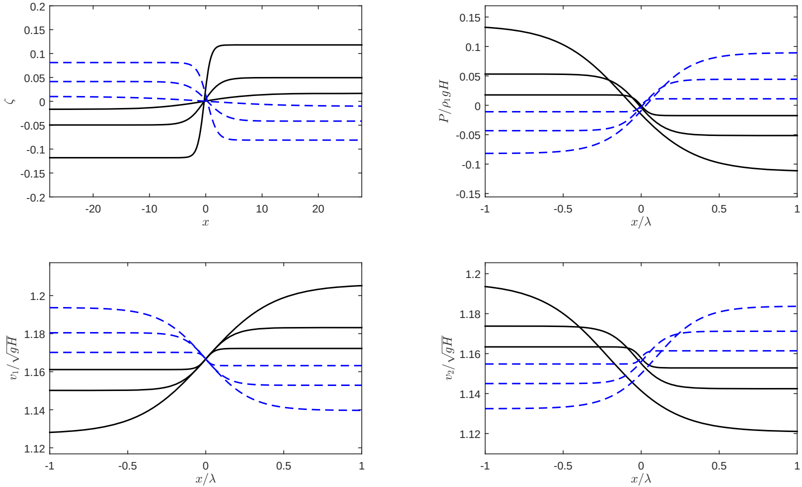
- Case 3, sample parameters and wavelengths for the kink/anti-kink solutions:

$$\hat{c} = 1, \quad h_1 = h_2 = 0.5 \text{ m}, \quad H = 1 \text{ m}, \quad g = 9.8 \text{ m/s}^2, \quad x_0 = t = 0, \quad S = 0.9;$$

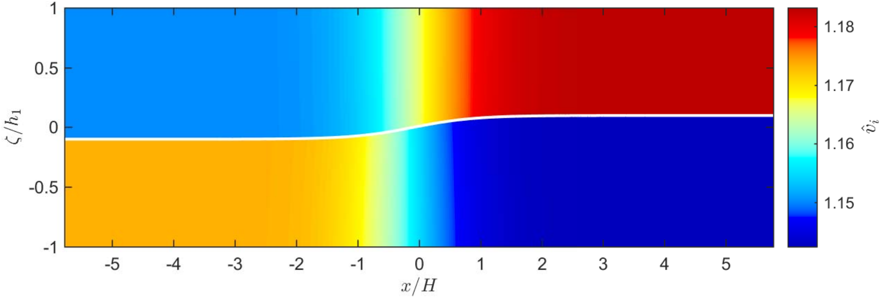
B_1	B_2	a	λ	$\epsilon = H/\lambda$
2	4.2361	0.8660	1.1547	0.8660
5	10.0990	0.3464	2.8868	0.3464
15	30.0333	0.1155	8.6603	0.1155
-3	-6.1623	0.5774	1.7321	0.5774
-6	-12.0828	0.2887	3.4641	0.2887
-24	-48.0208	0.2887	13.8564	0.0722

Kink/Anti-Kink Solutions – Sample Plots

- Solution plots: Black solid curves (large to small amplitude) correspond to the first tree rows of the table (kink solutions). Blue dashed curves (large to small amplitude) correspond to the rows 4-6 of the table (anti-kink solutions).

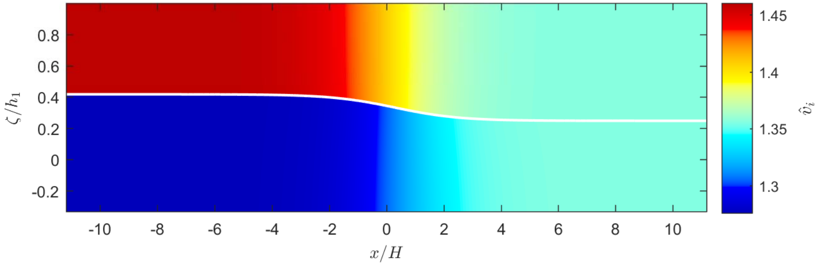


- Sample flood diagram, for the solution parameters in the second row of the table:



- Interface displacement: $\zeta(x,t) = h_1 - \dfrac{HB_1}{\tanh\left(\gamma\,\dfrac{x-ct}{H}\right) + B_2}$.
- Can require $|\Delta v| = |v_1 - v_2| \to 0$ as $x \to \infty$ or $x \to -\infty$.
- Example: flood diagram for

$$\hat{c} = 1, \quad h_1/h_2 = 3, \quad H = 1 \text{ m}, \quad g = 9.8 \text{ m/s}, \quad S = 0.6:$$



Summary: Results

- A natural dimensionless form of the Choi-Camassa model is derived, involving a single dimensionless physical parameter.
- Dimensionless traveling wave ODE; reduction of order via integrating factors.
- Exact traveling wave solutions of several important types, given by elementary explicit formulas:
  - periodic waves;
  - solitary waves;
  - kink/anti-kink.
- Wave properties are independent of the wave speed (Galilei invariance).
- The presented solutions are essentially different from semi-numerical solitary waves of the original CC paper, where *zero velocity shear at infinity* was assumed.

Summary: Methods

- Asymptotic approximation and dimension reduction / averaging:

$$(t, x, y, z) \;\rightarrow\; (t, x).$$

- Equivalence transformations / non-dimensionalization $\rightarrow$ single dimensionless parameter $S = \rho_1/\rho_2$.

- Symmetry reduction: PDE $\rightarrow$ ODE.

- Conservation laws / integrating factors: 3rd-order ODE $\rightarrow$ 1st-order ODE.

- Exact solutions.

- Galilei transformations $\rightarrow$ arbitrary wave speed c.

Some Further Questions

- Stability of the traveling wave solutions?
- Multi-layer generalization?
- R. Camassa: *"Quality of approximation by the CC model may be related to the conservation law structure similarity of the CC and Euler systems".*
- Conservation laws of PDE systems:

$$\mathrm{D}_t\,\Theta + \text{div}\ \boldsymbol{\Psi} = 0.$$

- Systematic conservation law construction, *direct method.*

Some references

Miyata, M. (1985)
An internal solitary wave of large amplitude. *La Mer* **23** (2), 43-48.

Choi, W., & Camassa, R. (1999)
Fully nonlinear internal waves in a two-fluid system. *JFM* **396**, 1–36.

Cheviakov, A. (2018)
Exact solutions of a fully nonlinear two-fluid model. *Phys. D*, accepted.

Miyata, M. (1985)
An internal solitary wave of large amplitude. *La Mer* **23** (2), 43-48.

Choi, W., & Camassa, R. (1999)
Fully nonlinear internal waves in a two-fluid system. *JFM* **396**, 1–36.

Cheviakov, A. (2018)
Exact solutions of a fully nonlinear two-fluid model. *Phys. D*, accepted.

Thank you for your attention!