Conservation Laws of Models of Biological Membranes in the Framework of Nonlinear Elastodynamics

Alexei Cheviakov, J.-F. Ganghoffer

University of Saskatchewan, Canada / Université de Lorraine, France

EUROMECH Colloquium 560

February 10, 2015

Outline

1. Local Conservation Laws
2. Fiber-Reinforced Materials; Governing Equations
3. Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves
4. Single Fiber Family, Ansatz 2 – 2D Shear Waves
5. Two Fiber Families, Planar Case
6. A Viscoelastic Model, Single Fiber Family, 1D Shear Waves
7. Discussion

Outline

1. Local Conservation Laws
2. Fiber-Reinforced Materials; Governing Equations
3. Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves
4. Single Fiber Family, Ansatz 2 – 2D Shear Waves
5. Two Fiber Families, Planar Case
6. A Viscoelastic Model, Single Fiber Family, 1D Shear Waves
7. Discussion

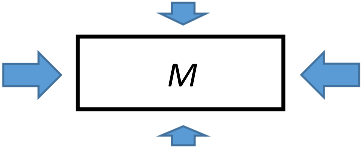
Introduction

Motivation

- Interesting mathematics!
- Study of fundamental properties of nonlinear elastodynamics equations arising in applications.

Notation

- $\mathrm{D}_t = \dfrac{\partial u}{\partial t} \equiv u_t.$



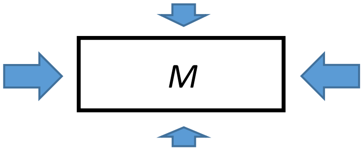
Global form

- Global quantity $M \in \mathcal{D}$ changes only due to boundary fluxes.

$$M = \int_{\mathcal{D}} \Theta \; dV; \qquad \frac{d}{dt} M = \oint_{\partial \mathcal{D}} \boldsymbol{\Psi} \cdot d\mathbf{S}.$$

- $\Theta[\mathbf{u}]$: conserved density; $\boldsymbol{\Psi}$: flux vector.

Conservation Laws



Global form

- Global quantity $M \in \mathcal{D}$ changes only due to boundary fluxes.

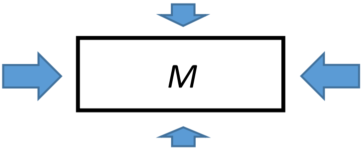
$$M = \int_{\mathcal{D}} \Theta \, dV; \qquad \frac{d}{dt} M = \oint_{\partial \mathcal{D}} \mathbf{\Psi} \cdot d\mathbf{S}.$$

- $\Theta[\mathbf{u}]$: conserved density; $\mathbf{\Psi}$: flux vector.

Local form

- A local conservation law: a divergence expression equal to zero, e.g.,

$$\mathrm{D}_t \, \Theta[\mathbf{u}] + \mathrm{D}_i \, \Psi^i[\mathbf{u}] = 0.$$



Global form

- Global quantity $M \in \mathcal{D}$ changes only due to boundary fluxes.

$$M = \int_{\mathcal{D}} \Theta \; dV; \qquad \frac{d}{dt} M = \oint_{\partial \mathcal{D}} \mathbf{\Psi} \cdot d\mathbf{S}.$$

- $\Theta[\mathbf{u}]$: conserved density; $\mathbf{\Psi}$: flux vector.

Global conserved quantity:

$$\frac{d}{dt} M = \mathrm{D}_t \int_V \Theta \; dV = 0 \qquad \text{when} \qquad \oint_{\partial V} \mathbf{\Psi} \cdot d\mathbf{S} = 0.$$

Applications of Conservation Laws

ODEs

- Constants of motion.
- Integration.

PDEs

- Rates of change of physical variables; constants of motion.
- Differential constraints.
- Analysis: existence, uniqueness, stability, integrability, linearization.
- Potentials, stream functions, etc.
- Conserved forms for numerical methods (finite volume, etc.).
- Numerical method testing.

For equations following from a variational principle:

- Can use Noether's theorem.
- Conservation laws are connected with variational symmetries.
- Technically difficult.

For equations following from a variational principle:

- Can use Noether's theorem.
- Conservation laws are connected with variational symmetries.
- Technically difficult.

For generic models: Direct conservation law construction method

- Conservation laws can be sought in the characteristic form $\Lambda_\sigma R^\sigma \equiv \mathrm{D}_i \Phi^i$.
- Systematically find the multipliers Λ_σ.
- Direct method is complete for a wide class of systems.
- Implemented in Maple/GeM: symbolic computations.

Outline

1 Local Conservation Laws

2 Fiber-Reinforced Materials; Governing Equations

3 Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves

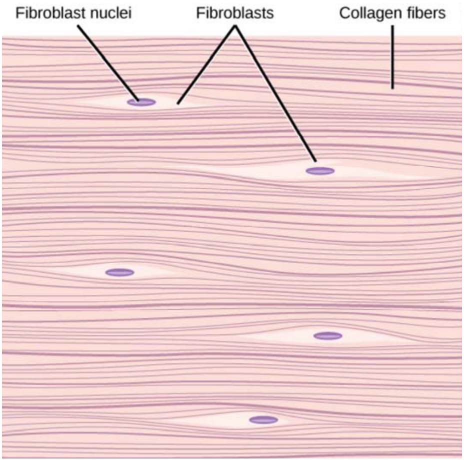
4 Single Fiber Family, Ansatz 2 – 2D Shear Waves

5 Two Fiber Families, Planar Case

6 A Viscoelastic Model, Single Fiber Family, 1D Shear Waves

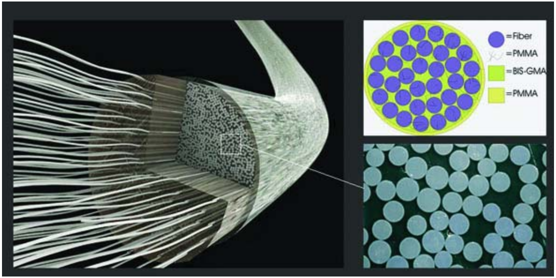
7 Discussion

Examples



Collagen fiber in tendons.

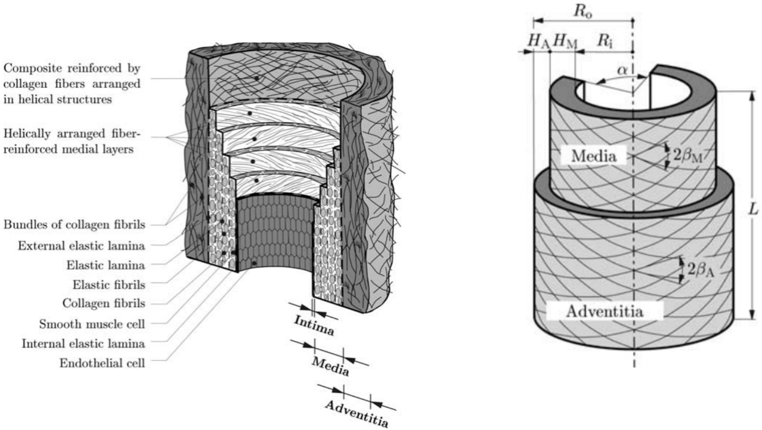
Single fiber family.



A fiber-reinforced composite in dentistry.

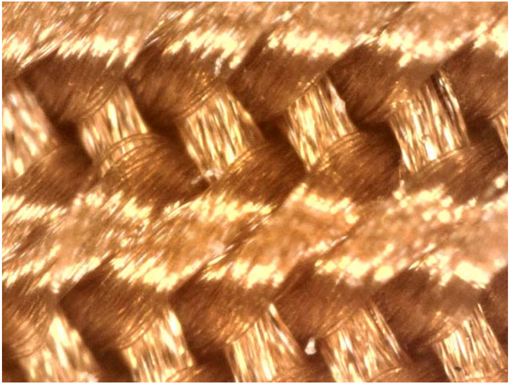
Single fiber family.

Examples



Arterial tissue (Holzapfel, Gasser, and Ogden, 2000).

Two helically arranged fiber families.



Fabric – two fiber families.

- **Appropriate framework:** incompressible hyperelasticity / viscoelasticity.

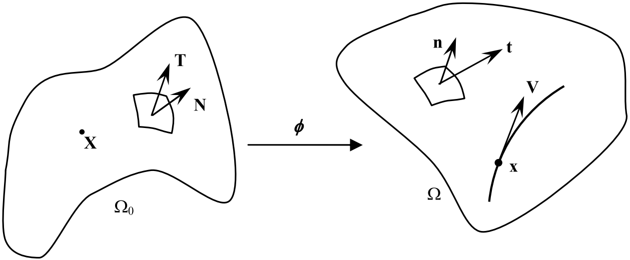


Fig. 1. Material and Eulerian coordinates.

Material picture

- Material points $\mathbf{X} \in \Omega_0$.
- Actual position of a material point: $\mathbf{x} = \phi(\mathbf{X}, t) \in \Omega$.
- Deformation gradient: $\mathbf{F}(\mathbf{X}, t) = \nabla\phi, \qquad F^i{}_j = \dfrac{\partial x^i}{\partial X^j}.$

Governing Equations

Incompressibility:

$$J = \det \mathbf{F} = \left| \frac{\partial x^i}{\partial X^j} \right| = 1, \qquad \rho = \rho_0 / J = \rho_0(\mathbf{X}).$$

Governing Equations

Incompressibility:

$$J = \det \mathbf{F} = \left| \frac{\partial x^i}{\partial X^j} \right| = 1, \qquad \rho = \rho_0 / J = \rho_0(\mathbf{X}).$$

Equations of motion:

$$\rho_0 \mathbf{x}_{tt} = \text{div}_{(X)} \mathbf{P} + \rho_0 \mathbf{R}, \qquad J = 1.$$

- $\mathbf{R} = \mathbf{R}(\mathbf{X}, t)$: total body force per unit mass; $\rho_0(\mathbf{X})$: density.
- Use $\mathbf{R} = 0$, $\rho_0 = \text{const}$.

Stress tensor (incompressible):

$$P^{ij} = -p\,(F^{-1})^{ji} + \rho_0 \frac{\partial W}{\partial F_{ij}}, \tag{1}$$

- W: scalar strain energy density; p: hydrostatic pressure.

Strain Energy Density

$$W = W_{iso} + W_{aniso}.$$

Isotropic Strain Energy Density

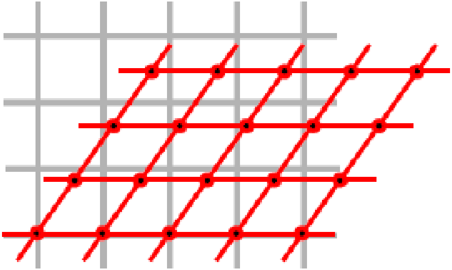
- Right Cauchy-Green strain tensor: $\mathbf{C} = \mathbf{F}^T\mathbf{F}$,

$$I_1 = \mathrm{Tr}\,\mathbf{C}, \qquad I_2 = \tfrac{1}{2}[(\mathrm{Tr}\,\mathbf{C})^2 - \mathrm{Tr}(\mathbf{C}^2)]. \tag{2}$$

- Mooney-Rivlin materials:

$$W_{iso} = a(I_1 - 3) + b(I_2 - 3), \quad a, b > 0.$$

Fiber Directions



Fiber directions

- Reference configuration: fibers along $\mathbf{A}$ $(|\mathbf{A}| = 1)$.
- Actual configuration: fibers along $\mathbf{a}$ $(|\mathbf{a}| = 1)$.
- Fiber stretch factor:
$$\lambda \mathbf{a} = \mathbf{F}\mathbf{A} \quad \Rightarrow \quad \lambda^2 = \mathbf{A}^T \mathbf{C}\, \mathbf{A}.$$

Anisotropic Strain Energy Density

- Fiber invariants:

$$I_4 = \mathbf{A}^T \mathbf{C} \mathbf{A}, \quad I_5 = \mathbf{A}^T \mathbf{C}^2 \mathbf{A}.$$

- General constitutive model:

$$W_{aniso} = f\left(I_4 - 1,\, I_5 - 1\right), \quad f(0,0) = 0.$$

- Standard reinforcement model: $W_{aniso} = q\left(I_4 - 1\right)^2$.

Anisotropic Strain Energy Density

- Fiber invariants:

$$I_4 = \mathbf{A}^T\mathbf{C}\mathbf{A}, \quad I_5 = \mathbf{A}^T\mathbf{C}^2\mathbf{A}.$$

- General constitutive model:

$$W_{aniso} = f\left(I_4 - 1,\ I_5 - 1\right), \quad f(0,0) = 0.$$

- Standard reinforcement model: $W_{aniso} = q\left(I_4 - 1\right)^2$.

Equations of motion:

$$\rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(X)}\mathbf{P}, \qquad J = \det\left[\frac{\partial x^i}{\partial X^j}\right] = 1,$$

$$P^{ij} = -p\left(F^{-1}\right)^{ji} + \rho_0 \frac{\partial W}{\partial F_{ij}}.$$

- Strain energy density, single fiber family:

$$W = W_{iso} + W_{aniso} = a(I_1 - 3) + b(I_2 - 3) + q\left(I_4 - 1\right)^2; \quad a, b, q > 0.$$

Outline

1. Local Conservation Laws
2. Fiber-Reinforced Materials; Governing Equations
3. Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves
4. Single Fiber Family, Ansatz 2 – 2D Shear Waves
5. Two Fiber Families, Planar Case
6. A Viscoelastic Model, Single Fiber Family, 1D Shear Waves
7. Discussion

Ansatz 1 Compatible with Incompressibility

Equilibrium and Displacements

- Equilibrium/no displacement: $\mathbf{x} = \mathbf{X}$, *natural state*.
- Time-dependent, with displacement: $\mathbf{x} = \mathbf{X} + \mathbf{G}, \quad \mathbf{G} = \mathbf{G}(\mathbf{X}, t)$.
- No linearization, or assumption of smallness of $\mathbf{G}$, etc.

Ansatz 1 Compatible with Incompressibility

Equilibrium and Displacements

- Equilibrium/no displacement: $\mathbf{x} = \mathbf{X}$, *natural state*.
- Time-dependent, with displacement: $\mathbf{x} = \mathbf{X} + \mathbf{G}, \quad \mathbf{G} = \mathbf{G}(\mathbf{X}, t)$.
- No linearization, or assumption of smallness of $\mathbf{G}$, etc.

Motions Transverse to a Plane

$$\mathbf{x} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Ansatz 1 Compatible with Incompressibility

Equilibrium and Displacements

- Equilibrium/no displacement: $\mathbf{x} = \mathbf{X}$, *natural state*.
- Time-dependent, with displacement: $\mathbf{x} = \mathbf{X} + \mathbf{G}, \quad \mathbf{G} = \mathbf{G}(\mathbf{X}, t)$.
- No linearization, or assumption of smallness of $\mathbf{G}$, etc.

Motions Transverse to a Plane

$$\mathbf{x} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Deformation gradient:

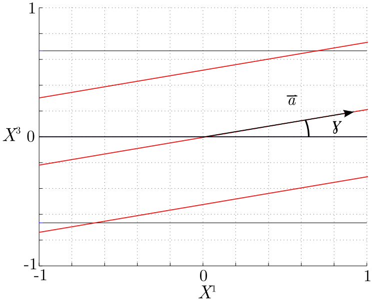
$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial G/\partial X_1 & 0 & 1 \end{bmatrix}, \qquad J = |\mathbf{F}| \equiv 1.$$

Equation of motion for one-dimensional displacements:

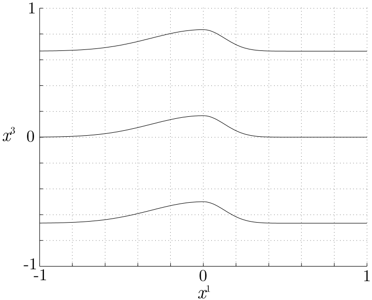
- Denote
$$X^1 = x, \quad G = G(x,t), \quad \alpha = 2(a+b) > 0, \quad \beta = 4q > 0.$$

- Single nonlinear PDE:
$$\boxed{G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\, (G_x)^2 + 6 \sin\gamma \cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.}$$

- Pressure is found explicitly:
$$p = \beta\rho_0 \cos^3\gamma\, (\cos\gamma\, G_x + 2\sin\gamma)\, G_x + f(t).$$



Reference Configuration



Actual Configuration

1D wave model in the case of a single fiber family

- Wave equation:

$$G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.$$

- General PDE class: $\boxed{G_{tt} = \left(A\,(G_x)^2 + BG_x + C\right) G_{xx},}$

$$A = 3\beta\cos^4\gamma > 0,$$
$$B = 6\beta\sin\gamma\,\cos^3\gamma, \qquad 0 \le \gamma < \pi/2.$$
$$C = \alpha + \tfrac{1}{2}\beta\sin^2(2\gamma) > 0,$$

Ansatz 1: Nonlinear Wave Equation and Its Properties

1D wave model in the case of a single fiber family

- Wave equation:

$$G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.$$

- General PDE class: $\boxed{G_{tt} = \left(A\,(G_x)^2 + BG_x + C\right) G_{xx},}$

$$A = 3\beta\cos^4\gamma > 0,$$
$$B = 6\beta\sin\gamma\,\cos^3\gamma, \qquad\qquad 0 \le \gamma < \pi/2.$$
$$C = \alpha + \tfrac{1}{2}\beta\sin^2(2\gamma) > 0,$$

Loss of hyperbolicity

- May occur when $B^2 - 4AC \ge 0$, i.e., $\sin^2(2\gamma) \ge \dfrac{4\alpha}{\beta}$.
- Can only happen for "strong" fiber contribution: $\beta \ge \dfrac{4\alpha}{\sin^2(2\gamma)}$.

Ansatz 1: Nonlinear Wave Equation and Its Properties

1D wave model in the case of a single fiber family

- Wave equation:

$$G_{tt} = \left(\alpha + \beta\cos^2\gamma\left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\,G_x + 2\sin^2\gamma\right)\right)G_{xx}.$$

- General PDE class: $\boxed{G_{tt} = \left(A\,(G_x)^2 + BG_x + C\right)G_{xx},}$

$$A = 3\beta\cos^4\gamma > 0,$$
$$B = 6\beta\sin\gamma\,\cos^3\gamma, \qquad 0 \le \gamma < \pi/2.$$
$$C = \alpha + \tfrac{1}{2}\beta\sin^2(2\gamma) > 0,$$

Variational structure

- Any nonlinear PDE of the above class follows from a variational principle, with the Lagrangian density (*up to equivalence*)

$$\mathcal{L} = \frac{1}{2}G_t^2 + \frac{A}{4}GG_x^2G_{xx} + \frac{B}{3}GG_xG_{xx} - \frac{C}{2}G_x^2.$$

Ansatz 1: Nonlinear Wave Equation and Its Properties

1D wave model in the case of a single fiber family

- Wave equation:

$$G_{tt} = \left(\alpha + \beta \cos^2\gamma \left(3\cos^2\gamma\,(G_x)^2 + 6\sin\gamma\cos\gamma\, G_x + 2\sin^2\gamma\right)\right) G_{xx}.$$

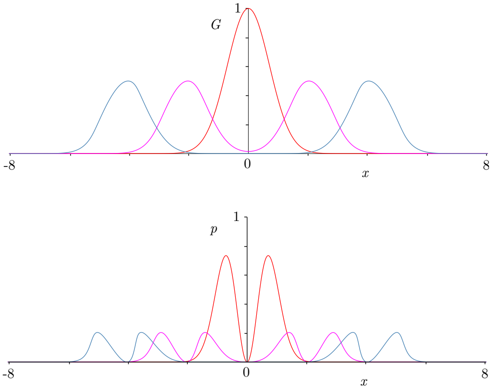
- General PDE class: $\boxed{G_{tt} = \left(A\,(G_x)^2 + BG_x + C\right) G_{xx},}$

$$A = 3\beta\cos^4\gamma > 0,$$
$$B = 6\beta\sin\gamma\,\cos^3\gamma, \qquad 0 \le \gamma < \pi/2.$$
$$C = \alpha + \tfrac{1}{2}\beta\sin^2(2\gamma) > 0,$$

Simplification

- Depending on the sign of $B^2 - 4AC$, equation can be transformed to

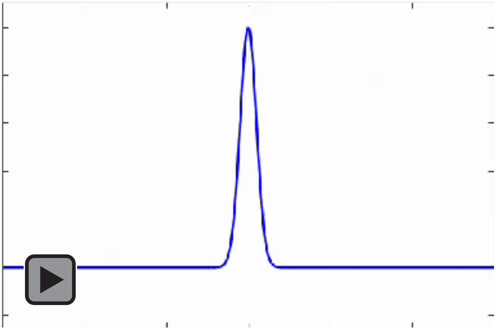
$$u_{tt} = \left((u_x)^2 + K\right) u_{xx}, \qquad K = 0, \pm 1.$$



- Wave speed dependent on u_x.
- Numerical instabilities.
- Wave breaking, applicability?

One-Dimensional Shear Waves

A numerical solution



Direct Construction of Conservation Laws for Ansatz 1

Find local CLs for the nonlinear wave equation

- Model: $u_{tt} = \left(u_x^2 + 1\right) u_{xx}$.
- Conserved form: $\Lambda[u] \left(u_{tt} - \left(u_x^2 + 1\right) u_{xx}\right) = \mathrm{D}_t \, \Theta + \mathrm{D}_x \, \Psi = 0$.
- Basic CLs:

Find local CLs for the nonlinear wave equation

- Model: $u_{tt} = \left(u_x^2 + 1\right) u_{xx}$.
- Conserved form: $\Lambda[u] \left(u_{tt} - \left(u_x^2 + 1\right) u_{xx}\right) = \mathrm{D}_t\, \Theta + \mathrm{D}_x\, \Psi = 0$.
- Basic CLs:

Eulerian momentum:

- $\Lambda = 1$,

$$\mathrm{D}_t(u_t) - \mathrm{D}_x \left[u_x \left(\frac{1}{3} u_x^2 + 1\right)\right] = 0.$$

Lagrangian momentum:

- $\Lambda = u_x$,

$$\mathrm{D}_t(u_x u_t) - \mathrm{D}_x \left(\frac{1}{2}(u_t^2 + u_x^2) + \frac{1}{4} u_x^4\right) = 0.$$

Find local CLs for the nonlinear wave equation

- Model: $u_{tt} = \left(u_x^2 + 1\right) u_{xx}$.
- Conserved form: $\Lambda[u] \left(u_{tt} - \left(u_x^2 + 1\right) u_{xx}\right) = \mathrm{D}_t \, \Theta + \mathrm{D}_x \, \Psi = 0$.
- Basic CLs:

Energy:

- $\Lambda = u_t$,

$$\mathrm{D}_t \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{1}{12} u_x^4\right) - \mathrm{D}_x \left[u_t u_x \left(\frac{1}{3} u_x^2 + 1\right)\right] = 0.$$

Center of mass theorem:

- $\Lambda = t$,

$$\mathrm{D}_t (t u_t - u) - \mathrm{D}_x \left[t u_x \left(\frac{1}{3} u_x^2 + 1\right)\right] = 0.$$

Direct Construction of Conservation Laws for Ansatz 1

Find local CLs for the nonlinear wave equation

- Model: $u_{tt} = \left(u_x^2 + 1\right) u_{xx}$.
- Conserved form: $\Lambda[u] \left(u_{tt} - \left(u_x^2 + 1\right) u_{xx}\right) = \mathrm{D}_t\, \Theta + \mathrm{D}_x\, \Psi = 0$.

Find local CLs for the nonlinear wave equation

- Model: $u_{tt} = \left(u_x^2 + 1\right) u_{xx}$.
- Conserved form: $\Lambda[u] \left(u_{tt} - \left(u_x^2 + 1\right) u_{xx}\right) = \mathrm{D}_t \, \Theta + \mathrm{D}_x \, \Psi = 0$.

An infinite family of conservation laws

- Multiplier: any function $\Lambda(u_t, u_x)$ satisfying

$$\Lambda_{u_x,u_x} = \left(u_x^2 + 1\right) \Lambda_{u_t,u_t}.$$

Linearization by a Legendre contact transformation:

$$y = u_x, \quad z = u_t, \quad w(y,z) = u(x,t) - xu_x - tu_t;$$

$$w_{yy} = \left(y^2 + 1\right) w_{zz}.$$

Find local CLs for the nonlinear wave equation

- Model: $u_{tt} = \left(u_x^2 + 1\right) u_{xx}$.
- Conserved form: $\Lambda[u] \left(u_{tt} - \left(u_x^2 + 1\right) u_{xx}\right) = \mathrm{D}_t\, \Theta + \mathrm{D}_x\, \Psi = 0$.

A more exotic, 2nd-order CL:

- For Λ depending on 3rd derivatives, can have, e.g.,

$$\mathrm{D}_t\, \frac{u_{xx}}{u_{tx} - (u_x^2 + 1)u_{xx}} + \mathrm{D}_x\, \frac{u_{tx}}{u_{tx}^2 - (u_x^2 + 1)u_{xx}^2} = 0.$$

Outline

1 Local Conservation Laws

2 Fiber-Reinforced Materials; Governing Equations

3 Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves

4 Single Fiber Family, Ansatz 2 – 2D Shear Waves

5 Two Fiber Families, Planar Case

6 A Viscoelastic Model, Single Fiber Family, 1D Shear Waves

7 Discussion

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \partial H/\partial X_1 & 1 & 0 \\ \partial G/\partial X_1 & 0 & 1 \end{bmatrix}, \qquad J = |\mathbf{F}| \equiv 1.$$

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Deformation gradient:

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \partial H/\partial X_1 & 1 & 0 \\ \partial G/\partial X_1 & 0 & 1 \end{bmatrix}, \qquad J = |\mathbf{F}| \equiv 1.$$

Governing PDEs:

- Denote $X^1 = x,\ G = G(x,t),\ H = H(x,t)$.

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Coupled nonlinear wave equations:

$$0 = p_x - 2\beta\rho_0 \cos^3\gamma \left[(\cos\gamma G_x + \sin\gamma)\, G_{xx} + \cos\gamma H_x H_{xx}\right],$$

$$H_{tt} = \quad \alpha H_{xx} + \beta\cos^3\gamma \left[\cos\gamma \left(\left[G_x^2 + H_x^2\right] H_{xx} + 2 G_x H_x G_{xx}\right) + 2\sin\gamma \frac{\partial}{\partial x}\left(G_x H_x\right)\right],$$

$$G_{tt} = \quad \alpha G_{xx} + \beta\cos^2\gamma \left[2\sin^2\gamma\ G_{xx} + \cos^2\gamma \left(2 G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2\right) G_{xx}\right)\right.$$
$$\left. + \sin 2\gamma \left(3 G_x G_{xx} + H_x H_{xx}\right)\right].$$

Ansatz 2 Compatible with Incompressibility

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + H\left(X^1, t\right) \\ X^3 + G\left(X^1, t\right) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \cos\gamma \\ 0 \\ \sin\gamma \end{bmatrix}.$$

Coupled nonlinear wave equations:

$$0 = p_x - 2\beta\rho_0 \cos^3\gamma \left[(\cos\gamma G_x + \sin\gamma)\, G_{xx} + \cos\gamma H_x H_{xx}\right],$$

$$H_{tt} = \alpha H_{xx} + \beta\cos^3\gamma \left[\cos\gamma\left(\left[G_x^2 + H_x^2\right] H_{xx} + 2 G_x H_x G_{xx}\right) + 2\sin\gamma \frac{\partial}{\partial x}\left(G_x H_x\right)\right],$$

$$G_{tt} = \alpha G_{xx} + \beta\cos^2\gamma \left[2\sin^2\gamma\; G_{xx} + \cos^2\gamma\left(2 G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2\right) G_{xx}\right) + \sin 2\gamma\left(3 G_x G_{xx} + H_x H_{xx}\right)\right].$$

Subcase 1: $\gamma = \pi/2$

$$H_{tt} = \alpha H_{xx}, \qquad G_{tt} = \alpha G_{xx}.$$

Ansatz 2 Compatible with Incompressibility

Subcase 2: $\gamma = 0$

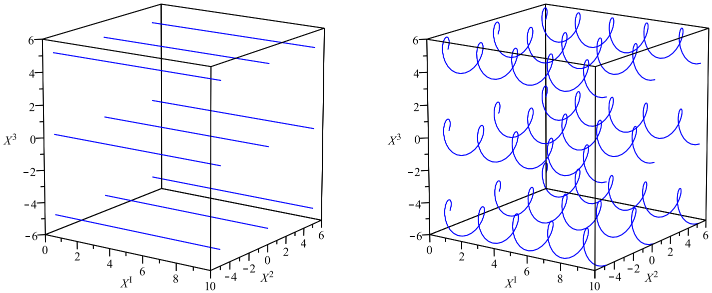
$$\begin{aligned} H_{tt} = &\quad \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2 \right] H_{xx} + 2G_x H_x G_{xx} \right) \right], \\ G_{tt} = &\quad \alpha G_{xx} + \beta \left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2 \right) G_{xx} \right) \right]. \end{aligned}$$

Subcase 2: $\gamma = 0$

$$H_{tt} = \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2 \right] H_{xx} + 2G_x H_x G_{xx} \right) \right],$$

$$G_{tt} = \alpha G_{xx} + \beta \left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2 \right) G_{xx} \right) \right].$$

- Exact traveling wave solutions can be derived [*A.C., J.-F.G., S.St.Jean (2015)*].
- e.g. Carrol-type nonlinear rotational shear waves



Compute local CLs for the coupled model

$$H_{tt} = \quad \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2 \right] H_{xx} + 2 G_x H_x G_{xx} \right) \right],$$

$$G_{tt} = \quad \alpha G_{xx} + \beta \left[\left(2 G_x H_x H_{xx} + \left(H_x^2 + 3 G_x^2 \right) G_{xx} \right) \right].$$

Compute local CLs for the coupled model

$$H_{tt} = \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2 \right] H_{xx} + 2G_x H_x G_{xx} \right) \right],$$

$$G_{tt} = \alpha G_{xx} + \beta \left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2 \right) G_{xx} \right) \right].$$

Linear momenta:

$$\Theta_1 = H_t, \qquad \Theta_2 = G_t,$$

x-components of the Lagrangian and the Angular momentum:

$$\Theta_3 = G_x G_t + G_x G_t, \qquad \Theta_4 = -GH_t + HG_t,$$

Direct Construction of Conservation Laws for Ansatz 2

Compute local CLs for the coupled model

$$H_{tt} = \alpha H_{xx} + \beta \left[\left(\left[3H_x^2 + G_x^2\right] H_{xx} + 2G_x H_x G_{xx}\right)\right],$$
$$G_{tt} = \alpha G_{xx} + \beta \left[\left(2G_x H_x H_{xx} + \left(H_x^2 + 3G_x^2\right) G_{xx}\right)\right].$$

Energy:

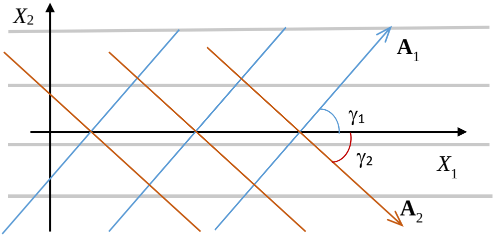
$$\Theta_5 = \frac{1}{2}(G_t^2 + H_t^2) + \frac{\alpha}{2}(G_x^2 + H_x^2) + \frac{\beta}{4}(G_x^2 + H_x^2)^2.$$

Center of mass theorem:

$$\Theta_6 = tG_t - G, \qquad \Theta_7 = tH_t - H.$$

Outline

1. Local Conservation Laws
2. Fiber-Reinforced Materials; Governing Equations
3. Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves
4. Single Fiber Family, Ansatz 2 – 2D Shear Waves
5. Two Fiber Families, Planar Case
6. A Viscoelastic Model, Single Fiber Family, 1D Shear Waves
7. Discussion

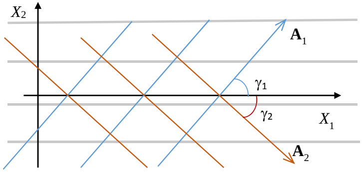


Fiber invariants:

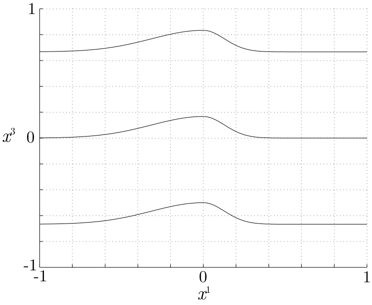
$$I_4 = \lambda_1^2 = \mathbf{A}_1^T \mathbf{C} \mathbf{A}_1, \qquad I_6 = \lambda_2^2 = \mathbf{A}_2^T \mathbf{C} \mathbf{A}_2, \qquad I_8 = (\mathbf{A}_1^T \mathbf{A}_2)(\mathbf{A}_1^T \mathbf{C} \mathbf{A}_2).$$

Strain energy density:

$$W = a(I_1 - 3) + b(I_2 - 3) + q_1 (I_4 - 1)^2 + q_2 (I_6 - 1)^2 + K_1 I_8^2 + K_2 I_8.$$



Reference Configuration



Actual Configuration

A Two-Fiber Planar Model, 1D Shear Waves

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + G\left(X^1, t\right) \end{bmatrix}, \qquad p = p(X^1, t).$$

A Two-Fiber Planar Model, 1D Shear Waves

Displacements transverse to an axis:

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 + G\left(X^1, t\right) \end{bmatrix}, \qquad p = p(X^1, t).$$

Equations:

- Denote $X^1 = x$.
- Incompressibility condition is again identically satisfied.
- $p(x, t)$ found explicitly.
- Displacement $G(x, t)$ satisfies a PDE from the same general class

$$\boxed{G_{tt} = \left(A\left(G_x\right)^2 + BG_x + C\right) G_{xx},}$$

where

$$A = A(K_1, q_{1,2}, \gamma_{1,2}), \quad B = B(K_1, q_{1,2}, \gamma_{1,2}), \quad C = C(K_{1,2}, q_{1,2}, \gamma_{1,2}),$$

Nonlinear wave equation

$$G_{tt} = \left(A\left(G_x\right)^2 + BG_x + C\right) G_{xx},$$

- Same conservation laws as found before!

A Two-Fiber Planar Model, 1D Shear Waves

Nonlinear wave equation

$$\boxed{G_{tt} = \left(A\left(G_x\right)^2 + BG_x + C\right) G_{xx},}$$

- Same conservation laws as found before!

Variational structure

- Any nonlinear PDE of the above class follows from a variational principle, with the Lagrangian density (*up to equivalence*)

$$\mathcal{L} = \frac{1}{2}G_t^2 + \frac{A}{4}GG_x^2G_{xx} + \frac{B}{3}GG_xG_{xx} - \frac{C}{2}G_x^2$$

A Two-Fiber Planar Model, 1D Shear Waves

Nonlinear wave equation

$$G_{tt} = \left(A\left(G_x\right)^2 + BG_x + C\right) G_{xx},$$

- Same conservation laws as found before!

Simplification

- Depending on the sign of $B^2 - 4AC$, PDE can be transformed to

$$u_{tt} = \left(\left(u_x\right)^2 \pm K\right) u_{xx}, \qquad K = 0, \pm 1.$$

Outline

1 Local Conservation Laws

2 Fiber-Reinforced Materials; Governing Equations

3 Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves

4 Single Fiber Family, Ansatz 2 – 2D Shear Waves

5 Two Fiber Families, Planar Case

6 A Viscoelastic Model, Single Fiber Family, 1D Shear Waves

7 Discussion

A Viscoelastic Planar Model

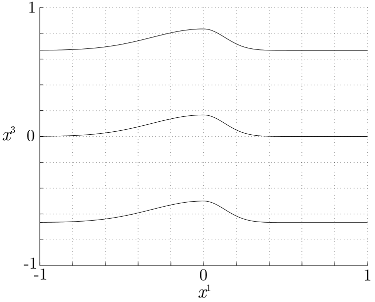
A hyper-viscoelastic model:

- An extra "invariant": $J_2 = \mathrm{Tr}(\dot{\mathbf{C}}^2)$.

Total potential, one fiber family:

$$W = a(I_1 - 3) + b(I_2 - 3) + q_1 (I_4 - 1)^2 + \frac{\eta}{4} J_2 (I_1 - 3).$$

One-Dimensional Viscoelastic Shear Waves

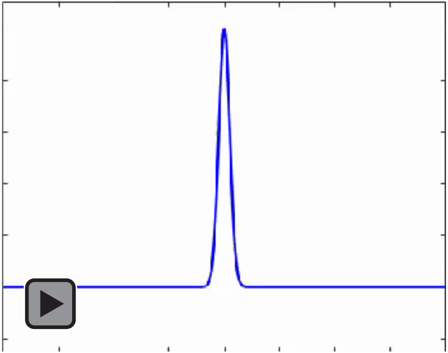


Equation of motion:

- Case: shear wave propagating along the fibers, X^1.
- Single nonlinear PDE:

$$G_{tt} = (\alpha + 3\beta G_x^2)G_{xx} + \eta \left[2(1+4G_x^2)G_x G_{tx} G_{xx} + (1+2G_x^2)G_x^2 G_{txx}\right].$$

- D'Alembert-type example: no wave breaking...



Compute local CLs for the coupled model

$$G_{tt} = (\alpha + 3\beta G_x^2) G_{xx} + \eta \left[2(1 + 4G_x^2) G_x G_{tx} G_{xx} + (1 + 2G_x^2) G_x^2 G_{txx} \right].$$

- $\alpha = \eta = 1.$

Compute local CLs for the coupled model

$$\boxed{G_{tt} = (\alpha + 3\beta G_x^2)G_{xx} + \eta\left[2(1+4G_x^2)G_x G_{tx} G_{xx} + (1+2G_x^2)G_x^2 G_{txx}\right].}$$

- $\alpha = \eta = 1.$

CL 1:

$$\mathrm{D}_t(u_t - (1+2u_x^2)u_x^2 u_{xx}) - \mathrm{D}_x((1+\beta u_x^2)u_x) = 0.$$

Potential system:

$$v_x = u_t - (1+2u_x^2)u_x^2 u_{xx}, \qquad v_t = (1+\beta u_x^2)u_x.$$

Evolution equations:

$$u_t = v_x + (1+2u_x^2)u_x^2 u_{xx},$$
$$v_t = (1+\beta u_x^2)u_x.$$

Compute local CLs for the coupled model

$$G_{tt} = (\alpha + 3\beta G_x^2)G_{xx} + \eta\left[2(1+4G_x^2)G_x G_{tx} G_{xx} + (1+2G_x^2)G_x^2 G_{txx}\right].$$

- $\alpha = \eta = 1.$

CL 2:

$$\mathrm{D}_t(tu_t - u - t(1+2u_x^2)u_x^2 u_{xx}) - \mathrm{D}_x\left[\left(t - \left(\frac{1}{3} - \beta t\right) + \frac{2}{5}u_x^4\right)u_x\right] = 0.$$

Outline

1 Local Conservation Laws

2 Fiber-Reinforced Materials; Governing Equations

3 Single Fiber Family, Ansatz 1 – One-Dimensional Shear Waves

4 Single Fiber Family, Ansatz 2 – 2D Shear Waves

5 Two Fiber Families, Planar Case

6 A Viscoelastic Model, Single Fiber Family, 1D Shear Waves

7 Discussion

Summary

Incompressible hyperelastic models

- Fundamental nonlinear equations for finite-amplitude waves are systematically obtained.
- Wave equations derived for one- and two-fiber-family cases.
- Variational structure is inherited in all models.
- Wave breaking in the one-dimensional case.
- Local conservation laws are computed.

Viscoelastic models

- A one-dimensional finite-amplitude nonlinear wave model is derived, for the two-fiber-family case.
- No wave breaking.
- Local conservation laws are considered.

Further research

- Consider different geometries of interest for applications (e.g., cylindrical, spherical,...).
- Use the derived local conservation laws for optimization and testing of numerical methods.

Anco, S. C. and Bluman, G. W. (2002)
Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications. *Eur. J. Appl. Math.* **13**, 545–566.

Bluman, G.W., Cheviakov, A.F., and Anco, S.C. (2010).
Applications of Symmetry Methods to Partial Differential Equations.
Springer: Applied Mathematical Sciences, Vol. 168.

Cheviakov, A. F. (2007)
GeM software package for computation of symmetries and conservation laws of differential equations. *Comput. Phys. Comm.* **176**, 48–61.

Cheviakov, A. F., Ganghoffer, J.-F., and St. Jean, S. (2015)
Fully nonlinear wave models in fiber-reinforced anisotropic incompressible hyperelastic solids. *Int. J. Nonlin. Mech.* **71**, 8–21.