# Conservation Laws of Fluid Dynamics Models

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# Outline


1. Fluid Dynamics Equations
2. CLs of Constant-Density Euler and N-S Equations
3. CLs of Helically Invariant Flows
4. CLs of An Inviscid Model in Gas Dynamics
5. CLs of a Surfactant Flow Model
6. Discussion

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# Definitions

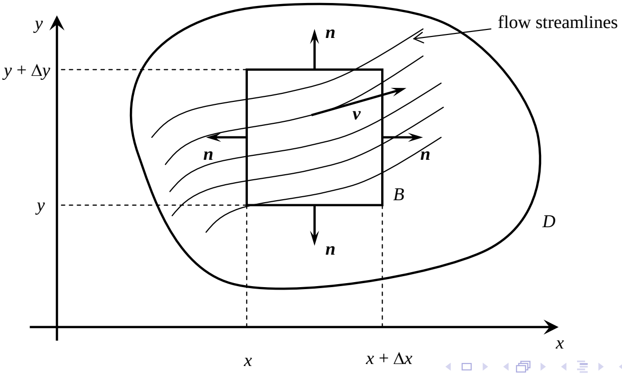
## Fluid/gas flow in 3D

- Independent variables: $t, x, y, z$.
- Dependent variables: $\mathbf{u} = (u^1, u^2, u^3) = (u, v, w)$; $p$; $\rho$.

## Fluid/gas flow in 3D

- Independent variables: $t, x, y, z$.
- Dependent variables: $\mathbf{u} = (u^1, u^2, u^3) = (u, v, w)$; $p$; $\rho$.

- 2D picture:



# Main Equations of Gas/Fluid Flow

- Euler equations:

$$\rho_t + \nabla \cdot (\rho\, \mathbf{u}) = 0,$$

$$\rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p = 0.$$

- Navier-Stokes equations (viscosity $\nu = \text{const}$):

$$\rho_t + \nabla \cdot (\rho\, \mathbf{u}) = 0,$$

$$\rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p - \nu\, \nabla^2 \mathbf{u} = 0.$$

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- 4 equations, 5 unknowns. Closure required.

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- Closure e.g. 1, homogeneous flow (e.g., water):

$$\rho = \mathrm{const}, \qquad \mathrm{div}\ \mathbf{u} = 0.$$

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- Closure e.g. 2, incompressible flow:

$$\operatorname{div}\,\mathbf{u} = 0,$$

$$\rho_t + \mathbf{u}\cdot\nabla\rho = 0.$$

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- Other closure choices: ideal gas/adiabatic, isothermal, polytropic (gas dynamics), etc...

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- Multiple other fluid models exist.

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2 CLs of Constant-Density Euler and N-S Equations

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# Conservation Laws of Constant-Density Euler Equations

## Constant-density Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

**Constant-density Euler equations:**

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

- CLs in a general setting.
- Additional CLs in a symmetric setting (e.g., axisymmetric).
- More additional CLs in a reduced setting (e.g., planar flow).

# Conservation Laws of Constant-Density Euler Equations

**Euler equations:**

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

- Some conservation laws known "forever", e.g., [Batchelor (2000)].
- Kovalevskaya form w.r.t. $x, y, z$.
- It remains an open problem to determine the upper bound of the CL order for the Euler system.
- Let us seek CLs using the Direct method, 2nd-order multipliers [C., Oberlack (2014)]:

$$\Lambda_\sigma = \Lambda_\sigma(45 \text{ variables});$$

$$\Lambda_\sigma R^\sigma \equiv \frac{\partial \Phi^i}{\partial x^i} = 0.$$

## Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

## Conservation of generalized momentum:

- $x$-direction:

$$\frac{\partial}{\partial t}(f(t)u^1) + \frac{\partial}{\partial x}\Big((u^1 f(t) - x f'(t))u^1 + f(t)p\Big)$$
$$+\frac{\partial}{\partial y}\Big((u^1 f(t) - x f'(t))u^2\Big) + \frac{\partial}{\partial z}\Big((u^1 f(t) - x f'(t))u^3\Big) = 0.$$

- Multipliers:

$$\Lambda_1 = f(t)u^1 - x f'(t), \qquad \Lambda_2 = f(t), \qquad \Lambda_3 = \Lambda_4 = 0.$$

- Arbitrary $f(t)$.
- Similar in $y$-, $z$-directions.

## Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

## Conservation of angular momentum $\mathbf{x} \times \mathbf{u}$:

- $x$-direction:

$$\frac{\partial}{\partial t}(zu^2 - yu^3) + \frac{\partial}{\partial x}\Big((zu^2 - yu^3)u^1\Big)$$
$$+\frac{\partial}{\partial y}\Big((zu^2 - yu^3)u^2 + zp\Big) + \frac{\partial}{\partial z}\Big((zu^2 - yu^3)u^3 - yp\Big) = 0.$$

- Multipliers:

$$\Lambda_1 = u_z^2 - u_y^3, \qquad \Lambda_2 = 0, \qquad \Lambda_3 = z, \qquad \Lambda_4 = -y.$$

- Similar in $y$-, $z$-directions.

### Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

### Conservation of kinetic energy:

- $x$-direction:

$$\frac{\partial}{\partial t} K + \nabla \cdot \Big( (K + p)\, \mathbf{u} \Big) = 0, \qquad K = \tfrac{1}{2} |\mathbf{u}|^2.$$

- Multipliers:

$$\Lambda_1 = K + p, \qquad \Lambda_i = u^i, \quad i = 1, 2, 3.$$

# Conservation Laws of Constant-Density Euler Equations

## Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

## Generalized continuity equation:

- For arbitrary $k(t)$:

$$\nabla \cdot (k(t)\,\mathbf{u}) = 0.$$

- Multipliers:

$$\Lambda_1 = k(t), \qquad \Lambda_2 = \Lambda_3 = \Lambda_4 = 0.$$

- Arbitrary $k(t)$.

# Conservation Laws of Constant-Density Euler Equations

## Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

## Conservation of helicity:

- Vorticity: $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$.
- Helicity: $h = \mathbf{u} \cdot \boldsymbol{\omega}$.
- Helicity conservation law:

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0,$$

  where $E = K + p$ is the total energy density.
- Topological significance/vortex line linkage.
- Multipliers:

$$\Lambda_1 = 0, \qquad \Lambda_i = \omega^i, \quad i = 1, 2, 3.$$

## Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

## Vorticity system: conservation of vorticity.

- Vorticity: $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$.
- Vorticity equations:

$$\operatorname{div} \boldsymbol{\omega} = 0, \qquad \boldsymbol{\omega}_t + \operatorname{curl}(\boldsymbol{\omega} \times \mathbf{u}) = 0.$$

# Conservation Laws of Constant-Density Euler Equations

## Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0.$$

## Vorticity system: potential vorticity.

- Vorticity equations:

$$\mathrm{div}\, \boldsymbol{\omega} = 0, \qquad \boldsymbol{\omega}_t + \mathrm{curl}\,(\boldsymbol{\omega} \times \mathbf{u}) = 0.$$

- CL:

$$(\boldsymbol{\omega} \cdot \nabla F)_t + \nabla \cdot (\boldsymbol{\beta} \times \nabla F - F_t\, \boldsymbol{\omega}) = 0, \qquad \boldsymbol{\beta} \equiv \boldsymbol{\omega} \times \mathbf{u}.$$

- Multipliers:

$$\Lambda_1 = -D_t F, \quad \Lambda_2 = D_x F, \quad \Lambda_2 = D_y F, \quad \Lambda_2 = D_z F,$$

  holding for an arbitrary differential function $F = F[\mathbf{u}, p]$.

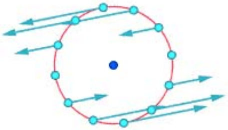
- Details [Müller (1995)], generalizations: [C. & Oberlack (2014)].

## Euler classical two-component plane flow:

$$u^z = \omega^x = \omega^y = 0; \qquad \frac{\partial}{\partial z} = 0.$$

$$\begin{cases} (u^x)_x + (u^y)_y = 0, \\ (u^x)_t + u^x(u^x)_x + u^y(u^x)_y = -p_x, \\ (u^y)_t + u^x(u^y)_x + u^y(u^y)_y = -p_y; \end{cases}$$

$$\begin{cases} \omega^z + (u^x)_y - (u^y)_x = 0, \\ (\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y = 0. \end{cases}$$



# Plane Euler Flows; Conservation of Enstrophy

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$$\begin{cases} \omega^z + (u^x)_y - (u^y)_x = 0, \\ (\omega^z)_t + u^x (\omega^z)_x + u^y (\omega^z)_y = 0. \end{cases}$$

## Enstrophy Conservation

- Enstrophy: $\mathcal{E} = |\boldsymbol{\omega}|^2 = (\omega^z)^2$.
- Material conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathrm{D}_t\, \mathcal{E} + \mathrm{D}_x\, (u^x \mathcal{E}) + \mathrm{D}_y\, (u^y \mathcal{E}) = 0.$$

- Was only known to hold for plane flows, $(2+1)$-dimensions.

# Plane Euler Flows; Conservation of Enstrophy

## Euler classical two-component plane flow:

$$u^z = \omega^x = \omega^y = 0; \qquad \frac{\partial}{\partial z} = 0.$$

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$$\begin{cases} \omega^z + (u^x)_y - (u^y)_x = 0, \\ (\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y = 0. \end{cases}$$

## Other Plane Flow CLs

- Several additional vorticity-related CLs known for plane flows (e.g., [Batchelor (2000)]);

## Navier-Stokes Equations equations in $3+1$ dimensions

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

Vorticity formulation:

$$\nabla \cdot \mathbf{u} = 0, \quad \boldsymbol{\omega} = \nabla \times \mathbf{u},$$
$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

## Basic conservation laws:

- Momentum / generalized momentum: $\Theta = f(t)u^i, \quad i = 1, 2, 3.$
- Angular momentum: $\Theta = (\mathbf{r} \times \mathbf{u})^i, \quad i = 1, 2, 3.$
- Vorticity: $\Theta = \omega^i, \quad i = 1, 2, 3.$
- Potential vorticity.

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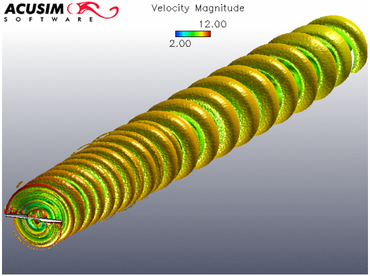
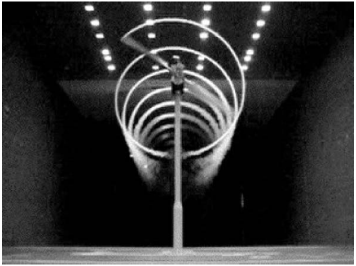
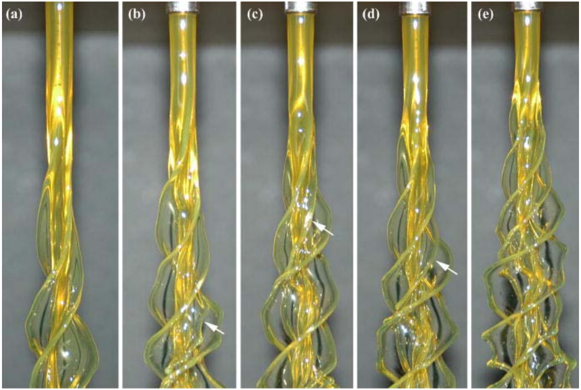
- Wind turbine wakes in aerodynamics [*Vermeer, Sorensen & Crespo, 2003*]



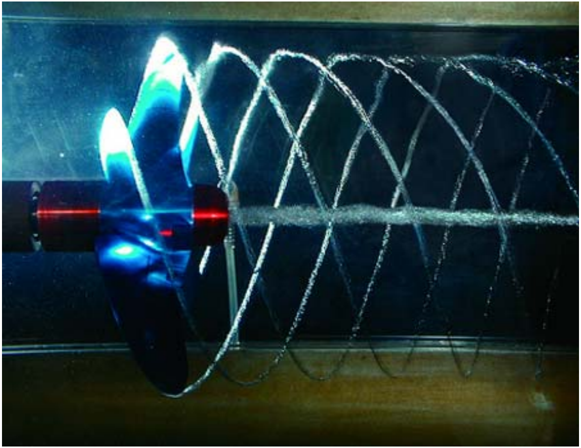



# Examples of Helical Flows in Nature

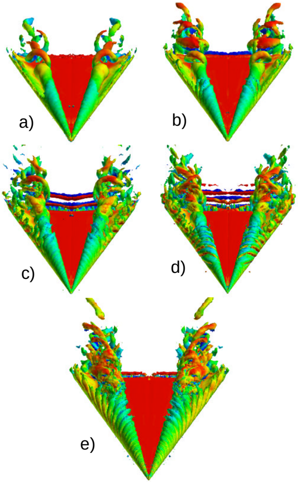
- Helical instability of rotating viscous jets [*Kubitschek & Weidman, 2007*]

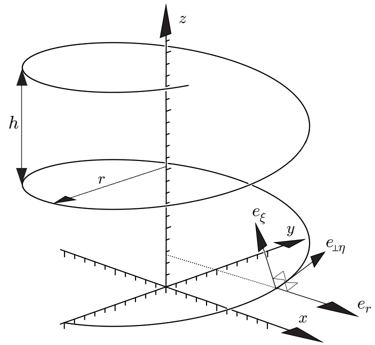


- Helical water flow past a propeller



- Wing tip vortices, in particular, on delta wings [*Mitchell, Morton & Forsythe, 1997*]

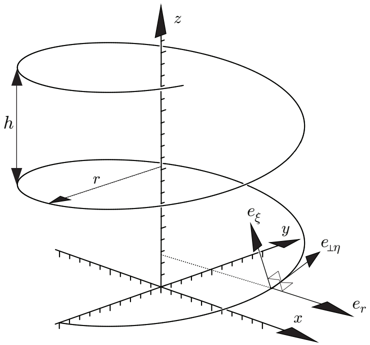





## Helical Coordinates

- Cylindrical coordinates: $(r, \varphi, z)$. **Helical coordinates:** $(r, \eta, \xi)$

$$\xi = az + b\varphi, \quad \eta = a\varphi - b\frac{z}{r^2}, \qquad a, b = \text{const}, \quad a^2 + b^2 > 0.$$

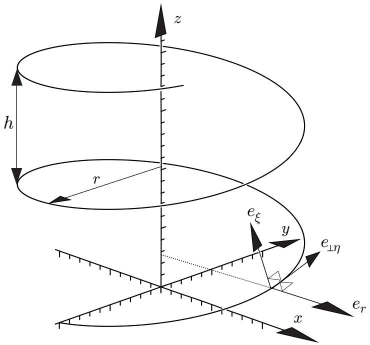



## Orthogonal Basis

$$\mathbf{e}_r = \frac{\nabla r}{|\nabla r|}, \quad \mathbf{e}_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad \mathbf{e}_{\perp\eta} = \frac{\nabla_\perp \eta}{|\nabla_\perp \eta|} = \mathbf{e}_\xi \times \mathbf{e}_r.$$

- Scaling factors: $H_r = 1, H_\eta = r, H_\xi = B(r), \qquad B(r) = \dfrac{r}{\sqrt{a^2 r^2 + b^2}}.$

# Helical Coordinates

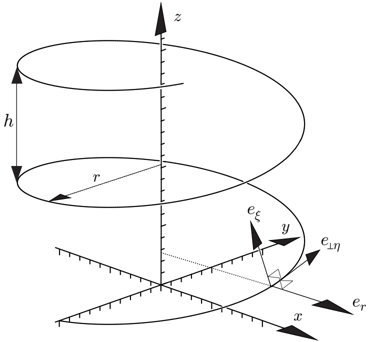



## Vector expansion

$$\mathbf{u} \;=\; u^r\mathbf{e}_r + u^\varphi\mathbf{e}_\varphi + u^z\mathbf{e}_z \;=\; u^r\mathbf{e}_r + u^\eta\mathbf{e}_{\perp\eta} + u^\xi\mathbf{e}_\xi.$$

$$u^\eta = \mathbf{u}\cdot\mathbf{e}_{\perp\eta} = B\left(au^\varphi - \frac{b}{r}u^z\right), \qquad u^\xi = \mathbf{u}\cdot\mathbf{e}_\xi = B\left(\frac{b}{r}u^\varphi + au^z\right).$$

# Helical Coordinates




## Helical invariance: generalizes axal and translational invariance

- Helical coordinates: $r, \quad \xi = az + b\varphi, \quad \eta = a\varphi - bz/r^2.$
- **General helical symmetry:** $f = f(r, \xi), \quad a, b \neq 0.$
- **Axial:** $a = 1,\ b = 0.$ **$z$-Translational:** $a = 0,\ b = 1.$

## Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

## Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

## Continuity:

$$\frac{1}{r} u^r + (u^r)_r + \frac{1}{B}(u^\xi)_\xi = 0$$

# Helically Invariant Navier-Stokes Equations

## Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

## $r$-momentum:

$$(u^r)_t + u^r (u^r)_r + \frac{1}{B} u^\xi (u^r)_\xi - \frac{B^2}{r}\left(\frac{b}{r}u^\xi + a u^\eta\right)^2 = -p_r$$
$$+ \nu \left[ \frac{1}{r}(r(u^r)_r)_r + \frac{1}{B^2}(u^r)_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2bB}{r^2}\left(a(u^\eta)_\xi + \frac{b}{r}(u^\xi)_\xi\right)\right]$$

**Navier-Stokes Equations:**

$$\nabla \cdot \mathbf{u} = 0,$$
$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \nu\,\nabla^2\mathbf{u} = 0.$$

**$\eta$-momentum:**

$$(u^\eta)_t + u^r (u^\eta)_r + \frac{1}{B} u^\xi (u^\eta)_\xi + \frac{a^2B^2}{r} u^r u^\eta$$
$$= \nu\left[\frac{1}{r}(r(u^\eta)_r)_r + \frac{1}{B^2}(u^\eta)_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2} u^\eta + \frac{2abB}{r^2}\left((u^r)_\xi - \left(Bu^\xi\right)_r\right)\right]$$

**Navier-Stokes Equations:**

$$\nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0.$$

**$\xi$-momentum:**

$$(u^\xi)_t + u^r (u^\xi)_r + \frac{1}{B} u^\xi (u^\xi)_\xi + \frac{2abB^2}{r^2} u^r u^\eta + \frac{b^2 B^2}{r^3} u^r u^\xi = -\frac{1}{B} p_\xi$$
$$+ \nu \left[ \frac{1}{r}(r(u^\xi)_r)_r + \frac{1}{B^2}(u^\xi)_{\xi\xi} + \frac{a^4 B^4 - 1}{r^2} u^\xi + \frac{2bB}{r} \left( \frac{b}{r^2}(u^r)_\xi + \left( \frac{aB}{r} u^\eta \right)_r \right) \right]$$

**Navier-Stokes Equations, Vorticity Formulation:**

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

# Helically Invariant Vorticity Formulation

## Navier-Stokes Equations, Vorticity Formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \ =: \ \boldsymbol{\omega} \ = \ \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times \ (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

## Vorticity definition:

$$\omega^r = -\frac{1}{B}(u^\eta)_\xi,$$

$$\omega^\eta = \frac{1}{B}(u^r)_\xi - \frac{1}{r}\left(r u^\xi\right)_r - \frac{2abB^2}{r^2}u^\eta + \frac{a^2B^2}{r}u^\xi,$$

$$\omega^\xi = (u^\eta)_r + \frac{a^2B^2}{r}u^\eta$$

## Navier-Stokes Equations, Vorticity Formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \; =: \; \boldsymbol{\omega} \; = \; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times \, (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

## $r$-Momentum:

$$(\omega^r)_t + u_r(\omega^r)_r + \frac{1}{B} u^\xi (\omega^r)_\xi = \omega^r (u^r)_r + \frac{1}{B} \omega^\xi (u^r)_\xi$$
$$+ \nu \left[ \frac{1}{r}(r(\omega^r)_r)_r + \frac{1}{B^2}(\omega^r)_{\xi\xi} - \frac{1}{r^2}\omega^r - \frac{2bB}{r^2}\left( a(\omega^\eta)_\xi + \frac{b}{r}(\omega^\xi)_\xi \right) \right]$$

## Navier-Stokes Equations, Vorticity Formulation:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

## $\eta$-Momentum:

$$(\omega^\eta)_t + u^r(\omega^\eta)_r + \frac{1}{B} u^\xi (\omega^\eta)_\xi$$

$$-\frac{a^2B^2}{r}(u^r\omega^\eta - u^\eta\omega^r) + \frac{2abB^2}{r^2}(u^\xi\omega^r - u^r\omega^\xi) = \omega^r(u^\eta)_r + \frac{1}{B}\omega^\xi(u^\eta)_\xi$$

$$+\nu\left[\frac{1}{r}(r(\omega^\eta)_r)_r + \frac{1}{B^2}(\omega^\eta)_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2}\omega^\eta + \frac{2abB}{r^2}\left((\omega^r)_\xi - \left(B\omega^\xi\right)_r\right)\right]$$

## Navier-Stokes Equations, Vorticity Formulation:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\nabla \times \mathbf{u} \;=:\; \boldsymbol{\omega} \;=\; \omega^r \mathbf{e}_r + \omega^\eta \mathbf{e}_{\perp\eta} + \omega^\xi \mathbf{e}_\xi,$$
$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0.$$

## $\xi$-Momentum:

$$\begin{aligned}(\omega^\xi)_t + u^r(\omega^\xi)_r + \frac{1}{B}u^\xi(\omega^\xi)_\xi \\ + \frac{1-a^2B^2}{r}(u^\xi\omega^r - u^r\omega^\xi) = \omega^r(u^\xi)_r + \frac{1}{B}\omega^\xi(u^\xi)_\xi \\ + \nu\left[\frac{1}{r}(r(\omega^\xi)_r)_r + \frac{1}{B^2}(\omega^\xi)_{\xi\xi} + \frac{a^4B^4-1}{r^2}\omega^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}(\omega^r)_\xi + \left(\frac{aB}{r}\omega^\eta\right)_r\right)\right]\end{aligned}$$

## For helically symmetric flows:

- Seek local conservation laws

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{\Phi} \equiv \frac{\partial \Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{1}{B}\frac{\partial \Phi^\xi}{\partial \xi} = 0$$

using divergence expressions

$$\frac{\partial \Gamma^1}{\partial t} + \frac{\partial \Gamma^2}{\partial r} + \frac{\partial \Gamma^3}{\partial \xi} = r\left[\frac{\partial}{\partial t}\left(\frac{\Gamma^1}{r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Gamma^2}{r}\right) + \frac{1}{B}\frac{\partial}{\partial \xi}\left(\frac{B}{r}\Gamma^3\right)\right] = 0,$$

i.e.,

$$\Theta \equiv \frac{\Gamma^1}{r}, \quad \Phi^r \equiv \frac{\Gamma^2}{r}, \quad \Phi^\xi \equiv \frac{B}{r}\Gamma^3.$$

- 1st-order multipliers in primitive variables.
- 0th-order multipliers in vorticity formulation.

# Conservation Laws for Helically Symmetric Inviscid Flows: $\nu = 0$

## Primitive variables - EP1 - Kinetic energy

$$\Theta = K, \quad \Phi^r = u^r(K+p), \quad \Phi^\xi = u^\xi(K+p), \qquad K = \frac{1}{2}|\mathbf{u}|^2.$$

## Primitive variables - EP2 - $z$-momentum

$$\Theta = B\left(-\frac{b}{r}u^\eta + au^\xi\right) = u^z, \quad \Phi^r = u^r u^z, \quad \Phi^\xi = u^\xi u^z + aBp.$$

## Primitive variables - EP3 - $z$-angular momentum

$$\Theta = rB\left(au^\eta + \frac{b}{r}u^\xi\right) = ru^\varphi, \quad \Phi^r = ru^r u^\varphi, \quad \Phi^\xi = ru^\xi u^\varphi + bBp.$$

## Primitive variables - EP4 - Generalized momenta/angular momenta

$$\Theta = F\left(\frac{r}{B}u^\eta\right), \quad \Phi^r = u^r F\left(\frac{r}{B}u^\eta\right), \quad \Phi^\xi = u^\xi F\left(\frac{r}{B}u^\eta\right),$$

where $F(\cdot)$ is an arbitrary function.

## Vorticity formulation - EV1 - Conservation of helicity

Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi .$$

The conservation law:

$$\Theta = h,$$

$$\Phi^r = \omega^r \left( E - (u^\eta)^2 - \left(u^\xi\right)^2 \right) + u^r \left(h - u^r \omega^r\right),$$

$$\Phi^\xi = \omega^\xi \left( E - (u^r)^2 - (u^\eta)^2 \right) + u^\xi \left(h - u^\xi \omega^\xi\right),$$

where

$$E = \frac{1}{2}|\mathbf{u}|^2 + p = \frac{1}{2}\left( (u^r)^2 + (u^\eta)^2 + \left(u^\xi\right)^2 \right) + p$$

is the total energy density. In vector notation:

$$\frac{\partial}{\partial t} h + \nabla \cdot (\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}) = 0.$$

## Vorticity formulation - EV2 - Generalized helicity

Helicity:

$$h = \mathbf{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi.$$

$$\frac{\partial}{\partial t}\left(h\,H\left(\frac{r}{B}u^\eta\right)\right) + \nabla \cdot \left[H\left(\frac{r}{B}u^\eta\right)\left[\mathbf{u} \times \nabla E + (\boldsymbol{\omega} \times \mathbf{u}) \times \mathbf{u}\right] + E u^\eta \mathbf{e}_{\perp\eta} \times \nabla H\left(\frac{r}{B}u^\eta\right)\right] = 0$$

for an arbitrary function $H = H(\cdot)$.

## Vorticity formulation - EV3 - Vorticity conservation laws

$$\Theta = \frac{Q(t)}{r}\omega^{\varphi},$$
$$\Phi^{r} = \frac{1}{r}\left(Q(t)[u^{r}\omega^{\varphi} - \omega^{r}u^{\varphi}] + Q'(t)u^{z}\right),$$
$$\Phi^{\xi} = -\frac{aB}{r}\left(Q(t)\left[u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}\right] + Q'(t)u^{r}\right),$$

where $Q(t)$ is an arbitrary function.

## Vorticity formulation - EV4 - Vorticity conservation law

$$\Theta = -rB\left(a^{3}\omega^{\eta} - \frac{b^{3}}{r^{3}}\omega^{\xi}\right),$$
$$\Phi^{r} = -2a^{2}u^{r}u^{z} - a^{3}Br\left(u^{r}\omega^{\eta} - u^{\eta}\omega^{r}\right) + \frac{Bb^{3}}{r^{2}}\left(u^{r}\omega^{\xi} - u^{\xi}\omega^{r}\right),$$
$$\Phi^{\xi} = a^{3}B\left[(u^{r})^{2} + (u^{\eta})^{2} - (u^{\xi})^{2} + r\left(u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}\right)\right] + \frac{2a^{2}bB}{r}u^{\eta}u^{\xi}.$$

# Conservation Laws for Helically Symmetric Inviscid Flows: $\nu = 0$

## Vorticity formulation - EV5 - Vorticity conservation law

$$\Theta = -\frac{B}{r^2}\left(\frac{b^2 r^2}{B^2}\omega^\xi + a^3 r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2 r^2}{B^2}\omega^\xi + \frac{a^3 r^4}{B}\omega^z\right),$$

$$\Phi^r = a^3 rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right) - \frac{a^4 r^4 + a^2 r^2 b^2 + b^4}{r\sqrt{a^2 r^2 + b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right),$$

$$\Phi^\xi = -a^3 bB\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4 rBu^\eta u^\xi.$$

## Vorticity formulation - EV6 - Vorticity conservation law

$$\nabla\cdot\mathbf{\Phi} = 0, \quad \Phi^r = N\omega^r - \frac{1}{B}N_\xi u^\eta, \quad \Phi^\xi = N\omega^\xi,$$

for an arbitrary $N(t,\xi)$.

- Generalization of the obvious divergence expression $\nabla\cdot(G(t)\boldsymbol{\omega}) = 0$.

## Primitive variables - NSP1 - $z$-momentum.

$$\Theta = u^z, \quad \Phi^r = u^r u^z - \nu (u^z)_r, \quad \Phi^\xi = u^\xi u^z + aBp - \frac{\nu}{B}(u^z)_\xi.$$

## Primitive variables - NSP2 - generalized momentum

$$\Theta = \frac{r}{B} u^\eta,$$

$$\begin{aligned}\Phi^r &= \frac{r}{B} u^r u^\eta - \nu \left[ -2aB \left( a u^\eta + 2\frac{b}{r} u^\xi \right) + \left( \frac{r}{B} u^\eta \right)_r \right] \\ &= \frac{r}{B} u^r u^\eta - \nu \left[ -2a u^\varphi + \left( \frac{r}{B} u^\eta \right)_r \right],\end{aligned}$$

$$\Phi^\xi = \frac{r}{B} u^\eta u^\xi - \nu \frac{1}{B} \left[ \frac{2abB^2}{r} u^r + \left( \frac{r}{B} u^\eta \right)_\xi \right].$$

## Vorticity formulation - NSV1 - Family of vorticity conservation laws

$$
\begin{aligned}
\Theta = \quad & \frac{Q(t)}{r} B\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right) = \frac{Q(t)}{r}\omega^{\varphi},\\
\Phi^{r} = \quad & \frac{1}{r}\left\{Q(t)\left[u^{r}B\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right) - \omega^{r}B\left(au^{\eta} + \frac{b}{r}u^{\xi}\right)\right] + Q'(t)B\left(-\frac{b}{r}u^{\eta} + au^{\xi}\right)\right.\\
& \left. -Q(t)\nu\left[\frac{aB}{r}\omega^{\eta} + \frac{b^{2}B}{r\left(a^{2}r^{2}+b^{2}\right)}\left(a\omega^{\eta} + \frac{b}{r}\omega^{\xi}\right) + B\left(a\omega^{\eta}_{r} + \frac{b}{r}\omega^{\xi}_{r}\right)\right]\right\},\\
\Phi^{\xi} = \quad & -\frac{B}{r}\left\{aQ(t)\left[u^{\eta}\omega^{\xi} - u^{\xi}\omega^{\eta}\right] + aQ'(t)u^{r}\right.\\
& \left. +\frac{Q(t)}{r^{3}}\nu\left[\frac{r^{3}}{B}\left(a\omega^{\eta}_{\xi} + \frac{b}{r}\omega^{\xi}_{\xi}\right) + 2br\omega^{r}\right]\right\},
\end{aligned}
$$

for an arbitrary function $Q(t)$.

## Vorticity formulation - NSV2 - Vorticity conservation law

$$\Theta = -rB\left(a^3\omega^\eta - \frac{b^3}{r^3}\omega^\xi\right),$$

$$\Phi^r = -\frac{B}{r^2}\left(a^3r^3\left(u^r\omega^\eta - u^\eta\omega^r\right) - b^3\left(u^r\omega^\xi - u^\xi\omega^r\right)\right) - 2a^2Bu^r\left(-\frac{b}{r}u^\eta + au^\xi\right)$$
$$-\frac{B}{r^2}\nu\left[\frac{r^2}{B^2}\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right) - r^3\left(a^3\omega^\eta_r - \frac{b^3}{r^3}\omega^\xi_r\right) + abB^2r\left(\frac{b^3}{r^3}\omega^\eta + a^3\omega^\xi\right)\right],$$

$$\Phi^\xi = a^3B\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + \frac{2a^2bB}{r}u^\eta u^\xi$$
$$+\frac{2a^2bB}{r}\nu\left[\left(1 - \frac{b^2}{a^2r^2}\right)\omega^r + \frac{r^2}{2a^2bB}\left(a^3\omega^\eta_\xi - \frac{b^3}{r^3}\omega^\xi_\xi\right)\right].$$

## Vorticity formulation - NSV3 - Vorticity conservation law

$$\Theta = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + a^3r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + \frac{a^3r^4}{B}\omega^z\right),$$

$$\begin{aligned}\Phi^r = \;& a^3rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right)\\ & -\frac{a^4r^4 + a^2r^2b^2 + b^4}{r\sqrt{a^2r^2+b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right)\\ & +\nu\left[4a^3B\left(au^\eta + \frac{b}{r}u^\xi\right) - a^3brB(\omega^\eta)_r + \frac{B}{r^3}\left(b^4 - a^4r^4 - \frac{a^6r^6}{a^2r^2+b^2}\right)\omega^\xi\right.\\ & \left. +\frac{B}{r^2}\left(a^4r^4 + a^2r^2b^2 + b^4\right)(\omega^\xi)_r + \frac{ab}{B}\left(2 + \frac{a^4r^4}{(a^2r^2+b^2)^2}\right)\omega^\eta\right],\end{aligned}$$

$$\begin{aligned}\Phi^\xi = \;& -a^3bB\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4rBu^\eta u^\xi\\ & +\nu\left[\frac{1}{r^2}\left(a^4r^4 + a^2r^2b^2 + b^4\right)(\omega^\xi)_\xi - a^3br(\omega^\eta)_\xi - \frac{4a^3bB}{r}u^r + \frac{2b^4B}{r^3}\omega^r\right].\end{aligned}$$

## Generalized enstrophy for inviscid plane flow (known)

$$\Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

## Generalized enstrophy for inviscid axisymmetric flow

$$\Theta = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right)$$

for arbitrary $S(\cdot)$.

## Generalized enstrophy for inviscid plane flow (known)

$$\Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z),$$

for an arbitrary $N(\cdot)$, equivalent to a material conservation law

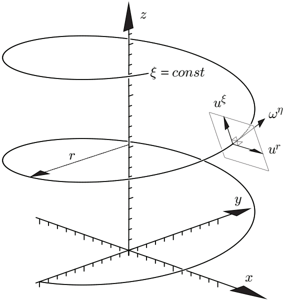
$$\frac{\mathrm{d}}{\mathrm{d}t} N(\omega^z) = 0.$$

## Generalized enstrophy for inviscid axisymmetric flow

$$\Theta = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right)$$

for arbitrary $S(\cdot)$.

- Several additional conservation laws arise for **plane** and **axisymmetric**, inviscid and viscous flows (details in paper).




## Generalized enstrophy for general inviscid helical 2-component flow

$$\Theta = T\left(\frac{B}{r}\omega^{\eta}\right), \quad \Phi^{r} = u^{r}\,T\left(\frac{B}{r}\omega^{\eta}\right), \quad \Phi^{\xi} = u^{\xi}\,T\left(\frac{B}{r}\omega^{\eta}\right),$$

for an arbitrary $T(\cdot)$, equivalent to a material conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\,T\left(\frac{B}{r}\omega^{\eta}\right) = 0.$$

# Summary for helical flows:

## Helically-Invariant Equations

- Full three-component Euler and Navier-Stokes equations written in helically-invariant form.
- Two-component reductions.

## Additional Conservation Laws

- Three-component Euler:
  - Generalized momenta. Generalized helicity. Additional vorticity CLs.
- Three-component Navier-Stokes:
  - Additional CLs in primitive and vorticity formulation.
- Two-component flows:
  - Infinite set of enstrophy-related vorticity CLs (inviscid case).
  - Additional CLs in viscous and inviscid case, for plane and axisymmetric flows.

# Outline


1 Fluid Dynamics Equations

2 CLs of Constant-Density Euler and N-S Equations

3 CLs of Helically Invariant Flows

4 CLs of An Inviscid Model in Gas Dynamics

5 CLs of a Surfactant Flow Model

6 Discussion

- Euler equations:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p = 0.$$

- A CL classification for 2D, 3D barotropic model:

$$p = p(\rho) \qquad (S = \text{const}).$$

[Anco & Dar (2010)].

# Outline


1. Fluid Dynamics Equations
2. CLs of Constant-Density Euler and N-S Equations
3. CLs of Helically Invariant Flows
4. CLs of An Inviscid Model in Gas Dynamics
5. CLs of a Surfactant Flow Model
6. Discussion

# Surfactants - Brief Overview

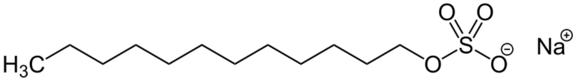
## Surfactants

- "Surfactant"="Surface active agent".
- Act as detergents, wetting agents, emulsifiers, foaming agents, and dispersants.
- Consist of a hydrophobic group (*tail*) and a hydrophilic group (*head*).

## Surfactants

- "Surfactant"="Surface active agent".
- Act as detergents, wetting agents, emulsifiers, foaming agents, and dispersants.
- Consist of a hydrophobic group (*tail*) and a hydrophilic group (*head*).

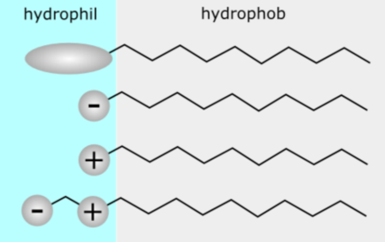
- Sodium lauryl sulfate:



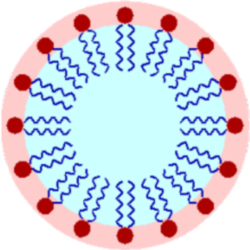
## Surfactants

- "Surfactant"="Surface active agent".
- Act as detergents, wetting agents, emulsifiers, foaming agents, and dispersants.
- Consist of a hydrophobic group (*tail*) and a hydrophilic group (*head*).

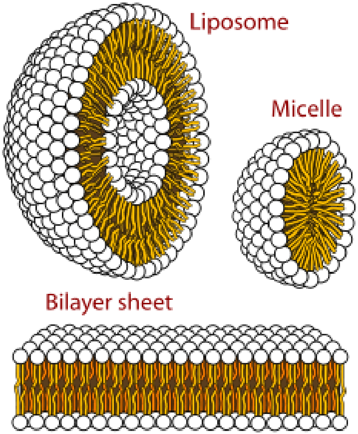
- Hydrophilic groups (heads) can have various properties:



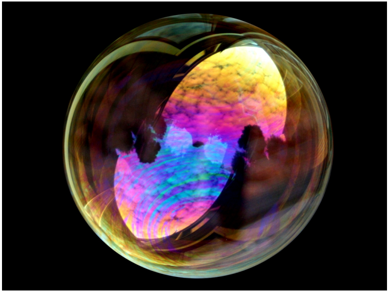
- Surfactant molecules adsorb at phase separation interfaces.
  - Stabilization of growth of bubbles / droplets.
  - Creation of emulsions of insoluble substances.
  - Multiple industrial and medical applications.

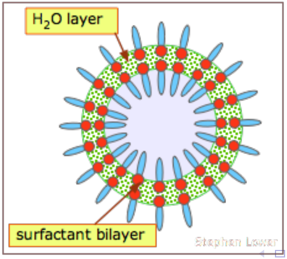


- Can form micelles, double layers, etc.



- Soap bubbles...

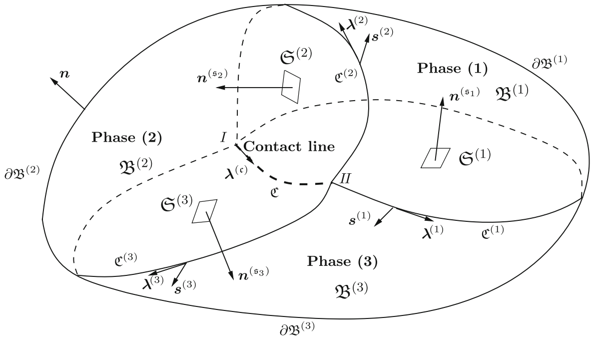




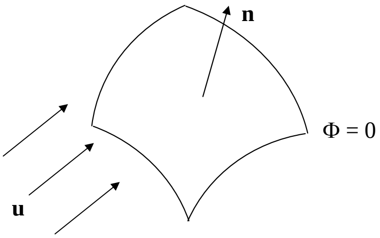
## Derivation:

Can be derived as a special case of multiphase flows with moving interfaces and contact lines:
*[Y.Wang, M. Oberlack, 2011]*

- Illustration:

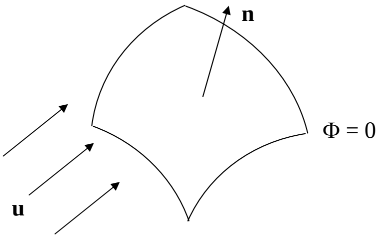


# Surfactant Transport Equations (ctd.)




## Parameters

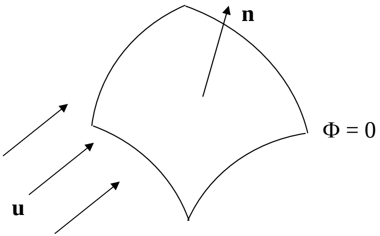
- Surfactant concentration $c = c(\mathbf{x}, t)$.
- Flow velocity $\mathbf{u}(\mathbf{x}, t)$.
- Two-phase interface: phase separation surface $\Phi(\mathbf{x}, t) = 0$.
- Unit normal: $\mathbf{n} = -\dfrac{\nabla \Phi}{|\nabla \Phi|}$.




## Surface gradient

- Surface projection tensor: $p_{ij} = \delta_{ij} - n_i n_j$.
- Surface gradient operator: $\nabla^s = \mathbf{p} \cdot \nabla = (\delta_{ij} - n_i n_j) \dfrac{\partial}{\partial x^j}$.
- Surface Laplacian:

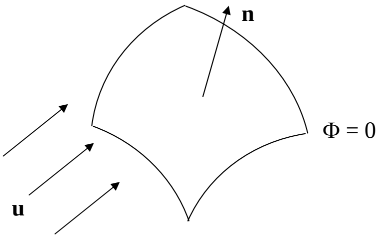
$$\Delta^s F = (\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_i n_k) \frac{\partial F}{\partial x^k} \right).$$




## Governing equations

- Incompressibility condition: $\nabla \cdot \mathbf{u} = 0.$
- Fluid dynamics equations: Euler or Navier-Stokes.
- Interface transport by the flow: $\Phi_t + \mathbf{u} \cdot \nabla \Phi = 0.$
- Surfactant transport equation:

$$c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha(\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.$$




## Fully conserved form

- Specific numerical methods (e.g., discontinuous Galerkin) require the system to be written in a fully conserved form.
- Straightforward for continuity, momentum, and interface transport equations.
- Can the surfactant transport equation be written in the conserved form?

$$c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha(\delta_{ij} - n_i n_j)\frac{\partial}{\partial x^j}\left((\delta_{ik} - n_i n_k)\frac{\partial c}{\partial x^k}\right) = 0.$$

## Governing equations ($\alpha = 0$)

$$R^1 = \frac{\partial u^i}{\partial x^i} = 0,$$

$$R^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0,$$

$$R^3 = c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} = 0.$$

# CLs of the Surfactant Dynamics Equations: The Convection Case

## Governing equations ($\alpha = 0$)

$$R^1 = \frac{\partial u^i}{\partial x^i} = 0,$$

$$R^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0,$$

$$R^3 = c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} = 0.$$

## Multiplier ansatz

$$\Lambda^i = \Lambda^i(t, \mathbf{x}, \Phi, c, \mathbf{u}, \partial\Phi, \partial c, \partial\mathbf{u}, \partial^2\Phi, \partial^2 c, \partial^2\mathbf{u}).$$

## Governing equations ($\alpha = 0$)

$$R^1 = \frac{\partial u^i}{\partial x^i} = 0,$$

$$R^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0,$$

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## Multiplier ansatz

$$\Lambda^i = \Lambda^i(t, \mathbf{x}, \Phi, c, \mathbf{u}, \partial\Phi, \partial c, \partial\mathbf{u}, \partial^2\Phi, \partial^2 c, \partial^2\mathbf{u}).$$

## Conservation Law Determining Equations

$$\mathrm{E}_{u^j}(\Lambda^\sigma R^\sigma) = 0, \quad j = 1, ..., 3; \qquad \mathrm{E}_{\Phi}(\Lambda^\sigma R^\sigma) = 0; \qquad \mathrm{E}_{c}(\Lambda^\sigma R^\sigma) = 0.$$

# CLs of the Surfactant Dynamics Equations: The Convection Case (ctd.)

## Governing equations ($\alpha = 0$)

$$R^1 = \frac{\partial u^i}{\partial x^i} = 0,$$

$$R^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0,$$

$$R^3 = c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} = 0.$$

## Principal Result 1 (multipliers)

- There exist an **infinite family** of multiplier sets with $\Lambda^3 \neq 0$, i.e., essentially involving $c$.
- Family of conservation laws with

$$\Lambda^3 = |\nabla \Phi| \, \mathcal{K}(\Phi, c|\nabla \Phi|).$$

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## Principal Result 1 (divergence expressions)

- Usual form:

$$\frac{\partial}{\partial t}\mathcal{G}(\Phi, c|\nabla\Phi|) + \frac{\partial}{\partial x^i}\left(u^i \mathcal{G}(\Phi, c|\nabla\Phi|)\right) = 0.$$

- Material form:

$$\frac{d}{dt}\mathcal{G}(\Phi, c|\nabla\Phi|) = 0.$$

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## Simplest conservation law with $c$-dependence

- Can take $\mathcal{G}(\Phi, c|\nabla\Phi|) = c|\nabla\Phi|$.

$$\frac{\partial}{\partial t}(c|\nabla\Phi|) + \frac{\partial}{\partial x^i}\left(u^i c|\nabla\Phi|\right) = 0.$$

## Governing equations ($\alpha \neq 0$)

$$R^1 = \frac{\partial u^i}{\partial x^i} = 0,$$

$$R^2 = \Phi_t + \frac{\partial (u^i \Phi)}{\partial x^i} = 0,$$

$$R^3 = c_t + u^i \frac{\partial c}{\partial x^i} - c n_i n_j \frac{\partial u^i}{\partial x^j} - \alpha(\delta_{ij} - n_i n_j) \frac{\partial}{\partial x^j} \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right) = 0.$$

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## Principal Result 2 (multipliers)

$$\Lambda^1 = \Phi \mathcal{F}(\Phi) \, |\nabla \Phi|^{-1} \left( \frac{\partial}{\partial x^j} \left( c \frac{\partial \Phi}{\partial x^j} \right) - c n_i n_j \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \right),$$

$$\Lambda^2 = -\mathcal{F}(\Phi) \, |\nabla \Phi|^{-1} \left( \frac{\partial}{\partial x^j} \left( c \frac{\partial \Phi}{\partial x^j} \right) - c n_i n_j \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \right),$$

$$\Lambda^3 = \mathcal{F}(\Phi) |\nabla \Phi|,$$

# The Convection-Diffusion Case (ctd.)

## Governing equations ($\alpha = 0$)

$$R^1 = \frac{\partial u^i}{\partial x^i} = 0,$$

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## Principal Result 2 (divergence expressions)

- An infinite family of conservation laws:

$$\frac{\partial}{\partial t} \left( c\, \mathcal{F}(\Phi)\, |\nabla \Phi| \right) + \frac{\partial}{\partial x^i} \left( A^i\, \mathcal{F}(\Phi)\, |\nabla \Phi| \right) = 0,$$

where

$$A^i = c u^i - \alpha \left( (\delta_{ik} - n_i n_k) \frac{\partial c}{\partial x^k} \right), \quad i = 1, 2, 3,$$

and $\mathcal{F}$ is an arbitrary sufficiently smooth function.

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## Simplest conservation law with $c$-dependence

- Can take $\mathcal{F}(\Phi) = 1$:

$$\frac{\partial}{\partial t} \left( c \, |\nabla \Phi| \right) + \frac{\partial}{\partial x^i} \left( A^i \, |\nabla \Phi| \right) = 0.$$

- Surfactant dynamics equations can be written in a fully conserved form.

# Outline


1. Fluid Dynamics Equations
2. CLs of Constant-Density Euler and N-S Equations
3. CLs of Helically Invariant Flows
4. CLs of An Inviscid Model in Gas Dynamics
5. CLs of a Surfactant Flow Model
6. Discussion

# Discussion

- Fluid & gas dynamics: a large number of general and specific models exist.
  - viscous and inviscid;
  - single and multi-phase;
  - non-Newtonian;
  - special reductions/geometries of interest;
  - asymptotic models (KdV, shallow water, etc.).

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- **A lot remains to be discovered!**
- Further applications: numerical simulations; development of specialized numerical methods, etc.

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