Local Conservation Laws for Nonlinear Models: Theory, Systematic Construction, and Computation Examples

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Outline

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Definitions

Variables:

- Independent: $\mathbf{x} = (x^1, x^2, ..., x^n)$ or $(t, x^1, x^2, ...)$ or $(t, x, y, ...)$.
- Dependent: $\mathbf{u} = (u^1(\mathbf{x}), u^2(\mathbf{x}), ..., u^m(\mathbf{x}))$ or $(u(\mathbf{x}), v(\mathbf{x}), ...)$.

Definitions

Partial derivatives:

- Notation:
$$\frac{\partial u^k}{\partial x^m} = u^k_{x^m} = \partial_{x^m} u^k.$$
- E.g.,
$$\frac{\partial}{\partial t} u(x, y, t) = u_t = \partial_t u.$$

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All first-order partial derivatives of $\mathbf{u}$: $\partial \mathbf{u}$.

- E.g.,
$$\mathbf{u} = (u^1(x,t), u^2(x,t)), \qquad \partial \mathbf{u} = \{u^1_x,\ u^1_t,\ u^2_x,\ u^2_t\}.$$

Higher-order partial derivatives

- Notation: for example,

$$\frac{\partial^2}{\partial x^2} u(x, y, z) = u_{xx} = \partial_x^2 u.$$

- All p^{th}-order partial derivatives: $\partial^p \mathbf{u}$.

Differential functions:

- A differential function is an expression that may involve independent and dependent variables, and derivatives of dependent variables to some order.

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Differential equations:

- A differential equation of order k:

$$R[\mathbf{u}] = R(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0.$$

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Example:

- The 1D diffusion equation for $u(x,t)$ can be written as

$$0 \;=\; u_t - u_{xx} \;=\; H(u, u_t, u_{xx}) \;=\; H[u].$$

The total derivative of a differential function:

- A basic chain rule.
- E.g., let $u = u(x, y)$, $g[u] = g(x, y, u, u_x, u_y)$. Then

$$\begin{aligned} \mathrm{D}_x g[u] &\equiv \frac{\partial}{\partial x} g(x, y, u, u_x, u_y)\Big|_{u=u(x,y)} \\ &= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} u_x + \frac{\partial g}{\partial u_x} u_{xx} + \frac{\partial g}{\partial u_y} u_{xy}. \end{aligned}$$

Definitions (ctd.)

Multi-indices:

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \in \mathbb{N} \cup \{0\}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n.$$

$$u^\sigma_\alpha \equiv \frac{\partial^{|\alpha|} u^\sigma}{\partial(x^1)^{\alpha_1} \ldots \partial(x^n)^{\alpha_n}}.$$

$$\delta_i \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$$

$$r\delta_i \equiv (0, \ldots, 0, \underset{(i)}{r}, 0, \ldots, 0), \quad r \in \mathbb{N}$$

$$\mathrm{D}_i \equiv \mathrm{D}_{x^i} = \partial_{x^i} + u^p_{\alpha+\delta_i} \partial_{u^p_\alpha}$$

$$\mathrm{D}^\alpha \equiv \mathrm{D}_1^{\alpha_1} \cdots \mathrm{D}_n^{\alpha_n}$$

Conservation laws

- A local conservation law: a divergence expression equal to zero,

$$\mathrm{D}_i \Psi^i[\mathbf{u}] \equiv \operatorname{div} \boldsymbol{\Psi}^{\mathbf{i}}[\mathbf{u}] = 0.$$

- For models involving time:

$$\mathrm{D}_t\, \Theta[\mathbf{u}] + \operatorname{div}_{\mathbf{x}}\, \boldsymbol{\Psi}[\mathbf{u}] = 0.$$

- $\Theta[\mathbf{u}]$: conserved density.
- $\boldsymbol{\Psi}[\mathbf{u}]$: flux vector.

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Example (PDE 1):

$$\mathbf{u}(x, y, z) = (u^1, u^2, u^3),$$

$$\mathrm{div}\ \mathbf{u} = \mathrm{D}_x u^1 + \mathrm{D}_y u^2 + \mathrm{D}_z u^3 = u^1_x + u^2_y + u^3_z = 0.$$

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Example (PDE 2):

$$u = u(x,t),$$

$$\mathrm{D}_t(u) - \mathrm{D}_x(u_x) = u_t - u_{xx} = 0.$$

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- A local conservation law $\Leftrightarrow$ a global conservation principle.

Globally Conserved Quantities

- Given: a local CL for a time-dependent system,

$$\mathrm{D}_t\, \Theta[\mathbf{u}] + \mathrm{div}_{\mathbf{x}}\ \mathbf{\Psi}[\mathbf{u}] = 0.$$

- Integrate in the spatial domain:

$$\int_V \mathrm{D}_t\, \Theta\ dV + \int_V (\mathrm{div}_{\mathbf{x}}\ \mathbf{\Psi})\ dV \ = \ \int_V \mathrm{D}_t\, \Theta\ dV + \oint_{\partial V} \mathbf{\Psi} \cdot d\mathbf{S} \ = \ 0.$$

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- When the total flux vanishes,

$$\oint_{\partial V} \mathbf{\Psi}[\mathbf{u}]\cdot d\mathbf{S} = 0,$$

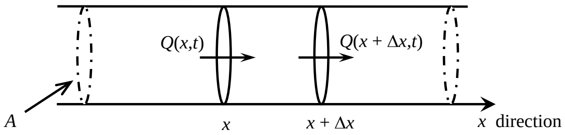
one has

$$\frac{d}{dt}\int_V \Theta[\mathbf{u}]\ dV = 0,$$

i.e., a global conserved quantity (an integral of motion):

$$\boxed{Q = \int_V \Theta\ dV = \mathrm{const.}}$$

- 1D advection equation:



$$\Delta M(t) = \int_x^{x+\Delta x} \rho(s,t)\, A\, ds \quad [\text{kg}], \qquad Q(x,t) = A\rho(x,t)v(x,t) \quad [kg/s].$$

$$\frac{d}{dt}\Delta M(t) = Q(x,t) - Q(x+\Delta x, t) \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\left(\rho(x,t)v(x,t)\right) = 0.}$$

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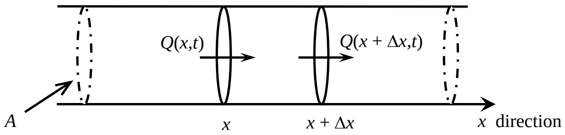

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- Similarly, other conservation principles for local continuum models with conservative/zero forcing yield local conservation laws.

ODE Models: Conserved Quantities

An ODE:

Dependent variable: $u = u(t)$;

A conservation law

$$\mathrm{D}_t F(t, u, u', \ldots) = \frac{d}{dt} F(t, u, u', \ldots) = 0$$

yields a conserved quantity (a constant of motion):

$$F(t, u, u', \ldots) = C = \mathrm{const.}$$

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Example: Harmonic oscillator, spring-mass system

Independent variable: t, dependent: $x(t)$.

ODE: $\ddot{x}(t) + \omega^2 x(t) = 0; \quad \omega^2 = k/m = \mathrm{const.}$

Conservation law: $\dfrac{d}{dt}\left(\dfrac{m\dot{x}^2(t)}{2} + \dfrac{kx^2(t)}{2}\right) = 0.$

Conserved quantity: energy.

Example: ODE integration

An ODE:

$$K'''(x) = \frac{-2\left(K''(x)\right)^2 K(x) - \left(K'(x)\right)^2 K''(x)}{K(x)K'(x)}.$$

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Three independent conserved quantities:

$$\frac{KK''}{(K')^2} = C_1, \quad \frac{KK'' \ln K}{(K')^2} - \ln K' = C_2, \quad \frac{xKK'' + KK'}{(K')^2} - x = C_3$$

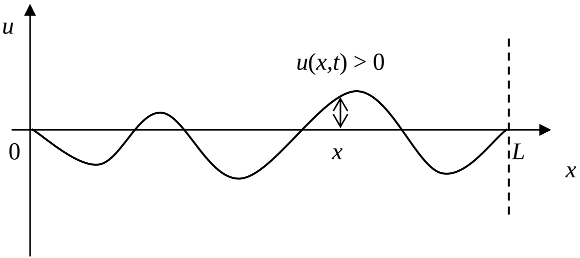
yield complete ODE integration.

Example:

- Small oscillations of a string (transverse) or a rod (longitudinal) ⇔ 1D wave equation:

$$u_{tt} = c^2 u_{xx}.$$

- Independent variables: x, t; dependent: $u(x, t)$.



- $c^2 = T/\rho$; $T, \rho = \text{const}$ (for a string).

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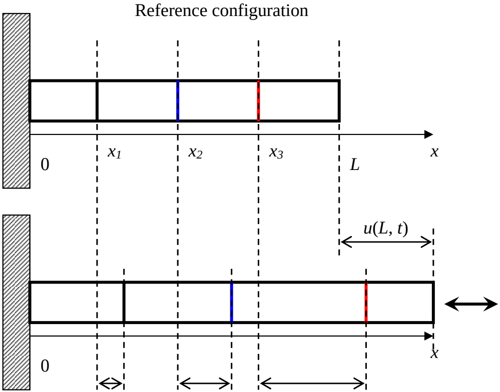

PDE Models

Example:

- Small oscillations of a string (transverse) or a rod (longitudinal) $\Leftrightarrow$ 1D wave equation:

$$u_{tt} = c^2 u_{xx}.$$

- Independent variables: x, t; dependent: $u(x, t)$.

Conservation of momentum:

- Local conservation law: $\mathrm{D}_t(\rho u_t) - \mathrm{D}_x(T u_x) = 0$;
- Global conserved quantity: total momentum

$$M = \int_a^b \rho u_t \, dx = \mathrm{const},$$

for Neumann homogeneous problems with $u_x(a, t) = u_x(b, t) = 0$.

PDE Models

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- Independent variables: x, t; dependent: $u(x, t)$.

Conservation of energy:

- Local conservation law:

$$\mathrm{D}_t \left(\frac{\rho u_t^2}{2} + \frac{T u_x^2}{2} \right) - \mathrm{D}_x (T u_t u_x) = 0;$$

- Global conserved quantity: total energy

$$E = \int_a^b \left(\frac{\rho u_t^2}{2} + \frac{T u_x^2}{2} \right) dx = \mathrm{const},$$

for both Neumann and Dirichlet homogeneous problems.

Example 2: Adiabatic motion of an ideal gas in 3D

Independent variables: t; $x = (x^1, x^2, x^3) \in \mathcal{D} \subset \mathbb{R}^3$.

Dependent: $\rho(x,t)$, $v^1(x,t)$, $v^2(x,t)$, $v^3(x,t)$, $p(x,t)$.

Equations:

$$\mathrm{D}_t \rho + \mathrm{D}_j(\rho v^j) = 0,$$

$$\rho(\mathrm{D}_t + v^j \mathrm{D}_j) v^i + \mathrm{D}_i p = 0, \quad i = 1, 2, 3,$$

$$\rho(\mathrm{D}_t + v^j \mathrm{D}_j) p + \gamma \rho p \mathrm{D}_j v^j = 0.$$

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Conservation laws:

- Mass: $\mathrm{D}_t \rho + \mathrm{D}_j(\rho v^j) = 0$,
- Momentum: $\mathrm{D}_t(\rho v^i) + \mathrm{D}_j(\rho v^i v^j + p\delta^{ij}) = 0, \quad i = 1,2,3,$
- Energy: $\mathrm{D}_t(E) + \mathrm{D}_j \left(v^j(E+p)\right) = 0, \quad E = \frac{1}{2}\rho|\mathbf{v}|^2 + \dfrac{p}{\gamma - 1}.$
- Angular momentum + more.

Applications to ODEs

- Constants of motion.
- Reduction of order; integration.

Applications to PDEs

- Rates of change of physical variables; constants of motion.
- Differential constraints (divergence-free or irrotational fields, etc.).
- Analysis: existence, uniqueness, stability.
- An infinite number of conservation laws may indicate integrability / linearization.
- Finite element/finite volume numerical methods may require conserved forms.
- Weak form of DEs for finite element numerical methods.
- Special numerical methods, conservation law-preserving methods (symplectic integrators, etc.).
- Numerical method testing.

Applications to PDEs

- Potentials, stream functions, etc.

- Magnetic vector potential:

$$\operatorname{div} \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \operatorname{curl} \mathbf{A}.$$

- Irrotational fluid flow, velocity potential:

$$\operatorname{curl} \mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{v} = \operatorname{grad} \Psi.$$

- Fluid flow, stream function in 2D:

$$\mathbf{v} = (u, v), \quad \operatorname{div} \mathbf{V} = u_x + v_y = 0, \quad \begin{cases} u = \Phi_y, \\ v = -\Phi_x. \end{cases}$$

Trivial Conservation Laws

Definition

A *trivial* local conservation law: a zero divergence expression that "does not carry a physical meaning".

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A trivial CL, Type 1:

- Density and all fluxes vanish on all solutions of the given PDE system.
- Example: consider a wave equation on $u(x,t)$: $u_{tt} = u_{xx}$. The conservation law

$$\mathrm{D}_t(u(u_{tt} - u_{xx})) + \mathrm{D}_x(2x(u_{xtt} - u_{xxx})) = 0$$

is a trivial conservation law of the first type.

Definition

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A trivial CL, Type 2:

- The conservation law vanishes as a differential identity.
- Example: for the wave equation on $u(x,t)$: $u_{tt} = u_{xx}$,

$$\mathrm{D}_t(u_{xx}) - \mathrm{D}_x(u_{xt}) \equiv 0$$

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Trivial Conservation Laws

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A *trivial* local conservation law: a zero divergence expression that "does not carry a physical meaning".

A trivial CL, Type 2:

- The conservation law vanishes as a differential identity.
- Another example:

$$\mathrm{div}(\mathrm{curl}\, \mathbf{\Phi}[\mathbf{u}]) \equiv 0.$$

Conservation Law Equivalence

Definition

Two conservation laws $\mathrm{D}_i \Phi^i[\mathbf{u}] = 0$ and $\mathrm{D}_i \Psi^i[\mathbf{u}] = 0$ are *equivalent* if $\mathrm{D}_i(\Phi^i[\mathbf{u}] - \Psi^i[\mathbf{u}]) = 0$ is a trivial conservation law. An *equivalence class* of conservation laws consists of all conservation laws equivalent to some given nontrivial conservation law.

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Definition

A set of ℓ conservation laws $\{\mathrm{D}_i\Phi^i_{(j)}[\mathbf{u}] = 0\}_{j=1}^{\ell}$ is *linearly dependent* if there exists a set of constants $\{a^{(j)}\}_{j=1}^{\ell}$, not all zero, such that the linear combination

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- In practice, one is interested in finding linearly independent sets of (nontrivial) conservation laws of a given PDE system.

Outline

Given:

- A *totally nondegenerate* PDE system $R^\sigma[\mathbf{u}] = 0$, $\sigma = 1, \dots, N$ [cf. *Olver (1993)*].
- A nontrivial local CL: $\mathrm{D}_i \Phi^i[\mathbf{u}] = 0$.
- Denote $G[\mathbf{U}] = \mathrm{D}_i \Phi^i[\mathbf{U}]$.

Hadamard lemma for differential functions:

A differential function $G[\mathbf{U}]$ vanishes on solutions of a PDE system $\mathbf{R}[\mathbf{u}] = 0$ if and only if it has the form

$$G[\mathbf{U}] = P_\sigma^\alpha[\mathbf{U}]\, \mathrm{D}_\alpha\, R^\sigma[\mathbf{U}].$$

Characteristic form of a CL:

Using the product rule, one has

$$G[\mathbf{U}] = \mathrm{D}_i\Phi^i[\mathbf{U}] = \Lambda_\sigma[\mathbf{U}]\, R^\sigma[\mathbf{U}] + \mathrm{div}\ \mathbf{H}[\mathbf{U}],$$

where $\mathbf{H}[\mathbf{U}]$ is linear in R^σ; $\mathrm{div}\ \mathbf{H}[\mathbf{u}] = 0$ is a trivial CL.

Hence every CL $\mathrm{D}_i\Phi^i[\mathbf{u}] = 0$ has an equivalent characteristic form

$$\mathrm{D}_i\tilde{\Phi}^i[\mathbf{u}] = \Lambda_\sigma[\mathbf{u}]\, R^\sigma[\mathbf{u}] = 0, \qquad \tilde{\Phi}^i = \Phi^i - H^i.$$

- CL multipliers (characteristics): $\{\Lambda_\sigma[\mathbf{u}]\}_{\sigma=1}^N$.

Result:

For most physical DE models, every local CL has an equivalent characteristic form

$$\mathrm{D}_i \Phi^i[\mathbf{u}] = \Lambda_\sigma[\mathbf{u}]\, R^\sigma[\mathbf{u}] = 0,$$

for some set of multipliers $\{\Lambda_\sigma[\mathbf{u}]\}$.

The Idea of the Direct Construction Method

Definition

The *Euler operator* with respect to U^j:

$$\mathrm{E}_{U^j} = (-D)^{\beta} \frac{\partial}{\partial U^j_{\beta}} = \frac{\partial}{\partial U^j} - \mathrm{D}_i \frac{\partial}{\partial U^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial U^j_{i_1 \ldots i_s}} + \cdots ,$$

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Theorem

Let $\mathbf{U}(\mathbf{x}) = (U^1, \ldots, U^m)$*. The equations*

$$\mathrm{E}_{U^j} F[\mathbf{U}] \equiv 0, \qquad j = 1, \ldots, m,$$

hold for arbitrary $\mathbf{U}(\mathbf{x})$ *if and only if*

$$F[\mathbf{U}] \equiv \mathrm{D}_i \Psi^i[\mathbf{U}]$$

for some functions $\{\Psi^i[\mathbf{U}]\}$*.*

Idea:

- Seek conservation laws in the characteristic form $\boxed{\mathrm{D}_i \Phi^i = \Lambda_\sigma R^\sigma = 0.}$
- Multiplier determining equations:

$$\mathrm{E}_{U^j}(\Lambda_\sigma R^\sigma) \equiv 0, \quad j = 1, \ldots, m.$$

The DCM Sequence

Consider a general system $\mathbf{R}[\mathbf{u}] = 0$ of N PDEs.

Direct Construction Method

- Specify dependence of multipliers: $\Lambda_\sigma[\mathbf{U}] = \Lambda_\sigma(\mathbf{x}, \mathbf{U}, ...)$, $\sigma = 1, ..., N$.

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- Solve the set of determining equations

$$\mathrm{E}_{U^j}(\Lambda_\sigma[\mathbf{U}]\, R^\sigma[\mathbf{U}]) \equiv 0, \quad j = 1, \ldots, m,$$

for arbitrary $\mathbf{U}(\mathbf{x})$ (off of the solution set!) to find all such sets of multipliers.

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- Find the corresponding fluxes $\Phi^i[\mathrm{U}]$ satisfying the identity

$$\Lambda_\sigma R^\sigma \equiv \mathrm{D}_i \Phi^i.$$

- Each set of fluxes, multipliers yields a local conservation law

$$\mathrm{D}_i \Phi^i[\mathbf{u}] = 0,$$

  holding on solutions $\mathbf{u}(\mathbf{x})$ of the given PDE system.

Example

The Korteweg-de Vries (KdV) equation

$$R[u] = u_t + uu_x + u_{xxx} = 0.$$

0th-order multipliers

- Determining equations:

$$\mathrm{E}_U\ \left(\Lambda(x,t,U)(U_t + UU_x + U_{xxx})\right) \equiv 0.$$

- Solution:

$$\Lambda_1 = 1, \quad \Lambda_2 = U, \quad \Lambda_3 = tU - x.$$

- Conservation laws:

$$\mathrm{D}_t(u) + \mathrm{D}_x\left(\tfrac{1}{2}u^2 + u_{xx}\right) = 0,$$

$$\mathrm{D}_t\left(\tfrac{1}{2}u^2\right) + \mathrm{D}_x\left(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2\right) = 0,$$

$$\mathrm{D}_t\left(\tfrac{1}{6}u^3 - \tfrac{1}{2}u_x^2\right) + \mathrm{D}_x\left(\tfrac{1}{8}u^4 - uu_x^2 + \tfrac{1}{2}(u^2u_{xx} + u_{xx}^2) - u_xu_{xxx}\right) = 0.$$

Example

The Korteweg-de Vries (KdV) equation

$$R[u] = u_t + uu_x + u_{xxx} = 0.$$

1st-order multipliers in x

- Form: $\Lambda = \Lambda(x, t, U, U_x)$
- Solution: no extra conservation laws.

Example

The Korteweg-de Vries (KdV) equation

$$R[u] = u_t + uu_x + u_{xxx} = 0.$$

2nd-order multipliers in x

- Form: $\Lambda = \Lambda(x, t, U, U_x, U_{xx})$
- Solution: one extra conservation law with

$$\Lambda_4 = U_{xx} + \tfrac{1}{2}U^2.$$

The Idea of the Direct Construction Method

Example

The Korteweg-de Vries (KdV) equation

$$R[u] = u_t + uu_x + u_{xxx} = 0.$$

2nd-order multipliers in x

- Form: $\Lambda = \Lambda(x, t, U, U_x, U_{xx})$
- Solution: one extra conservation law with

$$\Lambda_4 = U_{xx} + \frac{1}{2}U^2.$$

- For PDE with additional structure, infinite sets of CLs may exist, including CLs of arbitrary order.
- E.g., integrable systems, recursion operators, ...

Flux Computation Problem

Suppose for a given PDE system, a set of CL multipliers has been found, and one has

$$\Lambda_\sigma[\mathbf{u}]R^\sigma[\mathbf{u}] \equiv \mathrm{D}_i\Phi^i[\mathbf{u}] = 0.$$

- How does one compute $\{\Phi^i[\mathbf{u}]\}$?

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Some methods [cf. *Wolf (2002), Cheviakov (2010)*]:

- Direct
- Homotopy 1 [*Bluman & Anco (2002)*]
- Homotopy 2 [*Hereman et al (2005)*]
- Scaling (when a specific scaling symmetry is present) [*Anco (2003)*]

Flux Computation Methods [see *Cheviakov (2010)*]

Table: Comparison of Four Methods of Flux Computation

Method	Applicability	Computational complexity
Direct	Simpler multipliers/PDE systems, which may involve arbitrary functions.	Solution of an overdetermined linear PDE system for fluxes.
Homotopy 1	Complicated multipliers/PDEs, not involving arbitrary functions.	One-dimensional integration.
Homotopy 2	Complicated multipliers/PDEs, not involving arbitrary functions. The divergence expression must vanish for $U = 0$. For some conservation laws, this method can yield divergent integrals.	One-dimensional integration.
Scaling symmetry	Complicated multipliers/PDEs, may involve arbitrary functions. Scaling-homogeneous PDEs and multipliers. Noncritical conservation laws.	Repeated differentiation.

Some refs:

- Review: Hereman, *Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions*, Preprint, 2006.
- Mathematica: Temuerchaolu, *An algorithmic theory of reduction of differential polynomial systems.* Adv. Math. 32, 208–220 (in Chinese), 2003.
- Maple/RIF: Reid, Wittkopf, Boulton, *Reduction of systems of nonlinear partial differential equations to simplified involutive forms*, Eur. J. Appl. Math. 7, 604–635, 1996.
- REDUCE: T. Wolf, *Crack, LiePDE, ApplySym and ConLaw*, 2002.
- Maple: A.C., symmetry/conservation law analysis module (GeM), 2004-now.

Example of use of the GeM package for Maple for the KdV.

- Use the module: `read("d:/gem32_12.mpl"):`
- Declare variables: `gem_decl_vars(indeps=[x,t], deps=[U(x,t)]);`
- Declare the equation:

```
gem_decl_eqs([diff(U(x,t),t)=U(x,t)*diff(U(x,t),x)
    +diff(U(x,t),x,x,x)],
    solve_for=[diff(U(x,t),t)]);
```

- Generate determining equations:

```
det_eqs:=gem_conslaw_det_eqs([x,t, U(x,t),
    diff(U(x,t),x), diff(U(x,t),x,x)]):
```

- Reduce the overdetermined system:

```
CL_multipliers:=gem_conslaw_multipliers();
simplified_eqs:=DEtools[rifsimp](det_eqs, CL_multipliers, mindim=1);
```

Example of use of the GeM package for Maple for the KdV.

- Solve determining equations:

```
multipliers_sol:=pdsolve(simplified_eqs[Solved]);
```

- Obtain corresponding conservation law fluxes/densities:

```
gem_get_CL_fluxes(multipliers_sol, method=*****);
```

Consider a nonlinear telegraph system for $u^1 = u(x,t)$, $u^2 = v(x,t)$:

$$\begin{aligned} R^1[u,v] &= v_t - (u^2+1)u_x - u = 0,\\ R^2[u,v] &= u_t - v_x = 0. \end{aligned}$$

Multiplier ansatz: $\Lambda_1 = \xi(x,t,U,V)$, $\Lambda_2 = \phi(x,t,U,V)$.

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Determining equations:

$$\mathrm{E}_U\left[\xi(x,t,U,V)(V_t - (U^2+1)U_x - U) + \phi(x,t,U,V)(U_t - V_x)\right] \equiv 0,$$

$$\mathrm{E}_V\left[\xi(x,t,U,V)(V_t - (U^2+1)U_x - U) + \phi(x,t,U,V)(U_t - V_x)\right] \equiv 0.$$

Euler operators:

$$\mathrm{E}_U = \frac{\partial}{\partial U} - \mathrm{D}_x\frac{\partial}{\partial U_x} - \mathrm{D}_t\frac{\partial}{\partial U_t},$$
$$\mathrm{E}_V = \frac{\partial}{\partial V} - \mathrm{D}_x\frac{\partial}{\partial V_x} - \mathrm{D}_t\frac{\partial}{\partial V_t}.$$

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$$\mathrm{E}_V\left[\xi(x,t,U,V)(V_t - (U^2+1)U_x - U) + \phi(x,t,U,V)(U_t - V_x)\right] \equiv 0.$$

Split determining equations:

$$\begin{aligned} \phi_V - \xi_U &= 0, \qquad \phi_U - (U^2+1)\xi_V = 0,\\ \phi_x - \xi_t - U\xi_V &= 0, \qquad (U^2+1)\xi_x - \phi_t - U\xi_U - \xi = 0. \end{aligned}$$

Consider a nonlinear telegraph system for $u^1 = u(x,t)$, $u^2 = v(x,t)$:

$$R^1[u,v] = v_t - (u^2+1)u_x - u = 0,$$
$$R^2[u,v] = u_t - v_x = 0.$$

Multiplier ansatz: $\Lambda_1 = \xi(x,t,U,V)$, $\Lambda_2 = \phi(x,t,U,V)$.

Solution: five sets of multipliers $(\xi, \phi) =$

0	1
t	$x - \frac{1}{2}t^2$
1	$-t$
$e^{x+\frac{1}{2}U^2+V}$	$Ue^{x+\frac{1}{2}U^2+V}$
$e^{x+\frac{1}{2}U^2-V}$	$-Ue^{x+\frac{1}{2}U^2-V}$

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Multiplier ansatz: $\Lambda_1 = \xi(x,t,U,V)$, $\Lambda_2 = \phi(x,t,U,V)$.

Resulting five conservation laws:

$$\mathrm{D}_t u - \mathrm{D}_x v = 0,$$

$$\mathrm{D}_t[(x - \tfrac{1}{2}t^2)u + tv] + \mathrm{D}_x[(\tfrac{1}{2}t^2 - x)v - t(\tfrac{1}{3}u^3 + u)] = 0,$$

$$\mathrm{D}_t[v - tu] + \mathrm{D}_x[tv - (\tfrac{1}{3}u^3 + u)] = 0,$$

$$\mathrm{D}_t[e^{x+\frac{1}{2}u^2+v}] + \mathrm{D}_x[-ue^{x+\frac{1}{2}u^2+v}] = 0,$$

$$\mathrm{D}_t[e^{x+\frac{1}{2}u^2-v}] + \mathrm{D}_x[ue^{x+\frac{1}{2}u^2-v}] = 0.$$

- To obtain **further conservation laws**, extend the multiplier ansatz...

Consider a nonlinear telegraph system for $u^1 = u(x,t)$, $u^2 = v(x,t)$:

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Multiplier ansatz: $\Lambda_1 = \xi(x,t,U,V)$, $\Lambda_2 = \phi(x,t,U,V)$.

- Maple example:

Definition

A CL multiplier $\Lambda_\sigma[\mathbf{U}]$ is *singular* if it is a singular function when evaluated on solutions of the given PDE system.

- In practice, one is only interested in non-singular sets of multipliers.
- Singular multipliers lead to arbitrary divergence expressions that are not conservation laws of the given system.

- For example, for the KdV, $R[u] = u_t + uu_x + u_{xxx} = 0$, a multiplier

$$\Lambda_\sigma[U] = \frac{\mathrm{D}_i \Phi^i[U]}{U_t + UU_x + U_{xxx}}$$

is a singular multiplier... yielding a "false" divergence expression

$$\frac{\mathrm{D}_i \Phi^i[U]}{U_t + UU_x + U_{xxx}}(U_t + UU_x + U_{xxx}) = \mathrm{D}_i \Phi^i[U]$$

for arbitrary functions $\Phi^1[U], \ldots, \Phi^n[U]$.

- To avoid getting an infinite set of singular multipliers: need to exclude some leading derivative (e.g., U_t) and its differential consequences.

Completeness of the Direct CL Construction Method

Extended Kovalevskaya form

A PDE system $\mathbf{R}[\mathbf{u}] = 0$ is in *extended Kovalevskaya form* with respect to an independent variable x^j, if the system is solved for the highest derivative of each dependent variable with respect to x^j, i.e.,

$$\frac{\partial^{s_\sigma}}{\partial (x^j)^{s_\sigma}} u^\sigma = Q^\sigma(x, u, \partial u, \ldots, \partial^k u), \quad 1 \le s_\sigma \le k, \quad \sigma = 1, \ldots, m,$$

where all derivatives with respect to x^j appearing in the right-hand side of each PDE above are of lower order than those appearing on the left-hand side.

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Theorem [*M. Alonso (1979)*]

Let $\mathbf{R}[\mathbf{u}] = 0$ be a PDE system in the extended Kovalevskaya form. Then every its local conservation law has an equivalent conservation law in the characteristic form,

$$\Lambda_\sigma R^\sigma \equiv \mathrm{D}_i \Phi^i = 0,$$

such that $\{\Lambda_\sigma\}$ do not involve the leading derivatives or their differential consequences.

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where all derivatives with respect to x^j appearing in the right-hand side of each PDE above are of lower order than those appearing on the left-hand side.

Example

The KdV equation

$$R[u] = u_t + uu_x + u_{xxx} = 0$$

has the extended Kovalevskaya form with respect to t ($u_t = \ldots$) or x ($u_{xxx} = \ldots$).

- For systems in the extended Kovalevskaya form, DCM for non-singular multipliers is complete.

Completeness of the Direct CL Construction Method

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- For systems in the extended Kovalevskaya form, DCM for non-singular multipliers is complete.
- For systems in a solved form but not in the extended Kovalevskaya form, multipliers may involve leading derivatives/their differential consequences.
- In practice, even if the extended Kovalevskaya form exists for a given system, it may be too complex to work with.
- One may use the Direct method on non-Kovalevskaya systems to get partial CL classifications.

Consider a PDE system

$$R^{\sigma}[\mathbf{u}] = 0, \quad \sigma = 1, \ldots, N,$$

with n independent variables $\mathbf{x} = (x^1, \ldots, x^n)$ and m dependent variables $\mathbf{u} = (u^1, \ldots, u^m)$.

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with n independent variables $\mathbf{x} = (x^1, \ldots, x^n)$ and m dependent variables $\mathbf{u} = (u^1, \ldots, u^m)$.

Consider an invertible point transformation
$$\begin{array}{rcl} x^i & = & x^i(\mathbf{z}, \mathbf{w}), \quad i = 1, \ldots, n, \\ u^{\mu} & = & u^{\mu}(\mathbf{z}, \mathbf{w}), \quad \mu = 1, \ldots, m, \end{array}$$
where $\mathbf{z} = (z^1, \ldots, z^m)$, $\mathbf{w}(\mathbf{z}) = (w^1, \ldots, w^m)$.

Obtain an equivalent PDE system
$$S^{\sigma}[\mathbf{w}] = 0, \quad \sigma = 1, \ldots, N,$$

Theorem

To any local CL (equivalence class)

$$\mathrm{D}_{x^i} \Phi^i[\mathbf{u}] = 0$$

of a PDE system $\mathbf{R}[\mathbf{u}] = 0$ *there corresponds a CL (equivalence class)*

$$\tilde{\mathrm{D}}_{z^j} \Psi^j[\mathbf{w}] = 0$$

holding for the PDE system $\mathbf{S}[\mathbf{w}] = 0$.

In particular,

$$\mathrm{J}[\mathbf{w}]\mathrm{D}_i \Phi^i[\mathbf{u}] = \tilde{\mathrm{D}}_{z^j} \Psi^j[\mathbf{w}], \qquad \mathrm{J}[\mathbf{w}] = \frac{\mathrm{D}(x^1, \ldots, x^n)}{\mathrm{D}(z^1, \ldots, z^n)}.$$

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- Local conservation laws are coordinate-independent.

Outline

- Local symmetries and local conservation laws of DE systems are closely related.
- A specific well-known relationship: Noether's theorem for variational DE systems.

Action integral

$$J[\mathbf{U}] = \int_\Omega \mathcal{L}(\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \ldots, \partial^k\mathbf{U})\, dx.$$

Principle of extremal action

Variation of $\mathbf{U}$: $\mathbf{U}(\mathbf{x}) \to \mathbf{U}(\mathbf{x}) + \delta\mathbf{U}(\mathbf{x})$; $\quad \delta\mathbf{U}(\mathbf{x}) = \varepsilon\mathbf{v}(\mathbf{x})$; $\quad \delta\mathbf{U}(\mathbf{x})\big|_{\partial\Omega} = 0$.

Variation of action: $\delta J \equiv J[\mathbf{U} + \varepsilon\mathbf{v}] - J[\mathbf{U}] = \int_\Omega (\delta\mathcal{L})\, dx = o(\varepsilon)$.

Variation of the Lagrangian

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}(\mathbf{x}, \mathbf{U}+\varepsilon\mathbf{v}, \partial\mathbf{U}+\varepsilon\partial\mathbf{v}, \ldots, \partial^k\mathbf{U}+\varepsilon\partial^k\mathbf{v}) - \mathcal{L}(\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \ldots, \partial^k\mathbf{U}) \\
&= \varepsilon\left(\frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma} v^\sigma + \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma_j} v^\sigma_j + \cdots + \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma_{j_1\cdots j_k}} v^\sigma_{j_1\cdots j_k}\right) + O(\varepsilon^2) \\
&\overset{\text{by parts}}{=} \varepsilon\left(v^\sigma \mathrm{E}_{U^\sigma}(\mathcal{L}[\mathbf{U}])\right) + \mathrm{div}(\ldots) + O(\varepsilon^2)
\end{aligned}$$

Euler-Lagrange equations, Euler operators:

$$\begin{aligned}
\mathrm{E}_{U^\sigma}(\mathcal{L}[\mathbf{U}]) &= \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma} + \cdots + (-1)^k \mathrm{D}_{j_1}\cdots\mathrm{D}_{j_k} \frac{\partial\mathcal{L}[\mathbf{U}]}{\partial U^\sigma_{j_1\cdots j_k}} = 0, \\
\sigma &= 1, \ldots, m.
\end{aligned}$$

Definition

A DE system $\mathbf{R}[\mathbf{u}] = 0$ is variational if its equations are Euler-Lagrange equations for some variational principle:

$$R^\sigma[\mathbf{U}] \;=\; \mathrm{E}_{U^\sigma}(\mathcal{L}[\mathbf{U}]), \qquad \sigma = 1, \ldots, m.$$

- Example 1: Harmonic oscillator, $U = x = x(t)$

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$$\mathrm{E}_x = \frac{d}{dx} - \mathrm{D}_t \frac{d}{d\dot{x}}$$

$$\mathrm{E}_x \mathcal{L} = -m(\ddot{x} + \omega^2 x) = 0, \qquad \omega^2 = k/m$$

- Example 2: Wave equation for $U = u(x, t)$

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$$\mathrm{E}_u \mathcal{L} = -\rho(u_{tt} - c^2 u_{xx}) = 0, \qquad c^2 = T/\rho$$

Variational DE systems

1. A DE system $\mathbf{R}^\sigma[\mathbf{U}]$ is variational if and only if its linearization is self-adjoint.

   - Linearization:
   $$L^\sigma[\mathbf{u}]\,\mathbf{v}(\mathbf{x}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R^\sigma[\mathbf{u}+\epsilon\mathbf{v}] = \frac{\partial R^\sigma[\mathbf{u}]}{\partial u^p_\alpha} D^\alpha v^p = 0;$$
   - Adjoint linearization:
   $$L^*_\mu[\mathbf{u}]\,\mathbf{w}(\mathbf{x}) = (-D)^\alpha \left(\frac{\partial R^\sigma[\mathbf{u}]}{\partial u^\mu_\alpha} w_\sigma\right) = 0$$
   - Relationship:
   $$\mathbf{W}\cdot(\mathbf{L}[\mathbf{U}]\;\mathbf{V}) - (\mathbf{L}^*[\mathbf{U}]\;\mathbf{W})\cdot\mathbf{v} \overset{\text{by parts}}{\equiv} \mathrm{div}\;P;$$
   in components,
   $$W_\sigma\, L^\sigma[\mathbf{U}]\,\mathbf{V} - V^\mu\, L^*_\mu[\mathbf{U}]\,\mathbf{W} \equiv \mathrm{D}_i P^i.$$

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   - Adjoint linearization:

$$L^*_\mu[\mathbf{u}]\,\mathbf{w}(\mathbf{x}) = (-D)^\alpha\left(\frac{\partial R^\sigma[\mathbf{u}]}{\partial u^\mu_\alpha} w_\sigma\right) = 0$$

   - Relationship:

$$\mathbf{W}\cdot(\mathbf{L}[\mathbf{U}]\ \mathbf{V}) - (\mathbf{L}^*[\mathbf{U}]\ \mathbf{W})\cdot\mathbf{v} \overset{\text{by parts}}{\equiv} \mathrm{div}\ P;$$

   in components,

$$W_\sigma\, L^\sigma[\mathbf{U}]\,\mathbf{V} - V^\mu\, L^*_\mu[\mathbf{U}]\,\mathbf{W} \equiv \mathrm{D}_i P^i.$$

2. Homotopy Formula for a Lagrangian:

$$\boxed{\mathcal{L} = \int_0^1 \mathbf{u}\cdot\mathbf{R}[\lambda\mathbf{u}]\ d\lambda.}$$

Example: Wave equation for $u(x,t)$

$$R[u] = u_{tt} - c^2 u_{xx} = 0;$$

Linearization (already linear!)

$$L[u]\, v(x,t) = v_{tt} - c^2 v_{xx} = 0;$$

Adjoint linearization operator:

$$w(x,t)\; L[u]\, v(x,t) = w(v_{tt} - c^2 v_{xx}) = (w_{tt} - c^2 w_{xx})v(x,t) + (v_t w - v w_t)_t - c^2 (v_x w - v w_x)_x;$$

Result:

$$L^*[u]\, v(x,t) = L[u]\, v(x,t),$$

so $R[u]$ is self-adjoint.

Lagrangian:

$$\mathcal{L} = \frac{1}{2} {u_t}^2 - \frac{1}{2} c^2 {u_x}^2.$$

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- A number of important physical non-dissipative systems have a variational formulation.
- The vast majority of PDE systems do not have a variational formulation.
- Self-adjointness is coordinate-dependent; also depends on the writing of the system.
- It remains an open problem how to determine whether a given system has a variational formulation.
- Pseudo-Lagrangians can be constructed by appending adjoint equations to given ones.

Outline

Consider a general DE system

$$R^\sigma[\mathbf{u}] = \mathbf{R}^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N$$

with variables $\mathbf{x} = (x^1, ..., x^n)$, $\mathbf{u} = (u^1, ..., u^m)$.

Definition

A one-parameter Lie group of point transformations

$$\mathbf{x}^* = f(\mathbf{x}, \mathbf{u}; a) = \mathbf{x} + a\xi(\mathbf{x}, \mathbf{u}) + O(a^2),$$
$$\mathbf{u}^* = g(\mathbf{x}, \mathbf{u}; a) = \mathbf{u} + a\eta(\mathbf{x}, \mathbf{u}) + O(a^2)$$

(with the parameter a) is a *point symmetry* of $R^\sigma[\mathbf{u}]$ if it transforms solutions to solutions: $\mathbf{u}(\mathbf{x}) \to \mathbf{u}^*(\mathbf{x}^*)$.

Consider a general DE system

$$R^\sigma[\mathbf{u}] = \mathbf{R}^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N$$

with variables $\mathbf{x} = (x^1, ..., x^n)$, $\mathbf{u} = (u^1, ..., u^m)$.

Definition

A one-parameter Lie group of point transformations

$$\mathbf{x}^* = f(\mathbf{x}, \mathbf{u}; a) = \mathbf{x} + a\xi(\mathbf{x}, \mathbf{u}) + O(a^2),$$
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Example 1: translations

A translation

$$x^* = x + C, \quad t^* = t, \quad u^* = u \quad (C \in \mathbb{R})$$

leaves the KdV equation invariant:

$$u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^*x^*x^*}.$$

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Example 2: scalings

A scaling

$$x^* = \alpha x, \quad t^* = \alpha^3 t, \quad u^* = \alpha u \quad (\alpha \in \mathbb{R})$$

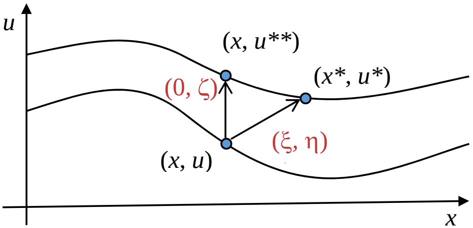
also leaves the KdV equation invariant:

$$u_t + uu_x + u_{xxx} = 0 = u^*_{t^*} + u^* u^*_{x^*} + u^*_{x^*x^*x^*}.$$

A symmetry (in 1D case)

$$x^* = f(x, u; a) = x + a\xi(x, u) + O(a^2),$$
$$u^* = g(x, u; a) = u + a\eta(x, u) + O(a^2).$$

maps a solution $u(x)$ into $u^*(x^*)$, changing both x and u.



In the evolutionary form, the same curve mapping does not change x:

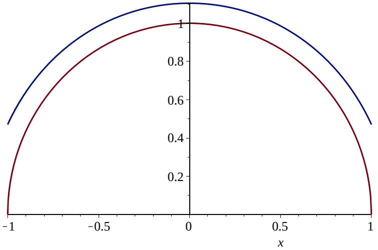
$$x^{**} = x, \quad u^{**} = u + a\,\zeta[u] + O(a^2),$$

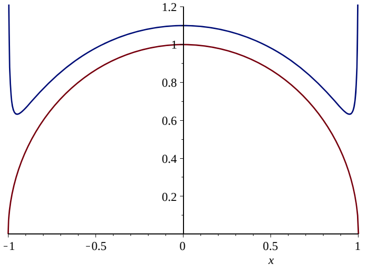
$$\boxed{\zeta[u] = \eta(x, u) - \frac{\partial u}{\partial x}\xi(x, u).}$$

Evolutionary Form of a Local Symmetry: Example

- Consider an ODE
$$y' = -\frac{x}{y} \quad \Leftrightarrow y^2 + x^2 = C = \text{const}.$$
- A scaling symmetry: $x^* = e^a x$, $y^* = e^a y$.
- Local form:
$$x^* = x + a\xi(x,y) + O(a^2), \quad y^* = y + a\eta(x,y) + O(a^2), \qquad \xi = x, \quad \eta = y.$$
- Evolutionary form: $\zeta[y] = \eta - y'(x)\,\xi = y + x^2/y$.
- Local transformation for the evolutionary form:
$$x^{**} = x,$$
$$u^{**} = u + a\left(y + \frac{x^2}{y}\right) + O(a^2).$$

- $a = 0.1$:





Consider a general DE system $\mathbf{R}^\sigma[\mathbf{u}] = 0$ that follows from a variational principle with

$$J[\mathbf{u}] = \int_\Omega \mathcal{L}[\mathbf{u}]\, dx$$

.

Definition

A local evolutionary symmetry of $\mathbf{R}^\sigma[\mathbf{u}] = 0$ is a **variational symmetry** if it preserves the action integral, or in other words, preserves $\mathcal{L}[\mathbf{u}]$ up to a divergence. [cf. *Olver (1993)*]

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Example 1: translations for the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad \mathcal{L} = \frac{1}{2}{u_t}^2 - \frac{c^2}{2}{u_x}^2.$$

The translation $x^* = x + C, \quad t^* = t, \quad u^* = u$ is a variational symmetry.

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Example 2: scaling for the wave equation

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Can show: the scaling $x^* = x, \quad t^* = t, \quad u^* = u/\alpha$ is not a variational symmetry.

Theorem

Given:

1. *a PDE system* $\mathbf{R}[\mathbf{u}] = 0$*, following from a variational principle;*
2. *a local variational symmetry in an evolutionary form:*

$$(x^i)^* = x^i, \quad (u^\sigma)^* = u^\sigma + a\,\zeta^\sigma[\mathbf{u}] + O(a^2).$$

Then the given DE system has a **local conservation law** $\mathrm{D}_i\Phi^i[\mathbf{u}] = 0$*.*
In particular,

$$\mathrm{D}_i\Phi^i[\mathbf{U}] = \Lambda_\sigma[\mathbf{U}]R^\sigma[\mathbf{U}],$$

where the multipliers are given by the evolutionary forms of symmetry components:

$$\Lambda_\sigma[\mathbf{U}] \equiv \zeta^\sigma[\mathbf{U}].$$

Example 1: time translation symmetry, harmonic oscillator

- **Equation:** $\ddot{x}(t) + \omega^2 x(t) = 0.$
- **Symmetry:**

$$\begin{aligned} t^* &= t + a, \quad & \xi &= 1; \\ x^* &= x, \quad & \eta &= 0, \end{aligned}$$

- **Multiplier** (integrating factor): $\Lambda = \eta - \dot{x}(t)\xi = -\dot{x}$;
- **Conservation law:**

$$\Lambda R = -\dot{x}(\ddot{x}(t) + \omega^2 x(t)) = -\frac{d}{dt}\left(\frac{\dot{x}^2(t)}{2} + \frac{\omega^2 x^2(t)}{2}\right) = 0.$$

Noether's Theorem: Examples

Example 2

- **Equation:** Wave equation $u_{tt} = c^2 u_{xx}, \quad u = u(x,t)$.
- **Space translation symmetry:**

$$\begin{aligned} t^* &= t, & \xi^t &= 0; \\ x^* &= x, & \xi^x &= 0, \\ u^* &= u + a, & \eta &= 1, \end{aligned}$$

- **Multiplier**: $\Lambda = \zeta = \eta - 0 \cdot u_x - 0 \cdot u_t = 1$;
- **Conservation law (Momentum):**

$$\Lambda R = 1(u_{tt} - c^2 u_{xx}) = D_t(u_t) - D_x\left(c^2 u_x\right) = 0.$$

Example 2

- **Equation:** Wave equation $u_{tt} = c^2 u_{xx}, \quad u = u(x,t)$.
- **Time translation symmetry:**

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- **Conservation law (Energy):**

$$\Lambda R = -u_t(u_{tt} - c^2 u_{xx}) = -\left[D_t \left(\frac{u_t^2}{2} + c^2 \frac{u_x^2}{2} \right) - D_x \left(c^2 u_t u_x \right) \right] = 0.$$

For a non-variational DE system $\mathbf{R}[\mathbf{u}] = 0$ of N PDEs:

- Local evolutionary symmetry components $\{\zeta^\sigma[\mathbf{u}]\}$ are solutions of the linearized system
$$L^\sigma[\mathbf{u}]\,\boldsymbol{\zeta}[\mathbf{u}]\Big|_{\mathbf{R}[\mathbf{u}]=0} = 0, \quad \sigma = 1, \ldots, m.$$
- Conservation law multipliers $\{\Lambda_\sigma[\mathbf{u}]\}$ are a subset of solutions of the adjoint linearized system:
$$L^*_\mu[\mathbf{u}]\,\boldsymbol{\Lambda}[\mathbf{u}]\Big|_{\mathbf{R}[\mathbf{u}]=0} = 0, \quad \mu = 1, \ldots, N.$$
- Classification examples show differences in symmetry and CL structure. [See, e.g., *Bluman and Temuerchaolu (2005)*.]
- Symmetries can be used to map local conservation laws into local conservation laws (new or known). [E.g., *Bluman, C., Anco (2010)* and refs therein.]
- In symmetric settings (planar, axial,...), physical systems often have extra conservation laws.

Outline

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- Generally, CLs can be obtained systematically through the Direct construction method:
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  - Noether's theorem for variational systems;
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- Noether's theorem is not a preferred way to derive unknown CLs.

Discussion

Some related topics not addressed in this talk:

- Trivial and equivalent CL multipliers [cf. *Olver (1993)*].
- Material CLs.
- Nonlocal CLs.
- Abnormal PDE systems; Noether's 2nd theorem.
- Upper bounds of CL order.
- Recursion operators.

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Next talk:

- Conservation law computations for fluid dynamics models.

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Outline

Peakon $b-$family:

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- Maple example: