New analytical solutions of static and dynamic plasma equilibrium equations in axial and helical symmetry

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Symmetry 2022 conference

December 14, 2022

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Collaborators

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Outline

1. MHD equations
2. Exact solutions with axial symmetry
3. Exact solutions with helical symmetry
4. Dynamic solutions
5. Discussion

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MHD equations

Ideal incompressible magnetohydrodynamics equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0,$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \rho \mathbf{V} \times \operatorname{curl} \mathbf{V} + \frac{1}{\mu} \operatorname{curl} \mathbf{B} \times \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2},$$

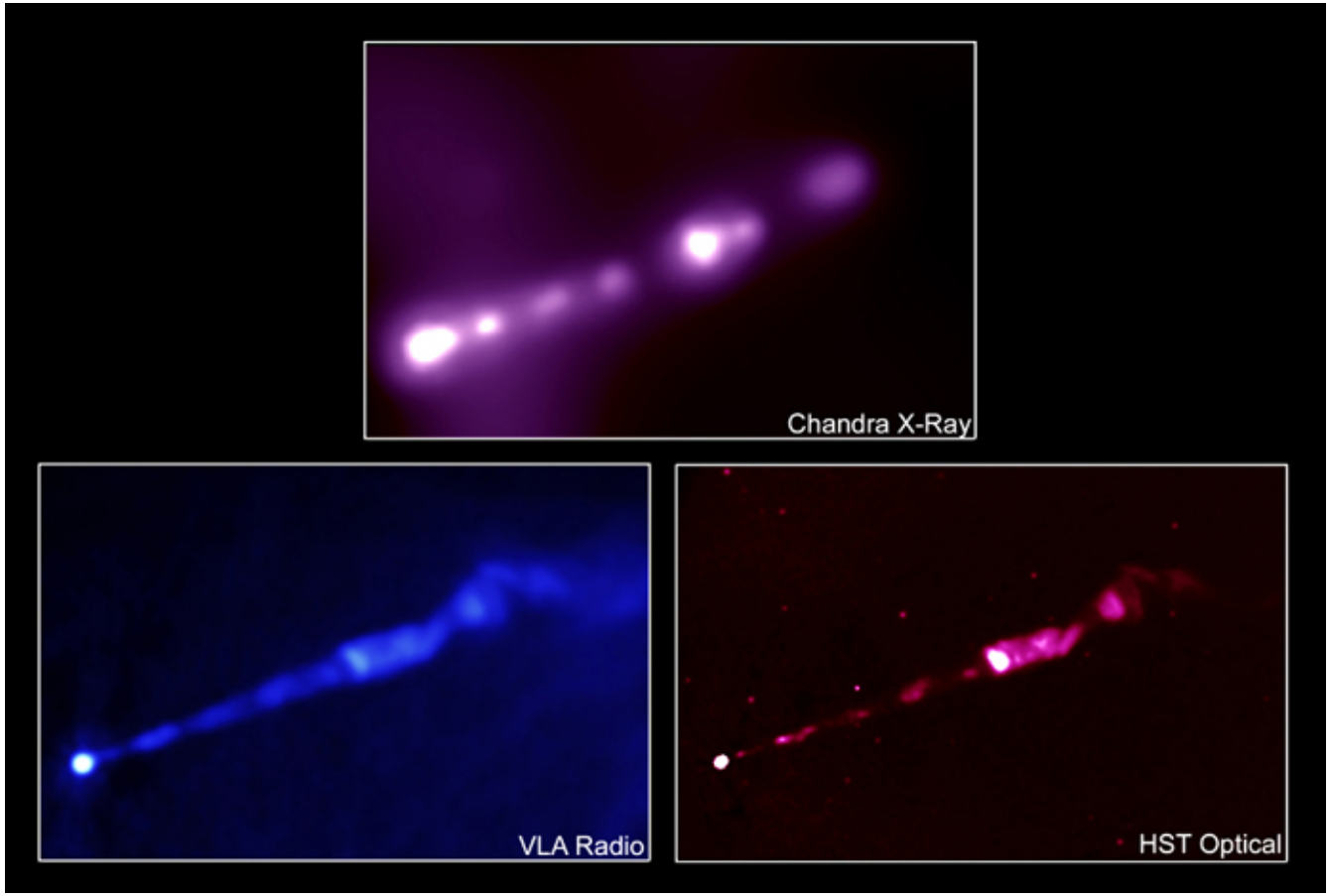
$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}),$$

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{div} \mathbf{V} = 0$$

Notation

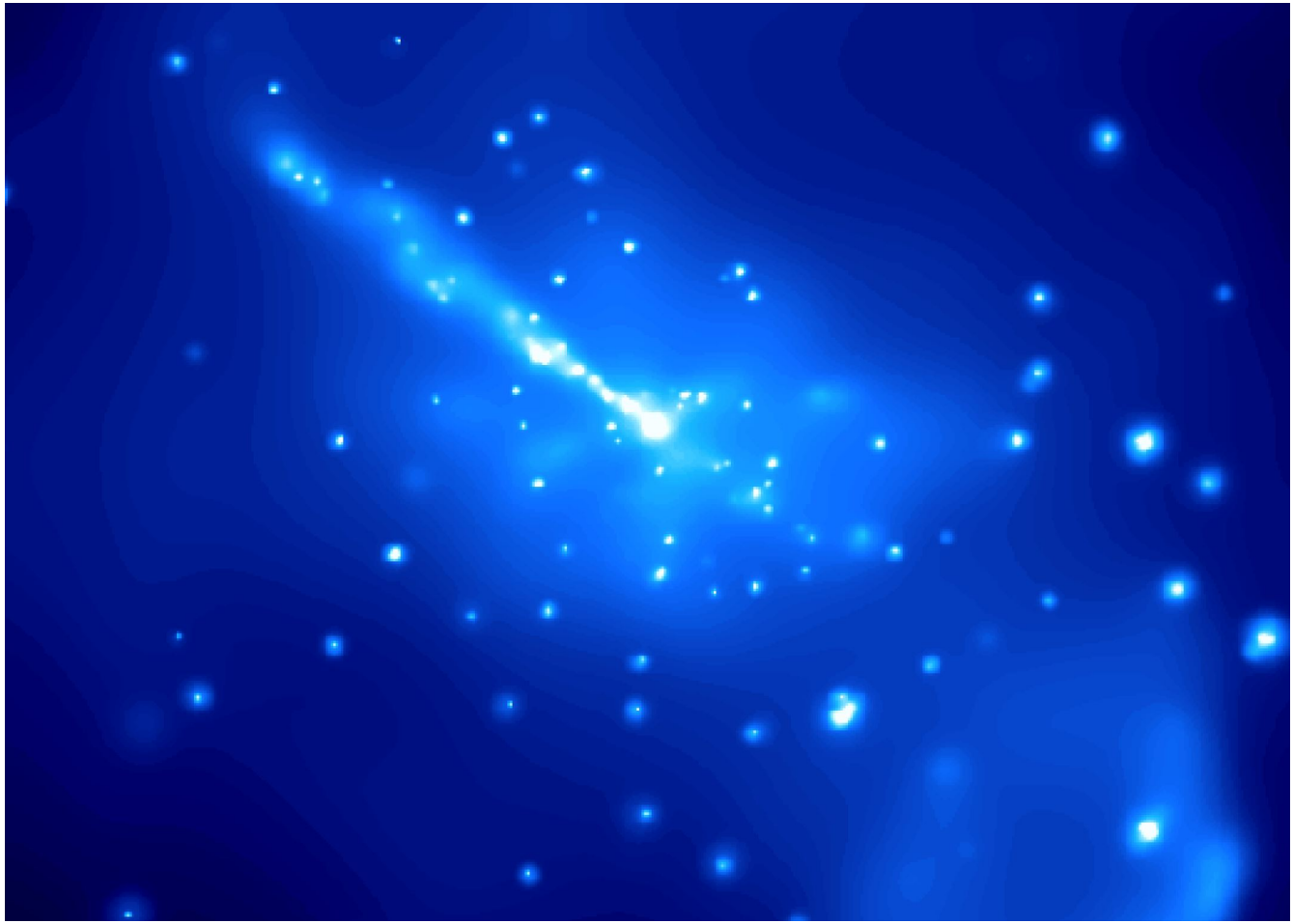
1. ρ plasma density
2. $\mathbf{B}$ magnetic induction vector
3. $\mathbf{V}$ plasma velocity vector
4. P scalar pressure
5. μ magnetic permeability coefficient

Applications



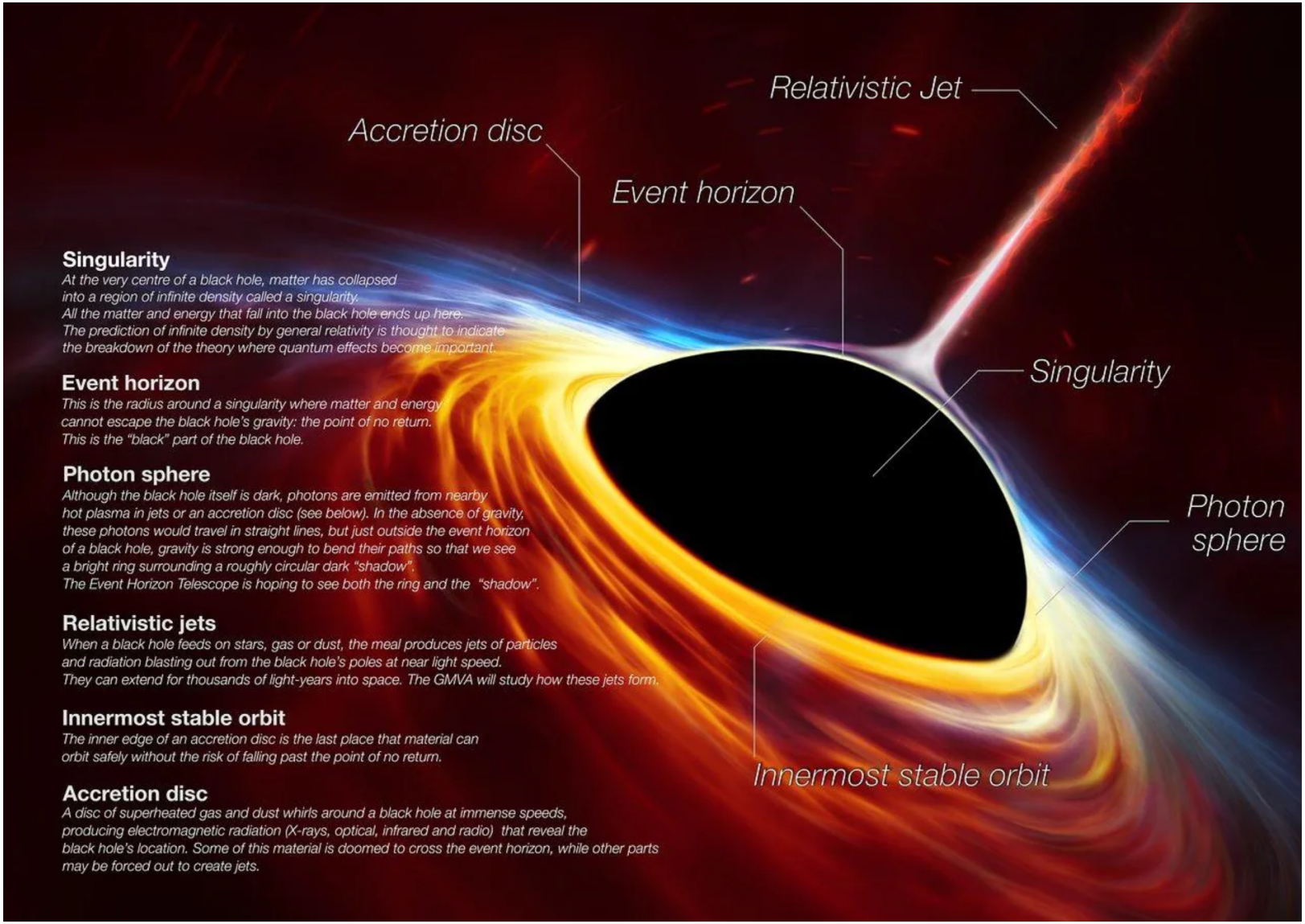
- Messier 87 supergiant elliptical galaxy jet, $\sim 5 \times 10^3$ light years

Applications

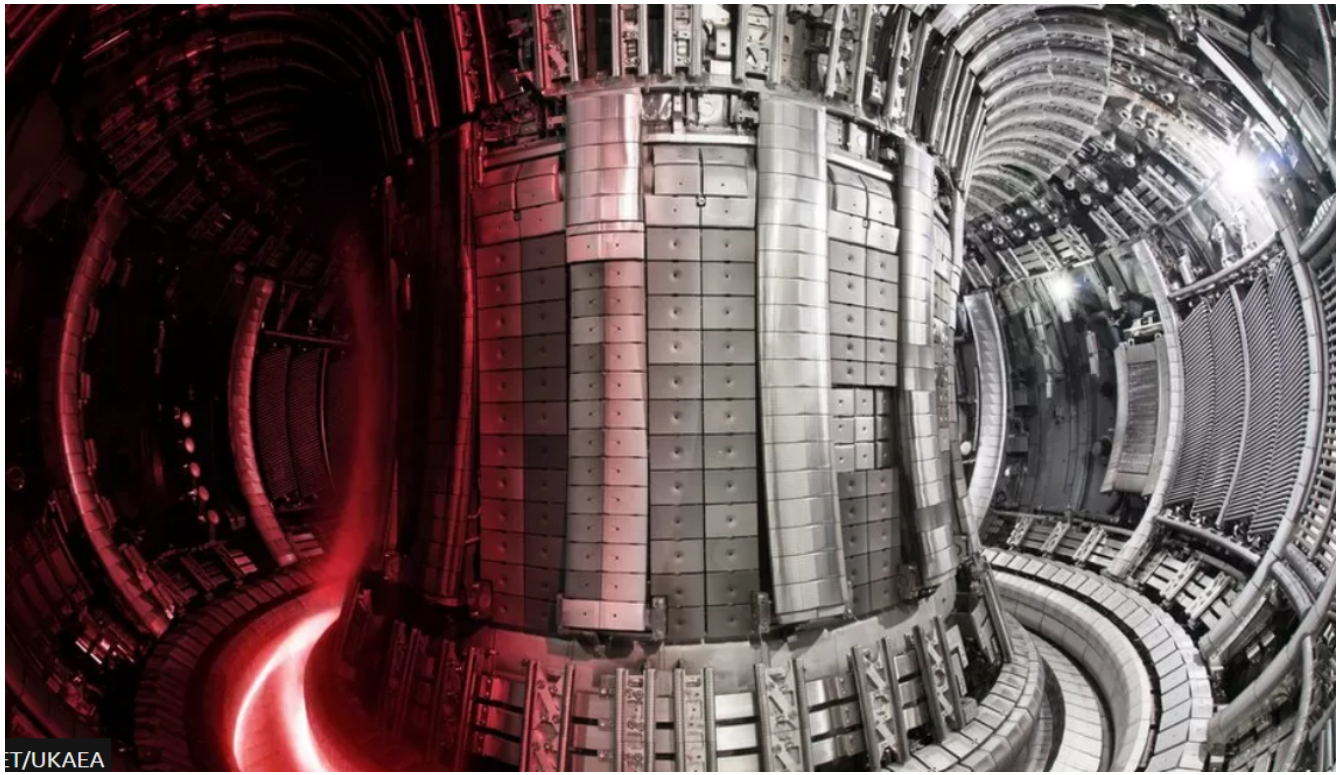


- Centaurus A: a relativistic jet from a supermassive black hole, $\sim 10^6$ light years

Applications



Applications



- Plasma discharge in the UK JET reactor

MHD equilibrium systems

Dynamic MHD equilibria in 3D

$$\operatorname{div} \rho \mathbf{V} = 0,$$

$$\rho \mathbf{V} \times \operatorname{curl} \mathbf{V} + \frac{1}{\mu} \operatorname{curl} \mathbf{B} \times \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} = 0,$$

$$\operatorname{curl} (\mathbf{V} \times \mathbf{B}) = 0,$$

$$\operatorname{div} \mathbf{B} = \operatorname{div} \mathbf{V} = 0$$

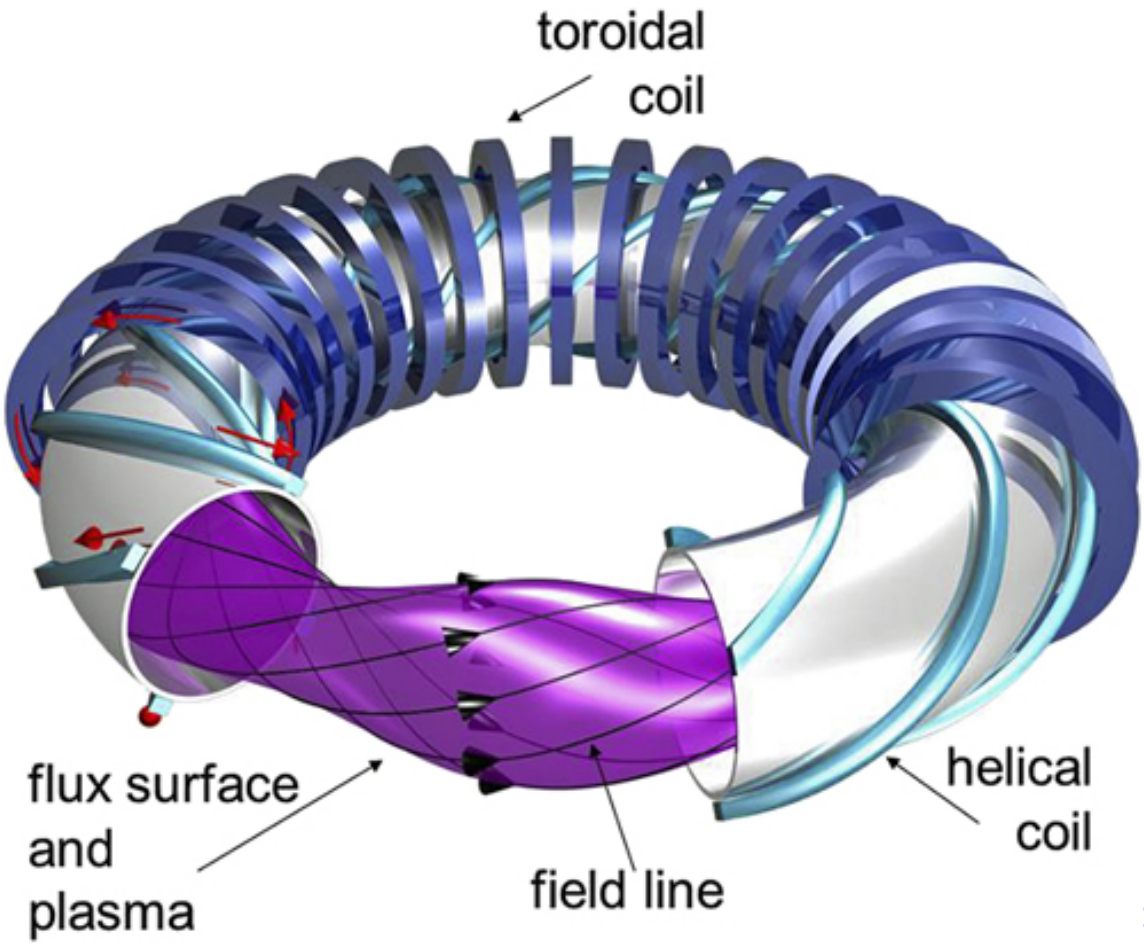
Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

MHD equilibrium topology

- Magnetic field lines/ streamlines go to infinity
- Magnetic field lines/ streamlines are tangent to *magnetic flux surfaces*
  - topological tori
  - constant pressure or flux function



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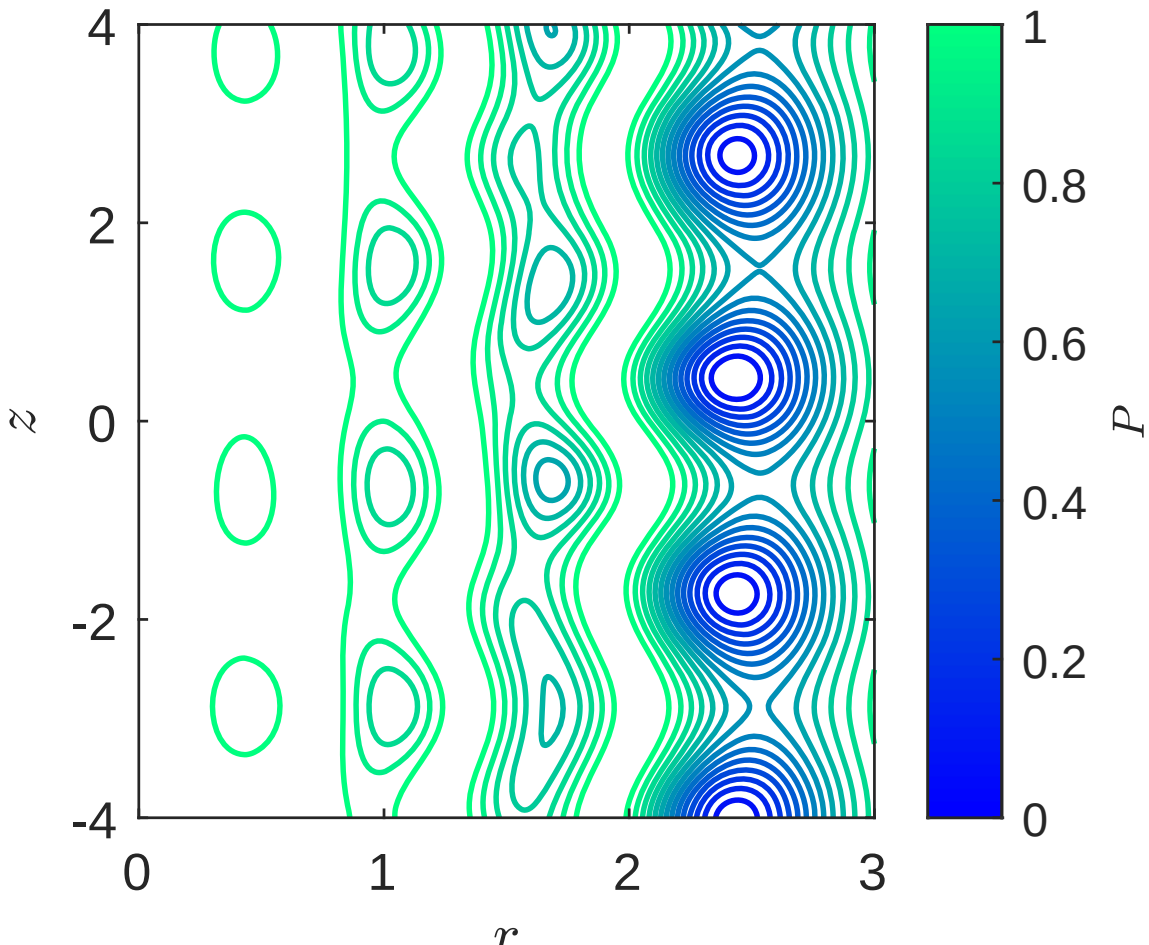


Physical requirements

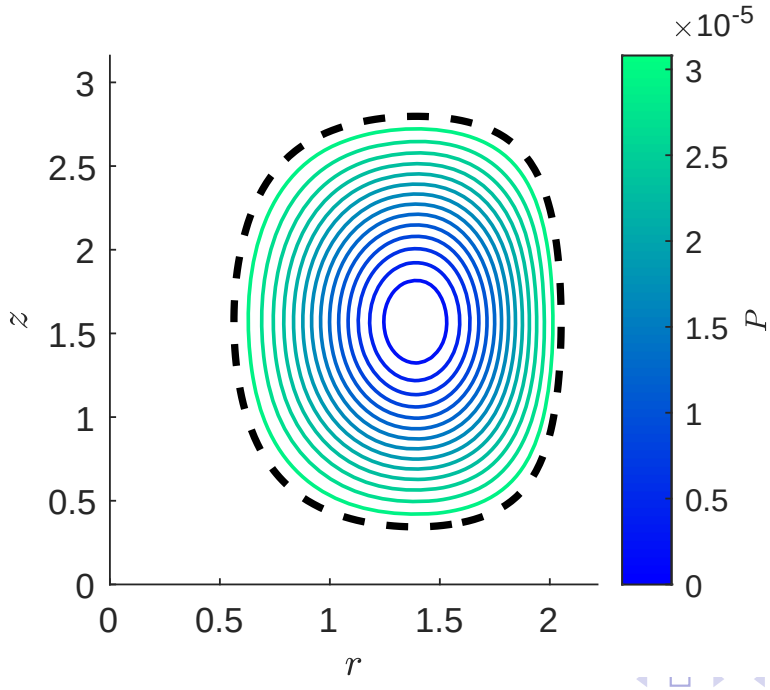
- Bounded kinetic and magnetic energy: total in $\mathcal{V}$, or per-layer

$$\int_{\mathcal{V}} |\mathbf{V}(\mathbf{x})|^2 \, dV \,, \qquad \int_{\mathcal{V}} |\mathbf{B}(\mathbf{x})|^2 \, dV \;\; < \;\; +\infty$$

- Pressure asymptotics: vacuum or ambient medium

$$P \to P_0 = \text{const} \;\; \text{as} \;\; |\mathbf{x}| \to +\infty \;\; \text{or on} \;\; \partial\mathcal{V}$$

- Current sheet for a "truncated" configuration: $\mathbf{K} = \dfrac{\mathbf{B}}{\mu} \times \mathbf{n}$ on $\partial\mathcal{V}$



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2. Exact solutions with axial symmetry
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Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Rotational symmetry $X = \partial/\partial\varphi$

- Magnetic field and pressure:

$$\mathbf{B} = B^r(r,z)\mathbf{e}_r + B^\varphi(r,z)\mathbf{e}_\varphi + B^z(r,z)\mathbf{e}_z, \qquad P = P(r,z)$$

Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Reduction to one PDE on one dependent variable (magnetic flux function):

$$\mathbf{B} = \frac{\psi_z}{r}\mathbf{e}_r + \frac{I(\psi)}{r}\mathbf{e}_\phi - \frac{\psi_r}{r}\mathbf{e}_z, \qquad P = P(\psi)$$

- Grad-Shafranov equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi)$$

- $I(\psi)$, $P(\psi)$ "free"
- Linear cases ($P(\psi)$ quadratic, $I(\psi)$ linear) have been considered by Bogoyavlenskij, Kaiser & Lortz, and others; interesting solutions were obtained

Linear homogeneous G-S

- Grad-Shafranov equation

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + I(\psi)I'(\psi) = -r^2 P'(\psi)$$

- Linear ansatz:

$$P(\psi) = P_0 + b\psi + \frac{1}{2}a\psi^2, \qquad I(\psi) = \alpha\psi$$

- Linear homogeneous case: $b = 0$,

$$\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} + \frac{\partial^2\psi}{\partial z^2} + (\alpha^2 + ar^2)\psi = 0$$

- Separation of variables:

$$Z'' = \lambda Z, \qquad R'' - \frac{1}{r}R' + (\alpha^2 + ar^2 + \lambda)R = 0$$

- Z-solutions for stretched configurations:

$$Z = C_3 \sin(kz) + C_4 \cos(kz)$$

The first family of axially symmetric solutions

- R-ODE:

$$R'' - \frac{1}{r}R' + (\alpha^2 + ar^2 + \lambda)R = 0$$

- Substitution $s = qr^2$, $R(r) = S(s)$ gives

$$S'' + \left(-\frac{1}{4} + \frac{\alpha^2 - k^2}{4qs}\right) S = 0,$$

which is related to the Whittaker ODE

$$y''(s) + \left(-\frac{1}{4} + \frac{\delta}{s} + \frac{1/4 - \nu^2}{s^2}\right) y(s) = 0$$

when $\delta = \alpha^2 - k^2/4q$, $\nu = 1/2$.

The first family of axially symmetric solutions

- Separated solutions of the GS:

$$\psi_k(r,z) = \left(C_1 W_M\left(\delta, \frac{1}{2}, qr^2\right) + C_2 W_W\left(\delta, \frac{1}{2}, qr^2\right)\right)(C_3 \sin kz + C_4 \cos kz)$$

- Globally regular if and only if $\delta \in \mathbb{N}$; regularity is not necessary under proper BCs

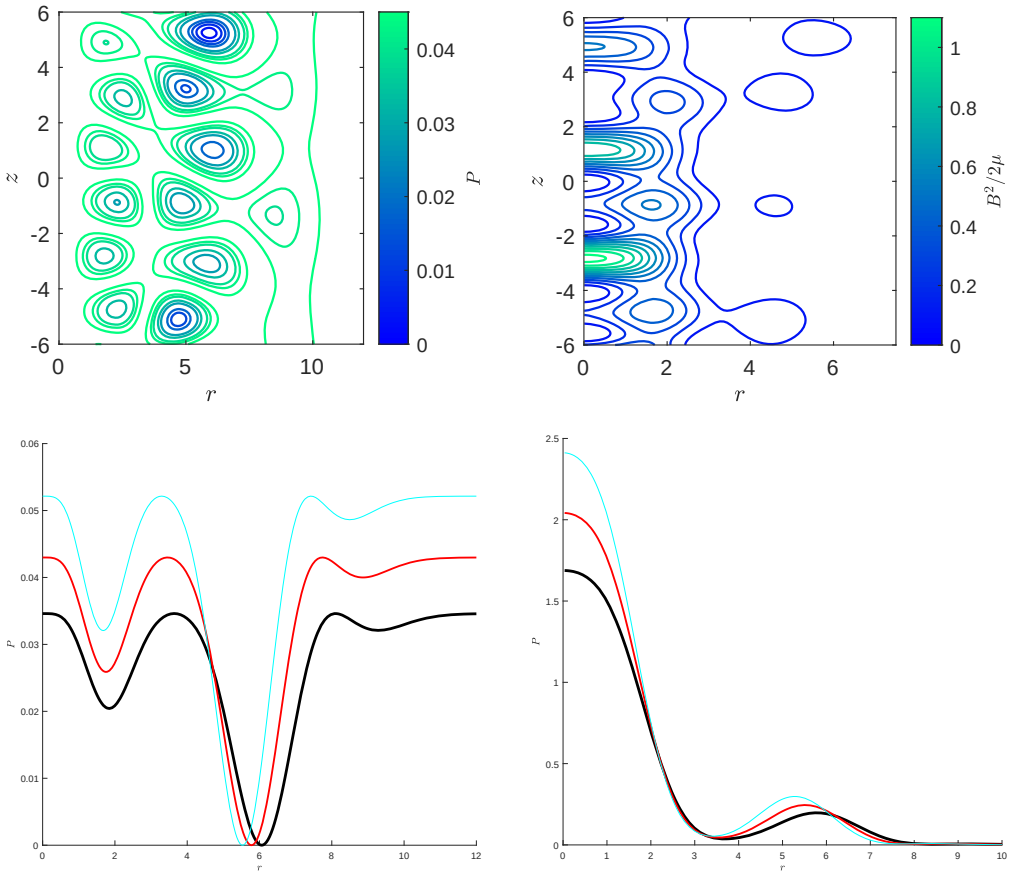
- A general linear combination: $\Psi(r,z) = \int_{-\infty}^{\infty} \psi_k(r,z)\, dk$

The first family of axially symmetric solutions

- **Example** [Bogoyavlenskij solutions]: pressure and magnetic energy

$$\psi(r,z) = e^{-\beta r^2}\left(a_N L_N^*(2\beta r^2) + \sum_{n=1}^{N-1} a_n \sin(\omega_n z + b_n) L_n^*(2\beta r^2)\right)$$

- Externally supported

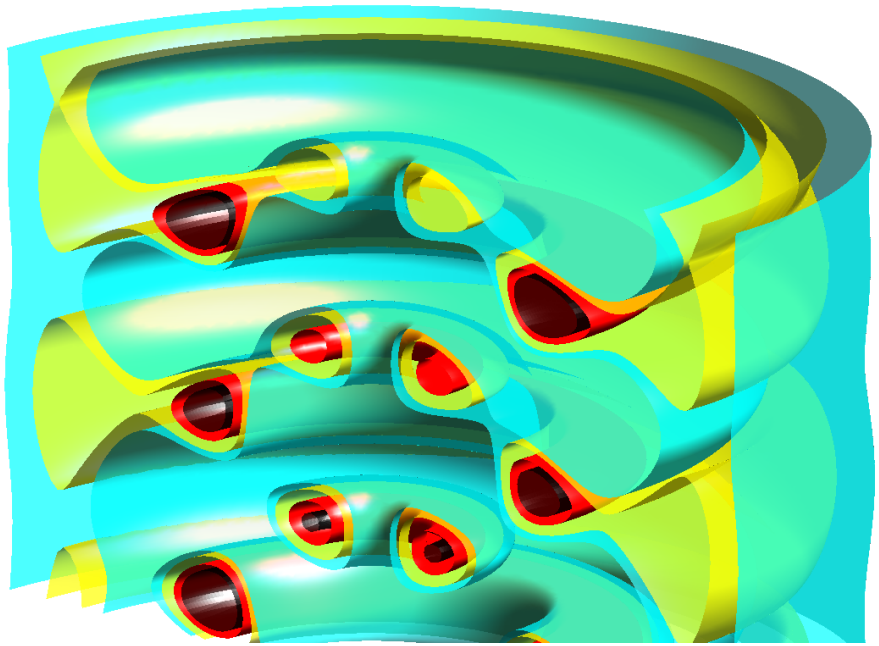


The first family of axially symmetric solutions

- **Example** [Bogoyavlenskij solutions]: pressure and magnetic energy

$$\psi(r,z) = e^{-\beta r^2}\left(a_N L_N^*(2\beta r^2) + \sum_{n=1}^{N-1} a_n \sin(\omega_n z + b_n) L_n^*(2\beta r^2)\right)$$

- Magnetic surfaces:

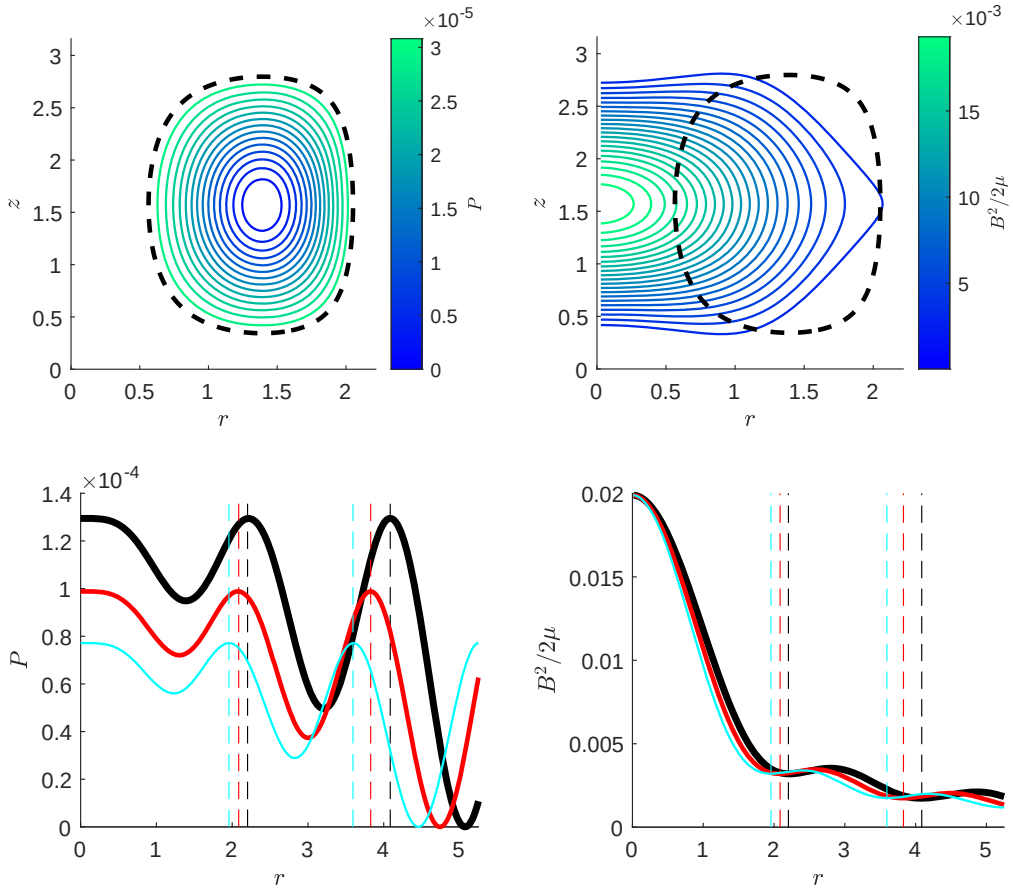


The first family of axially symmetric solutions

- **Example 2**: Whittaker functions

$$\psi_k(r,z) = \left(C_1 W_M\left(\delta, \frac{1}{2}, qr^2\right) + C_2 W_W\left(\delta, \frac{1}{2}, qr^2\right)\right)(C_3 \sin kz + C_4 \cos kz)$$

- Also externally supported



The second family of axially symmetric solutions

- **Second axial family**
- Pressure: $P(\psi) = P_0 + \frac{1}{2} q^2 \psi^2$, positive inside $\mathcal{V}$, zero outside $\mathcal{V}$
- R-ODE:

$$R'' + \left(-\frac{1}{4} + \frac{k^2 - \alpha^2}{4qx} i \right) R = 0$$

- Whittaker functions of complex index $\rightarrow$ Coulomb ODE:

$$y''(s) + \left(1 - \frac{2\sigma}{s} - \frac{L(L+1)}{s^2} \right) y(s) = 0$$

- Radial solution component:

$$R(r) = C_1 \mathcal{C}_F \left(0, -\delta, \frac{q}{2} r^2 \right) + C_2 \mathcal{C}_G \left(0, -\delta, \frac{q}{2} r^2 \right)$$

in terms of Coulomb special functions.

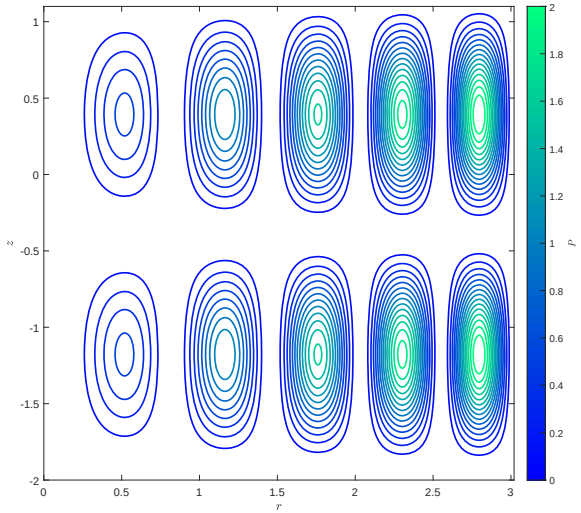
The second family of axially symmetric solutions

- Separated solutions of the GS:

$$\psi_k(r,z) = \left(C_1\, \mathcal{C}_F\left(0, -\delta, \frac{q}{2} r^2\right) + C_2\, \mathcal{C}_G\left(0, -\delta, \frac{q}{2} r^2\right)\right) (C_3 \sin kz + C_4 \cos kz)$$

- A general linear combination: $\Psi(r,z) = \int_{-\infty}^{\infty} \psi_k(r,z)\, dk$

- **Example:** magnetic surfaces



The second family of axially symmetric solutions

- Separated solutions of the GS:

$$\psi_k(r,z) = \left(C_1\, \mathcal{C}_F\left(0, -\delta, \frac{q}{2} r^2\right) + C_2\, \mathcal{C}_G\left(0, -\delta, \frac{q}{2} r^2\right)\right) (C_3 \sin kz + C_4 \cos kz)$$

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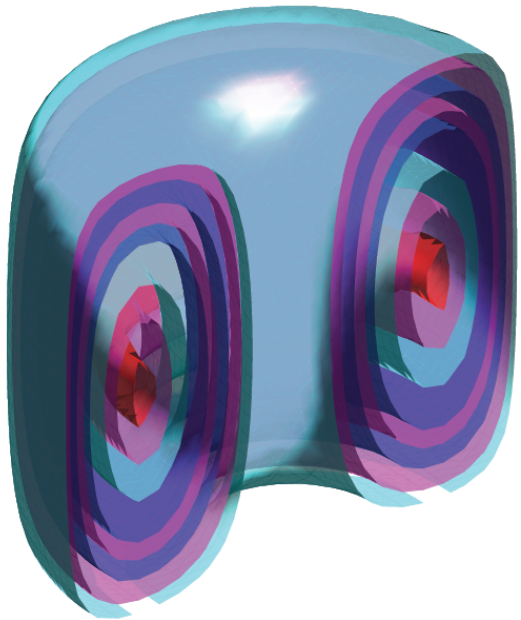
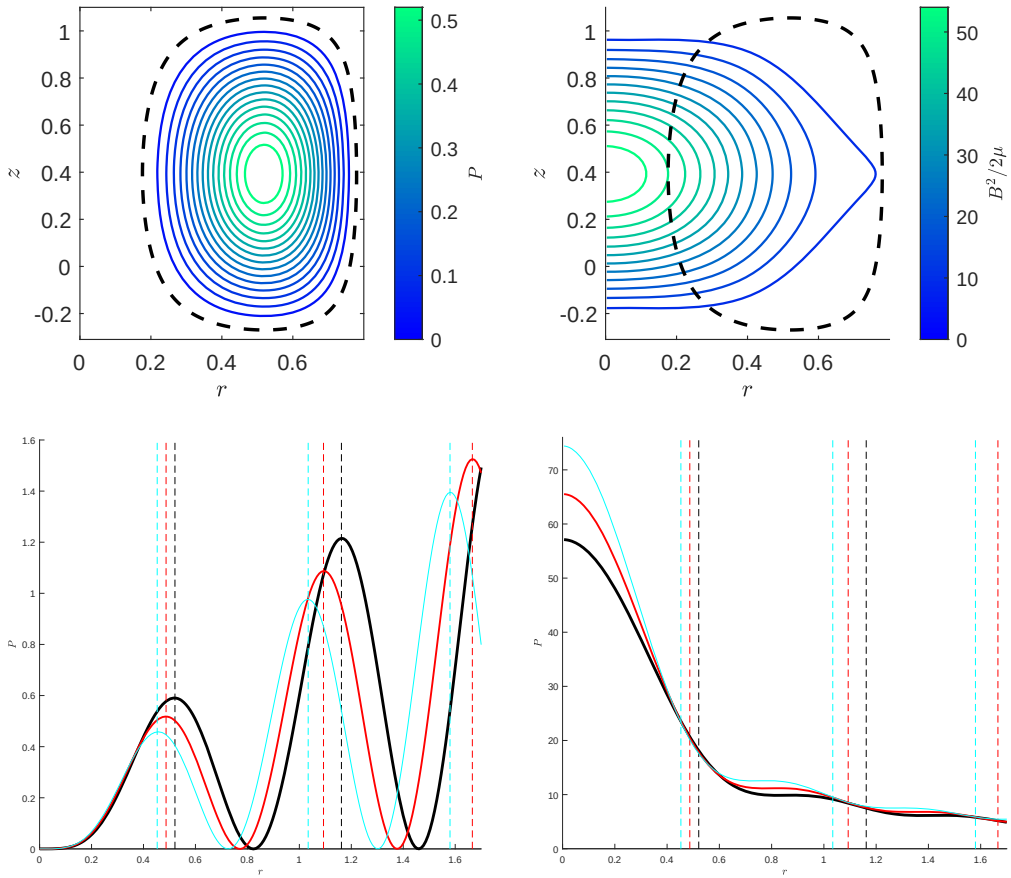


The second family of axially symmetric solutions

- Configuration with lower external pressure and a current sheet



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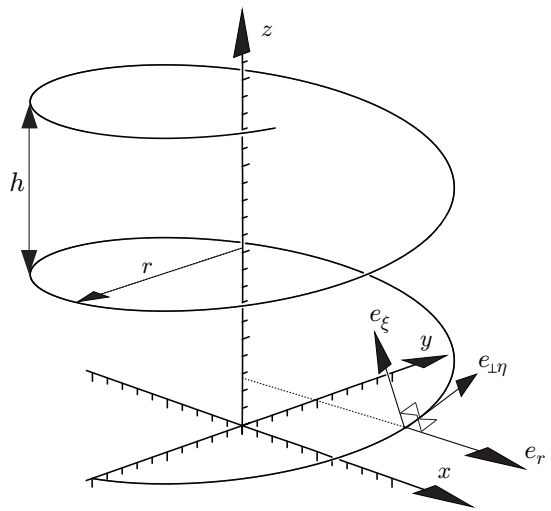
Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Rotational and translational symmetry $X_1 = \partial/\partial\varphi$, $X_2 = \partial/\partial z$
- Helical coordinates (r, η, ξ) in terms of cylindrical coordinates (r, φ, z):

$$r, \qquad \eta = \varphi + \gamma z/r^2, \qquad \xi = z - \gamma\varphi$$



Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Reduction to one PDE on one dependent variable (magnetic flux function):

$$\mathbf{B} = \frac{\psi_\xi}{r}\mathbf{e}_r + \frac{rI(\psi) + \gamma\psi_r}{r^2+\gamma^2}\mathbf{e}_\varphi + \frac{\gamma I(\psi) - r\psi_r}{r^2+\gamma^2}\mathbf{e}_z$$

- Johnson-Frieman-Kulsrud-Oberman (JFKO) equation

$$\frac{\psi_{\xi\xi}}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2+\gamma^2}\psi_r\right) + \frac{I(\psi)I'(\psi)}{r^2+\gamma^2} + \frac{2\gamma I(\psi)}{(r^2+\gamma^2)^2} = -\mu P'(\psi)$$

- $I(\psi)$, $P(\psi)$ "free"
- Consider again the linear homogeneous case: $P(\psi) = P_0 + \frac{1}{2}\sigma\psi^2$, $I(\psi) = \alpha\psi$

Static MHD equilibria

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P$$

- Linear JFKO:

$$\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2+\gamma^2}\frac{\partial \psi}{\partial r}\right) + \frac{\alpha^2 \psi}{r^2+\gamma^2} + \frac{2\gamma\alpha\psi}{(r^2+\gamma^2)^2} + 4\sigma\psi = 0$$

- Separated solutions $\psi(r,u) = R(r)\Xi(\xi)$

$$\Xi'' = \lambda\Xi, \quad \lambda = -\omega^2 < 0 \quad \rightarrow \quad \Xi(\xi) = C_3 \sin(\omega\xi) + C_4 \cos(\omega\xi)$$

$$r\left(\frac{r}{r^2+\gamma^2}R'\right)' + \left(\frac{\alpha^2 r^2}{r^2+\gamma^2} + \frac{2\gamma\alpha r^2}{(r^2+\gamma^2)^2} + 4ar^2\right)R = -\lambda R$$

The first family of helically symmetric solutions

- $\sigma = -\kappa^2 < 0$
- R-ODE:

$$r\Big(\frac{r}{r^2+\gamma^2}R'\Big)' + \Big(\frac{\alpha^2 r^2}{r^2+\gamma^2} + \frac{2\gamma\alpha r^2}{(r^2+\gamma^2)^2} + 4\kappa^2 r^2\Big)R = \omega^2 R.$$

- Solution:

$$R(r) = e^{-\kappa r^2}\Big(C_1 r^b \mathcal{H}_C(a,b,-2,c,d,-r^2/\gamma^2) + C_2 r^{-b}\mathcal{H}_C(a,-b,-2,c,d,-r^2/\gamma^2)\Big)$$

$$a = \kappa\gamma^2, \qquad b = \gamma\omega, \qquad c = \frac{\gamma^2(\gamma^2\kappa^2 - \alpha^2 + \omega^2)}{4},$$

$$d = 1 - \frac{\kappa^2\gamma^2}{4} + \frac{\alpha^2-\omega^2}{4}\gamma^2 + \frac{\alpha\gamma}{2},$$

where $\mathcal{H}_C$ satisfies the confluent Heun ODE

$$y'' - \frac{(-x^2 a + (-b+a)x + b + 1)}{x(x-1)}y' - \frac{((-ba-2c)x + (b+1)a + b - 2d + 2)}{2x(x-1)}y = 0$$

The first family of helically symmetric solutions

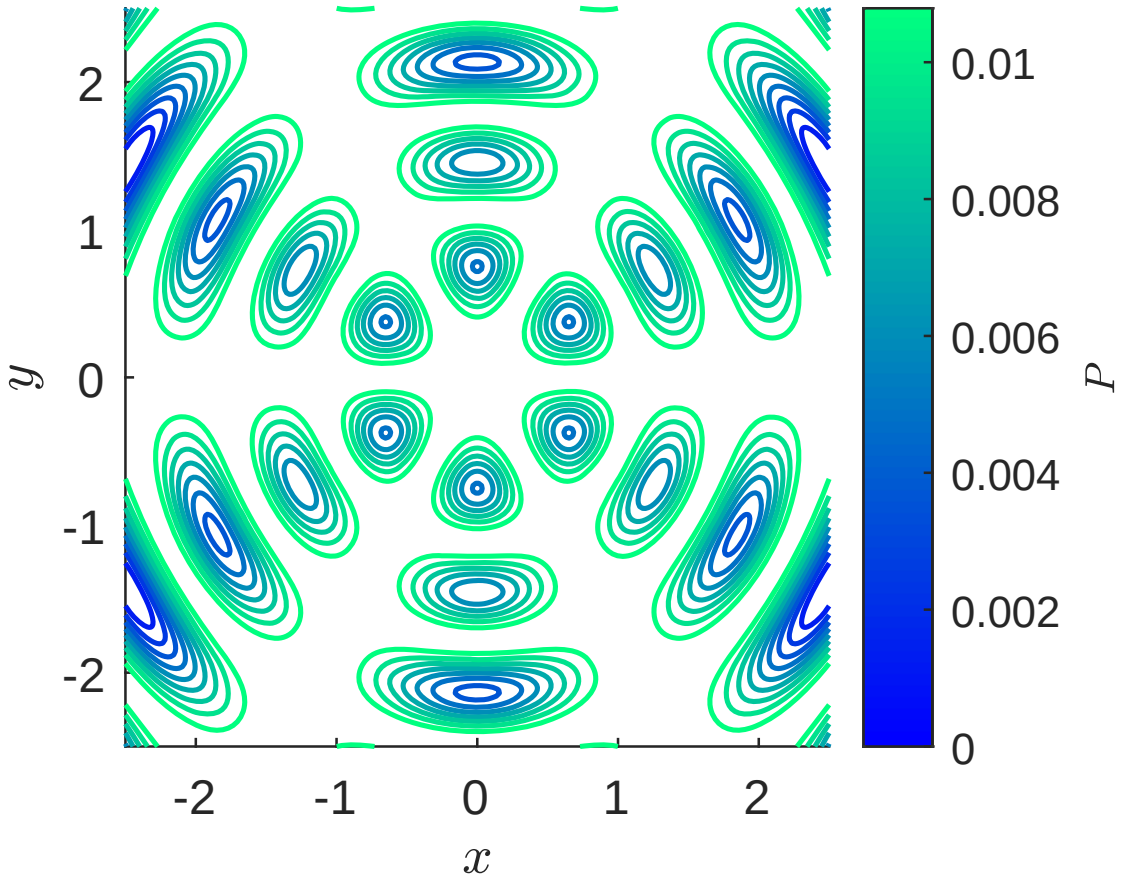
- Separated solutions of the JFKO:

$$\psi_\omega(r,\xi) = e^{-\kappa r^2} r^b \mathcal{H}_C(a,b,-2,c,d,-r^2/\gamma^2)(C_3\sin(\omega\xi) + C_4\cos(\omega\xi)).$$

- A general linear combination: $\psi(r,\xi) = \int_{-\infty}^{\infty} \psi_\omega(r,\xi)\, d\omega$

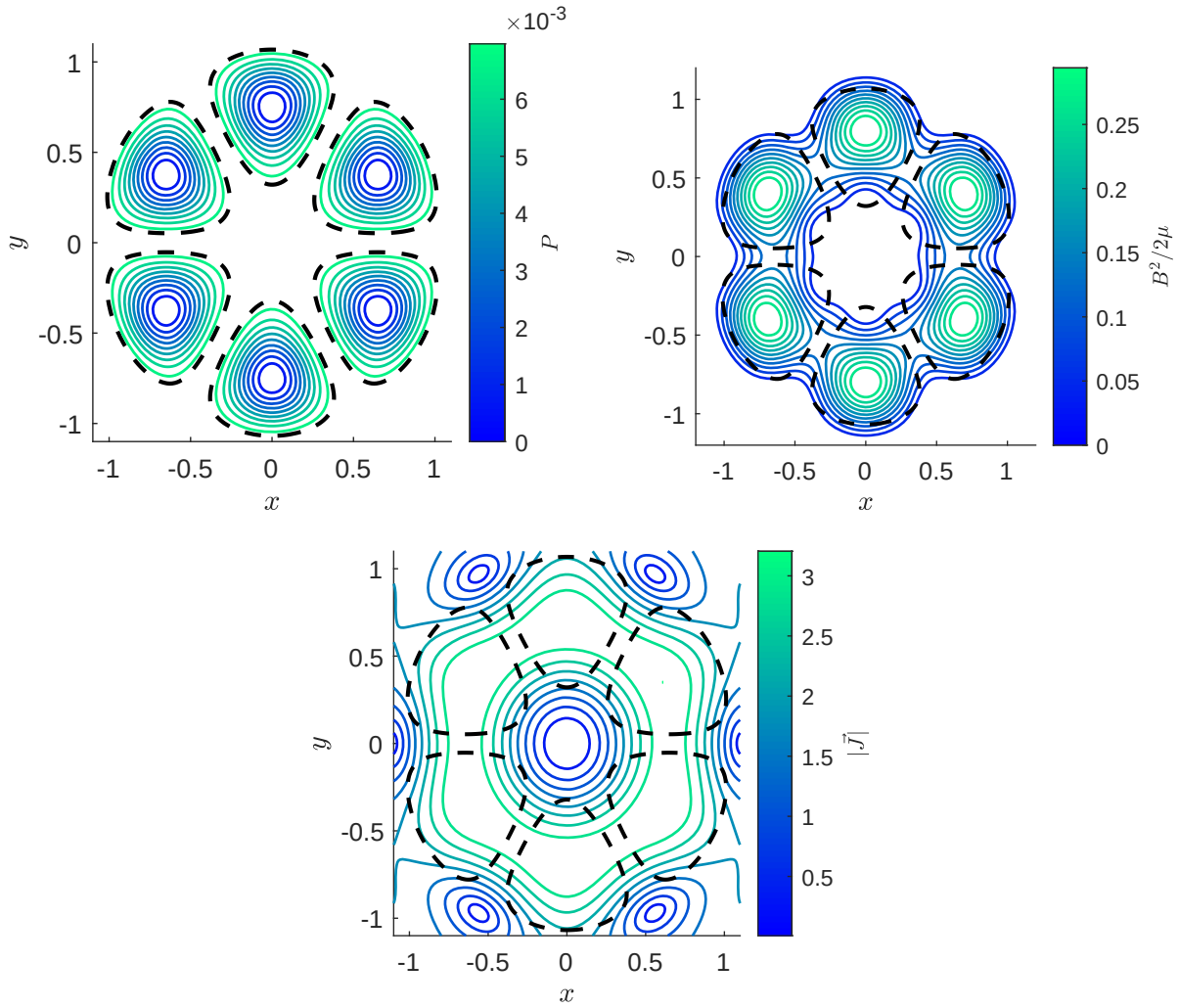
The first family of helically symmetric solutions

- **First helical family: an example**
- Magnetic surfaces: global



The first family of helically symmetric solutions

- **First helical family: an example**
- Magnetic surfaces (local/bounded), magnetic energy, current density



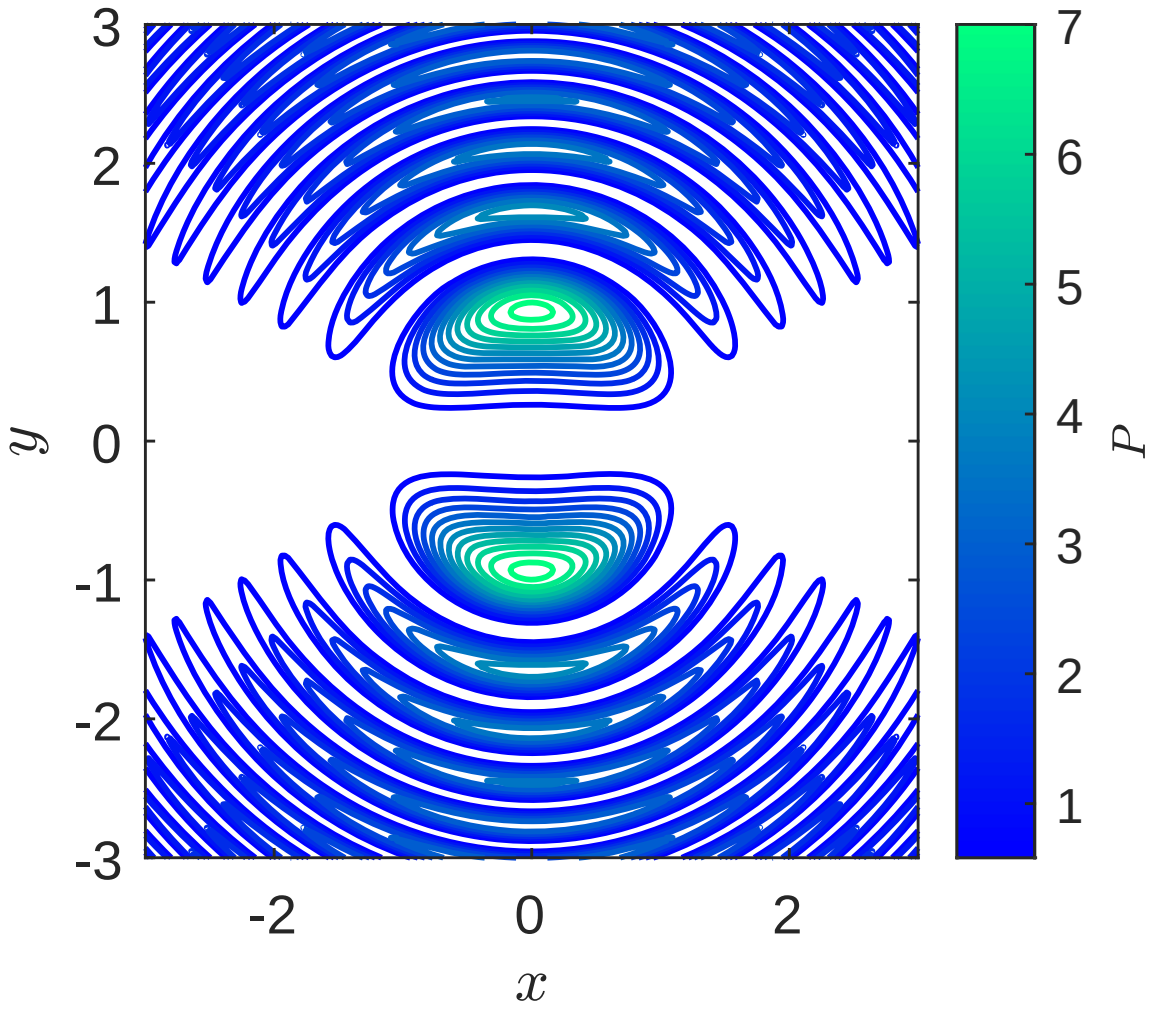
The second family of helically symmetric solutions

- $\sigma = \kappa^2 > 0$
- Real-valued separated solutions:

$$\psi_\omega(r,\xi) = e^{-i\kappa r^2} r^b \mathcal{H}_C(ia, b, -2, c, d, -r^2/\gamma^2) \left(C_1 \sin(\omega\xi) + C_2 \cos(\omega\xi)\right)$$

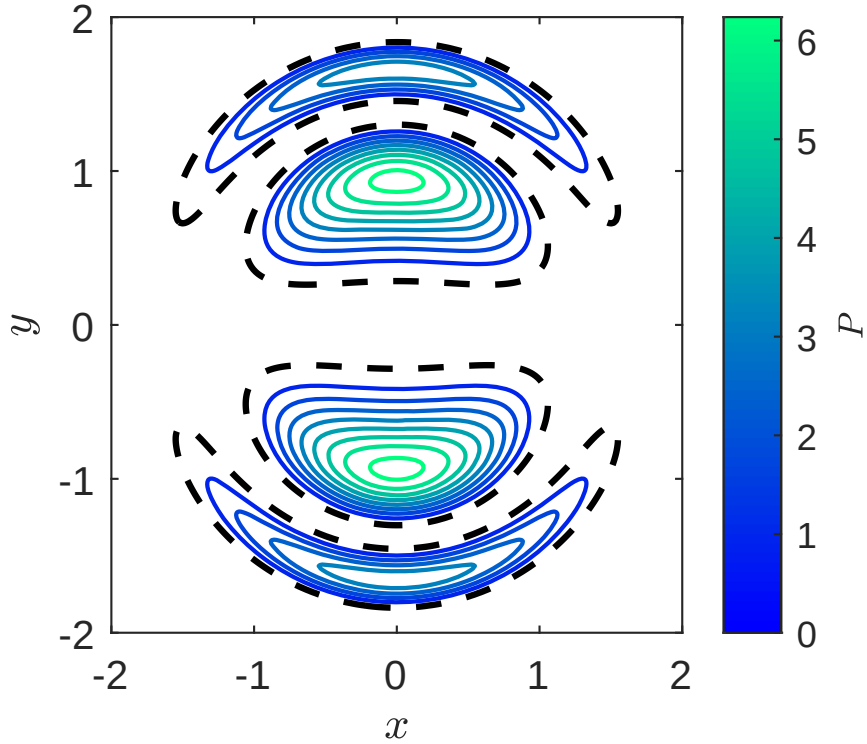
The second family of helically symmetric solutions

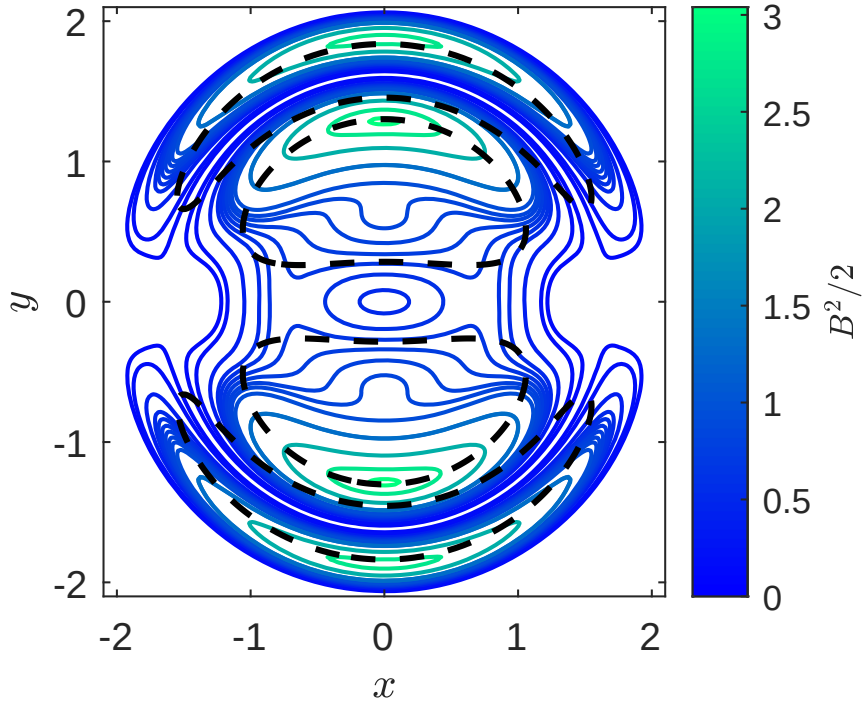
- **Second helical family: an example**
- Magnetic surfaces: global



The second family of helically symmetric solutions

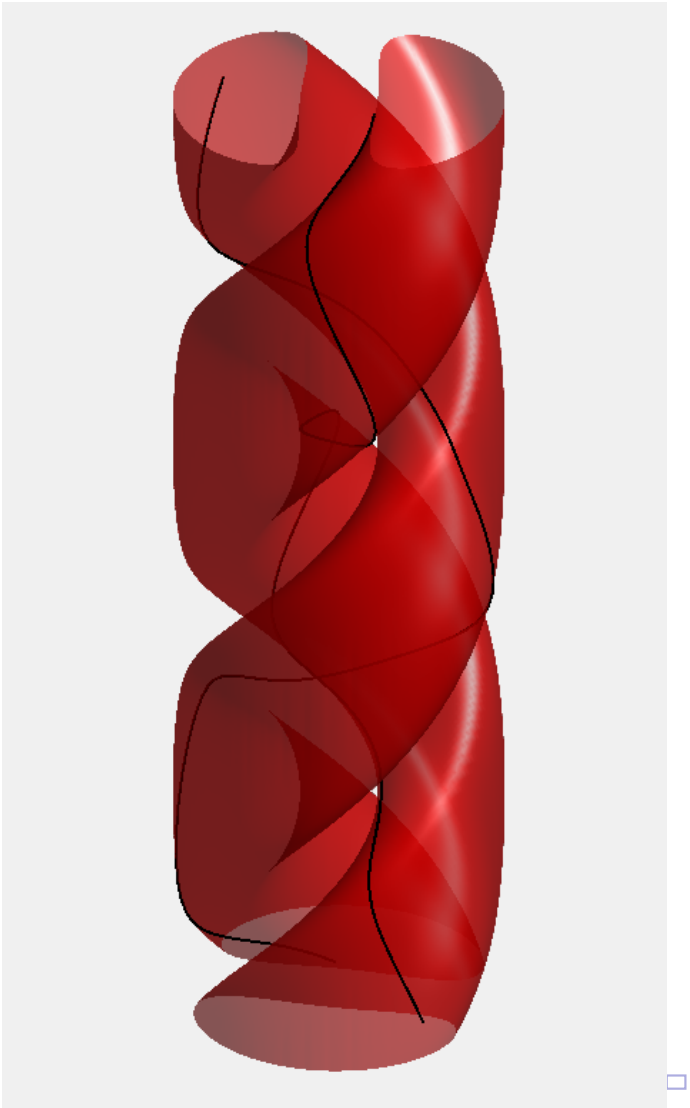
- **Second helical family: an example**
- Magnetic surfaces (bounded), magnetic energy





The second family of helically symmetric solutions

- **Second helical family: an example**
- A magnetic surface and tangent magnetic field lines:



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Dynamic solutions

- Galas-Bogoyavlenskij transformations (potential symmetries of equilibrium MHD):

$$\mathbf{B}_1 = b(\psi)\mathbf{B} + c(\psi)\sqrt{\mu\rho}\mathbf{V},$$

$$\mathbf{V}_1 = \frac{c(\psi)}{a(\psi)\sqrt{\mu\rho}}\mathbf{B} + \frac{b(\psi)}{a(\psi)}\mathbf{V},$$

$$P_1 = CP + \frac{C\mathbf{B}^2 - \mathbf{B}_1^2}{2\mu},$$

$$\rho_1 = a^2(\psi)\rho$$

- $a = a(\psi)$, $b = (\psi)$, $c = c(\psi)$ are constant on both magnetic fields lines and streamlines, and

$$b^2(\psi) - c^2(\psi) = C = \text{const}.$$

Dynamic solutions

- Starting from a static equilibrium, get field-aligned solutions

$$\mathbf{B}_1 = \sqrt{C + c^2(\psi)}\mathbf{B},$$

$$\mathbf{V}_1 = \frac{c(\psi)}{a(\psi)\sqrt{\mu\rho}}\mathbf{B},$$

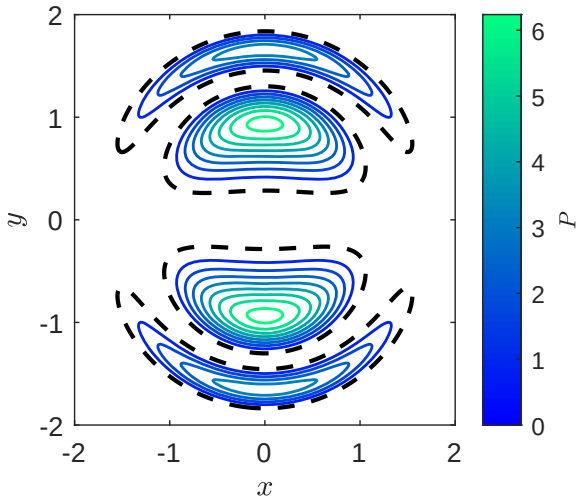
$$P_1 = CP + -\frac{c^2(\psi)\mathbf{B}^2}{2\mu},$$

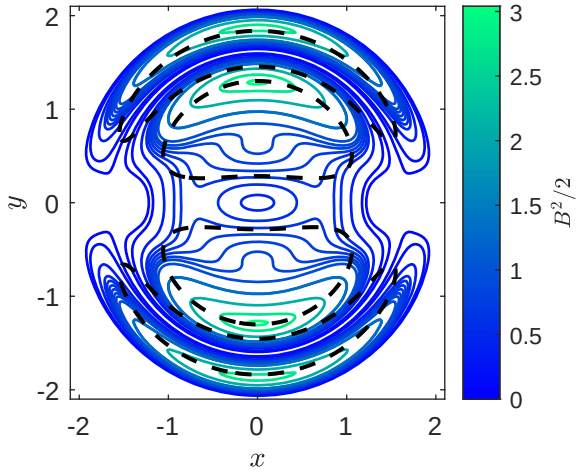
$$\rho_1 = a^2(\psi)\rho$$

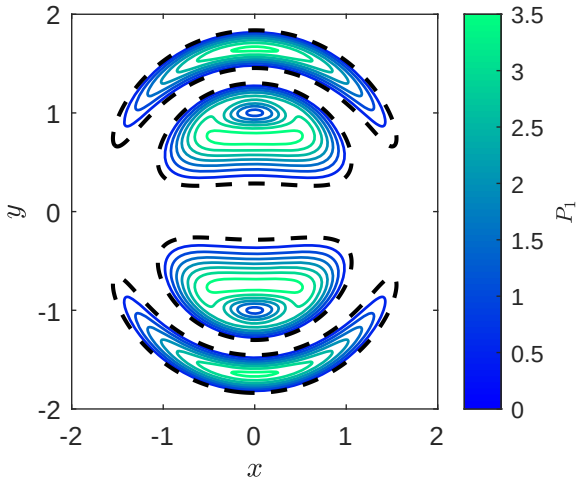
- Can apply to all of the above solutions!

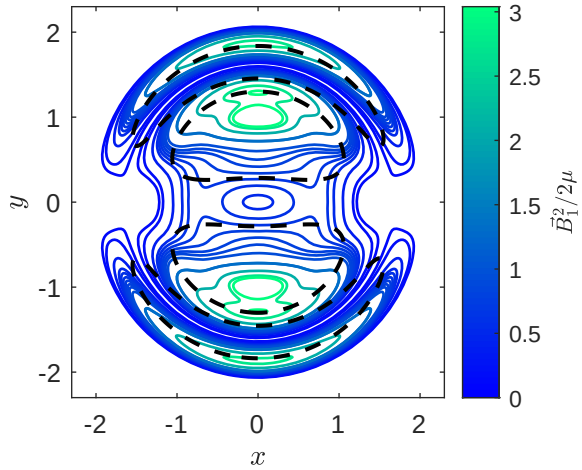
Dynamic solutions

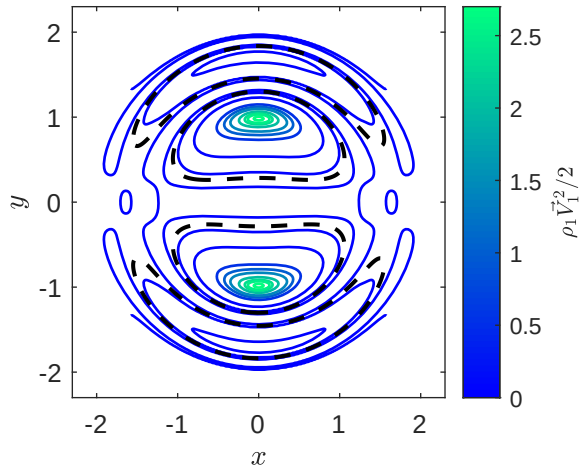
- **Second helical family: an example**
- Before and after: magnetic surfaces (bounded), magnetic energy, kinetic energy











Outline

1. MHD equations
2. Exact solutions with axial symmetry
3. Exact solutions with helical symmetry
4. Dynamic solutions
5. Discussion

Discussion

- New closed-form exact solutions in terms of special functions, in axial and helical symmetry, modeling prolonged static and dynamic plasma equilibria
- Field lines are not axially (or helically) symmetric
- Linear combinations of basic solutions $\rightarrow$ no z-periodicity
- Galas-Bogoyavlenskij symmetries: a rare example of useful nonlocal symmetries in multi-dimensions
- Linear and nonlinear stability? Numerical solutions?
- Heun functions are gaining popularity; implemented in numerical software
- Any physical invariant solutions from the time-dependent helical system [Dierkes and Oberlack (2017)]?

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