Analytical properties of nonlinear partial differential equations in shallow water theory and beyond

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Outline

Outline

Euler and Navier-Stokes equations

Navier-Stokes equations with forcing

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0\,,$$
$$\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\operatorname{grad} P + \mu\,\Delta\mathbf{v} + \mathbf{M}$$

Euler equations

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0\,,$$
$$\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\operatorname{grad} P$$

Euler equations: formulations and special cases

Euler equations

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$$\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\operatorname{grad} P$$

Incompressible flows

- Local incompressibility: $\operatorname{div}\mathbf{v} = 0$
- Constant density (additional): $\rho = \mathrm{const}$

Vorticity formulation

- Vorticity: $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$
- Incompressible Euler, vorticity formulation:

$$\operatorname{div}\mathbf{v} = 0,$$
$$\operatorname{curl}\mathbf{v} = \boldsymbol{\omega},$$
$$\boldsymbol{\omega}_t + \operatorname{curl}(\boldsymbol{\omega}\times\mathbf{v}) = 0$$

Euler equations: formulations and special cases

Euler equations

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0\,,$$
$$\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\operatorname{grad} P$$

Incompressible flows

- Local incompressibility: $\operatorname{div}\mathbf{v} = 0$
- Constant density (additional): $\rho = \mathrm{const}$

Vorticity-stream function formulation

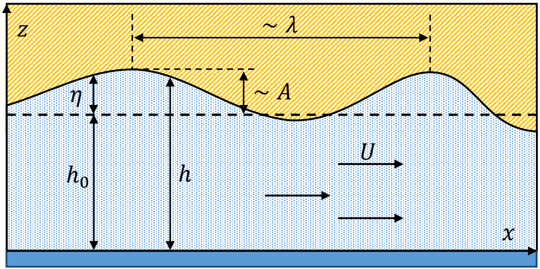
- Vorticity: $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$
- Stream function: $\operatorname{div}\mathbf{v} = 0 \quad \to \quad \mathbf{v} = \operatorname{curl}\boldsymbol{\psi}$
- Euler equations in vorticity-stream function formulation:

$$\operatorname{curl}(\operatorname{curl}\boldsymbol{\psi}) = \boldsymbol{\omega}\,,$$
$$\boldsymbol{\omega}_t + \operatorname{curl}(\boldsymbol{\omega}\times\operatorname{curl}\boldsymbol{\psi}) = 0$$

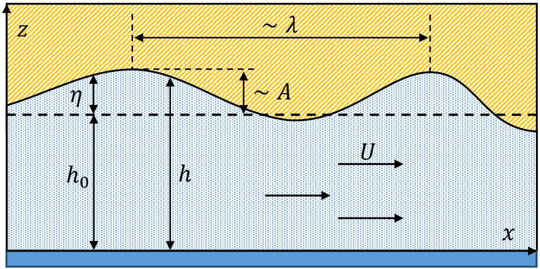
- Irrotational (potential) flows: $\boldsymbol{\omega} = 0$

Outline

(1+1)-dimensional fluid wave setup



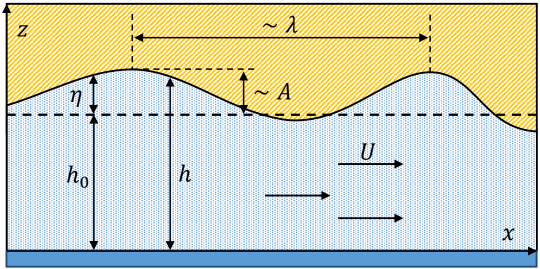
- free surface elevation $\eta(x, t)$
- total fluid depth $h = h(x, t) = h_0 + \eta$
- more generally: variable bottom topography $h_0 = h_0(x)$



- Euler problem in a water channel with free surface:

$$
\begin{aligned}
&U_x + W_z = 0\\
&U_t + UU_x + WU_z = -\frac{1}{\rho} p_x\\
&W_t + UW_x + WW_z = -\frac{1}{\rho} p_z\\
&W = 0 \quad \text{at} \quad z = 0\\
&W = \eta_t + U\eta_x, \qquad p = \rho g \eta \quad \text{at} \quad z = h(x,t)
\end{aligned}
$$

(1+1)-dimensional fluid wave setup



- Irrotational case: $\phi = \phi(x, z, t)$, $U = \phi_x$, $W = \phi_z$
- Laplace problem in a time-dependent domain:

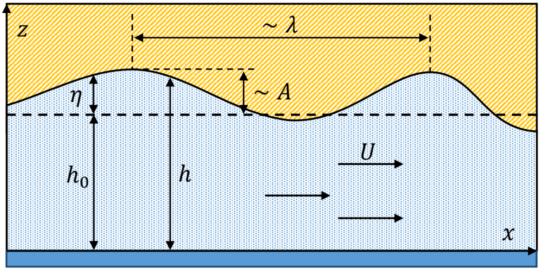
$$\phi_{xx} + \phi_{zz} = 0\,, \qquad 0 < z < h(x,t)$$

$$\phi_z = 0 \quad \text{at} \quad z = 0$$

$$\eta_t + \phi_x \eta_x - \phi_z = 0 \quad \text{at} \quad z = h(x,t)$$

$$\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \quad \text{at} \quad z = h(x,t)$$

(1+1)-dimensional fluid wave setup

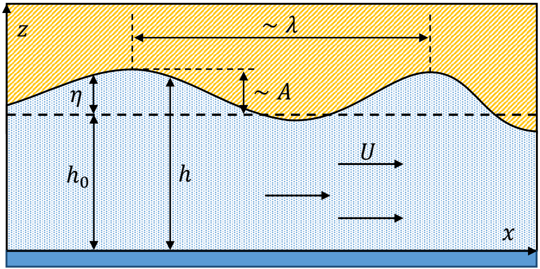


- Small parameters: dispersion parameter $\delta = h_0/\lambda$, amplitude parameter $\varepsilon = A/h_0$

Physical setups:

- Weakly nonlinear, dispersionless: $\delta^2 \ll \varepsilon \ll 1$
- Weakly nonlinear, weakly dispersive: Boussinesq regime $\delta^2 \sim \varepsilon \ll 1$
- Strongly nonlinear, weakly dispersive: $\delta^2 \ll 1$, $\varepsilon = O(1)$

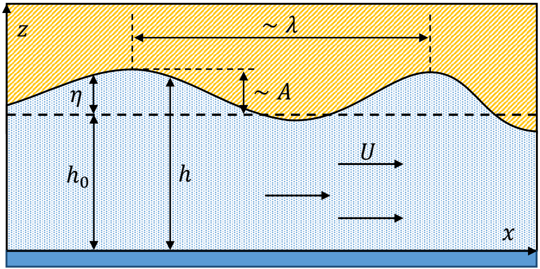
(1+1)-dimensional fluid wave setup



- Shallow water models are written in terms of dimensionless versions of surface elevation $\eta(x,t)$ and the depth-averaged horizontal velocity

$$u(x,t) = \frac{1}{h(x,t)} \int_0^{h(x,t)} U(x,z,t)\, dz\,.$$

(1+1)-dimensional fluid wave setup



Non-dimensionalization:

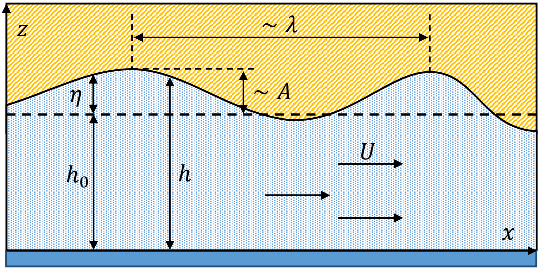
$$t = \frac{\lambda}{c_0}\, t^*\,, \qquad x = \lambda\, x^*\,, \qquad z = h_0\, z^*\,,$$

$$\eta = A\,\eta^*\,, \qquad h = h_0\, h^*\,,$$

$$U = \varepsilon c_0\, U^*\,, \qquad W = \varepsilon\delta c_0\, W^*\,, \qquad u = \varepsilon c_0\, u^*\,,$$

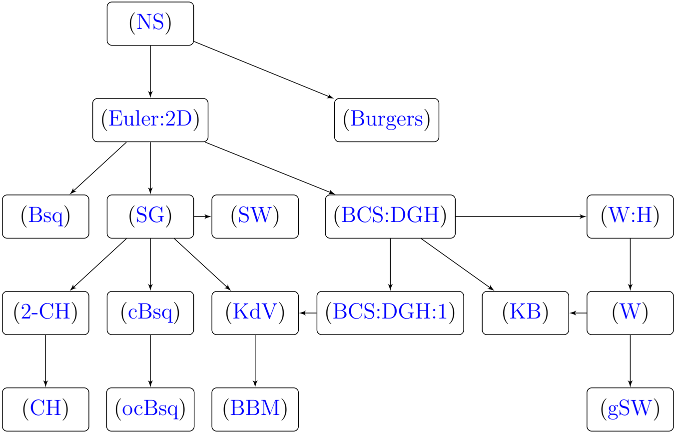
$$\phi = \varepsilon c_0 \lambda\, \phi^*\,, \qquad \psi = \varepsilon c_0 h_0\, \psi^*\,, \qquad p = \varepsilon\,\rho\, c_0^2\, p^*$$

(1+1)-dimensional fluid wave setup

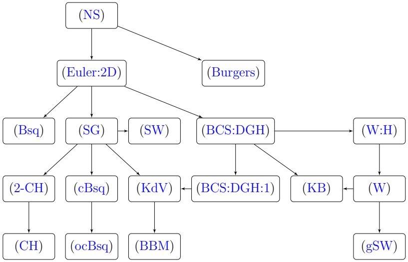


- Zeroth-order approximation: $\eta^*_{t^*} = -u^*_{x^*} + O(\varepsilon, \delta^2)$, $u^*_{t^*} = -\eta^*_{x^*} + O(\varepsilon, \delta^2)$
- Linear wave equations: $\eta_{tt} \simeq c_0^2\, \eta_{xx}$, $u_{tt} \simeq c_0^2\, u_{xx}$, $c_0 = \sqrt{gh_0}$
- d'Alembert solution: $\eta^* = F(x^* - t^*) + G(x^* + t^*) + O(\varepsilon, \delta^2)$
- Unidirectional right- and left-propagating: $u^* = \eta^* \pm O(\varepsilon, \delta^2)$ (Burn's condition)

Some most common SW models



- starred: dimensionless, explicit ε, δ dependence
- canonical: dimensional and/or simplest forms

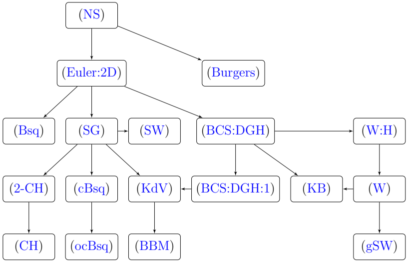


- Su-Gardner equations (SG) (also Serre or Green-Naghdi)

$$u^*_{t^*} + \varepsilon u^* u^*_{x^*} + \eta^*_{x^*} = \frac{\delta^2}{3h^*}\left((h^*)^3\left(u^*_{x^*t^*} + \varepsilon u^* u^*_{x^*x^*} - \varepsilon(u^*_{x^*})^2\right)\right)_{x^*}$$
$$h^*_{t^*} + \varepsilon(h^* u^*)_{x^*} = 0$$

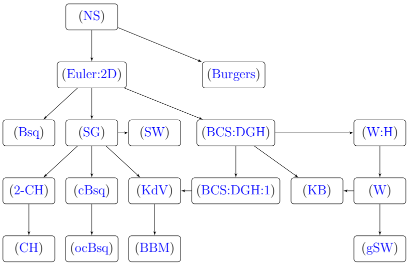
- ε not assumed small; bidirectional waves
- Dimensional:

$$u_t + uu_x + gh_x = \frac{1}{3h}\left(h^3\left(u_{xt} + uu_{xx} - (u_x)^2\right)\right)_x$$
$$h_t + (hu)_x = 0$$

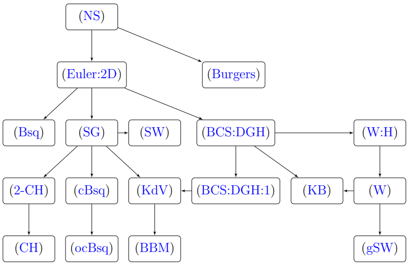


- The Boussinesq equation $\eta^*_{t^*t^*} = \eta^*_{x^*x^*} + \frac{3}{2}\varepsilon\left(\eta^{*2}\right)_{x^*x^*} + \frac{\delta^2}{3}\eta^*_{x^*x^*x^*x^*}$
- Holds in the Boussinesq regime $\delta^2 \sim \varepsilon \ll 1$; bidirectional waves
- Dimensional version: $\eta_{tt} = c_0^2\left(\eta + \frac{3}{2}\frac{\eta^2}{h_0} + \frac{1}{3}h_0^2\eta_{xx}\right)_{xx}$
- Canonical form: $u_{tt} = \alpha u_{xx} + \beta\left(u^2\right)_{xx} + \gamma u_{xxxx}$; S-integrable
- Regularized/ "Bogolubsky" PDE: $u_{tt} = \alpha u_{xx} + \beta\left(u^2\right)_{xx} + \gamma u_{xxtt}$
  same asymptotic approximation, non-integrable

Some most common SW models

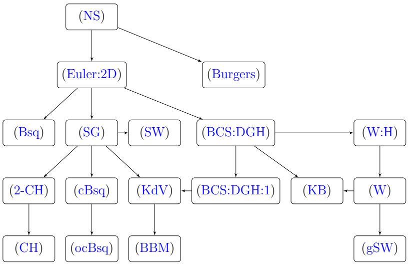


- Korteweg-de Vries (KdV) $\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\,\eta^*\eta^*_{x^*} + \frac{\delta^2}{6}\,\eta^*_{x^*x^*x^*} = 0$
- Unidirectional flow; Boussinesq regime
- Canonical form: $u_t + 6uu_x + u_{xxx} = 0$
- S-integrable; multi-soliton solutions



- Benjamin-Bona-Mahony (BBM) $\eta^*_{t^*} + \eta^*_{x^*} + \dfrac{3}{2}\varepsilon\eta^*\eta^*_{x^*} - \dfrac{\delta^2}{6}\eta^*_{x^*x^*t^*} = 0$
- Coincides with the KdV in the Boussinesq regime approximation order: $\eta^*_{t^*} = -\eta^*_{x^*} + O(\varepsilon, \delta^2)$
- Canonical form: $u_t + u_x + uu_x - u_{xxt} = 0$
- Non-integrable, not Galilei-invariant

Some most common SW models



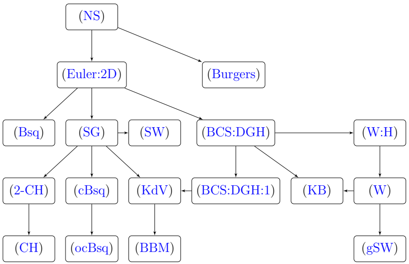
- Kaup-Boussinesq

$$u_t + uu_x + h_x = 0$$
$$h_t + (hu)_x + \beta^2 u_{xxx} = 0$$

- Dimensionless form:

$$\hat{u}^*_{t^*} + \varepsilon \hat{u}^* \hat{u}^*_{x^*} + \eta^*_{x^*} = 0\,,$$
$$\eta^*_{t^*} + \left((1+\varepsilon\eta^*)\hat{u}^*\right)_{x^*} + \frac{\delta^2}{3}\hat{u}^*_{x^*x^*x^*} = 0$$

- Bidirectional; Boussinesq regime; S-integrable



- Shallow water equations

$$u_t + uu_x + h_x = 0$$
$$h_t + (hu)_x = 0$$

- Dimensionless form:

$$u^*_{t^*} + \eta^*_{x^*} + \varepsilon u^* u^*_{x^*} = 0$$
$$h^*_t + (h^* u^*)_{x^*} = 0$$

- Bidirectional; dispersionless regime $\delta \to 0$; C-integrable

Where do different SW models come from?

- Approximation regime: relations between ε and δ
- Order of asymptotic approximation
- Substitutions within the same order, such as $\eta^*_{t^*} = -u^*_{x^*} + O(\varepsilon, \delta^2)$
- Use of a velocity variable different from the depth-averaged horizontal velocity

$$u(x,t) = \frac{1}{h(x,t)} \int_0^{h(x,t)} U(x,z,t)\,dz$$

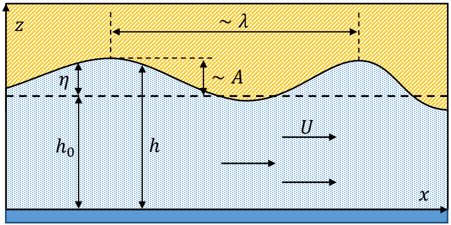
  for example, velocity at a fixed depth: $u(x,t) = U(x, z_0, t)$

- Use of "artificial physics" for PDEs obtained from other considerations, such as integrability requirement, e.g., the Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

  whose derivation was math-motivated

Outline



- The generalized shallow water equation

$$\eta_t + \eta_x - \alpha\eta\eta_t + \beta\eta_x\partial_x^{-1}\eta_t - \eta_{xxt} = 0$$

with

$$\partial_x^{-1}\eta_t = \int_x^{\infty} \eta_t(s,t)\,ds$$

- Local (potential) form: $\eta = u_x$,

$$u_{xt} + u_{xx} - \alpha u_x u_{xt} - \beta u_{xx} u_t - u_{xxxt} = 0$$

- S-integrable if and only if $\alpha/\beta = 2$ or $\alpha/\beta = 1$

A higher-order model with tension

- The combined Bona-Chen-Saut-Dullin-Gottwald-Holm system:

$$\tilde{u}^*_{t^*} + \eta^*_{x^*} + \varepsilon \tilde{u}^* \tilde{u}^*_{x^*} + \frac{\delta^2}{2}(\theta^2 - 1)\tilde{u}^*_{x^*x^*t^*} - \delta^2 \sigma^* \eta^*_{x^*x^*x^*}$$

$$- \frac{\varepsilon\delta^2}{2}\left(2(\eta^* \tilde{u}^*_{x^*t^*})_x - (\theta^2+1)\tilde{u}^*_{x^*}\tilde{u}^*_{x^*x^*} - (\theta^2-1)\tilde{u}^*\tilde{u}^*_{x^*x^*x^*}\right)$$

$$+ \frac{5}{24}\delta^4(\theta^2-1)\left(\theta^2 - \frac{1}{5}\right)\tilde{u}^*_{x^*x^*x^*x^*t^*} = O(\delta^6)$$

$$\eta^*_{t^*} + ((1+\varepsilon\eta^*)\tilde{u}^*)_{x^*} + \frac{\delta^2}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{u}^*_{x^*x^*x^*} + \frac{\varepsilon\delta^2}{2}(\theta^2-1)(\eta^*\tilde{u}^*_{x^*x^*})_{x^*}$$

$$+ \frac{5}{24}\delta^4\left(\theta^2 - \frac{1}{5}\right)^2 \tilde{u}^*_{x^*x^*x^*x^*x^*} = O(\delta^6)$$

- Nonzero surface tension coefficient σ^*
- Horizontal velocity $\tilde{u}$ measured at an arbitrary dimensionless elevation $\theta \in [0,1]$ above the flat bottom
- Gives rise to many simpler SW models

A higher-order model with tension

- A higher-order single-PDE unidirectional version

- Can exclude $\tilde{u}^*$:

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\eta^*\eta^*_{x^*} + \frac{\delta^2}{6}(1-3\sigma^*)\eta^*_{x^*x^*x^*} - \frac{3}{8}\varepsilon^2(\eta^*)^2\eta^*_{x^*}$$

$$+ \frac{\varepsilon\delta^2}{24}\left((23+15\sigma^*)\eta^*_{x^*}\eta^*_{x^*x^*} + 2(5-3\sigma^*)\eta^*\eta^*_{x^*x^*x^*}\right)$$

$$+ \frac{\delta^4}{360}(19-30\sigma^* - 45(\sigma^*)^2)\eta^*_{x^*x^*x^*x^*x^*} = O(\delta^6)$$

- Generalizes the KdV

Outline

What is integrability?

- Many definitions, with unclear relationships
- C-integrability
- S-integrability
- Symmetry- and conservation law-integrability: existence of higher-order symmetries and conservation laws.
- Integrability in the Hirota sense: multi-soliton solutions
- Liouville ("complete") integrability: a Hamiltonian PDE system that has an infinite number of conserved densities in pairwise involution. True for bi-Hamiltonian PDE systems
- Formal integrability: existence of a recursion operator
- Painlevé integrability: pass the Painlevé test
- ...

Related analytical structures

- Lax pairs and zero-curvature representation
- Infinite-parameter and/or higher-order Lie-type symmetries
- Infinite-parameter and/or higher-order local conservation laws
- Painlevé property
- Variational principles and Lagrangian structure
- Hamiltonian and bi-Hamiltonian structure, recursion operators

C-integrability

C-integrability: "*A nonlinear partial differential equation is called C-integrable if it can be solved by a Change of variables*"

Example: the shallow water (SW) system

$$u_t + uu_x + gh_x = 0$$
$$h_t + (hu)_x = 0$$

- a point hodograph transformation

$$x = x(h,u)\,, \qquad t = t(h,u)\,, \qquad h(x,t) = h, \qquad u(x,t) = u$$

- invertible linearization:

$$x_u - u\,t_u + h\,t_h = 0$$
$$x_h - u\,t_h + g\,t_u = 0$$

Another famous example: the Burgers equation $u_t + uu_x = \nu u_{xx}$

- the Hopf-Cole transformation $u = -2\,\nu\,w_x/w$
- w satisfies the heat equation $w_t = \nu\,w_{xx}$

S-integrability: "*the possibility of construction of solutions of a given PDE system using a Spectral transform technique*"

- Main ingredient: a nontrivial Lax pair (L, P), isospectral flow

$$\mathrm{L}\Psi = \lambda\Psi, \qquad \Psi_t = \mathrm{P}\Psi,$$

$$\mathrm{L}_t = [\mathrm{P}, \mathrm{L}] \equiv \mathrm{PL} - \mathrm{LP}$$

- More generally: a zero-curvature representation with matrix operators U, V

$$\hat{\Psi}_x = \mathrm{U}\hat{\Psi}\,, \qquad \hat{\Psi}_t = \mathrm{V}\hat{\Psi}\,,$$

$$\mathrm{U}_t - \mathrm{V}_x + [\mathrm{U}, \mathrm{V}] = 0$$

Main application: exact solutions

- Inverse scattering
- Darboux transformation can be used to iterate solutions
- Multi-soliton solutions through Hirota's bilinear method

S-integrability

Lax pairs:

- Art or inspiration?
- Existence hinted by other analytical structure (bi-Hamiltonian form, symmetries, etc.)
- May be systematically constructed in WTC/Painlevé analysis

Example: Kaup-Boussinseq model

$$u_t + uu_x + h_x = 0$$
$$h_t + (hu)_x + \frac{1}{4} u_{xxx} = 0$$

- Lax pair:

$$\Psi_{xx} = \left(\lambda^2 - \lambda u - h + \frac{1}{4}u^2\right)\Psi, \qquad \Psi_t = -\left(\lambda + \frac{1}{2}u\right)\Psi_x + \frac{u_x}{4}\Psi$$

- ZCR: $\hat{\Psi}_x = \mathrm{U}\hat{\Psi}$, $\hat{\Psi}_t = \mathrm{V}\hat{\Psi}$ where

$$\mathrm{U} = \begin{pmatrix} 0 & 1 \\ \lambda^2 - \lambda u - h + \frac{1}{4}u^2 & 0 \end{pmatrix}, \qquad \mathrm{V} = \frac{1}{4}\begin{pmatrix} u_x & -2(u+2\lambda) \\ u_{xx} & -u_x - 2(u+2\lambda)\,\mathrm{D}_x \end{pmatrix}$$

Outline

Symmetries

- Symmetries are transformations of variables that preserve the solution set of the model
- Point symmetries:

$$(z^*)^i = f^i(z,u;\epsilon) = z^i + \epsilon\xi^i(z,u) + O(\epsilon^2)\,, \qquad i = 1,\dots,n$$
$$(u^*)^\mu = g^\mu(z,u;\epsilon) = u^\mu + \epsilon\eta^\mu(z,u) + O(\epsilon^2)\,, \qquad \mu = 1,\dots,m$$

- More general: local

$$\hat{z}^i = z^i, \quad i = 1,\dots,n,$$
$$\hat{u}^\mu = u^\mu + \epsilon\zeta^\mu[u] + O(\epsilon^2), \quad \mu = 1,\dots,m,$$

- Infinitesimal generators:

$$\mathrm{X} = \xi^i\,\partial_{z^i} + \eta^\mu\,\partial_{u^\mu}\,, \qquad \hat{\mathrm{X}} = \zeta^\mu[u]\,\partial_{u^\mu}\,.$$

- Lie-type symmetries (and extensions like contact, higher-order, nonlocal, approximate, ...) can be systematically sought using Lie's algorithm

Symmetries

Some basic one-parameter Lie groups of point symmetries:

- Translations in space and time, generated by $\mathrm{X}_1 = \partial_x$, $\mathrm{X}_2 = \partial_t$, with global groups

$$x^* = x + \epsilon\,, \qquad t^* = t\,, \qquad u^* = u\,,$$
$$x^* = x\,, \qquad t^* = t + \epsilon\,, \qquad u^* = u\,.$$

- Scaling symmetries, generated by, for example, $\mathrm{X} = Ax\partial_x + Bt\,\partial_t + Cu\,\partial_u$, where A, B, C are constants, with global group

$$x^* = xe^{A\epsilon}\,, \qquad t^* = te^{B\epsilon}\,, \qquad u^* = ue^{C\epsilon}\,.$$

- The Galilei symmetry, generated by, for example, $\mathrm{X} = t\,\partial_x + \partial_u$, with global group

$$x^* = x + \epsilon t\,, \qquad t^* = t\,, \qquad u^* = u + \epsilon\,.$$

Main applications:

- Construction of invariant reductions, invariant solutions
- Mapping structures such as conservation laws; iterate exact solutions
- Can hint linearizing invertible transformations for C-integrable models
- Eequivalence transforms: mappings relating PDE families, parameter reduction

Symmetry families parameterized by arbitrary functions can lead to invertible linearization (C-integrability)

Example: An infinite-dimensional set of point symmetries of the SW equations

$$\mathrm{X}_\infty = A\,\partial_x + B\,\partial_t\,,$$

where $(A, B) = (A(u,h), B(u,h))$ is an arbitrary solution of the linear PDE system

$$uA_u + hA_h - (u^2 - gh)B_u = 0\,, \qquad uB_u - hB_h - A_u = 0\,.$$

Infinite countable hierarchies of higher-order symmetries of increasing order are associated with S-integrability (no guarantees!)

Example: Korteweg-de Vries equation $u_t + 6uu_x + u_{xxx} = 0$:

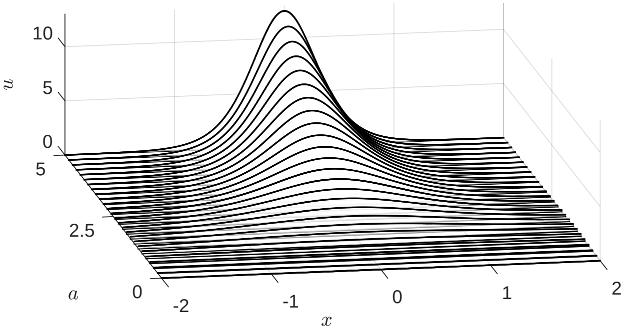
$$\hat{\mathrm{X}}_i = K_i\,\partial_u, \qquad K_0 = u_x\,, \qquad K_1 = 6uu_x + u_{xxx}\,,$$
$$K_2 = 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx}\,,$$
$$\ldots$$

Outline

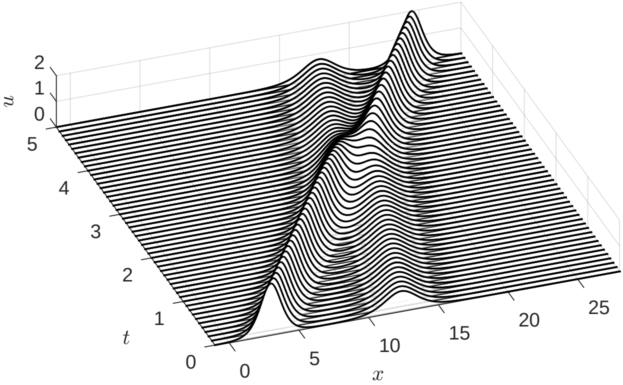
- Symmetry-invariant: e.g., traveling wave $u(x,t) = u(x - ct)$

- Example: Korteweg-de Vries equation $u_t + 6uu_x + u_{xxx} = 0$, single-soliton solution

$$u(x,t) = \frac{a^2}{2}\,\mathrm{sech}^2\left(\frac{a}{2}(x - x_0 - ct)\right) + u_0$$

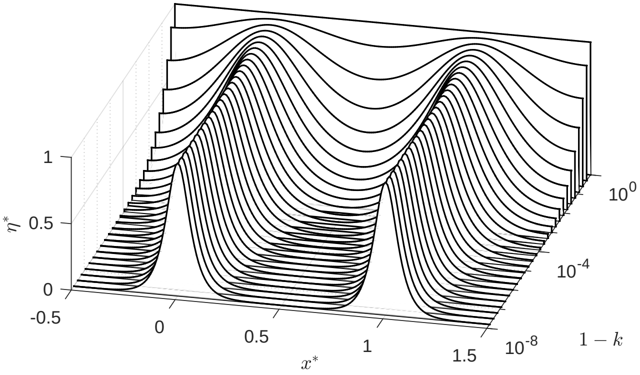


- Multi-soliton solutions
- For example, using Hirota's bilinear method



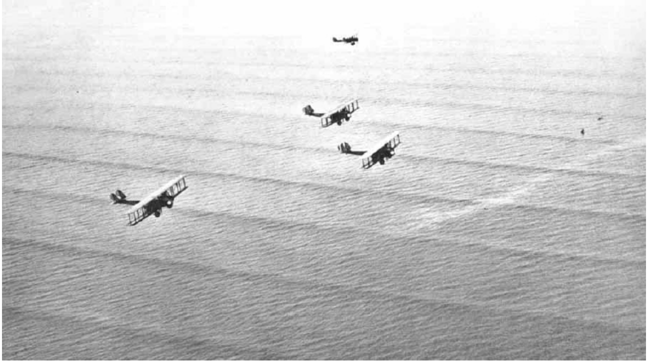
- Cnoidal traveling waves for Su-Gardner equations

$$u_t + uu_x + gh_x = \frac{1}{3h}\left(h^3\left(u_{xt} + uu_{xx} - (u_x)^2\right)\right)_x ,$$
$$h_t + (hu)_x = 0$$



- Cnoidal traveling waves for Su-Gardner equations

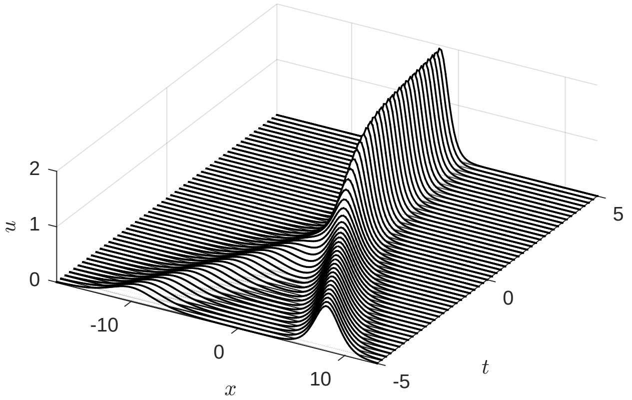
$$u_t + uu_x + gh_x = \frac{1}{3h}\left(h^3\left(u_{xt} + uu_{xx} - (u_x)^2\right)\right)_x ,$$
$$h_t + (hu)_x = 0$$



- Solutions using "nonclassical symmetries":

- Example: merging solitons for the generalized shallow water equation

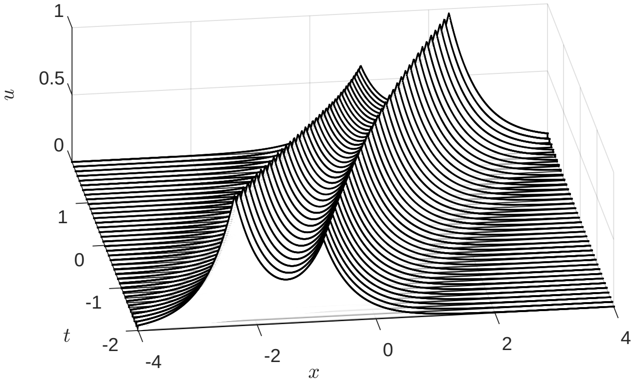
$$\eta_t + \eta_x - \alpha\eta\eta_t + \beta\eta_x\partial_x^{-1}\eta_t - \eta_{xxt} = 0$$



- Weak peakon solutions for the Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

- A "close encounter" of two peakons



Outline

Conservation laws

- Conservation laws provide local and global conserved quantities
- (1+1)-D:
$$\mathrm{D}_t\Theta[u] + \mathrm{D}_x\Phi[u] = 0\,.$$
- Global quantity:
$$\frac{d}{dt}\,\mathcal{C}[u] = \frac{d}{dt}\int_a^b \Theta[u]\,dx = -\Phi[u]\Big|_a^b,$$
- Divergence expressions can be systematically obtained using the multiplier method & Euler operators
$$\mathrm{D}_i\,Z^i[u] \equiv \Lambda_\sigma[u]\,R^\sigma[u] = 0,$$
$$\mathrm{E}_{u^j}\left(\Lambda_\sigma[u]R^\sigma[u]\right) \equiv 0, \quad j = 1,\ldots,m$$

Main applications:

- Conservative numerical methods
- Relations to C- and S-integrability
- Relations to symmetries (Noether's theorem)
- A tool to systematically construct nonlocally related (such as potential) PDE systems that lead to new results

Conservation laws

- Example: conservation laws for Su-Gardner equations

$$u^*_{t^*} + \varepsilon u^* u^*_{x^*} + \eta^*_{x^*} = \frac{\delta^2}{3h^*}\left((h^*)^3\left(u^*_{x^*t^*} + \varepsilon u^* u^*_{x^*x^*} - \varepsilon(u^*_{x^*})^2\right)\right)_{x^*}$$

$$h^*_{t^*} + \varepsilon(h^* u^*)_{x^*} = 0$$

$$\mathrm{D}_t\, h + \mathrm{D}_x(hu) = 0\,,$$

$$\mathrm{D}_t(hu) + \mathrm{D}_x\Big(hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^3\big(u_x^2 - u_{xt} - uu_{xx}\big)\Big) = 0\,,$$

$$\mathrm{D}_t\Big(\frac{1}{2}h\Big(u^2 + gh + \frac{1}{3}h^2u_x^2\Big)\Big)$$
$$+\mathrm{D}_x\Big(hu\Big(\frac{1}{2}u^2 + gh + \frac{1}{2}h^2u_x^2 - \frac{1}{3}h^2(u_{xt} + uu_{xx})\Big)\Big) = 0\,,$$

$$\mathrm{D}_t\Big(u - hh_xu_x - \frac{1}{3}h^2u_{xx}\Big) + \mathrm{D}_x\Big(\frac{1}{2}u^2 + gh + hh_tu_x + \frac{1}{2}h^2u_x^2 - \frac{1}{3}h^2uu_{xx}\Big) = 0\,,$$

$$\mathrm{D}_t\Big(h(t\,u - x)\Big) + \mathrm{D}_x\Big(h\,u(t\,u - x) + \frac{1}{2}g\,t\,h^2 + \frac{1}{3}t\,h^3(u_x^2 - u_{xt} - u\,u_{xx})\Big) = 0$$

Conservation laws

- Example: conservation laws for the KdV $u_t + 6uu_x + u_{xxx} = 0$
- An infinite set of polynomial higher-order CLs related to S-integrability

$$
\begin{aligned}
&\mathrm{D}_t\, u + \mathrm{D}_x(3u^2 + u_{xx}) = 0\,,\\
&\mathrm{D}_t\left(\frac{1}{2}u^2\right) + \mathrm{D}_x\left(2u^3 - \frac{1}{2}u_x^2 + uu_{xx}\right) = 0\,,\\
&\mathrm{D}_t\left(u^3 - \frac{1}{2}u_x^2\right) + \mathrm{D}_x\left(\frac{9}{2}u^4 + u_x u_t + 3u^2 u_{xx} + \frac{1}{2}u_{xx}^2\right) = 0\,,\\
&\vdots
\end{aligned}
$$

Conservation laws

- dimensionless Korteweg-de Vries (KdV)

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\,\eta^*\eta^*_{x^*} + \frac{\delta^2}{6}\,\eta^*_{x^*x^*x^*} = 0$$

- versus Benjamin-Bona-Mahony (BBM):

$$\eta^*_{t^*} + \eta^*_{x^*} + \frac{3}{2}\varepsilon\eta^*\eta^*_{x^*} - \frac{\delta^2}{6}\eta^*_{x^*x^*t^*} = 0$$

- same order of asymptotic approximation!

- KdV is S-integrable; it has an infinite hierarchy of higher-order local CLs

- BBM is not integrable; it has exactly three local CLs

Outline

A Hamiltonian evolution equation:

$$u_t = K[u] = \mathcal{D} \cdot \frac{\delta H}{\delta u}$$

Main applications:

- Conserved Hamiltonian density, Casimirs
- Hamiltonian version of Noether's theorem
- Bi-Hamiltonian & multi-Hamiltonian structures; recursion operators

A bi-Hamiltonian system:

$$u_t = K_1[u] = \mathcal{D} \cdot \frac{\delta H_1}{\delta u} = \mathcal{E} \cdot \frac{\delta H_0}{\delta u}$$

- Recursion operator:

$$\mathcal{R} = \mathcal{E} \cdot \mathcal{D}^{-1}$$

Hamiltonian structure

- Example: a tri-Hamiltonian structure for the SW system

$$u_t + uu_x + gh_x = 0\,,$$
$$h_t + (hu)_x = 0$$

$$H_0 = \frac{1}{2}\left(hu^2 + gh^2\right), \qquad H_1 = hu\,, \qquad H_2 = h\,,$$

$$\mathcal{D}_0 = -\begin{pmatrix} 0 & \mathrm{D}_x \\ \mathrm{D}_x & 0 \end{pmatrix}, \qquad \mathcal{D}_1 = -\frac{1}{2}\begin{pmatrix} 2g\mathrm{D}_x & u\mathrm{D}_x + u_x \\ u\mathrm{D}_x & h\mathrm{D}_x + \mathrm{D}_x h \end{pmatrix},$$

$$\mathcal{D}_2 = -\begin{pmatrix} g(u\mathrm{D}_x + \mathrm{D}_x u) & \left(\frac{1}{2}u^2 + 2gh\right)\mathrm{D}_x + uu_x + gh_x \\ \left(\frac{1}{2}u^2 + 2gh\right)\mathrm{D}_x + gh_x & uh\mathrm{D}_x + \mathrm{D}_x uh \end{pmatrix}$$

- Recursion operators:

$$\mathcal{R}_1 = \mathcal{D}_1 \cdot \mathcal{D}_0^{-1}, \qquad \mathcal{R}_2 = \mathcal{D}_2 \cdot \mathcal{D}_0^{-1}, \qquad \mathcal{R}_3 = \mathcal{D}_2 \cdot \mathcal{D}_1^{-1},$$

Outline

- Stability of equilibrium
- Linearization about $u = u_0(x)$:

$$u^\mu(x,t) = u_0^\mu + \epsilon u_1^\mu e^{i(kx-\omega t)}, \quad \mu = 1\ldots,m,$$

- Dispersion relations:

$$\omega = \omega(k), \qquad c = \frac{\omega}{k} = c(k).$$

- Stability: $\operatorname{Im}\omega = 0$ for all k
- Dispersion: linearized waves of different wavelengths travel at different speeds

- Full water wave problem:

$$\phi_{xx} + \phi_{zz} = 0\,, \qquad 0 < z < h(x,t),$$

$$\phi_z = 0 \quad \text{at} \quad z = 0\,,$$

$$\eta_t + \phi_x \eta_x - \phi_z = 0\,, \quad \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \quad \text{at} \quad z = h(x,t)$$

- Perturbation of zero state:

$$h = h_0 + \eta, \qquad \eta = \epsilon \eta_0 \, e^{i(kx - \omega t)}, \qquad \phi = \epsilon f(z)\, e^{i(kx-\omega t)}$$

- Dispersion relation:

$$\omega^2 = gk \tanh kh_0\,, \qquad c = c_0 \sqrt{\frac{\tanh kh_0}{kh_0}}$$

Dispersion relations

Compare:

- Full water wave dispersion relation:

$$\omega^2 = gk \tanh kh_0, \qquad c = c_0 \sqrt{\frac{\tanh kh_0}{kh_0}}$$

- SG:

$$\omega = \frac{c_0\, k}{\sqrt{1 + h_0^2\, k^2/3}}, \qquad c = \frac{c_0}{\sqrt{1 + h_0^2\, k^2/3}}$$

- KdV:

$$\omega = c_0\, k \left(1 - \frac{1}{6} h_0^2 k^2\right), \qquad c = c_0 \left(1 - \frac{1}{6} h_0^2 k^2\right)$$

- BBM:

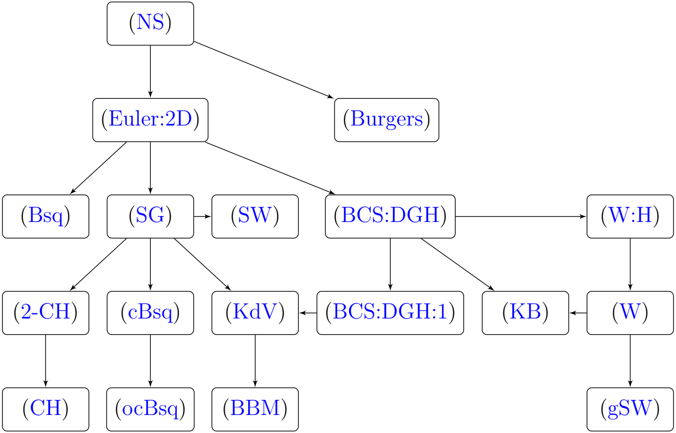
$$\omega = \frac{c_0 k}{1 + h_0^2\, k^2/6}, \qquad c = \frac{c_0}{1 + h_0^2\, k^2/6}$$

- SW: $c \neq c(k)$ so no dispersion

$$\omega^2 = c_0^2\, k^2, \qquad c = c_0 = \sqrt{gh_0}$$

Outline

A diagram of physical relations between some shallow water models:



PDE naming and the Arnold's principle

Arnold's principle

If a model bears a name, it is not the name of the person who discovered it

Examples:

- Korteweg-de Vries $\rightarrow$ Boussinesq (25 years earlier)
- Su-Gardner (Green-Naghdi) $\rightarrow$ Serre (13 and 21 years earlier)
- Camassa-Holm $\rightarrow$ Fokas and Fuchssteiner (12 years earlier)

Too many things left out...

- Variational/Lagrangian structure: self-adjointness of linearization
- Painlevé property: all linear equations pass; related to integrability; Camassa-Holm as counterexample
- Solution existence, uniqueness, stability
- Numerical aspects
- MANY extended PDE models
- Multi-dimensional versions
- ... and more ...

Thank you for your attention!