Analytical properties of nonlinear partial differential equations in shallow water theory and beyond

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Outline

1. Euler and Navier-Stokes
2. Shallow water models
3. Higher-order and nonlocal models
4. Integrability
5. Symmetries
6. Exact solutions
7. Conservation laws
8. Hamiltonian and multi-Hamiltonian structure
9. Dispersion relations
10. Conclusions
Navier-Stokes equations with forcing

\[
\begin{align*}
\rho_t + \text{div} \left( \rho v \right) &= 0, \\
\rho \left( v_t + (v \cdot \nabla)v \right) &= -\text{grad} \, P + \mu \Delta v + M
\end{align*}
\]

Euler equations

\[
\begin{align*}
\rho_t + \text{div} \left( \rho v \right) &= 0, \\
\rho \left( v_t + (v \cdot \nabla)v \right) &= -\text{grad} \, P
\end{align*}
\]
Euler equations

\[ \rho_t + \text{div} (\rho \mathbf{v}) = 0, \]
\[ \rho (\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\text{grad} \, P \]

Incompressible flows

- Local incompressibility: \( \text{div} \, \mathbf{v} = 0 \)
- Constant density (additional): \( \rho = \text{const} \)

Vorticity formulation

- Vorticity: \( \mathbf{\omega} = \text{curl} \, \mathbf{v} \)
- Incompressible Euler, vorticity formulation:
  \[ \text{div} \, \mathbf{v} = 0, \]
  \[ \text{curl} \, \mathbf{v} = \mathbf{\omega}, \]
  \[ \mathbf{\omega}_t + \text{curl} \,(\mathbf{\omega} \times \mathbf{v}) = 0 \]
Euler equations: formulations and special cases

Euler equations

\[ \rho_t + \text{div} (\rho \mathbf{v}) = 0, \]
\[ \rho (\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\text{grad} \, P \]

Incompressible flows

- Local incompressibility: \( \text{div} \, \mathbf{v} = 0 \)
- Constant density (additional): \( \rho = \text{const} \)

Vorticity-stream function formulation

- Vorticity: \( \omega = \text{curl} \, \mathbf{v} \)
- Stream function: \( \text{div} \, \mathbf{v} = 0 \rightarrow \mathbf{v} = \text{curl} \, \psi \)
- Euler equations in vorticity-stream function formulation:
  \[ \text{curl} \, (\text{curl} \, \psi) = \omega, \]
  \[ \omega_t + \text{curl} \, (\omega \times \text{curl} \, \psi) = 0 \]
- Irrotational (potential) flows: \( \omega = 0 \)
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free surface elevation $\eta(x, t)$

total fluid depth $h = h(x, t) = h_0 + \eta$

more generally: variable bottom topography $h_0 = h_0(x)$
(1+1)-dimensional fluid wave setup

- Euler problem in a water channel with free surface:

\[
\begin{align*}
U_x + W_z &= 0 \\
U_t + UU_x + WW_z &= \frac{1}{\rho} p_x \\
W_t + UW_x + WW_z &= \frac{1}{\rho} p_z \\
W &= 0 \quad \text{at} \quad z = 0 \\
W &= \eta_t + U\eta_x, \quad p = \rho g \eta \quad \text{at} \quad z = h(x, t)
\end{align*}
\]
Irrotational case: $\phi = \phi(x, z, t)$, $U = \phi_x$, $W = \phi_z$

Laplace problem in a time-dependent domain:

$$\phi_{xx} + \phi_{zz} = 0, \quad 0 < z < h(x, t)$$

$$\phi_z = 0 \quad \text{at} \quad z = 0$$

$$\eta_t + \phi_x \eta_x - \phi_z = 0 \quad \text{at} \quad z = h(x, t)$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \quad \text{at} \quad z = h(x, t)$$
(1+1)-dimensional fluid wave setup

Small parameters: dispersion parameter $\delta = h_0/\lambda$, amplitude parameter $\varepsilon = A/h_0$

Physical setups:
- Weakly nonlinear, dispersionless: $\delta^2 \ll \varepsilon \ll 1$
- Weakly nonlinear, weakly dispersive: Boussinesq regime $\delta^2 \sim \varepsilon \ll 1$
- Strongly nonlinear, weakly dispersive: $\delta^2 \ll 1$, $\varepsilon = O(1)$
Shallow water models are written in terms of dimensionless versions of surface elevation $\eta(x, t)$ and the depth-averaged horizontal velocity

$$u(x, t) = \frac{1}{h(x, t)} \int_0^{h(x, t)} U(x, z, t) \, dz.$$
(1+1)-dimensional fluid wave setup

Non-dimensionalization:

\[
\begin{align*}
\eta &= A \eta^*, \\
h &= h_0 h^*, \\
U &= \varepsilon c_0 U^*, \\
W &= \varepsilon \delta c_0 W^*, \\
u &= \varepsilon c_0 u^*, \\
\phi &= \varepsilon c_0 \lambda \phi^*, \\
\psi &= \varepsilon c_0 h_0 \psi^*, \\
p &= \varepsilon \rho c_0^2 p^*
\end{align*}
\]
(1+1)-dimensional fluid wave setup

Zeroth-order approximation: \( \eta_t^* = -u_x^* + O(\varepsilon, \delta^2) \), \( u_t^* = -\eta_x^* + O(\varepsilon, \delta^2) \)

Linear wave equations: \( \eta_{tt} \approx c_0^2 \eta_{xx} \), \( u_{tt} \approx c_0^2 u_{xx} \), \( c_0 = \sqrt{gh_0} \)

d’Alembert solution: \( \eta^* = F(x^* - t^*) + G(x^* + t^*) + O(\varepsilon, \delta^2) \)

Unidirectional right- and left-propagating: \( u^* = \eta^* \pm O(\varepsilon, \delta^2) \) (Burn’s condition)
11) The Camassa-Holm equation (Section 3.12)

\[ u_t - u_{xxt} + 3uu_x - 2uxuxx - uuxxx = 0 . \] (CH)

12) The two-component Camassa-Holm equations (Section 3.13)

\[ u_t - u_{xxt} + 3uu_x - 2uxuxx - uuxxx + \varrho \varrho_x = 0, \]
\[ \varrho_t + (\varrho u)_x = 0. \] (2-CH)

There are multiple ways to derive PDE systems mentioned in this work. Figure 3.1 shows a possible set of relationships as per the corresponding Derivation sections.

Figure 3.1: Shallow water models: a relationship diagram.

3.2 Euler equations in two dimensions

The Euler equations describing the dynamics of an incompressible inviscid fluid under the action of an external volumetric force \( F \) are given by

\[ \text{div} \ v = 0, \]
\[ \rho (v_t + (v \cdot \nabla) v) = -\text{grad} \ P + M. \] (3.2.1)

In (3.2.1), the density \( \rho \) may in fact be variable, satisfying the advection equation (1.2.5); however, a common assumption for fluids such as water is \( \rho = \text{const.} \) We additionally assume the external force is conservative, satisfying (1.2.7) (such as the gravity force, \( M = \rho g = 68 \)).

- starred: dimensionless, explicit \( \varepsilon, \delta \) dependence
- canonical: dimensional and/or simplest forms
Some most common SW models

- **Su-Gardner equations (SG)** (also Serre or Green-Naghdi)
  \[
  u_{t}^* + \varepsilon u^* u_x^* + \eta_x^* = \frac{\delta^2}{3h^*} (h^*)^3 \left( u_{x}^* {t}^* + \varepsilon u^* u_{x}^* x^* - \varepsilon (u_{x}^* )^2 \right) x^* \\
  h_{t}^* + \varepsilon ( h^* u^* ) x^* = 0
  \]
  - \( \varepsilon \) not assumed small; bidirectional waves
  - Dimensional:
    \[
    u_{t} + uu_{x} + gh_{x} = \frac{1}{3h} \left( h^3 \left( u_{x} + uu_{xx} - (u_{x})^2 \right) \right) _{x} \\
    h_{t} + (hu)_x = 0
    \]
Some most common SW models

- The Camassa-Holm equation (Section 3.12)
  \[ u_{tt} - u_{xxt} + 3uu_x - 2u_{xx}u_{xx} - uu_{xxx} = 0 . \]  
  (CH)

- The two-component Camassa-Holm equations (Section 3.13)
  \[ u_{tt} - u_{xxt} + 3uu_x - 2u_{xx}u_{xx} - uu_{xxx} + \rho \rho_x = 0 , \]
  \[ \rho_t + (\rho u)_x = 0 . \]  
  (2-CH)

There are multiple ways to derive PDE systems mentioned in this work. Figure 3.1 shows a possible set of relationships as per the corresponding Derivation sections.

- The Boussinesq equation
  \[ \eta_{t^* t^*} = \eta^{*}_{x^* x^*} + \frac{3}{2} \varepsilon (\eta^{*2})_{x^* x^*} + \frac{\delta^2}{3} \eta^{*}_{x^* x^* x^* x^*} \]

- Holds in the Boussinesq regime \( \delta^2 \sim \varepsilon \ll 1; \) bidirectional waves

- Dimensional version:
  \[ \eta_{tt} = c_0^2 \left( \eta + \frac{3}{2} \frac{\eta^2}{h_0} + \frac{1}{3} h_0^2 \eta_{xx} \right)_{xx} \]

- Canonical form:
  \[ u_{tt} = \alpha u_{xx} + \beta (u^2)_{xx} + \gamma u_{xxxx} ; \) S-integrable

- Regularized/“Bogolubsky” PDE:
  \[ u_{tt} = \alpha u_{xx} + \beta (u^2)_{xx} + \gamma u_{xxtt} \]
  same asymptotic approximation, non-integrable
Some most common SW models

- **Korteweg-de Vries (KdV)**
  \[ \eta^*_t + \eta^*_x + \frac{3}{2} \varepsilon \eta^*_x \eta^*_x + \frac{\delta^2}{6} \eta^*_x \eta^*_x \eta^*_x = 0 \]

- Unidirectional flow; Boussinesq regime

- Canonical form: \( u_t + 6uu_x + u_{xxx} = 0 \)

- S-integrable; multi-soliton solutions
Some most common SW models

11) The Camassa-Holm equation (Section 3.12)
\[ u_t - u_{xxx} + 3u u_x - 2u_x u_{xx} - uu_{xxx} = 0. \] (CH)

12) The two-component Camassa-Holm equations (Section 3.13)
\[ u_t - u_{xxx} + 3u u_x - 2u_x u_{xx} - uu_{xxx} + \rho \rho_x = 0, \]
\[ \rho_t + (\rho u) x = 0. \] (2-CH)

There are multiple ways to derive PDE systems mentioned in this work. Figure 3.1 shows a possible set of relationships as per the corresponding Derivation sections.

- **Benjamin-Bona-Mahony (BBM)**
  \[ \eta_{t*} + \eta_{x*} + \frac{3}{2} \varepsilon \eta_{x*} \eta_{x*} - \frac{\delta^2}{6} \eta_{x* x* t*} = 0 \]

- Coincides with the KdV in the Boussinesq regime approximation order:
  \[ \eta_{t*} = -\eta_{x*} + O(\varepsilon, \delta^2) \]

- Canonical form: \[ u_t + u_x + uu_x - u_{xxt} = 0 \]

- Non-integrable, not Galilei-invariant
Some most common SW models

- **Kaup-Boussinesq**

  \[ u_t + uu_x + h_x = 0 \]

  \[ h_t + (hu)_x + \beta^2 u_{xxx} = 0 \]

- **Dimensionless form:**

  \[ \hat{u}^*_{t^*} + \varepsilon \hat{u}^* \hat{u}_{x^*} + \eta_{x^*} = 0, \]

  \[ \eta_{t^*} + ((1 + \varepsilon \eta^*) \hat{u}^*)_{x^*} + \frac{\delta^2}{3} \hat{u}_{x^*x^*x^*} = 0 \]

- **Bidirectional; Boussinesq regime; S-integrable**
Some most common SW models

- **Shallow water equations**
  \[ u_t + uu_x + h_x = 0 \]
  \[ h_t + (hu)_x = 0 \]

- **Dimensionless form:**
  \[ u^*_t + \eta^*_x + \varepsilon u^* u^*_x = 0 \]
  \[ h^*_t + (h^* u^*)_x = 0 \]

- **Bidirectional; dispersionless regime** \( \delta \to 0; \text{C-integrable} \)
Where do different SW models come from?

- Approximation regime: relations between $\varepsilon$ and $\delta$
- Order of asymptotic approximation
- Substitutions within the same order, such as $\eta^* = -u^*_x + O(\varepsilon, \delta^2)$
- Use of a velocity variable different from the depth-averaged horizontal velocity

$$u(x, t) = \frac{1}{h(x, t)} \int_0^{h(x, t)} U(x, z, t) \, dz$$

for example, velocity at a fixed depth: $u(x, t) = U(x, z_0, t)$

- Use of “artificial physics” for PDEs obtained from other considerations, such as integrability requirement, e.g., the Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0$$

whose derivation was math-motivated
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The generalized shallow water equation

\[ \eta_t + \eta_x - \alpha \eta \eta_t + \beta \eta_x \partial_x^{-1} \eta_t - \eta_{xxt} = 0 \]

with

\[ \partial_x^{-1} \eta_t = \int_x^\infty \eta_t(s, t) \, ds \]

Local (potential) form: \( \eta = u_x \),

\[ u_{xt} + u_{xx} - \alpha u_x u_{xt} - \beta u_{xx} u_t - u_{xxx} = 0 \]

S-integrable if and only if \( \alpha/\beta = 2 \) or \( \alpha/\beta = 1 \)
A higher-order model with tension

- The combined Bona-Chen-Saut-Dullin-Gottwald-Holm system:

\[
\tilde{u}_t^* + \eta_x^* + \varepsilon \tilde{u}^* \tilde{u}_x^* + \frac{\delta^2}{2} (\theta^2 - 1) \tilde{u}_{xx}^* - \delta^2 \sigma^* \eta_{xxx}^*
\]

\[- \frac{\varepsilon \delta^2}{2} \left( 2(\eta^* \tilde{u}_{xx}^*)_x - (\theta^2 + 1) \tilde{u}_x^* \tilde{u}_x^* - (\theta^2 - 1) \tilde{u}^* \tilde{u}_{xx}^* \right)
\]

\[+ \frac{5}{24} \delta^4 (\theta^2 - 1) \left( \theta^2 - \frac{1}{5} \right) \tilde{u}_{xx}^* = O(\delta^6)
\]

\[\eta_t^* + ((1 + \varepsilon \eta^*) \tilde{u}^*)_x + \frac{\delta^2}{2} \left( \theta^2 - \frac{1}{3} \right) \tilde{u}_{xx}^* + \frac{\varepsilon \delta^2}{2} (\theta^2 - 1)(\eta^* \tilde{u}_{xx}^*)_x
\]

\[+ \frac{5}{24} \delta^4 \left( \theta^2 - \frac{1}{5} \right)^2 \tilde{u}_{xx}^* = O(\delta^6)
\]

- Nonzero surface tension coefficient \(\sigma^*\)

- Horizontal velocity \(\tilde{u}\) measured at an arbitrary dimensionless elevation \(\theta \in [0, 1]\) above the flat bottom

- Gives rise to many simpler SW models
A higher-order model with tension

- A higher-order single-PDE unidirectional version

- Can exclude $\tilde{u}^*$:

$$
\eta_{t^*} + \eta_{x^*} + \frac{3}{2} \varepsilon \eta^* \eta_{x^*} + \frac{\delta^2}{6} (1 - 3 \sigma^*) \eta_{x^* x^* x^*} - \frac{3}{8} \varepsilon^2 (\eta^*)^2 \eta_{x^*} \\
+ \frac{\varepsilon \delta^2}{24} ((23 + 15 \sigma^*) \eta_{x^* x^*} \eta_{x^*} + 2 (5 - 3 \sigma^*) \eta^* \eta_{x^* x^* x^*}) \\
+ \frac{\delta^4}{360} (19 - 30 \sigma^* - 45 (\sigma^*)^2) \eta_{x^* x^* x^* x^* x^*} = O(\delta^6)
$$

- Generalizes the KdV
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What is integrability?

- Many definitions, with unclear relationships
- C-integrability
- S-integrability
- Symmetry- and conservation law-integrability: existence of higher-order symmetries and conservation laws.
- Integrability in the Hirota sense: multi-soliton solutions
- Liouville (“complete”) integrability: a Hamiltonian PDE system that has an infinite number of conserved densities in pairwise involution. True for bi-Hamiltonian PDE systems
- Formal integrability: existence of a recursion operator
- Painlevé integrability: pass the Painlevé test
- ...
Related analytical structures

- Lax pairs and zero-curvature representation
- Infinite-parameter and/or higher-order Lie-type symmetries
- Infinite-parameter and/or higher-order local conservation laws
- Painlevé property
- Variational principles and Lagrangian structure
- Hamiltonian and bi-Hamiltonian structure, recursion operators
C-integrability

C-integrability: “A nonlinear partial differential equation is called C-integrable if it can be solved by a Change of variables”

Example: the shallow water (SW) system

\[
\begin{align*}
    u_t + uu_x + gh_x &= 0 \\
    h_t + (hu)_x &= 0
\end{align*}
\]

- a point hodograph transformation

\[
\begin{align*}
    x &= x(h, u), \\
    t &= t(h, u), \\
    h(x, t) &= h, \\
    u(x, t) &= u
\end{align*}
\]

- invertible linearization:

\[
\begin{align*}
    x_u - u t_u + h t_h &= 0 \\
    x_h - u t_h + g t_u &= 0
\end{align*}
\]

Another famous example: the Burgers equation \( u_t + uu_x = \nu u_{xx} \)

- the Hopf-Cole transformation \( u = -2 \nu w_x / w \)
- \( w \) satisfies the heat equation \( w_t = \nu w_{xx} \)
S-integrability: “the possibility of construction of solutions of a given PDE system using a Spectral transform technique”

- **Main ingredient:** a nontrivial Lax pair \((L, P)\), isospectral flow
  \[
  L \psi = \lambda \psi, \quad \psi_t = P \psi,
  \]
  \[
  L_t = [P, L] \equiv PL - LP
  \]

- **More generally:** a zero-curvature representation with matrix operators \(U, V\)
  \[
  \hat{\psi}_x = U \hat{\psi}, \quad \hat{\psi}_t = V \hat{\psi},
  \]
  \[
  U_t - V_x + [U, V] = 0
  \]

**Main application: exact solutions**

- **Inverse scattering**
- **Darboux transformation** can be used to iterate solutions
- **Multi-soliton solutions through Hirota’s bilinear method**
S-integrability

Lax pairs:
- Art or inspiration?
- Existence hinted by other analytical structure (bi-Hamiltonian form, symmetries, etc.)
- May be systematically constructed in WTC/Painlevé analysis

Example: Kaup-Boussinseq model

\[
\begin{align*}
  u_t + uu_x + h_x &= 0 \\
  h_t + (hu)_x + \frac{1}{4}u_{xxx} &= 0
\end{align*}
\]

Lax pair:

\[
\psi_{xx} = \left( \lambda^2 - \lambda u - h + \frac{1}{4}u^2 \right) \psi, \quad \psi_t = -\left( \lambda + \frac{1}{2}u \right) \psi_x + \frac{u_x}{4} \psi
\]

ZCR: \( \hat{\psi}_x = U \hat{\psi}, \hat{\psi}_t = V \hat{\psi} \) where

\[
U = \begin{pmatrix} 0 & 1 \\ \lambda^2 - \lambda u - h + \frac{1}{4}u^2 & 0 \end{pmatrix}, \quad V = \frac{1}{4} \begin{pmatrix} u_x & -2(u + 2\lambda) \\ u_{xx} & -u_x - 2(u + 2\lambda) D_x \end{pmatrix}
\]
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Symmetries

- **Symmetries** are transformations of variables that preserve the solution set of the model.

- **Point symmetries:**

  \[(z^*)^i = f^i(z, u; \epsilon) = z^i + \epsilon \xi^i(z, u) + O(\epsilon^2), \quad i = 1, \ldots, n\]

  \[(u^*)^\mu = g^\mu(z, u; \epsilon) = u^\mu + \epsilon \eta^\mu(z, u) + O(\epsilon^2), \quad \mu = 1, \ldots, m\]

- More general: local

  \[\hat{z}^i = z^i, \quad i = 1, \ldots, n,\]

  \[\hat{u}^\mu = u^\mu + \epsilon \zeta^\mu[u] + O(\epsilon^2), \quad \mu = 1, \ldots, m,\]

- Infinitesimal generators:

  \[X = \xi^i \partial_{z^i} + \eta^\mu \partial_{u^\mu}, \quad \hat{X} = \zeta^\mu[u] \partial_{u^\mu} .\]

- **Lie-type symmetries** (and extensions like contact, higher-order, nonlocal, approximate, ...) can be systematically sought using Lie’s algorithm.
Symmetries

Some basic one-parameter Lie groups of point symmetries:

- Translations in space and time, generated by $X_1 = \partial_x$, $X_2 = \partial_t$, with global groups
  \[
  x^* = x + \epsilon, \quad t^* = t, \quad u^* = u,
  \]
  \[
  x^* = x, \quad t^* = t + \epsilon, \quad u^* = u.
  \]

- Scaling symmetries, generated by, for example, $X = Ax \partial_x + Bt \partial_t + Cu \partial_u$, where $A$, $B$, $C$ are constants, with global group
  \[
  x^* = xe^{A\epsilon}, \quad t^* = te^{B\epsilon}, \quad u^* = ue^{C\epsilon}.
  \]

- The Galilei symmetry, generated by, for example, $X = t \partial_x + \partial_u$, with global group
  \[
  x^* = x + \epsilon t, \quad t^* = t, \quad u^* = u + \epsilon.
  \]

Main applications:

- Construction of invariant reductions, invariant solutions
- Mapping structures such as conservation laws; iterate exact solutions
- Can hint linearizing invertible transformations for C-integrable models
- Equivalence transforms: mappings relating PDE families, parameter reduction
Symmetry families parameterized by arbitrary functions can lead to invertible linearization (C-integrability)

Example: An infinite-dimensional set of point symmetries of the SW equations

\[ X_\infty = A \partial_x + B \partial_t , \]

where \((A, B) = (A(u, h), B(u, h))\) is an arbitrary solution of the linear PDE system

\[ u A_u + h A_h - (u^2 - gh) B_u = 0 , \quad u B_u - h B_h - A_u = 0 . \]

Infinite countable hierarchies of higher-order symmetries of increasing order are associated with S-integrability (no guarantees!)

Example: Korteweg-de Vries equation \( u_t + 6uu_x + u_{xxx} = 0 \):

\[ \hat{X}_i = K_i \partial_u , \quad K_0 = u_x , \quad K_1 = 6uu_x + u_{xxx} , \]
\[ K_2 = 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{xxxx} , \]
\[ \ldots \]
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Symmetries

- **Symmetry-invariant:** e.g., traveling wave \( u(x, t) = u(x - ct) \)

- **Example:** Korteweg-de Vries equation \( u_t + 6uu_x + u_{xxx} = 0 \), single-soliton solution
  \[
  u(x, t) = \frac{a^2}{2} \operatorname{sech}^2 \left( \frac{a}{2}(x - x_0 - ct) \right) + u_0
  \]
Symmetries

- Multi-soliton solutions

- For example, using Hirota’s bilinear method
Symmetries

- **Cnoidal traveling waves** for Su-Gardner equations

\[
    u_t + uu_x + gh_x = \frac{1}{3h} \left( h^3 (u_{xt} + uu_{xx} - (u_x)^2) \right)_x ,
\]

\[
    h_t + (hu)_x = 0
\]
Cnoidal traveling waves for Su-Gardner equations

\[ u_t + uu_x + gh_x = \frac{1}{3h} \left( h^3 (u_{xt} + uu_{xx} - (u_x)^2) \right)_x , \]
\[ h_t + (hu)_x = 0 \]
Symmetries

- Solutions using “nonclassical symmetries”:

- Example: merging solitons for the generalized shallow water equation

\[
\eta_t + \eta_x - \alpha \eta \eta_t + \beta \eta_x \partial_x^{-1} \eta_t - \eta_{xxt} = 0
\]
Symmetries

- Weak peakon solutions for the Camassa-Holm equation

\[ u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \]

- A “close encounter” of two peakons
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Conservation laws

- **Conservation laws** provide local and global conserved quantities

- (1+1)-D:
  \[ D_t \Theta[u] + D_x \Phi[u] = 0. \]

- Global quantity:
  \[ \frac{d}{dt} C[u] = \frac{d}{dt} \int_a^b \Theta[u] \, dx = -\Phi[u] \bigg|_a^b, \]

- Divergence expressions can be **systematically obtained using the multiplier method & Euler operators**
  \[ D_i Z^i[u] \equiv \Lambda_\sigma[u] R^\sigma[u] = 0, \]
  \[ E_{\omega j} (\Lambda_\sigma[u] R^\sigma[u]) \equiv 0, \quad j = 1, \ldots, m \]

**Main applications:**

- **Conservative numerical methods**
- Relations to C- and S-integrability
- Relations to symmetries (Noether’s theorem)
- A tool to systematically construct **nonlocally related** (such as potential) PDE systems that lead to new results
Conservation laws

- Example: conservation laws for Su-Gardner equations

\[
\begin{align*}
    u_t^* + \varepsilon u^* u_x^* + \eta_x^* &= \frac{\delta^2}{3h^*} \left( (h^*)^3 \left( u_{x^* t^*}^* + \varepsilon u^* u_{x^* x^*}^* - \varepsilon (u_{x^*}^*)^2 \right) \right)_{x^*}, \\
    h_t^* + \varepsilon (h^* u^*)_{x^*} &= 0
\end{align*}
\]

\[
\begin{align*}
    D_t h + D_x (hu) &= 0, \\
    D_t (hu) + D_x \left( hu^2 + \frac{1}{2} gh^2 + \frac{1}{3} h^3 (u_x^2 - u_{xt} - uu_{xx}) \right) &= 0, \\
    D_t \left( \frac{1}{2} h \left( u^2 + gh + \frac{1}{3} h^2 u_x^2 \right) \right) + D_x \left( hu \left( \frac{1}{2} u^2 + gh + \frac{1}{2} h^2 u_x^2 - \frac{1}{3} h^2 (u_{xt} + uu_{xx}) \right) \right) &= 0, \\
    D_t \left( u - hh_x u_x - \frac{1}{3} h^2 u_{xx} \right) + D_x \left( \frac{1}{2} u^2 + gh + hh_t u_x + \frac{1}{2} h^2 u_x^2 - \frac{1}{3} h^2 uu_{xx} \right) &= 0, \\
    D_t \left( h(t u - x) \right) + D_x \left( hu(t u - x) + \frac{1}{2} g t h^2 + \frac{1}{3} t h^3 (u_x^2 - u_{xt} - uu_{xx}) \right) &= 0
\end{align*}
\]
Conservation laws

- **Example:** conservation laws for the KdV \( u_t + 6uu_x + u_{xxx} = 0 \)
- **An infinite set** of polynomial higher-order CLs related to \( S \)-integrability

\[
D_t u + D_x(3u^2 + u_{xx}) = 0,
\]
\[
D_t \left(\frac{1}{2}u^2\right) + D_x \left(2u^3 - \frac{1}{2}u_x^2 + uu_{xx}\right) = 0,
\]
\[
D_t \left(u^3 - \frac{1}{2}u_x^2\right) + D_x \left(\frac{9}{2}u^4 + u_xu_t + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2\right) = 0,
\]
\[\vdots\]
Conservation laws

- dimensionless Korteweg-de Vries (KdV)

\[
\eta_{t^*} + \eta_{x^*} + \frac{3}{2} \varepsilon \eta \eta_{x^*} + \frac{\delta^2}{6} \eta_{x^* x^* x^*} = 0
\]

- versus Benjamin-Bona-Mahony (BBM):

\[
\eta_{t^*} + \eta_{x^*} + \frac{3}{2} \varepsilon \eta \eta_{x^*} - \frac{\delta^2}{6} \eta_{x^* x^* x^* t^*} = 0
\]

- same order of asymptotic approximation!

- KdV is **S-integrable**; it has an **infinite hierarchy** of higher-order local CLs

- BBM is **not integrable**; it has **exactly three** local CLs
Outline

1. Euler and Navier-Stokes
2. Shallow water models
3. Higher-order and nonlocal models
4. Integrability
5. Symmetries
6. Exact solutions
7. Conservation laws
8. **Hamiltonian and multi-Hamiltonian structure**
9. Dispersion relations
10. Conclusions
Hamiltonian structure

A Hamiltonian evolution equation:

\[ u_t = K[u] = \mathcal{D} \cdot \frac{\delta H}{\delta u} \]

Main applications:

- Conserved Hamiltonian density, Casimirs
- Hamiltonian version of Noether's theorem
- Bi-Hamiltonian & multi-Hamiltonian structures; recursion operators

A bi-Hamiltonian system:

\[ u_t = K_1[u] = \mathcal{D} \cdot \frac{\delta H_1}{\delta u} = \mathcal{E} \cdot \frac{\delta H_0}{\delta u} \]

Recursion operator:

\[ \mathcal{R} = \mathcal{E} \cdot \mathcal{D}^{-1} \]
Example: a tri-Hamiltonian structure for the SW system

\[ u_t + uu_x + gh_x = 0, \]
\[ h_t + (hu)_x = 0 \]

\[ H_0 = \frac{1}{2} \left( hu^2 + gh^2 \right), \quad H_1 = hu, \quad H_2 = h, \]

\[ D_0 = - \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}, \quad D_1 = -\frac{1}{2} \begin{pmatrix} 2gD_x & uD_x + u_x \\ uD_x & hD_x + D_xh \end{pmatrix}, \]
\[ D_2 = - \begin{pmatrix} g(uD_x + D_xu) & \left( \frac{1}{2}u^2 + 2gh \right)D_x + uu_x + gh_x \\ \left( \frac{1}{2}u^2 + 2gh \right)D_x + gh_x & uhD_x + D_xuh \end{pmatrix} \]

Recursion operators:

\[ R_1 = D_1 \cdot D_0^{-1}, \quad R_2 = D_2 \cdot D_0^{-1}, \quad R_3 = D_2 \cdot D_1^{-1}, \]
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Dispersion relations

- Stability of equilibrium

- Linearization about \( u = u_0(x) \):

  \[
  u^\mu(x, t) = u_0^\mu + \epsilon u_1^\mu e^{i(kx - \omega t)}, \quad \mu = 1 \ldots, m,
  \]

- Dispersion relations:

  \[
  \omega = \omega(k), \quad c = \frac{\omega}{k} = c(k).
  \]

- **Stability:** \( \text{Im} \omega = 0 \) for all \( k \)

- **Dispersion:** linearized waves of different wavelengths travel at different speeds
Dispersion relations

- Full water wave problem:

\[
\phi_{xx} + \phi_{zz} = 0, \quad 0 < z < h(x, t),
\]

\[
\phi_z = 0 \quad \text{at} \quad z = 0,
\]

\[
\eta_t + \phi_x \eta_x - \phi_z = 0, \quad \phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + g \eta = 0 \quad \text{at} \quad z = h(x, t)
\]

- Perturbation of zero state:

\[
h = h_0 + \eta, \quad \eta = \epsilon \eta_0 e^{i(kx - \omega t)}, \quad \phi = \epsilon f(z) e^{i(kx - \omega t)}
\]

- Dispersion relation:

\[
\omega^2 = gk \tanh kh_0, \quad c = c_0 \sqrt{\frac{\tanh kh_0}{kh_0}}
\]
Dispersion relations

Compare:

- **Full water wave dispersion relation:**
  \[ \omega^2 = gk \tanh kh_0, \quad c = c_0 \sqrt{\frac{\tanh kh_0}{kh_0}} \]

- **SG:**
  \[ \omega = \frac{c_0 k}{\sqrt{1 + h_0^2 k^2 / 3}}, \quad c = \frac{c_0}{\sqrt{1 + h_0^2 k^2 / 3}} \]

- **KdV:**
  \[ \omega = c_0 k \left(1 - \frac{1}{6} h_0^2 k^2\right), \quad c = c_0 \left(1 - \frac{1}{6} h_0^2 k^2\right) \]

- **BBM:**
  \[ \omega = \frac{c_0 k}{1 + h_0^2 k^2 / 6}, \quad c = \frac{c_0}{1 + h_0^2 k^2 / 6} \]

- **SW:** \( c \neq c(k) \) so no dispersion
  \[ \omega^2 = c_0^2 k^2, \quad c = c_0 = \sqrt{gh_0} \]
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Some most common SW models

A diagram of physical relations between some shallow water models:

- (NS)
- (Euler:2D)
- (Burgers)
- (BSq)
- (SG)
- (SW)
- (BCS:DGH)
- (2-CH)
- (cBSq)
- (KdV)
- (BCS:DGH:1)
- (KB)
- (W:H)
- (CH)
- (ocBSq)
- (BBM)
- (W)
- (gSW)
PDE naming and the Arnold’s principle

Arnold’s principle
If a model bears a name, it is not the name of the person who discovered it

Examples:

- Korteweg-de Vries $\rightarrow$ Boussinesq (25 years earlier)
- Su-Gardner (Green-Naghdi) $\rightarrow$ Serre (13 and 21 years earlier)
- Camassa-Holm $\rightarrow$ Fokas and Fuchssteiner (12 years earlier)
Too many things left out...

- **Variational/Lagrangian structure**: self-adjointness of linearization
- **Painlevé property**: all linear equations pass; related to integrability; Camassa-Holm as counterexample
- Solution existence, uniqueness, stability
- Numerical aspects
- MANY extended PDE models
- Multi-dimensional versions
- ... and more ...
Thank you for your attention!