SYMMETRIES AND EXACT SOLUTIONS OF PLASMA EQUILIBRIUM EQUATIONS

by

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Abstract

The description of plasma as a continuous medium is employed in many applications, including thermonuclear fusion studies and astrophysics. The most widely used continuum plasma models are the isotropic magnetohydrodynamics (MHD) and the anisotropic (tensor-pressure) Chew-Goldberger-Low (CGL) equations.

In this work, time-independent (equilibrium) plasma flows and static configurations are considered.

MHD and CGL equilibrium systems are essentially non-linear. Due to their complexity, the knowledge of their analytical structure is very limited. Only several exact solutions have been found so far; most of these apply to the static isotropic equilibrium case and have generally non-physical behaviour. The knowledge of exact particular solutions and other analytical properties of these systems is desirable for both direct modeling and effective numerical simulations.

In the current work, we perform an analytical study of symmetries and other properties of the MHD and CGL equilibrium systems and develop methods of construction of exact solutions.

A correspondence is established between Bogoyavlenskij symmetries [1, 2] of the MHD equilibrium equations and Lie point transformations of these equations. We show that certain non-trivial Lie point transformations (that are obtained by direct application of the Lie method) are equivalent to Bogoyavlenskij symmetries.

Also, an infinite-dimensional set of transformations between solutions of isotropic (MHD) and anisotropic (CGL) equilibria is presented. These transformations depend on the topology of the original solution and allow the building of a wide class of anisotropic plasma equilibrium solutions with different geometries and topologies, including 3D solutions with no geometrical symmetries. The solutions obtained from the transformations satisfy necessary physical applicability and stability conditions. Examples are given.

We show that anisotropic (CGL) plasma equilibria possess topology-dependent infinite-dimensional symmetries that generalize Bogoyavlenskij symmetries for the isotropic case. The symmetries can be used to construct new anisotropic plasma equilibrium solutions from known ones.

A representation of the static MHD equilibrium system in coordinates connected with magnetic surfaces is suggested. It is used for producing families of non-trivial exact solutions of isotropic and anisotropic plasma equilibria with and without dynamics, and often without geometrical symmetries. The ways of finding coordinates in which exact equilibria can be constructed are discussed; examples and their applications as physical models are presented.

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Chapter 1

Introduction

1.1 Plasma in nature. Phenomena and applications involving plasma

To describe *plasma*, a highly ionized gas, a term ”the fourth state of matter” is often used. It follows from the idea that as heat is added to a solid, it undergoes a phase transition to a new state, usually liquid. If heat is added to a liquid, a phase transition to the gaseous state takes place. The addition of more energy to the gas results in the ionization of some of the atoms. At a temperature above 100 000 K most matter exists in an ionized state.

The majority of visible astrophysical objects exist in the plasma state. Plasma makes up stars, quasars and jets; it exists in the Earth ionosphere; farther out from the Earth, plasma is trapped in the earth's magnetic field in the near vacuum of space. Plasma streams toward the Earth from the Sun (the solar wind), and fills many regions of interstellar space, forming the medium through which outer space is viewed.

Among the most important potential and (in some cases) existing practical uses of *man-made plasmas* one can name nuclear-fusion devices, magnetohydrodynamic (MHD) generators, plasma propulsion systems, gaseous lasers, arc jets, and fluorescent tubes.

Reliable and precise *models of plasma* describing behaviour that would give good agreement with experimental data would not only create and simplify applications, but also serve to understand the core of many important phenomena in astrophysics, atmospherical sciences and other areas of great interest and importance.

Several examples of MHD applications are given below.

Nuclear fusion. The nuclear-fusion devices could have the greatest impact from a practical standpoint, but they have also proved to be the most difficult to develop. The basic difficulty in achieving fusion is that the process requires the interacting particles to approach within a distance of order 10^{-14} m [3]. Therefore fusion will generally not occur in great numbers until the temperatures come close to 10^5 eV, or 10^9 K. (Actually, since at any temperature, some of the particles have energies well above average, some fusion occurs at temperatures as low as about $4{\cdot}10^7$ K.)

Important fusion reactions are:

$$H^2 + H^2 \Rightarrow He^3 + n^1 \quad +3.27\ \text{MeV}$$

$$H^2 + H^2 \Rightarrow H^3 + H^1 \quad +4.03\ \text{MeV}$$

$$H^2 + H^3 \Rightarrow He^4 + n^1 \quad +17.58\ \text{MeV}$$

$$H^2 + He^3 \Rightarrow He^4 + H^1 \quad +18.34\ \text{MeV}$$

For example, the third reaction (between deuterium and tritium) effectively goes under the temperatures of about $(1-2)\cdot 10^8$ K when the so-called Lawson criterium holds: $n\tau > 10^{14}$, where τ is plasma lifetime (in seconds), and n the number of particles per unit volume (cm^3).

The amount of energy released in these fusion reactions is very high. For comparison, the energy released in a typical fission reaction is about 200 MeV, but as the atomic weight of the fuel (U^{235} or Pu^{239}) is about 240, the energy release per unit mass is actually lower than in the above fusion reactions. Another example is given by Bishop [4], who notes that as much energy is obtained from the fusion of the deuterium in 1 gal of ordinary water as is obtained from the combustion of 300 gal of gasoline.

A possible way to achieve the temperatures necessary to ignite the fusion reaction is to heat confined plasma. The problem of plasma confinement may be said to be the most industrially important application of MHD studies.

Astrophysical applications. Astrophysics is one of the original and most wide areas of plasma research. Here we list several directions where magnetohydrodynamics (MHD) framework described below is extensively used for description and modelling purposes.

(i) Astrophysical jets. The evidence for highly collimated (cone angle $\leq 20°$) *jets* in astrophysics goes back to the early radio observations of twin lobes in extended radio galaxies [5]. These jets were discovered to emit also in the optical, X-, and γ-ray bands, and their relationship with very high-energy phenomena originating in the deep cores of active galactic nuclei (AGNs) was definitively established. Modelling of collimated outflows from AGNs has been one of the most challenging problems in astrophysics in recent years.

Magnetohydrodynamics, especially analytically derived solutions and dependencies, is extensively used in modelling the phenomenon of astrophysical jets. Appropriate references are [5]-[9].

(ii) Star formation. Gravitational and plasma effects play the determining role in the process of the formation of stars from molecular clouds. Different parts of plasma theory, in particular, the classical MHD equations, have been employed to describe various aspects of star formation. For example, the quasistationary phase can be effectively modelled by ideal (infinitely conductive) MHD equilibrium system. The finite plasma conductivity effects - ambipolar diffusion and the resulting magnetic cloud collapse - are taken into account by using models with non-zero resistivity.

Many papers devoted to the description and modelling of star formation use the MHD approximation. Examples are [10]-[14].

(iii) Earth magnetosheath. The anisotropic formulation of classical magnetohydrodynamics has recently been applied in the studies of the Earth magnetosphere, in particular, of the Earth magnetosheath under the solar wind pressure. Experimental data, empiric relations between plasma parameters and their explanation using the combined numerical and analytical approach can be found in [15, 16].

Ball lightning models. The phenomenon of ball lightning has been reported to have many remarkable and mysterious properties; it always attracted great attention of both experimental (e.g. [17]) and theoretical researchers. Many authors try to explain ball lightning using the MHD framework. For the lack of a commonly accepted theory, it has been proposed that the problem be split into three different partial problems [18], viz., the problem of confining gas or plasma stably for a few seconds (or even minutes) to a finite volume, the question of what mechanism is responsible for the observed electrical properties, and the problem of the energy source, which produces heat and light. Concerning the first partial problem, most answers are based on Hill's vortex [19] in fluid dynamics or on spheromak-like configurations in plasma physics (see e.g. [20, 21]). All MHD equilibrium fireball models that have a magnetic field decaying at infinity, in fact, have a vanishing magnetic field outside a simply connected plasma region and are confined by the atmospheric pressure only.

In several papers (e.g. [22, 23]) an attempt has been made to model a ball lightning utilizing *purely magnetic*, force-free configurations as opposed to those having non-constant plasma pressure.

1.2 Plasma description models

In this work, we write all equations in SI units, which appear to be the most natural system for plasma physics.

The most general and precise way to describe motion of gas is the statistical approach, involving *Boltzmann equation*.

Consider number dn_α of particles of type α in a differential volume $d\mathbf{r}d\mathbf{v}$ of the phase space: $dn_\alpha = f_\alpha\ (\mathbf{r},\mathbf{v},t)d\mathbf{r}d\mathbf{v}$ (here $d\mathbf{r}$ is a volume, $d\mathbf{v}$ – volume in velocity space). Then the probability distribution function $f_\alpha\ (\mathbf{r},\ \mathbf{v},\ t)$ satisfies the Boltzmann equation, which is merely the conservation relation for particles [24, 25]:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot \mathrm{grad}_{\mathbf{r}} f_\alpha + \mathbf{a}\cdot \mathrm{grad}_{\mathbf{v}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_{coll}, \tag{1.1}$$

here $\mathbf{a}$ is acceleration, $(\partial f_\alpha/\partial t)_{coll}$ is a collision term, i.e. time rate of change of f_α due to collisions.

Macroscopically observable quantities are found from the velocity momenta of the distribution function f.

We note that the Boltzmann equation is a one-body reduction of the Liouville equation for N-body distribution function $F(\mathbf{r}_1,...,\mathbf{r}_N,\ \mathbf{v}_1,...,\mathbf{v}_N,\ t)$:

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial F}{\partial \mathbf{r}_i}\mathbf{v}_i + \frac{\partial F}{\partial \mathbf{v}_i}\mathbf{a}_i^T\right) = 0,$$

where $\mathbf{a}_i^T$ is the total acceleration of particle i due to external and interparticle forces.

Under the assumptions that no new particles are born in the collisions and that collisions preserve total momentum, it appears that macroscopic mass and momentum conservation equations do not depend on the collision term $(\partial f_\alpha/\partial t)_{coll}$.

1.2.1 Isotropic MHD equations

To derive the ***macroscopic isotropic plasma equations*** for ordinary physical parameters, such as gas density, mean velocity, pressure, electric current, electric and magnetic field, one should use Maxwell's electromagnetic field equations and some natural simplifying assumptions [24, 25]:

- The plasma is nearly isotropic;
- The plasma is neutral;
- $m_e/m_i << 1$;
- Number of particles and momenta are conserved in collisions (i.e. the plasma is highly ionized);
- The interchange of momentum between electrons and ions is proportional to current:
- $\int m_e \mathbf{v} \left(\partial f_e/\partial t\right)_{coll\ ei} d\mathbf{v} = -n_e q_e \eta \mathbf{J}$ (here η is resistivity).

Macroscopic parameters are then expressed through microscopic ones, e.g. the density is $\rho = n_e m_e + n_i m_i$, the current density, $\mathbf{J} = n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i$, and velocity averaging is carried out.

Here m is mass, n, concentration; q, charge; $\mathbf{v}$, velocity. The lower index i denotes ions; e, electrons.

Finally, the ***system of Isotropic Magnetohydrodynamic (MHD) equations*** takes the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0, \tag{1.2}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = \rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} \ P - \rho \ \operatorname{grad} \frac{\mathbf{V}^2}{2} + \mu_1 \triangle \mathbf{V}, \tag{1.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta \triangle \mathbf{B}; \quad \eta = \frac{1}{\sigma \mu}, \tag{1.4}$$

$$\operatorname{div} \mathbf{B} = 0, \quad \mathbf{J} = \frac{1}{\mu} \operatorname{curl} \mathbf{B}. \tag{1.5}$$

Here $\mathbf{V}$ is plasma velocity, $\mathbf{B}$ is magnetic field, $\mathbf{J}$, electric current density, ρ, plasma density, μ, the magnetic permeability of free space, σ, conductivity coefficient, μ_1, the plasma viscosity coefficient; η, resistivity coefficient. The usual scalar Laplace operator is denoted by $\triangle$.

Remark 1. For a vanishing magnetic field, $\mathbf{B} = 0$, the above system is reduced to Navier-Stokes equations of motion of a viscous fluid.

Remark 2. The MHD system must be closed with an additional equation of state. In the case of incompressible plasmas, the equation

$$\operatorname{div} \ \mathbf{V} = 0$$

is added to the above system; for compressible cases an appropriate equation of state must be chosen. For example, it can be the ideal gas equation of state coupled to the adiabatic process:

$$P = \rho^\gamma \exp(S/c_v), \ \frac{\partial S}{\partial t} + \mathbf{V} \cdot \operatorname{grad} \ S = 0. \tag{1.6}$$

Here c_v is the heat capacity at constant volume; γ, the adiabatic exponent; and S, entropy density.

Remark 3. The system (1.2)-(1.5) can be derived independently, using only Maxwell equations and conservation principles (see e.g. [3]). But the Boltzmann equation lets one as easily obtain, for example, a two-fluid plasma theory that treats electrons and ions as independent fluids [25], or the anisotropic (tensor-pressure) model (see the next subsection).

1.2.2 Anisotropic magnetohydrodynamics (CGL) equations

As described above, the isotropic MHD approximation employs scalar pressure P and is valid when the mean free path of plasma particles is much less than the typical scale of the problem, so that the picture is maintained nearly isotropic via frequent collisions.

On the other hand, when the mean free path for particle collisions is long compared to Larmor radius (for instance, in strongly magnetized or rarified plasmas), the Chew-Goldberger-Low (CGL) anisotropic magnetohydrodynamics model should be used.

The CGL model is derived from Boltzmann and Maxwell equations under isotropy assumptions different from those for the MHD model [26]. Instead of expanding the density function in Boltzmann equation (1.1) in the powers of the mean free path, in the CGL approach the expansion is made in the powers of m_i/e, which is equivalent to the expansion in the powers of the Larmor radius.

The resulting system is anisotropic, because it has a distinguished direction – the direction of the magnetic field $\mathbf{B}$ [26]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0, \tag{1.7}$$

$$\rho\frac{\partial \mathbf{V}}{\partial t} = \rho\mathbf{V} \times \text{curl } \mathbf{V} - \frac{1}{\mu}\mathbf{B} \times \text{curl } \mathbf{B} - \text{div } \mathbb{P} - \rho \text{ grad } \frac{\mathbf{V}^2}{2} + \mu_1 \triangle \mathbf{V}, \tag{1.8}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl}(\mathbf{V} \times \mathbf{B}) + \eta \triangle \mathbf{B}; \quad \eta = \frac{1}{\sigma\mu}, \tag{1.9}$$

$$\text{div}\,\mathbf{B} = 0, \quad \mathbf{J} = \frac{1}{\mu}\,\text{curl}\,\mathbf{B}. \tag{1.10}$$

where $\mathbb{P}$ is a 3×3 pressure tensor with two independent components: the pressure along the magnetic field $p_{\|}$ and in the transverse direction $p_{\perp}$.

$$\mathbb{P} = \mathbb{I}p_{\perp} + \frac{p_{\|} - p_{\perp}}{\mathbf{B^2}}(\mathbf{BB}). \tag{1.11}$$

Here $\mathbb{I}$ is a unit tensor.

In the limit $p_{\perp} = p_{\|} = p$, CGL and MHD models coincide.

For the above system to be closed, one needs to add to it two equations of state. The original work by Chew, Goldberger and Low contains the "double-adiabatic" equations, which have been obtained in the assumption of vanishing of the pressure-transport tensor:

$$\frac{d}{dt}\left(\frac{p_{\perp}}{\rho B}\right) = 0, \quad \frac{d}{dt}\left(\frac{p_{\|}B^2}{\rho^3}\right) = 0. \tag{1.12}$$

However, if these equations are adopted, the CGL system does not reduce to the general MHD system in the limit $p_{\perp} = p_{\|} = p$, but contains an unnatural connection of magnetic field and density along the streamlines. Also, generally no plausible argument is available about why and when the pressure-transport tensor should vanish.

Experimental observations of anisotropic plasmas yield different empirical relations. For example, in the studies of the solar wind flow in the Earth magnetosheath, the relation

$$p_{\perp}/p_{\|} = 1 + 0.847(B^2/(2p_{\|})) \tag{1.13}$$

is proposed [27]. In the Compact Helical System (CHS) plasma confinement device, the anisotropy factors $p_{\|}/p_{\perp} \propto 3$ have been measured [28].

Therefore generally solutions to anisotropic plasma dynamics and equilibrium configurations with any physically reasonable equations of state would be of interest.

1.2.3 Properties of time-dependent MHD systems

In this section we list the most remarkable properties of the systems of isotropic and anisotropic magnetohydrodynamic equations: the MHD system (1.2)-(1.5) and the CGL system (1.7)-(1.10).

Unless explicitly stated otherwise, the following properties are true for both MHD and CGL systems.

Frozen-in magnetic field

Consider the evolution equation for the magnetic field - equations (1.4) and (1.9) of the MHD and CGL systems respectively - in the case of large magnetic Reynolds numbers $R_m = (vL)/\eta \gg 1$ (v, average speed; L, size of the system) i.e. when the term with magnetic permeability is negligible. Then, integrating this equation over some surface, one gets

$$\int\limits_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} = \int\limits_S \mathrm{curl}(\mathbf{V}\times\mathbf{B}) d\mathbf{S} = \oint\limits_{\partial S} (\mathbf{V}\times\mathbf{B}) dl = -\oint\limits_{\partial S} \mathbf{B}(\mathbf{V}\times d\mathbf{l}),$$

last equality is due to cyclic identity for triple product: $\mathbf{a}(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{c}\times\mathbf{a})$.

For a surface S moving with velocity $\mathbf{V}$, the change of $\mathbf{B}$-flux through S is due to both change of $\mathbf{B}$ and the surface:

$$\frac{d}{dt}\left(\int\limits_S \mathbf{B} d\mathbf{S}\right) = \int\limits_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} + \oint\limits_{\partial S} \mathbf{B}(\mathbf{V}\times d\mathbf{l}),$$

which is equal to zero by previous equation. Therefore

$$\left(\int_S \mathbf{B} d\mathbf{S} \right) = \text{const}$$

for any surface S moving with the fluid. This is known as a *frozen-in magnetic field condition*, meaning that the magnetic field itself is carried by the fluid. This statement has the name of *Kelvin's theorem.*

Lagrangian structure of the MHD equations

Note. From now on, unless specified otherwise, we will restrict ourselves to the case of nonviscous infinitely-conducting plasmas: $\mu_1 = \eta = 0$.

It is known that the nonviscous hydrodynamic equations of motion (Euler equations) can be derived from a variational principle. It turns out that the same applies to nonviscous infinitely-conducting plasma (the property of the frozenness of the magnetic field is essential here).

William Newcomb [29] has shown that *incompressible* isotropic plasmas (1.2)-(1.5), $\mu_1 = \eta = 0$, as well as the corresponding *adiabatic* isotropic system with

$$\frac{\partial S}{\partial t} + \mathbf{V} \cdot \text{grad } \ S = 0$$

and an arbitrary equation of state $U = U(S, \rho)$, admit variational formulations.

For the adiabatic flow case, the Lagrangian function is

$$L = T - W = \frac{\rho \mathbf{V}^2}{2} - \left(\rho U(S, \rho) + \frac{\mathbf{B}^2}{2\mu} \right).$$

and the corresponding Euler-Lagrange equations that follow from the Hamilton's variation principle give rise respectively to three projections of the momentum conservation equation (1.3), provided that all the other equations of the system are treated as constraints.

In [29] it is also shown that a similar Lagrangian formulation is admissible for the anisotropic Chew-Goldberger-Low system (1.7) - (1.10) with the double-adiabatic equations of state (1.12).

Conservation of helicity. Force-free fields

Magnetic field $\mathbf{B}$ is a solenoidal field $(\operatorname{div}\mathbf{B}) = 0$, therefore for any smooth magnetic field in a star-shaped domain a *vector potential* $\mathbf{A}$ exists: curl $\mathbf{A} = \mathbf{B}$. The *magnetic helicity*, the term introduced by Woltjer [30] in 1958, is a scalar (generally depending on time) quantity

$$H(t) = \int_V \mathbf{A} \cdot \text{curl } \mathbf{A}\, dV, \tag{1.14}$$

the integral is taken over the volume where the magnetic field exists. It can be defined this way only if the integral converges (e.g. when the system has finite size, or the magnetic field decreases rapidly enough at infinity.)

The following results obtained by Woltjer are worth mentioning here.

Theorem 1.1 (Woltjer) *Helicity (1.14) of time-dependent MHD configurations (1.2)-(1.5) without viscosity and of infinite volume is time-invariant.*

Definition. An equilibrium *force-free magnetic field* is a field satisfying the equations

$$\text{curl}\,\mathbf{B} = \alpha(\mathbf{r})\mathbf{B}, \quad \text{div}\,\mathbf{B} = 0, \tag{1.15}$$

Theorem 1.2 (Woltjer) *Among all the plasma equilibrium configurations with the same helicity H, the minimum of magnetic energy $E_m = \int_V \frac{\mathbf{B}^2}{2\mu} dV$ is achieved at the force-free configuration* curl $\mathbf{B} = \alpha\mathbf{B}$, *with* $\alpha = \text{const}$.

1.3 Isotropic and anisotropic plasma equilibrium equations

1.3.1 Isotropic ideal MHD equilibrium equations

The *equilibrium states* of isotropic moving plasmas are described by the system of *MHD Equilibrium equations*, which is obtained from (1.2)-(1.5) by imposing the time-independence on all variables. Under the assumptions of infinite conductivity and negligible viscosity, which is the case when Reynolds and magnetic Reynolds numbers are large ($R = (vL\rho)/\mu_1 \gg 1$, $R_m = (vL)/\eta \gg 1$; v, average speed; L, size of the system), the equations take the form [3]

$$\rho \mathbf{V} \times \text{curl } \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \text{curl } \mathbf{B} - \text{grad } P - \rho \text{ grad} \frac{\mathbf{V}^2}{2} = 0, \tag{1.16}$$

$$\text{div } \rho \mathbf{V} = 0, \quad \text{curl}(\mathbf{V} \times \mathbf{B}) = 0, \quad \text{div } \mathbf{B} = 0. \tag{1.17}$$

This system, as the dynamic one, must be closed with an additional equation of state. In the case of incompressible plasmas, the equation

$$\text{div } \mathbf{V} = 0 \tag{1.18}$$

is used; in the adiabatic ideal gas approximation, the time-independent reduction of (1.6) is applicable:

$$P = \rho^\gamma \exp(S/c_v), \ \mathbf{V} \cdot \text{grad } S = 0. \tag{1.19}$$

In the static case $\mathbf{V} = 0$, the above MHD equilibrium equations take the form

$$\text{curl } \mathbf{B} \times \mathbf{B} = \mu \text{ grad } P, \quad \text{div } \mathbf{B} = 0. \tag{1.20}$$

This system is often referred to as *"the system of plasma equilibrium equations"*.

Remark 1. If $\rho = \text{const}$, $\mathbf{V} = c\mathbf{B}$, $c = \text{const}$, then the dynamic equilibrium system (1.16)-(1.17) can be evidently algebraically reduced to the static system (1.20).

Remark 2. When the pressure $P = \text{const}$, the plasma equilibrium is called *force-free*, and the magnetic field satisfies (1.15).

Definition. A force-free field (1.15) with $\alpha(\mathbf{r}) = \text{const}$ is called a *Beltrami flow*.

1.3.2 Anisotropic (CGL) plasma equilibria

Analogously to the isotropic case, equilibria of anisotropic plasmas are described by the system obtained from (1.7)-(1.10) by dropping the time-dependence.

Introducing the *anisotropy factor*

$$\tau = \frac{p_\parallel - p_\perp}{\mathbf{B^2}} \tag{1.21}$$

and using vector calculus identities to write the divergence of the pressure tensor (1.11) explicitly, one finds

$$\text{div } \mathbb{P} = \text{grad} p_\perp + \tau \text{ curl } \mathbf{B} \times \mathbf{B} + \tau \text{grad} \frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \text{grad}\tau).$$

The resulting *CGL equilibrium* system thus takes the form

$$\begin{aligned} \rho \mathbf{V} \times \text{curl } \mathbf{V} - \left(\frac{1}{\mu} - \tau\right) \mathbf{B} \times \text{curl } \mathbf{B} = \; & \text{grad } \; p_\perp + \rho \; \text{grad} \frac{\mathbf{V}^2}{2} \\ & + \tau \, \text{grad} \frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \text{grad } \tau), \end{aligned} \tag{1.22}$$

$$\text{div } \rho \mathbf{V} = 0, \quad \text{curl}(\mathbf{V} \times \mathbf{B}) = 0, \quad \text{div } \mathbf{B} = 0. \tag{1.23}$$

For the *static case*, the above CGL equilibrium system rewrites as

$$\left(\frac{1}{\mu} - \tau\right) \text{curl } \mathbf{B} \times \mathbf{B} = \text{grad } \; p_\perp + \tau \; \text{grad} \frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \text{grad } \tau), \quad \text{div } \mathbf{B} = 0. \tag{1.24}$$

1.3.3 Possible equilibrium topologies

In this subsection, we briefly describe possible topologies of MHD and plasma equilibria.

Definition. The *magnetic field lines* of a given magnetic field $\mathbf{B}(\mathbf{r})$ are defined as parametric curves $(x(t), y(t), z(t))$ that are solutions to the dynamical system

$$dx/dt = B_1(x, y, z), \quad dy/dt = B_2(x, y, z), \quad dz/dt = B_3(x, y, z). \tag{1.25}$$

The same way *plasma streamlines* are defined as curves tangent to the plasma velocity vector field $\mathbf{V}(\mathbf{r})$.

We now separately consider several cases that require particular attention.

1. Isotropic and anisotropic equilibria with non-parallel V and B.

In this case, both MHD and CGL equilibrium systems contain an equation $\operatorname{curl}(\mathbf{V}\times\mathbf{B}) = 0$, therefore the field $\mathbf{V}\times\mathbf{B}$ in a simply connected domain is potential, i.e.

$$\mathbf{V}\times\mathbf{B} = \operatorname{grad}\Psi(\mathbf{r}), \quad \mathbf{V}\cdot\operatorname{grad}\Psi(\mathbf{r}) = 0, \quad \mathbf{B}\cdot\operatorname{grad}\Psi(\mathbf{r}) = 0.$$

This means that there exists a family of ***magnetic surfaces*** (or a foliation) $\Psi(\mathbf{r}) = \text{const}$, to which both $\mathbf{V}$ and $\mathbf{B}$ are tangent, and thus magnetic field lines and plasma streamlines lie on these surfaces. (The function $\Psi(\mathbf{r})$ introduced here may indeed be multivalued.)

2. Isotropic incompressible plasma equilibria with parallel V and B.

For such plasmas, combining the first equation of (1.17) and the incompressibility condition (1.18), one gets $(\operatorname{grad}\rho(\mathbf{r})\cdot\mathbf{V}) = 0$.

On the other hand, from the collinearity requirement, $\mathbf{V} = f(\mathbf{r})\mathbf{B}$, and hence from the solenoidality of $\mathbf{B}$ and $\mathbf{V}$ one also has $(\operatorname{grad} f(\mathbf{r})\cdot\mathbf{V}) = 0$.

Thus both $f(\mathbf{r})$ and $\rho(\mathbf{r})$ are constant on plasma streamlines and magnetic field lines (which coincide). By the application of transformations given in sec. 1.3.4 below, the isotropic MHD equilibrium system under consideration is reducible in this case to the static isotropic plasma equilibrium system (1.20).

3. Static isotropic plasma equilibria.

We now enumerate the possible topologies of static isotropic plasma equilibria, described by the system (1.20) [31, 32].

If (curl $\mathbf{B}$) and $\mathbf{B}$ are non-parallel, then the pressure P is non-constant, and there exists a family of magnetic surfaces enumerated by values of pressure:

$$\operatorname{curl}\mathbf{B}\cdot\operatorname{grad}P = 0, \quad \mathbf{B}\cdot\operatorname{grad}P = 0.$$

Consider now the remaining case (curl $\mathbf{B}$) || $\mathbf{B}$, which means $\mathbf{B}$ is a force-free field (1.15). Taking the divergence of both sides of the equation $\operatorname{curl}\mathbf{B} = \alpha(\mathbf{r})\mathbf{B}$, we obtain

$$\mathbf{B}\cdot\operatorname{grad}\alpha(\mathbf{r}) = 0, \quad \operatorname{curl}\mathbf{B}\cdot\operatorname{grad}\alpha(\mathbf{r}) = 0,$$

i.e. again (curl $\mathbf{B}$) and $\mathbf{B}$ are tangent to a family of magnetic surfaces, which are now surfaces of constant level of $\alpha(\mathbf{r})$.

The most "degenerate" case of isotropic plasma equilibria occurs when $\alpha(\mathbf{r}) = \alpha = \text{const}$, i.e. for *Beltrami flows*, as defined above in sec. 1.3. For these flows, generally no magnetic surfaces exist, and numerical evidence shows that magnetic field lines can be dense in some 3D region (e.g. *ABC-flows*).

4. Anisotropic plasma equilibria with parallel V and B.

A particular case of incompressible CGL equilibria (1.22)-(1.23) when $\mathbf{V}$ || $\mathbf{B}$ (or $\mathbf{V}$=0) and the anisotropy factor τ is constant on magnetic field lines (and streamlines) requires special consideration. In this case, using the theory presented in Chapter 3 below, it is also possible to show that lines of $\mathbf{V}$ and $\mathbf{B}$ are tangent to 2D magnetic surfaces. (It is not the case only when the magnetic field $\mathbf{B}$ is a Beltrami field.)

Conclusion.

All isotropic ((1.16)-(1.17)) and anisotropic ((1.22)-(1.23)) equilibrium configurations with $\mathbf{V}$ and $\mathbf{B}$ non-parallel, and all incompressible (div $\mathbf{V} = 0$) isotropic MHD equilibrium configurations with $\mathbf{V} \parallel \mathbf{B}$ (excluding Beltrami flows), have magnetic field lines and plasma streamlines that lie on 2D magnetic surfaces and therefore *are not dense in any 3D region.*

The same is true about anisotropic (CGL) plasma equilibria with $\mathbf{V} \parallel \mathbf{B}$ (or $\mathbf{V}$=0) and the anisotropy factor τ constant on magnetic field lines.

For all these configurations, there exist two possibilities [1]:

(a). All magnetic field lines go to infinity or are closed curves - then the magnetic surfaces are *not* uniquely defined;

(b). The magnetic field lines are dense on some magnetic surfaces. As remarked in [31], if a magnetic surface has no edges, and if the magnetic field $\mathbf{B}$ nowhere vanishes on it, then by a known theorem [33] it must be a toroid (a topological torus) or a Klein bottle. The latter cannot be realized in $\mathbb{R}^3$. Thus magnetic surfaces are generically topologically equivalent to tori $\mathbb{T}^2$.

We note that other particular magnetic surface configurations are possible, for example, ones with spherical magnetic surfaces, as will be shown in subsequent chapters.

Hence, in the *isotropic* case, *only* (i) Beltrami flows and (ii) compressible MHD equilibria with $\mathbf{V} \parallel \mathbf{B}$ can have magnetic field lines and plasma streamlines that are dense in some 3-dimensional domains.

In the *anisotropic* case, only configurations with $\mathbf{V} \parallel \mathbf{B}$ which are (i) compressible, or (ii) have the anisotropy factor τ non-constant on magnetic field lines, or (iii) Beltrami flows can have magnetic field dense in 3-dimensional domains.

1.3.4 Symmetries of the MHD and CGL equilibrium equations

In this subsection, we list known symmetries of the MHD and CGL equilibrium equations.

Reflection symmetry

The general system of equations of compressible MHD and CGL equilibria (1.16)-(1.17), (1.22)-(1.23) admit the following two independent reflection symmetries:

$$\mathbf{V} \to -\mathbf{V}, \;\; \mathbf{B} \to -\mathbf{B}.$$

("Symmetry" here means that the system of differential equations is invariant under a certain change of variables).

"Interchange symmetry"

If the density $\rho = \text{const}$, then by a scaling transform $\mathbf{V}_1 = \sqrt{\rho}\mathbf{V}$, $\mathbf{B}_1 = \sqrt{1/\mu}\mathbf{B}$ the MHD equilibrium system (1.16)-(1.17) can be rewritten in the invariant form [34]:

$$\mathbf{V}_1 \times \operatorname{curl} \mathbf{V}_1 - \mathbf{B}_1 \times \operatorname{curl} \mathbf{B}_1 - \operatorname{grad} P_1 = 0, \tag{1.26}$$

$$\operatorname{curl}(\mathbf{V_1} \times \mathbf{B}_1) = 0, \quad \operatorname{div} \mathbf{B}_1 = 0, \quad \operatorname{div} \mathbf{V}_1 = 0.$$

Here $P_1 = \left(P - {\mathbf{V}_1}^2/2\right)$.

Note 1. Under the additional assumption $\tau = \text{const}$, $\tau < 1/\mu$, the CGL equilibrium system (1.22)-(1.23) is also brought to the form (1.26).

If a solution to the system (1.26) $\{\mathbf{V}_1, \mathbf{B}_1, P_1\}$ is known, then evidently $\{\mathbf{B}_1, \mathbf{V}_1, -P_1\}$ is also a solution, i.e. the system is invariant under the transformation

$$\mathbf{V} \leftrightarrow \mathbf{B}, \quad P \rightarrow -P.$$

Note 2. CGL equilibria with $\tau = \mathrm{const}$, $\tau > 1/\mu$, evidently admit a similar symmetry

$$\mathbf{V} \leftrightarrow \mathbf{B}, \quad P \rightarrow P.$$

Therefore we conclude that a generic ideal MHD equilibrium has a group of symmetries $Z_2 \oplus Z_2$; ideal MHD equilibria with constant density and ideal CGL equilibria with constant density and τ have a group of symmetries $Z_2 \oplus Z_2 \oplus Z_2$.

3. Infinite symmetries of ideal isotropic MHD equilibria. Recently, O. I. Bogoyavlenskij [1, 2] has shown that the ideal isotropic MHD equilibrium equations (1.16)-(1.17), (1.18) also have the following families of intrinsic symmetries.

Let $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ be a solution of (1.16)-(1.17), where the density $\rho(\mathbf{r})$ is constant on both magnetic field lines and streamlines. Then $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), P_1(\mathbf{r}), \rho_1(\mathbf{r})\}$ is also a solution, where

$$\begin{aligned} &\mathbf{B}_1 = b(\mathbf{r})\mathbf{B} + c(\mathbf{r})\sqrt{\mu\rho}\mathbf{V}, \\ &\mathbf{V}_1 = \frac{c(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho}}\mathbf{B} + \frac{b(\mathbf{r})}{a(\mathbf{r})}\mathbf{V}, \\ &\rho_1 = a^2(\mathbf{r})\rho, \quad P_1 = CP + (C\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu). \end{aligned} \tag{1.27}$$

Here $b^2(\mathbf{r}) - c^2(\mathbf{r}) = C = \mathrm{const}$, and $a(\mathbf{r}), b(\mathbf{r}), c(\mathbf{r})$ are functions constant on both magnetic field lines and streamlines (i.e. on magnetic surfaces $\Psi = \mathrm{const}$, when they exist).

Transformations (1.27) are obviously applicable if magnetic surfaces exist (see sec. 1.3.3). In this case the functions $a(\mathbf{r})$, $b(\mathbf{r})$, $c(\mathbf{r})$ and the plasma density $\rho(\mathbf{r})$ must be constant on each of the surfaces.

As we have observed in sec. 1.3.3, for the majority of MHD equilibrium configurations (including the most general case of nonparallel $\mathbf{V}$ and $\mathbf{B}$) magnetic surfaces *do* exist, therefore these symmetries can be applied.

If magnetic surfaces *do not* exist, which happens when $\mathbf{V}$ and $\mathbf{B}$ are parallel and their field lines are dense in some 3D region, then to use the above symmetries all $a(\mathbf{r})$, $b(\mathbf{r})$, $c(\mathbf{r})$ and $\rho(\mathbf{r})$ must be constants in that region.

If magnetic field lines and plasma streamlines coincide and are closed or go to infinity, then the values of $a(\mathbf{r})$, $b(\mathbf{r})$, $c(\mathbf{r})$ can be chosen different on every magnetic field line.

The transformations (1.27) form an infinite-dimensional Abelian group [2]

$$G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2, \tag{1.28}$$

where R^+ is a multiplicative group of positive numbers, and A_m is an additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on magnetic field lines and plasma streamlines. The group G_m has eight connected components.

4. Infinite symmetries of compressible isotropic MHD equilibria. Another transformation, which is applicable to *compressible* MHD equilibria (1.16)-(1.17), is given by the following formulas [1, 2]:

$$\rho_1 = a^2(\mathbf{r})\rho, \quad \mathbf{B}_1 = b\mathbf{B}, \quad \mathbf{V}_1 = \frac{b}{a(\mathbf{r})}\mathbf{V}, \quad P_1 = b^2 P, \tag{1.29}$$

where $a(\mathbf{r})$ is an arbitrary smooth function that is constant on both magnetic field lines and streamlines, and $b \neq 0$ is a constant.

For the case of ideal gas undergoing an adiabatic process (1.19), this transformation changes the entropy as follows:

$$S_1 = S + 2c_v \left(\ln|b| - \gamma \ln|a(\mathbf{r})|\right). \tag{1.30}$$

The symmetries (1.29)-(1.30) form the subgroup

$$G_{0m} = A_m \oplus R^+ \oplus Z_2 \oplus Z_2, \tag{1.31}$$

which has four connected components.

1.3.5 Important reductions of the plasma equilibrium equations

The Grad-Shafranov (GS) equation

In 1958, Grad and Rubin [35] and Shafranov [36] have independently shown that the system (1.20) of static isotropic plasma equilibrium equations having axial symmetry (independent of the polar angle) is equivalent to one scalar equation, called *Grad-Shafranov equation*:

$$\Psi_{rr} - \frac{\Psi_r}{r} + \Psi_{zz} + I(\Psi)I'(\Psi) = -\mu r^2 P'(\Psi), \tag{1.32}$$

Here $P(\Psi)$ is plasma pressure, which in the axially symmetric case depends only on the unknown flux function $\Psi = \Psi(r, z)$; $I(\Psi)$ is an arbitrary function, and primes denote derivatives.

The magnetic field $\mathbf{B}$ here has the form

$$\mathbf{B} = \frac{\Psi_z}{r}\mathbf{e}_r + \frac{I(\Psi)}{r}\mathbf{e}_\varphi - \frac{\Psi_r}{r}\mathbf{e}_z,$$

and $(r,\ \varphi,\ z)$ are cylindrical coordinates.

Surfaces $\Psi = \mathrm{const}$ are the magnetic surfaces.

Remark. An analogue of Grad-Shafranov equation for anisotropic plasmas, the axially symmetric reduction of the CGL dynamic equilibrium equations (1.22)-(1.23) with double-adiabatic CGL equations of state (1.12), was constructed in [37]. However, compared to the original Grad-Shafranov equation, this equation has such a complicated form that hardly any non-trivial solutions can be found from it.

The JFKO equation

This is another reduction of the system of static isotropic plasma equilibrium equations (1.20), which describes helically symmetric plasma equilibrium configurations, i.e. configurations invariant with respect to the helical transformations

$$z \to z + \gamma h, \;\; \varphi \to \varphi + h, \;\; r \to r, \tag{1.33}$$

where $(r,\, \varphi,\, z)$ are cylindrical coordinates. Solutions to the JFKO equation therefore depend only on $(r,\, u)$, where $u = z - \gamma\varphi$.

The equation was obtained by Johnson *et al* [38] in 1958 and may be written in the form

$$\frac{\Psi_{uu}}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2+\gamma^2}\Psi_r\right) + \frac{I(\Psi)I'(\Psi)}{r^2+\gamma^2} + \frac{2\gamma I(\Psi)}{\left(r^2+\gamma^2\right)^2} = -\mu P'(\Psi). \tag{1.34}$$

The helically symmetric magnetic field is

$$\mathbf{B}_h = \frac{\psi_u}{r}\mathbf{e}_r + B_1\mathbf{e}_z + B_2\mathbf{e}_\phi, \quad B_1 = \frac{\gamma I(\Psi) - r\psi_r}{r^2+\gamma^2}, \quad B_2 = \frac{rI(\Psi) + \gamma\psi_r}{r^2+\gamma^2},$$

$(r,\, \varphi,\, z)$ are cylindrical coordinates. $I(\Psi)$ and $P(\Psi)$ are arbitrary functions.

As in the Grad-Shafranov reduction, the magnetic surfaces here are enumerated by the values $\Psi = \Psi(r,u) = \text{const}$.

1.3.6 Necessary properties for physical relevance of exact plasma equilibrium solutions

Though the first applications of the MHD and CGL equilibrium systems were astrophysical problems and plasma confinement for controlled thermonuclear fusion, the majority of exact solutions obtained so far have generally non-physical behaviour, e.g. they unboundedly grow at infinity, or have singularities, or their total energy is infinite.

Such solutions usually have only very restricted applicability to astrophysics, if they have applications at all.

Here we list several properties that physically relevant plasma equilibrium solutions should possess.

For solutions in a bounded domain $\mathcal{D}$ with the boundary $\partial\mathcal{D}$, one should demand

$$\begin{aligned}
&0 \leq P|_{\mathcal{D}} \leq \mathcal{P}_{max} \quad \text{(for anisotropic plasmas, } 0 \leq p_{\parallel}|_{\mathcal{D}},\ p_{\perp}|_{\mathcal{D}} \leq \mathcal{P}_{max});\\
&0 \leq \mathbf{B}^2|_{\mathcal{D}} \leq \mathcal{B}^2_{max}; \quad 0 \leq \mathbf{V}^2|_{\mathcal{D}} \leq \mathcal{V}^2_{max}; \quad 0 \leq \rho|_{\mathcal{D}} \leq \rho_{max};\\
&\mathbf{n}\cdot\mathbf{B}|_{\partial\mathcal{D}} = 0;\\
&\mathbf{n}\cdot\mathbf{V}|_{\partial\mathcal{D}} = 0 \ \text{ or } \ \mathbf{V}|_{\partial\mathcal{D}} = 0.
\end{aligned} \tag{1.35}$$

For an unbounded domain $\mathcal{D}$, the natural conditions are

$$\begin{aligned}
&0 \leq P|_{\mathcal{D}} \leq \mathcal{P}_{max} \quad \text{(for anisotropic plasmas, } 0 \leq p_{\parallel}|_{\mathcal{D}},\ p_{\perp}|_{\mathcal{D}} \leq \mathcal{P}_{max});\\
&0 \leq \mathbf{B}^2|_{\mathcal{D}} \leq \mathcal{B}^2_{max}; \quad 0 \leq \mathbf{V}^2|_{\mathcal{D}} \leq \mathcal{V}^2_{max}; \quad 0 \leq \rho|_{\mathcal{D}} \leq \rho_{max};\\
&P \ \ (\text{or } p_{\parallel}, p_{\perp}),\ \mathbf{B}^2,\ \mathbf{V}^2,\ \rho \ \rightarrow \ \text{const} \ \text{ at } \ |\mathbf{r}| \rightarrow \infty.
\end{aligned} \tag{1.36}$$

For localized solutions in vacuum, the asymptotic constants must be zero, and the magnetic field $\mathbf{B}$ and the velocity $\mathbf{V}$ should decrease at infinity quickly enough to give *finite total energy*

$$\int\limits_{\mathcal{D}} \left(\frac{\rho\mathbf{V}^2}{2} + \frac{\mathbf{B}^2}{2\mu} \right) dV < \infty; \tag{1.37}$$

For solutions in vacuum that are infinitely stretched in one dimension z (e.g. models of astrophysical jets), the above relations the should be satisfied in every layer $z_1 < z < z_2$. All magnetic field lines and plasma current lines should be bounded in the cylindrical radial variable r.

Several representative examples of available exact analytical solutions are given below.

1.4 Examples of analytical plasma equilibria

The systems of isotropic and anisotropic plasma equilibria (1.16)-(1.17), (1.22)-(1.23) are essentially non-linear systems of partial differential equations, for which no general methods of building solutions of boundary value problems are known.

Only several exact solutions to these systems have been found; the majority for the static isotropic case (1.20).

The anisotropic equations received even less attention in terms of studying their analytical properties; however substantially many numerical models employ these equations. The author is not aware of any non-trivial physically interesting analytical solutions for these equations for any equation of state.

In this section we review several classical solutions of the isotropic plasma equilibrium system (1.16)-(1.17).

1. Kadomtsev solution. An example of an isotropic static plasma equilibrium solution that grows unboundedly at infinity is a helically symmetric solution obtained by Kadomtsev in 1960 [39]. He studied plasma equilibrium with $\mathbf{B}$ and P depending on r and $\xi = kz\text{-}m\varphi$, with m integer, i.e. exactly in the JFKO reduction.

Taking $I(\Psi) = \text{const}$, he obtained the following analytical solution:

$$\Psi = -\frac{ar^2}{8}\left(k^2r^2 + 2m^2\right) - \frac{kr^2 I}{2m} + A\left(\frac{k^2r^2}{2} + m^2 \ln r\right) + \Psi_1,$$

$$\Psi_1 = (kr)\left(BI'_m(kr) + CK'_m(kr)\right)\sin\xi,$$

$$P = P_0 + a\Psi.$$

Here I_m, K_m are Bessel functions of the imaginary argument of first and second kind.

Magnetic field B is calculated the same way as in sec. 1.3.5, with $\gamma = m/\ k$.

Though this solution has non-trivial spatial structure, e.g. it is not translationally symmetric, both pressure and magnetic field magnitude grow infinitely at zero and infinity in the radial variable r. This reduces the practical value of this solution.

Unfortunately, the feature of unbounded growth is inherent to many other analytical solutions as well.

2. A ball lightning model. The following is an example of solution to static isotropic plasma equilibrium system (1.20) that vanishes outside of a *finite volume*, but it appears to be non-smooth. This solution was suggested by Kaiser and Lortz in [18] and represents a model of a ball lightning supported by atmospherical pressure.

The authors solve the Grad-Shafranov equation (1.32) with linear profile functions

$$I = \lambda\Psi/\mu, \quad P = P_0 - \delta\Psi/\mu; \quad \lambda, \delta = \text{const}.$$

Then the equation in spherical coordinates (r, θ, φ) takes the form

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \lambda^2\right]\Psi = \delta r^2 \sin^2\theta.$$

For the solution to occupy finite volume, it is easy to see that the boundary separating the system from the "outer world" must be a magnetic surface. This condition is necessary and sufficient for $\mathbf{B}$ and $\mathbf{J} = (1/\mu \text{ curl } \mathbf{B})$ to be tangent to the boundary. This ensures reasonable boundary conditions: normal components of magnetic field and electric current continuously go to zero at the boundary and are precisely zero outside.

The solution with these boundary conditions is then

$$\Psi = \begin{cases} CW(r)\sin^2\theta, & r < R \\ 0, & r \geqslant R \end{cases}$$

$$W(r) = \lambda r J_1(\lambda r) - (\lambda r)^2 \frac{J_1(\lambda R)}{\lambda R}.$$

Here J_1(x) is the Bessel function of order 1; R is selected so that λR is any one of the countable set of solutions to an equation

$$J_2(\lambda R) = 0.$$

The family of magnetic surfaces corresponding to the first root of this equation is shown in Fig. 1-1.

It is easy to see that plasma pressure (assumed positive-definite) reaches a positive value P_0 at the boundary of the plasma domain, therefore a ball lightning described by this model can only exist in atmosphere that confines it.

The estimates of parameter magnitudes in this model are in agreement with those that a natural fireball may possess (e.g. $B_{max} \sim 1$ Tesla).

Having described this solution, we should also mention a solution by Bobnev [21], who also has solved the Grad-Shafranov equation in spherical coordinates, having obtained a solution different from above, but with very similar properties, including that the plasma ball should be also atmospherically confined.

Note. In both of the above solutions modelling an atmospherical fireball a common negative feature is present. At the boundary, the flux function Ψ goes to zero contin-

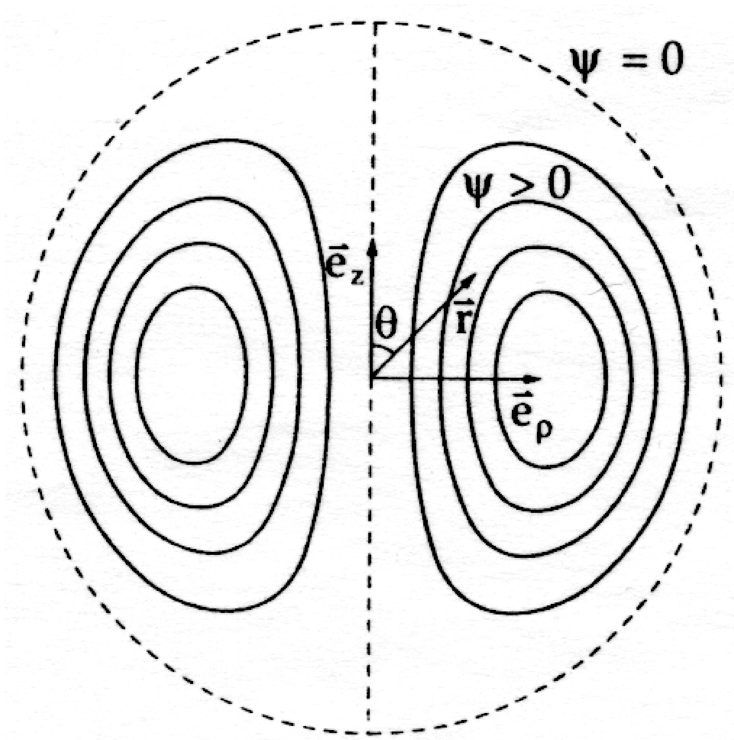

Figure 1-1: Magnetic surfaces in the compact ball lightning model.

The section of a family of magnetic surfaces $\Psi = \text{const}$ in the compact ball lightning model [18], which is a static isotropic axially symmetric plasma equilibrium, derived as a solution to Grad-Shafranov equation written in spherical coordinates. All magnetic surfaces are axially symmetric tori.

uously with its first derivative, and so does the pressure, whereas some components of magnetic field $\mathbf{B}$ are non-smooth at the boundary.

3. 3-dimensional unbounded isotropic plasma equilibria without symmetries. In papers [40]-[43], Kaiser, Salat and Tataroins presented several families of unbounded solutions to isotropic static plasma equilibrium equations (1.20) that have *no spatial symmetries.*

Authors search for the solution representing magnetic field through *Euler potentials*:

$$\mathbf{B} = \operatorname{grad} F \times \operatorname{grad} G(z),$$

and obtain a solution depending on 2 arbitrary functions of variable z, which can be selected so that the configuration has no symmetries.

This particular solution has both $|\mathbf{B}|$ and P infinitely growing in cylindrical radius. The same is true about other families of solutions constructed by these authors in the works cited above. In spite of this fact, exact solutions without symmetries are extremely significant.

An example of non-symmetric magnetic surfaces of one of the solutions in the family constructed in [41] is shown on Fig. 1-2.

4. Global isotropic plasma equilibria with axial symmetry. Symmetry breaking. Recently Bogoyavlenskij has found families of exact isotropic plasma equilibria possessing axial [44, 45] and helical [2] symmetries. All these solutions have important properties – they have *finite magnetic energy* in every layer $c_1 < z < c_2$, and everywhere-finite values of pressure. These properties make them applicable for modelling astrophysical objects, e.g. jets. None of the solutions that were found before has such properties.

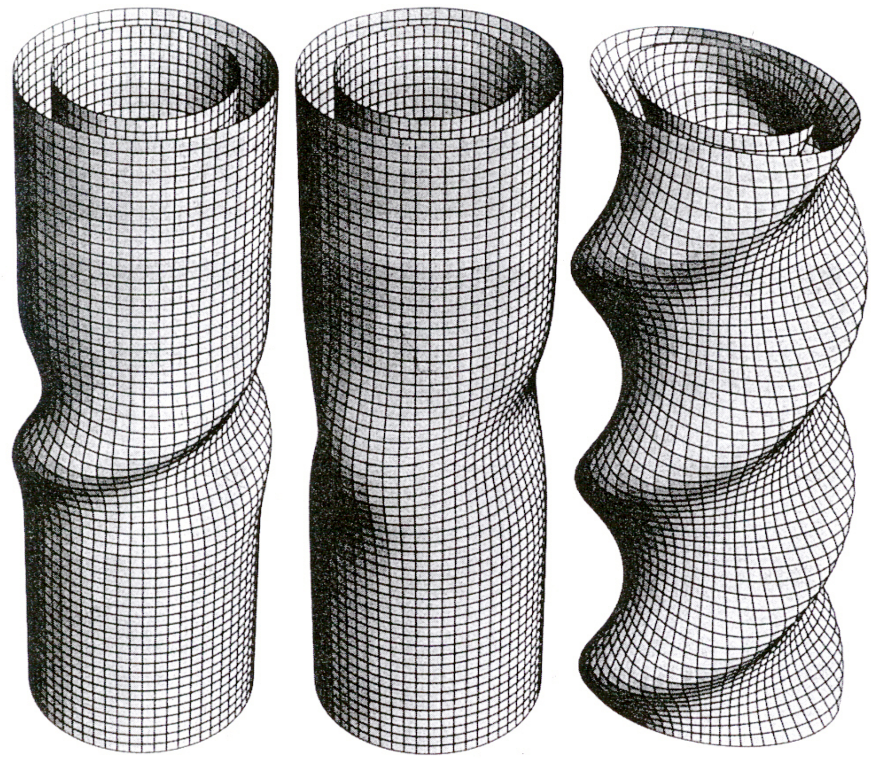

Figure 1-2: Example of non-compact magnetic surfaces without symmetries.

An example of non-symmetric magnetic surfaces of one of the solutions in the family constructed in [41]. These solutions have non-compact cylinder-like magnetic surfaces; $|\mathbf{B}|$ and P grow infinitely with cylindrical radius.

The solution families with axial symmetry were found solving the Grad-Shafranov equation (1.32) under assumptions

$$I = \alpha\Psi, \qquad P = P_1 - 2\beta^2\Psi^2/\mu.$$

Then the equation becomes *linear*, and solutions are linear combinations of

$$\Psi_n(r,z) = e^{-\beta r^2} L_n^*(2\beta r^2)(a_n \cos(\omega_n z) + b_n \sin(\omega_n z)), \tag{1.38}$$

where $\omega_n = \sqrt{\alpha^2 - 8\beta N}$, $N = \left[\dfrac{\alpha^2}{8\beta}\right]$, and $L_n^*(x)$ are polynomials.

Frequencies ω_n are generically not rational multiples of each other, therefore a linear combination of the above solutions gives a quasiperiodic solution to the GS equation, and consequently a quasiperiodic plasma configuration. (Fig. 1-3 shows a characteristic quasiperiodic family of axially symmetric magnetic surfaces).

In the radial direction, both $|\mathbf{B}|$ and $|P - P_1|$ decrease exponentially, therefore the magnetic energy in every layer $c_1 < z < c_2$ is finite, and the outside pressure required to maintain such equilibrium is also finite.

Symmetry breaking. All axially symmetric solutions constructed by the procedure described above as particular cases of (1.38) have a domain $0 < r < r_0$ where magnetic surfaces are not defined uniquely (e.g. for solution shown on Fig. 1-3 we may take $r_0 = 1$). Magnetic field lines are not dense in that domain. Therefore if the transformations (1.27) are applied to such solutions, then transformation parameters $a(\mathbf{r})$, $b(\mathbf{r})$, $c(\mathbf{r})$ in the domain $0 < r < r_0$can be chosen to have *different values on every field line*, thus being arbitrary functions of two variables $(r,\ \theta)$ (or $(x,\ y)$). Therefore a field-aligned MHD equilibria obtained as a result of such transformation will have *no spatial symmetry.*

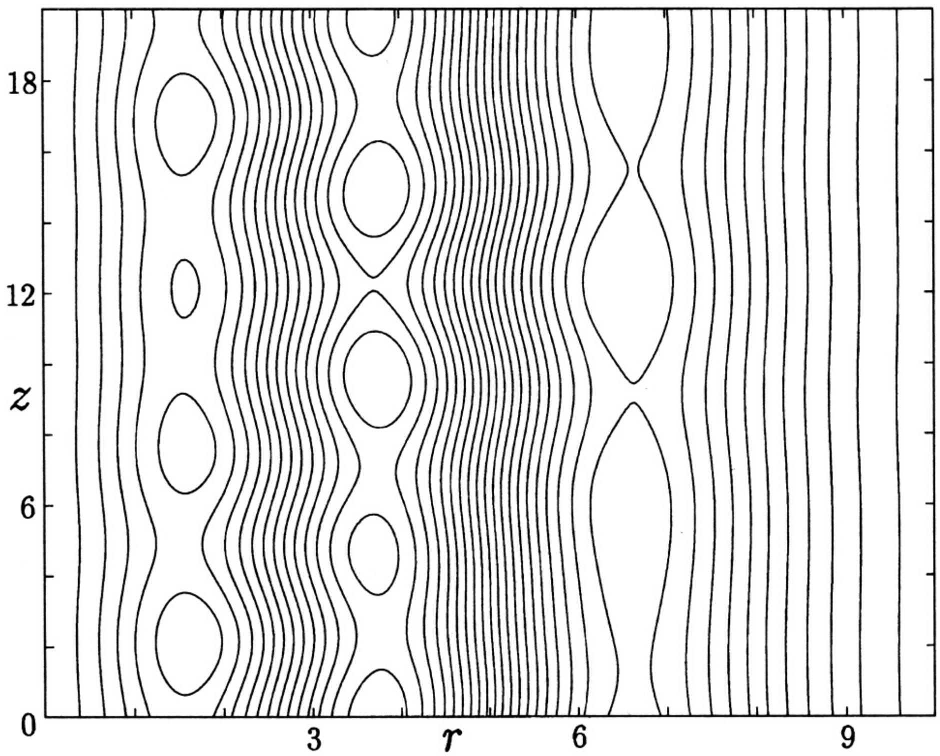

Figure 1-3: Quasiperiodic axially symmetric magnetic surfaces.

The section of magnetic surfaces of a sample quasiperiodic axially symmetric plasma equilibrium solution. This class of solutions constructed in [44, 45] models axially symmetric astrophysical jets. The magnetic field, pressure and magnetic energy in every layer $c_1 < z < c_2$ are finite.

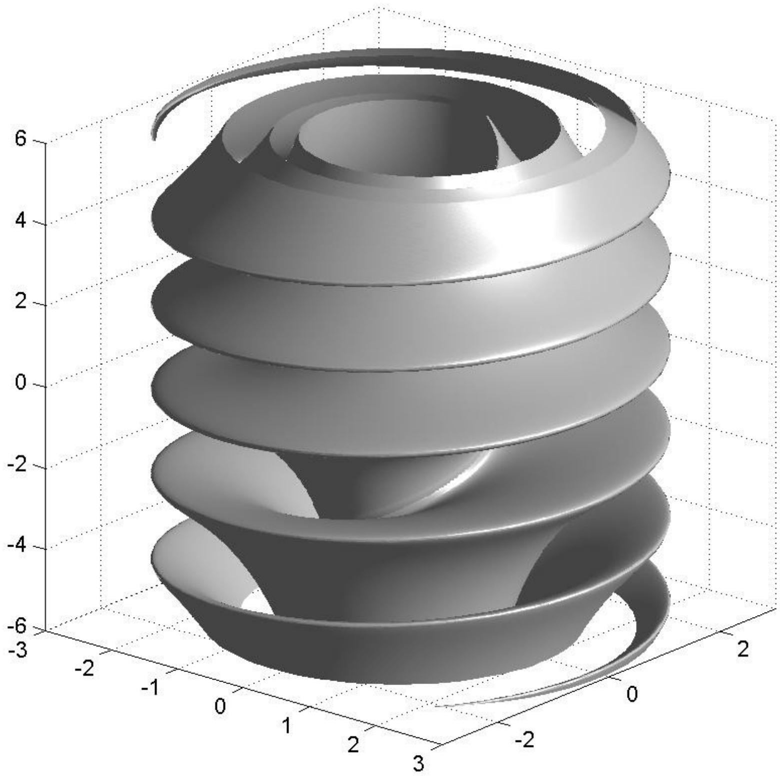

Figure 1-4: A magnetic surface without symmetries obtained by symmetry breaking.

An example of an absolutely non-symmetric magnetic surface $\Psi = \text{const}$ obtained by the symmetry breaking procedure (via the application of Bogoyavlenskij symmetries (1.27) to a plasma equilibrium configuration with magnetic lines going to infinity.)

An example of an absolutely non-symmetric magnetic surface obtained this way is given on Fig. 1-4.

5. Global helically-symmetric isotropic plasma equilibria. In [8], an infinite family of helically-symmetric isotropic plasma equilibria was found by Bogoyavlenskij. The solutions depend on several arbitrary functions and constant parameters. The magnetic field quickly decreases with cylindrical radius, and magnetic energy in every layer $c_1 < z < c_2$ is finite.

In these solutions, the magnetic surface function $\Psi(\mathbf{r}) = \Psi(r, u)$ depends only on the helical variable u and cylindrical radius r, and thus is a solution to the JFKO equation (1.34).

These solutions model helically symmetric astrophysical jets, and have no axial or translational symmetry.

1.5 Stability and existence of plasma equilibria

In this section, the results concerning stability of isotropic and anisotropic plasma equilibria are discussed, as well as the question of existence of solutions to the corresponding systems of partial differential equations.

1.5.1 Plasma stability

Plasma is known to be extremely unstable. This is the main difficulty in building a satisfactory confinement device. As an example we give illustrations for two common instabilities that occur in laboratory devices [24].

On Fig. 1-5, the development of the "sausage" instability in a simple plasma pinch is shown.

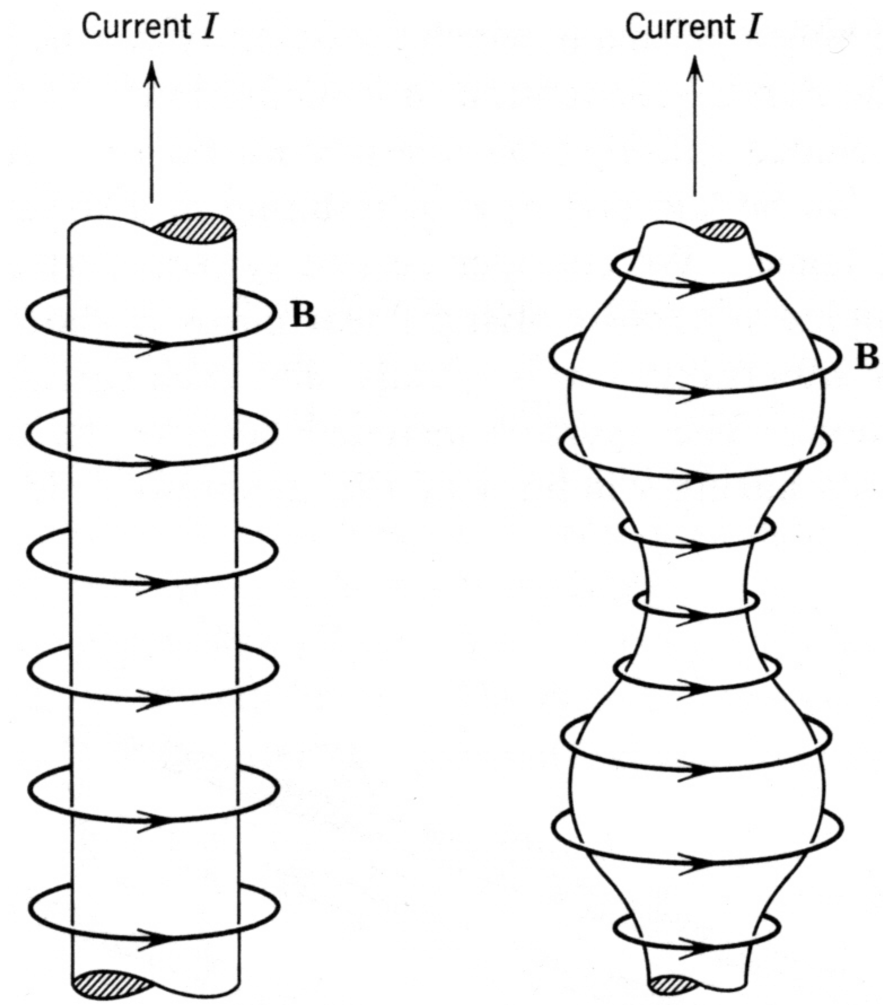

Figure 1-5: "Sausage" instability.

The development of the "sausage" instability in a simple plasma pinch. Here I is electric current, **B**, the magnetic field.

Fig. 1-6 shows the development of the "ripple" instability, which is also called "interchange" or "flute" instability.

Various laboratory techniques have been developed to fight these and other common instabilities, but the general stability problem is not solved – the laboratory configurations that maintain plasma in static or dynamic equilibria or quasi-equilibria for satisfactory times do not exist.

Vast theoretical research in the problem of plasma stability is also being conducted since the middle of the 20^{th} century, and several general theorems were proven, mostly based on energy functional variation approach – the "energy principle", originally introduced by Bernstein *et al* [46] (also see, Newcomb [29]). This technique has advantages over the method of small perturbations, where a *linearized* MHD system has to be explicitly solved.

To prove that a given equilibrium configuration is stable by the energy principle, a positive-definiteness of a certain energy functional with respect to all perturbations must be shown. This task usually can not be completely performed, neither analytically nor numerically, if the configuration is not degenerate. But even in some general cases, for certain types of perturbations it was possible to prove the *indefiniteness* of the energy functional, and hence the instability.

Among important instability results of this type, results we would like to mention a paper by Friedlander and Vishik [47], where the authors show that for *isotropic MHD equilibria with constant density* the following theorem is true:

Theorem 1.3 (Friedlander, Vishik) *An isotropic MHD equilibrium (1.16)-(1.17) is* ***unstable*** *when either (a) there exists a point where both* $\boldsymbol{V}$ *and (curl* $\boldsymbol{V}$*) are nonzero and non-parallel to* $\boldsymbol{B}$*, or (b)* $\boldsymbol{B}$ *is parallel to* $\boldsymbol{V}$ *and* $|\mathbf{B}| < |\mathbf{V}|$*. (The choice* $\mu = \rho = 1$ *is assumed).*

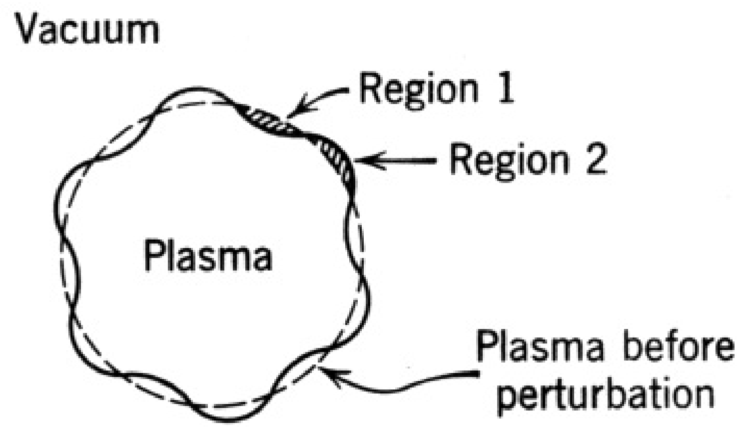

Figure 1-6: "Ripple" instability.

The development of the "ripple" instability in a simple plasma pinch.

For *anisotropic* (Chew-Goldberger-Low) plasmas, under the assumption of double-adiabatic equation of state (1.12), two explicit instability conditions are known (see, e.g., [48].)

The *fire-hose* instability takes place when

$$p_{\|} - p_{\perp} > \frac{\mathbf{B}^2}{\mu}, \tag{1.39}$$

(or, equivalently, $\tau > 1/\mu$), and the *mirror* instability - when

$$p_{\perp}\left(\frac{p_{\perp}}{6p_{\|}} - 1\right) > \frac{\mathbf{B}^2}{2\mu}. \tag{1.40}$$

1.5.2 Existence of equilibrium configurations. Grad's hypothesis

Studying stability means studying the behaviour of a system in a neighbourhood of an equilibrium. Stability research is therefore based on the assumption that equilibrium configurations do exist.

In a series of papers including [49, 50], Grad argues that static isotropic equilibrium states in plasma (system (1.20)) do not exist, except certain particular cases. In [49], he writes: "*Almost all stability analyses are predicated on the existence of equilibrium state that is then subject to perturbation. But a more primitive reason than instability for lack of confinement is the* <u>*absence of an appropriate equilibrium state*</u>."

Grad states that there are exactly *four* known types of symmetry where toroidal plasma equilibria with nested magnetic surfaces can exist. They are: (a) two-dimensional cases; (b) axial symmetry; (c) helical symmetry; (d) reflection symmetry. In [50] he then writes: "*...No additional exceptions have arisen since 1967, when it* *was conjectured that toroidal existence...of smooth solutions with simple nested p-surfaces admits only these...exceptions. ... The proper formulation of the nonexistence statement is that, other than stated symmetric exceptions, there are no families of solutions depending smoothly on a parameter.*"

We believe that it is necessary to shed some light on the formulation and possible validity of Grad's conjecture.

First, we remark that the conjecture may relate only to *classical static isotropic equilibrium system* (1.20) with *compact magnetic surfaces.* (For non-compact magnetic surfaces, smooth *infinite families* of non-symmetric solutions were found, e.g. [40, 41, 44, 45]).

Second, we note that Grad's requirements (a) to (c) represent reductions to two dimensions, and (d) (if the reflection plane is not a magnetic surface itself) generally means that the magnetic field lines are closed loops. Thus the claim is the *non-existence of compact 3-dimensional static isotropic plasma equilibria (1.20) without geometrical symmetries.*

In our view, Grad's arguments in support of his hypothesis (such as references to numerical simulations and the fact that analytical non-symmetric toroidal solutions have not yet been found, [49, 50]) cannot be accepted as a proof. Nevertheless, by part of the research community working in the area of plasma physics the Grad's hypothesis is referred to as a known fact.

Neither the proof of Grad's hypothesis, nor a counterexample (a localized 3D non-symmetric non-degenerate static plasma equilibrium) is currently available.

However, in several cases of *more general* or *more specific* plasma equilibria than the static isotropic case considered by Grad, analytical 3D non-symmetric equilibria have been constructed.

For example, it is possible to build totally non-symmetric static isotropic equilibrium solutions in the particular case of Beltrami flows (e.g. [51]). For non-Beltrami force-free fields, non-symmetric plasma equilibria with spherical magnetic surfaces (with singularities) are constructed (see Chapter 4 of this work.) In more general plasma equilibrium set-ups, localized toroidal non-symmetric anisotropic static equilibrium solutions and isotropic dynamic equilibrium solutions have been found (see Chapter 3).

In Chapter 4 below, a different coordinate representation of the system of interest is presented; we believe that its detailed study and the extension of methods introduced in that chapter could result in the construction of an exact counterexample to Grad's hypothesis. (The difficulty in building non-symmetric static isotropic equilibrium solutions, in our understanding, is mainly due to the difficulty of treatment of a complicated system of non-linear PDEs depending on all three variables.)

Though Grad's conjecture in its present formulation, for static isotropic MHD equilibria, might be correct, its value for applications is questionable, because exact on-symmetric equilibria in more physically relevant cases (with dynamics and pressure anisotropy) have been found.

1.6 Open problems in Magnetohydrodynamics

The above-described continuum plasma descriptions, MHD (1.2)-(1.5) and CGL (1.7)-(1.10), are currently the most widely used plasma models in theoretical investigations, experiments and modelling in astrophysics, thermonuclear fusion research, experimental chemistry, geological sciences and other areas.

However, as can be seen from this review, because of the complexity of these systems, the knowledge of their analytical structure is very limited. The general existence, uniqueness and stability of solutions of initial and boundary value problems are unknown; methods of construction of particular solutions have not been developed.

Due to these reasons, most MHD modelling is currently done numerically, usually under dimension reductions. Fully 3-dimensional dynamic MHD simulations have been named as one of the important applications of multiprocessor supercomputers, which illustrates their exceptional computational complexity.

In numerical simulations, the existence and uniqueness of solutions of interest are usually assumed, which generally might not be the case.

Indeed, the existence, uniqueness and regularity of solutions of the 3D initial/boundary value problem for the motion of incompressible viscous fluid (the Navier-Stokes system, NS) has been named by Clay Mathematical Institute as one of the seven most challenging problems of contemporary mathematics. The positive result is available for 2-dimensional fluid flow only, and is due to Ladyzhenskaya; it was obtained by considering generalized solutions to the NS system, using methods described in, e.g., [52, 53].

The full MHD (1.2)-(1.5) and CGL (1.7)-(1.10) systems are significantly compressible and reduce to the NS system only in a particular case. No general results on existence, uniqueness and regularity of their solutions have been proven.

The importance of knowledge of analytical properties of systems of partial differential equations, such as exact particular solutions, conservation laws and symmetries, can not be overestimated. Such information is desirable for both direct modelling and the simplification of numerical simulations.

Concerning the MHD systems, even for the substantially simpler ideal time-independent (equilibrium) isotropic equations (1.16)-(1.17), until recently, only several exact solutions have been found (by *ad hoc* methods), and few of them satisfy necessary physical conditions. Before the works of Bogoyavlenskij [1, 44, 45], no methods for constructing families of non-trivial equilibrium solutions were available. For the anisotropic equilibrium system (1.22)-(1.23), only a few trivial exact solutions are known (see, e.g., [37]).

Among the methods of construction of exact solutions to a system of non-linear partial differential equations, the following can be named:

1. Dimension reduction;

2. Special assumptions about the solution – self-similarity etc.

3. Transformations from solutions of other equations;

4. Symmetries - transformations of solutions into solutions.

5. Methods specific to the equations under consideration.

Important simplifications following from the first approach have been discussed above in Section 1.3.5 of this chapter.

In the current work, we perform an analytical study of symmetries and other properties of isotropic and anisotropic, dynamic and static ideal plasma equilibria and develop methods of construction of analytic solutions, following the methods 3 – 5 of the above list.

Chapter 2

Bogoyavlenskij symmetries of ideal MHD equilibria as Lie point transformations

2.1 Introduction.

As was discussed in Chapter 1, in the recent papers [1, 2] Bogoyavlenskij introduced new symmetry transforms (1.27), (1.29) of incompressible and compressible isotropic MHD equilibrium equations. In certain classes of plasma configurations, Bogoyavlenskij symmetries break geometrical symmetry, thus giving rise to important classes of non-symmetric equilibrium solutions.

In this chapter we study whether such complex intrinsic symmetries of systems of partial differential equations such as Bogoyavlenskij symmetries can be obtained by applying a general method.

The goal of the current chapter is to prove that the Bogoyavlenskij symmetries are contained in particular Lie groups of point transformations, which are found independently using the classical Lie approach. The results presented in this chapter follow our recent publication [54].

It is shown that the Bogoyavlenskij symmetries can be found as Lie point transformations of the MHD equilibrium system *only* if the general solution topology (the existence of magnetic surfaces to which vector fields $\mathbf{B}$ and $\mathbf{V}$ are tangent) and the incompressibility condition are explicitly taken into account in the form of additional constraints:

$$\rho(\mathbf{r}) = \rho(\Psi(\mathbf{r})), \quad \mathrm{grad}(\Psi(\mathbf{r})) \cdot \mathbf{B} = 0, \quad \mathrm{grad}(\Psi(\mathbf{r})) \cdot \mathbf{V} = 0.$$

Here $\Psi(\mathbf{r})$ is a magnetic surface function (or, more generally, a function constant on magnetic field lines and plasma streamlines.)

The Bogoyavlenskij symmetries form an infinite-dimensional Abelian group of transformations with eight connected components in the case of incompressible plasmas (1.28), and four connected components in the case of compressible gas plasmas (1.31). In Section 2.3 of the current chapter, we prove that the system of isotropic MHD equilibrium equations in compressible and incompressible cases possesses specific infinite-dimensional Lie groups of point transformations, which are equivalent to Bogoyavlenskij symmetries.

The Lie symmetry method [55] used in this work is generally capable of detecting both simple geometric symmetries of systems of PDEs (e.g., rotations, scaling transforms and translations), and more complicated ones. It is known that infinite-dimensional transformations for partial differential equations can be obtained by Lie point symmetry method; the example is a generalized Kadomtsev - Petviashvili equation [56]. When the Lie transformations are found, they can be used to build particular solutions of the system under consideration, to reduce the order and to obtain invariants. Self-similar solutions constructed from Lie symmetries often have transparent physical meaning. Many appropriate examples can be found in [57].

We remark, however, that not all symmetries of a given system can be found by the Lie method, but only continuous symmetries with Lie group structure.

Continuous Lie symmetries can also be used to obtain discrete symmetries of differential equations. One of the simplest ways of finding discrete symmetries is the complexification of the parameter (an example is Lemma C.1 in Appendix C of this chapter). A recently developed more powerful algorithm [58]-[60] enables the user to obtain *all* discrete point symmetries of systems of ordinary and partial differential equations. The algorithm proceeds by classifying the adjoint actions of discrete point symmetries on the Lie algebra of Lie point symmetry generators. This method is easy to apply and for simple systems does not require any computer algebra.

The advantage of the Lie group analysis procedure is that it can be applied directly to any system of equations (provided that all involved functions are sufficiently smooth).

On the other hand, the application of the Lie symmetry method is almost always extremely resource-demanding - it requires a lot of algebraic manipulation and the solution of large systems of dependent linear partial differential equations. This makes the analysis of systems of several PDEs in several variables "by hand" practically impossible. Due to this difficulty many important results obtained by the Lie method were discovered earlier using much less general techniques.

However, the use of modern analytical computation software often significantly facilitates the computations. Recently developed methods using Gröbner bases [61]-[64] and characteristic sets [65]-[66] to handle large overdetermined systems of partial differential equations, such as those arising from the Lie group analysis procedure, make it possible to perform complete or partial group analysis of many complicated systems.

A review of analytical computation software employing these ideas is given in [67].

In this work, the most involved algebraic manipulations were done on Waterloo Maple using `Rif` package for PDE systems reduction. This package is an extended version of well-known Standard Form package developed by Reid and Wittkopf [68].

Another widely used software package for Maple is `diffgrob2` developed by E.L.Mansfield [69].

In Appendix D, we perform the potential symmetry analysis [70, 71] of the static isotropic plasma equilibrium system (1.20), and on its basis we conclude that this system can not be linearized by an invertible transformation.

2.2 Lie group formalism for the MHD equilibrium equations

A system of l first-order partial differential equations

$$\begin{aligned}&\mathbf{E}(\mathbf{x},\mathbf{u},\underset{1}{\mathbf{u}})=0,\\ &\mathbf{E}=(E^1,\ldots,E^l),\ \mathbf{x}=(x^1,\ldots,x^n)\in X,\ \mathbf{u}=(u^1,\ldots,u^m)\in U,\\ &\underset{1}{\mathbf{u}}=\left(\frac{\partial u^j}{\partial x^i}|\ \ i=1,\ldots,n;j=1,\ldots,m\right)\in U_1\end{aligned} \tag{2.1}$$

corresponds to a manifold Ω in $(m+n)$ - dimensional space $X\times U$, and a manifold Ω^1 in $(m+n+mn)$ - dimensional prolonged (jet) space $X\times U\times U_1$ of dependent and independent variables together with partial derivatives [55].

Studying ideal isotropic MHD equilibria, one should take into account that generally the plasma domain is spanned by nested 2-dimensional *magnetic surfaces* - surfaces on which magnetic field lines and plasma streamlines lie [31] (see Chapter 1, sec. 1.3.3).

In the case of adiabatic compressible MHD equilibrium equations, one has $n = 3$ independent and $m = 10$ dependent variables:

$$\mathbf{x} = (x, y, z), \quad \mathbf{u} = (V_1, V_2, V_3, B_1, B_2, B_3, \Psi, P, \rho, S). \tag{2.2}$$

Here Ψ is a function constant on magnetic field lines and plasma streamlines, i.e. on magnetic surfaces, when they exist:

$$\operatorname{grad}(\Psi(\mathbf{r})) \cdot \mathbf{B} = 0, \quad \operatorname{grad}(\Psi(\mathbf{r})) \cdot \mathbf{V} = 0. \tag{2.3}$$

The Lie method of seeking one-parametric groups of transformations that map solutions of (2.1) into solutions consists in finding the Lie algebra of vector fields tangent to the solution manifold Ω^1 in the jet space. These vector fields serve as infinitesimal generators for a Lie symmetry group with representation

$$\begin{aligned} (x')^i &= f^i(\mathbf{x}, \mathbf{u}, a) \quad (i = 1, \ldots, n), \\ (u')^j &= g^j(\mathbf{x}, \mathbf{u}, a) \quad (j = 1, \ldots, m), \end{aligned} \tag{2.4}$$

and have the form

$$\mathbf{h} = \sum_i \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_k \eta^k(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^k} + \sum_{i,\,k} \xi_i^k(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}) \frac{\partial}{\partial u_i^k}. \tag{2.5}$$

Components of these tangent vector fields are expressed through the group representation as follows:

$$\xi^i(\mathbf{x}, \mathbf{u}) = \frac{\partial f^i(\mathbf{x}, \mathbf{u}, a)}{\partial a}\Big|_{a=0}, \quad \eta^j(\mathbf{x}, \mathbf{u}) = \frac{\partial g^j(\mathbf{x}, \mathbf{u}, a)}{\partial a}\Big|_{a=0}, \tag{2.6}$$

$$i = 1, \ldots, n, \quad j = 1, \ldots, m.$$

The variables ξ_i^k in (2.5) are the coordinates of the prolonged tangent vector field corresponding to the derivatives u_i^k:

$$\xi_i^j(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}) = D_i \eta^j - \sum_{k=1}^{n} u_k^j D_i \xi^k, \quad D_i \equiv \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} u_i^j \frac{\partial}{\partial u^j}. \tag{2.7}$$

We remark that relation (2.7) defines an isomorphism between tangent vector fields (2.5) and infinitesimal operators

$$X = \sum_i \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_k \eta^k(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^k}. \tag{2.8}$$

The explicit reconstruction of the transformations (2.4) from a generator (2.5) is done by solving the initial value problem

$$\frac{\partial f^i(a)}{\partial a} = \xi^i(\mathbf{f}, \mathbf{g}), \quad \frac{\partial g^k(a)}{\partial a} = \eta^k(\mathbf{f}, \mathbf{g}), \tag{2.9}$$

$$f^i(0) = x^i, \quad g^k(0) = u^k.$$

To find all Lie group generators admissible by the original system (2.1), one needs to solve the determining equations

$$\mathbf{hE}(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}})|_{\mathbf{E}(\mathbf{x},\mathbf{u},\underset{1}{\mathbf{u}})=0} = 0. \tag{2.10}$$

All l determining equations (2.10) are linear partial differential equations with respect to $m+n$ unknown functions (2.6) of $m+n$ variables (2.2).

According to the formula (2.10), the determining equations are obtained as follows. First, one applies the operator $\mathbf{h}$ to the original equations (2.1). Second, using the original equations as true equalities, one eliminates from this intermediate result some terms (usually the highest order partial derivatives).

To solve the determining equations and obtain the tangent vector field coordinates (2.6), one should use the fact that the latter do not depend on derivatives u_i^k. Therefore in all l determining equations coefficients at different derivatives must equal zero. Thus the system (2.7) splits into $N \leq l(mn+1)$ simpler linear partial differential equations. In the case of adiabatic isotropic compressible plasma equilibria, for example, this generally leads to a system of 188 linear PDEs on 13 unknown functions.

It is not realistic to solve such a system "by hand"; however, computer algebra algorithms mentioned above can sometimes be successfully applied to reduce the system of equations and to exclude dependence of tangent vector field coordinates ξ^i, η^k on some variables. It is shown in the proof of Theorem 2.1 how such a simplification can significantly reduce the system of determining equations to the point when it can be processed manually.

2.3 Correspondence between Bogoyavlenskij symmetries and Lie transformations of the MHD equilibrium equations

In this section we answer the question about the possibility of obtaining the Bogoyavlenskij symmetries (1.27) and (1.29)-(1.30) of the isotropic MHD equilibrium equations using the Lie group formalism. This question was raised soon after the discovery of the symmetries.

Theorem 2.1 shows that the application of the Lie group formalism to the MHD equilibrium system (1.16)-(1.17) yields certain groups of Lie point transformations, some of which are *infinite-dimensional*.

Theorem 2.2 proves that these Lie point transformations are equivalent to the groups G_m and G_{0m} of Bogoyavlenskij symmetries (for incompressible and compressible plasmas respectively).

Theorem 2.1 *(i) Consider the incompressible isotropic MHD equilibrium system of equations (1.16)-(1.18), where the density $\rho(\mathbf{r})$ is constant on both magnetic field lines* *and streamlines. This system admits the infinitesimal operators*

$$X^{(1)} = M(\mathbf{r})\left(\sum_{k=1}^{3} \frac{B_k}{\mu\rho}\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k \frac{\partial}{\partial B_k} - \frac{1}{\mu}(\mathbf{V}\cdot\mathbf{B})\frac{\partial}{\partial P}\right) , \tag{2.11}$$

$$X^{(2)} = \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2P\frac{\partial}{\partial P} , \tag{2.12}$$

$$X^{(3)} = N(\mathbf{r})\left(2\rho\frac{\partial}{\partial \rho} - \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k}\right) , \tag{2.13}$$

$$X^{(4)} = \frac{\partial}{\partial P}. \tag{2.14}$$

These operators form a basis of the Lie algebra of infinitesimal operators in the class of Lie point transformations $\{\mathbf{x}' = \mathbf{x},\ \mathbf{u}' = \mathbf{g}(\mathbf{u}, a)\}$. *Here* $M(\mathbf{r}), N(\mathbf{r})$ *are arbitrary smooth functions constant on both magnetic field lines and streamlines.*

(ii) Compressible isotropic ideal MHD equilibrium equations (1.16)-(1.17) with ideal gas state equation (1.19), for arbitrary density, admit the infinitesimal operators

$$X^{(5)} = \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2P\frac{\partial}{\partial P} + 2c_v\frac{\partial}{\partial S} , \tag{2.15}$$

$$X^{(6)} = N(\mathbf{r})\left(2\rho\frac{\partial}{\partial \rho} - \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} - 2c_v\gamma\frac{\partial}{\partial S}\right) , \tag{2.16}$$

where $N(\mathbf{r})$ *is an arbitrary smooth function constant on both magnetic field lines and streamlines.*

The proof of Theorem 2.1 is given in the Appendix A. It directly follows the Lie group analysis procedure. In the alternative proof of the theorem, operators (2.11)-(2.13), (2.15)-(2.16) are obtained by direct differentiation of Bogoyavlenskij symmetries (1.27), (1.29) with respect to a properly chosen parameter, as shown in Appendix B. This alternative proof is simpler, but it is based on the knowledge of the precise form of Bogoyavlenskij symmetries, while the original proof does not require it.

Also, the alternative proof does not contain the proof of the fact that the operators (2.11)-(2.14) form a basis of the Lie algebra of the operators of the whole class of Lie point transformations $\{\mathbf{x}' = \mathbf{x},\ \mathbf{u}' = \mathbf{g}(\mathbf{u}, a)\}$.

Remark.

Let us explicitly write down the transformations contained in the infinitesimal operators (2.11)-(2.16). According to the reconstruction procedure (2.9), for the operator (2.11), we have

$$\rho_1 = \rho, \quad \mathbf{x}_1 = \mathbf{x},$$

and need to solve the linear initial value problem

$$\begin{aligned} &\frac{\partial \mathbf{V}_1}{\partial \tau} = \mathbf{B}_1 \frac{M(\mathbf{r})}{\mu\rho}, \quad \frac{\partial \mathbf{B}_1}{\partial \tau} = \mathbf{V}_1 M(\mathbf{r}), \quad \frac{\partial P_1}{\partial \tau} = -\frac{M(\mathbf{r})}{\mu}(\mathbf{V}_1 \cdot \mathbf{B}_1), \\ &\mathbf{V}_1(\tau = 0) = \mathbf{V}, \quad \mathbf{B}_1(\tau = 0) = \mathbf{B}, \quad P_1(\tau = 0) = P. \end{aligned} \tag{2.17}$$

The solution is

$$\begin{aligned} &\mathbf{B}_1 = \cosh\left(\frac{M(\mathbf{r})\tau}{\sqrt{\mu\rho}}\right)\mathbf{B} + \sinh\left(\frac{M(\mathbf{r})\tau}{\sqrt{\mu\rho}}\right)\sqrt{\mu\rho}\mathbf{V}, \\ &\mathbf{V}_1 = \sinh\left(\frac{M(\mathbf{r})\tau}{\sqrt{\mu\rho}}\right)\frac{\mathbf{B}}{\sqrt{\mu\rho}} + \cosh\left(\frac{M(\mathbf{r})\tau}{\sqrt{\mu\rho}}\right)\mathbf{V}, \\ &P_1 = P + (\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu), \quad \rho_1 = \rho. \end{aligned} \tag{2.18}$$

The infinitesimal operator (2.11) thus contains the possibility of "mixing" the components of the vector fields **B** and **V** of the original solution into a new solution.

The same way by solving a corresponding initial value problem (2.9) we find that transformations contained in the operator (2.12) are scalings

$$\rho_1 = \rho, \quad \mathbf{B}_1 = \exp(\tau)\mathbf{B}, \quad \mathbf{V}_1 = \exp(\tau)\mathbf{V}, \quad P_1 = \exp(2\tau)P\ ; \tag{2.19}$$

the operator (2.13) corresponds to infinite-dimensional scalings

$$\rho_1 = \exp(2N(\mathbf{r})\tau)\rho, \quad \mathbf{B}_1 = \mathbf{B}, \quad \mathbf{V}_1 = \exp(-N(\mathbf{r})\tau)\mathbf{V}, \quad P_1 = P \ ; \tag{2.20}$$

the operator (2.14) - to translations

$$\rho_1 = \rho, \quad \mathbf{B}_1 = \mathbf{B}, \quad \mathbf{V}_1 = \mathbf{V}, \quad P_1 = P + \tau \ . \tag{2.21}$$

The transformations provided by the operators (2.15) and (2.16) are respectively

$$\rho_1 = \rho, \quad \mathbf{B}_1 = \exp(\tau)\mathbf{B}, \quad \mathbf{V}_1 = \exp(\tau)\mathbf{V}, \quad P_1 = \exp(2\tau)P, \quad S_1 = S + 2c_v\tau \tag{2.22}$$

and

$$\rho_1 = \exp(2N(\mathbf{r})\tau)\rho, \quad \mathbf{B}_1 = \mathbf{B}, \quad \mathbf{V}_1 = \exp(N(\mathbf{r})\tau)\mathbf{V}, P_1 = P, \quad S_1 = S - 2c_v\gamma N(\mathbf{r})\tau. \tag{2.23}$$

Theorem 2.2 *(i) Lie point transformations (2.18)-(2.20) are equivalent to the group G_m of Bogoyavlenskij transformations (1.27), (1.28).*

(ii) Lie point transformations (2.22)- (2.23) are equivalent to the group G_{0m} of Bogoyavlenskij transformations (1.29)-(1.30), (1.31).

The proof of Theorem 2.2 is presented in the Appendix C.

2.4 Conclusion

It is remarkable that the infinite-dimensional groups of Bogoyavlenskij symmetries (1.27), (1.29)-(1.30) of the isotropic MHD equilibrium equations (1.16)-(1.17), the richest known class of transformations for these equations, is implied by the Lie point transformations of these equations.

Bogoyavlenskij symmetries form infinite-dimensional Abelian groups: $G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2$ in the incompressible case and $G_{0m} = A_m \oplus R^+ \oplus Z_2 \oplus Z_2$ in the compressible case. G_m has eight connected components, and G_{0m} has four. In this chapter we have shown that the groups G_m and G_{0m} are equivalent to Lie point transformations generated by infinitesimal operators (2.11)-(2.13) and (2.15)-(2.16) respectively.

Thus Bogoyavlenskij symmetries are obtained from the standard procedure of Lie group analysis that is applicable to any system of PDEs with sufficiently smooth coefficients.

The Lie point transformations that correspond to Bogoyavlenskij symmetries were found by direct application of the Lie procedure to the MHD equilibrium equations (1.16)-(1.17) in incompressible (1.18) and compressible (1.19) cases.

The Lie procedure in application to the MHD system is described in Section 2.2. Every system of PDEs with n variables and m unknown functions represents a manifold Ω^1 in $(m + n + mn)$ - dimensional jet space $X \times U \times U_1$ of independent and dependent variables $\mathbf{x}, \mathbf{u}$ (2.2) and partial derivatives u_i^k (2.1). The Lie procedure consists in finding vector fields $\mathbf{h}$ (2.5) tangent to Ω^1. These vector fields serve as infinitesimal transformation group generators. Their components ξ^i, η^j (2.6) are functions of all independent and dependent variables. The equations (2.10) for determining the tangent vector field components are the conditions of invariance of the solution manifold Ω^1 under the action of $\mathbf{h}$.

It is known that generally the ideal plasma domain is spanned by nested 2-dimensional magnetic surfaces - surfaces tangent to plasma velocity and magnetic field [31]. In the group analysis procedure, this fact was taken into account by explicitly introducing a function $\Psi(\mathbf{r})$ (2.3) constant on every magnetic surface (or on magnetic field lines and plasma streamlines, if the surfaces seize to exist.) Introducing this function enables one to find Lie symmetries depending on functions constant magnetic surfaces.

The determining equations (2.10) are linear first-order partial differential equations. They are solved by employing the fact that the tangent vector field components do not depend on partial derivatives. Thus for the case of incompressible isotropic MHD equilibrium the determining system splits into 150 equations on 11 unknown functions, in the compressible case - into 188 equations on 13 unknown functions. Handling these systems, even with the help of computer symbolic manipulation software described in the introduction, puts extremely high demands on computer resources. Therefore we restricted our study to a subgroup of Lie point transformations of the type $\{\mathbf{x}' = \mathbf{x};\;\; \mathbf{u}' = \mathbf{g}(\mathbf{u}, a)\}$. These transformations preserve spatial variables and do not depend on them. In this case we got 141 determining equations for the incompressible case, and 187 - for the compressible case. These systems are substantially simpler than those arising from the general Lie procedure. Using Maple with `Rif` package, the systems were reduced respectively to 21 and 10 equations (in the compressible case, additional simplifying assumptions had to be used). Solving them, we obtained the transformation generators (2.11)-(2.14) for incompressible MHD equilibria, and (2.15)-(2.16) for compressible MHD equilibria.

The operators (2.11)-(2.14) admissible by incompressible MHD equilibria form a basis of the Lie algebra of infinitesimal operators corresponding to the subgroup $\{\mathbf{x}' = \mathbf{x},\;\; \mathbf{u}' = \mathbf{g}(\mathbf{u}, a)\}$ of the group (2.4) of all Lie point transformations.

Theorem 2.2 stated above shows that the transformations generated by operators (2.11)-(2.13), (2.15)-(2.16) are equivalent to Bogoyavlenskij symmetries G_m, G_{0m}.

This result illustrates that the general Lie approach of analyzing systems of partial differential equations is capable of revealing highly non-trivial intrinsic transformations, that may have great importance in applications, as is the case for Bogoyavlenskij symmetries.

In Appendix D, the potential symmetry analysis [70, 71] of the static isotropic plasma equilibrium system (1.20) is performed, and it is proven that an invertible linearization of this system or its auxiliary system is not possible.

Chapter 3

Symmetries and exact solutions of the Anisotropic Plasma Equilibrium system

3.1 Introduction

In the current chapter we present an *infinite-dimensional* set of transformations between isotropic (MHD) and anisotropic (CGL) plasma equilibria. These transformations can be applied to any static plasma equilibrium and to a wide class of dynamic equilibria to yield physically interesting anisotropic equilibrium solutions.

The topology of the original isotropic plasma equilibrium is essential for the transformations. It is well-known that all *compact* isotropic non-viscous incompressible MHD equilibria (except Beltrami flows) have a special topology - the plasma domain is spanned by nested 2-dimensional magnetic surfaces - surfaces on which magnetic field lines and plasma streamlines lie [1, 2, 31, 32] (see sec. 1.3.3 of Chapter 1). The new transformations explicitly depend on two arbitrary functions constant on magnetic surfaces. If the magnetic surfaces are not uniquely defined (for instance, for unbounded configurations with magnetic field lines going to infinity, or for configurations with closed magnetic field lines), the arbitrary functions of the transformations only have to be constant on magnetic field lines and plasma streamlines.

Subsection 3.2.1 contains a general theorem that describes the transformations of dynamic isotropic plasma equilibria to dynamic anisotropic equilibria, whereas subsection 3.2.2 deals with the restriction to the static case.

The new family of transformations has features different from those for Bäcklund transforms for soliton equations. Unlike Bäcklund transforms, the new transformations are explicit and depend on all three spatial variables.

In subsection 3.2.3 we show that the new transformations allow the building of anisotropic plasma equilibria that possess necessary conditions to be physically relevant and stable with respect to fire-hose and mirror instabilities.

Using the new transformations, we construct several analytical examples of localized and non-localized anisotropic plasma equilibria with different pressure profiles and different topologies.

In section 3.3, we give several exact plasma equilibrium examples: a closed non-symmetric anisotropic plasma flux tube (subsection 3.3.1), an anisotropic plasma configuration with no magnetic surfaces (subsection 3.3.2), and a model of anisotropic astrophysical jets (subsection 3.3.3).

The new transformations can also be applied to other known analytical isotropic MHD models, such as Hill-vortex-like solutions [18, 21] (cf. Chapter 1, sec. 1.4), to produce corresponding anisotropic plasma equilibria with the same topology.

The results presented in this chapter prove that in the case of anisotropic plasmas Grad's conjecture (Chapter 1, sec. 1.5.2) does not hold. Indeed, the transformations described here allow the construction of various examples of solutions of different topologies with no geometrical symmetries, with magnetic field lines not necessarily closed.

An important property of the ideal (non-viscous) isotropic MHD equilibrium equations is that they possess an infinite-dimensional abelian group of symmetries G_m (1.27), (1.28), which were recently found by Bogoyavlenskij in [1, 2]. These symmetries preserve the solution topology, but can break the geometrical symmetry. In section 3.4 of this chapter we show that these symmetries can be generalized onto the case of incompressible non-viscous *anisotropic* plasmas. The new symmetries for anisotropic plasmas also form an infinite-dimensional abelian group G, and the original group Bogoyavlenskij symmetries G_m constitutes a subgroup of G. The group G has sixteen connected components, whereas G_m has eight [2].

In sec. 3.4.3 it is shown that, similarly to the original Bogoyavlenskij symmetries, the new symmetries for anisotropic plasmas can be found as Lie point transformations from the classical Lie group analysis procedure, if the general solution topology and the incompressibility condition are explicitly taken into account in the form of additional constraints.

3.2 Transformations between MHD and CGL equilibria

In this section we present an infinite-dimensional family of transformations that map isotropic (MHD) plasma equilibrium solutions into anisotropic (CGL) ones.

Subsection 3.2.1 deals with the dynamic equilibrium case ($\mathbf{V}^2 > 0$); subsection 3.2.2 presents the transformations for static equilibria.

In subsection 3.2.3, we discuss conditions that CGL plasma equilibrium solutions must satisfy to model real phenomena, and study the stability of anisotropic equilibria that arise from the introduced transformations.

3.2.1 Transformations for dynamic equilibria

Equilibrium states of isotropic moving plasmas are described by the system of MHD equilibrium equations, which under the assumptions of infinite conductivity and negligible viscosity have the form (1.16)-(1.17) [3]

$$\rho\mathbf{V} \times \text{curl } \mathbf{V} - \frac{1}{\mu}\mathbf{B} \times \text{curl } \mathbf{B} - \text{grad } P - \rho \text{ grad}\frac{\mathbf{V}^2}{2} = 0,$$

$$\text{div } \rho\mathbf{V} = 0, \quad \text{curl}(\mathbf{V} \times \mathbf{B}) = 0, \quad \text{div } \mathbf{B} = 0.$$

In the case of incompressible plasma, the equation (1.18)

$$\text{div } \mathbf{V} = 0$$

is added to the above system; for a compressible case an appropriate equation of state must be chosen. For example, it can be the adiabatic ideal gas equation of state (1.19):

$$P = \rho^{\gamma} \exp(S/c_v), \ \mathbf{V} \cdot \text{grad } S = 0.$$

In this chapter we restrict our consideration to incompressible plasmas.

The incompressibility condition is widely used in the modelling of plasma media. For example, it is a good approximation for subsonic plasma flows with low Mach numbers $M \ll 1$, $M^2 = \mathbf{V}^2/(\gamma P/\rho)$. For incompressible plasma the continuity equation div $\rho\mathbf{V} = 0$ implies $\mathbf{V} \cdot \text{grad } \rho = 0$; hence density is constant on plasma streamlines.

It is known (cf. Chapter 1, sec. 1.3.3) that all compact incompressible MHD equilibrium configurations, except the Beltrami case curl $\mathbf{B} = \alpha\mathbf{B}, \quad \alpha = \text{const}$, are spanned by two-dimensional magnetic surfaces - the vector fields $\mathbf{B}$ and $\mathbf{V}$ are in every point tangent to magnetic surfaces.

When $\mathbf{V} \parallel \mathbf{B}$, magnetic surfaces may not be uniquely defined for unbounded configurations with magnetic field lines going to infinity, as well as for configurations with closed magnetic field lines.

For *anisotropic* plasmas with Larmor radius small compared to characteristic dimensions of the system, the corresponding set of equilibrium equations is (1.22)-(1.23) [26]:

$$\rho\mathbf{V}\times\text{curl }\mathbf{V} - \left(\frac{1}{\mu} - \tau\right)\mathbf{B}\times\text{curl }\mathbf{B} = \text{grad } p_{\perp} + \rho \text{ grad}\frac{\mathbf{V}^2}{2} + \tau \text{ grad}\frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B}\cdot\text{grad } \tau),$$

$$\text{div } \mathbf{V} = 0, \quad \text{div } \mathbf{B} = 0, \quad \text{curl}(\mathbf{V}\times\mathbf{B}) = 0.$$

Here τ is the anisotropy factor (1.21):

$$\tau = \frac{p_{\parallel} - p_{\perp}}{\mathbf{B}^2}.$$

The following theorem shows that there exist infinite-dimensional transformations that map solutions of incompressible MHD equilibrium equations to incompressible anisotropic (CGL) equilibria.

Theorem 3.1 *Let* $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ *be a solution of the system (1.16)-(1.18) of incompressible MHD equilibrium equations, where the density* $\rho(\mathbf{r})$ *is constant on both magnetic field lines and plasma streamlines (i.e. on magnetic surfaces* $\Psi = \text{const}$*, if they exist.)*

Then $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), p_{\perp 1}(\mathbf{r}), p_{\| 1}(\mathbf{r}), \rho_1(\mathbf{r})\}$ *is a solution to incompressible CGL plasma equilibria (1.22)-(1.23), where*

$$\mathbf{B}_1(\mathbf{r}) = f(\mathbf{r})\mathbf{B}(\mathbf{r}), \quad \mathbf{V}_1(\mathbf{r}) = g(\mathbf{r})\mathbf{V}(\mathbf{r}), \quad \rho_1 = C_0\rho(\mathbf{r})\mu/g(\mathbf{r})^2,$$

$$p_{\perp 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 + (C_0 - f(\mathbf{r})^2/\mu)\ \mathbf{B}(\mathbf{r})^2/2, \tag{3.1}$$

$$p_{\| 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 - (C_0 - f(\mathbf{r})^2/\mu)\ \mathbf{B}(\mathbf{r})^2/2,$$

and $f(\mathbf{r}),\ g(\mathbf{r})$ *are arbitrary functions constant on the magnetic field lines and streamlines.* C_0, C_1 *are arbitrary constants.*

The proof is given in the Appendix E.

Remark 1. Under the conditions of the theorem, the anisotropy factor

$$\tau_1 \equiv (p_{\| 1} - p_{\perp 1})/\mathbf{B_1}^{\mathbf{2}} = 1/\mu - C_0/f(\mathbf{r})^2 \tag{3.2}$$

is also constant on the magnetic field lines and streamlines, and the following relations hold:

$$p_{\perp 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 - \tau_1(\mathbf{r})\ \mathbf{B_1}(\mathbf{r})^2/2, \tag{3.3}$$

$$p_{\| 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 + \tau_1(\mathbf{r})\ \mathbf{B_1}(\mathbf{r})^2/2,$$

Remark 2. The structure of functions $f(\mathbf{r}),\ g(\mathbf{r})$.

The structure of the undefined functions $f(\mathbf{r}),\ g(\mathbf{r})$ in the transformations (3.1) depends on the topology of the original MHD equilibrium configuration $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ (see Sec. 1.3.3 of Chapter 1).

(i). If the magnetic field $\mathbf{B}$ and velocity $\mathbf{V}$ of the original MHD equilibrium configuration are not collinear, then the vector fields $\mathbf{B}$ and $\mathbf{V}$ are in every point

tangent to magnetic surfaces [31, 32], and therefore the functions $f(\mathbf{r}),\ g(\mathbf{r})$ must be constant on each of these surfaces.

(ii). Magnetic field and velocity are collinear, $\mathbf{B} = k(\mathbf{r})\mathbf{V}$ ($k(\mathbf{r})$ is some smooth function in $\mathbb{R}^3$), and each field line is dense on a 2-dimensional magnetic surface. Then $f(\mathbf{r}),\ g(\mathbf{r})$ have to be constant on every such surface.

(iii). Magnetic field and velocity are collinear, and field lines are closed loops or go to infinity. Then the functions $f(\mathbf{r}),\ g(\mathbf{r})$ only have to be constant on the plasma streamlines.

(iv). Magnetic field and velocity are collinear, and their field lines are dense in some 3D domain $\mathcal{D}$. This situation may only occur if both $\mathbf{B}$ and $\mathbf{V}$ satisfy Beltrami equations $\text{curl}\ \mathbf{B} = \alpha\mathbf{B},\ \text{curl}\ \mathbf{V} = \beta\mathbf{V},\ \ \alpha, \beta = \text{const}$. Then the functions $f(\mathbf{r}),\ g(\mathbf{r})$ are constant in $\mathcal{D}$.

Remark 3. We note that the transformations (3.1) preserve the topology of plasma configurations. All CGL solutions obtained from non-Beltrami MHD equilibria using Theorem 3.1 have the same magnetic surfaces as the original MHD equilibrium.

3.2.2 Transformations for static equilibria

It is useful to rewrite the above theorem for the case of static plasma equilibria. In the case $\mathbf{V} = 0$, the MHD equilibrium equations (1.16)-(1.18) take the form (1.20)

$$\text{curl}\ \mathbf{B} \times \mathbf{B} = \mu\ \text{grad}\ P, \quad \text{div}\ \mathbf{B} = 0,$$

and the CGL equations can be rewritten as (1.24):

$$\left(\frac{1}{\mu} - \tau\right) \text{curl}\ \mathbf{B} \times \mathbf{B} = \text{grad}\ p_{\perp} + \tau\ \text{grad}\frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \text{grad}\ \tau), \quad \text{div}\ \mathbf{B} = 0.$$

From Theorem 3.1 follows

Corollary 3.1 *Let* $\{\mathbf{B}(\mathbf{r}), P(\mathbf{r})\}$ *be a solution of the static isotropic plasma equilibrium system (1.20). Then* $\mathbf{B}_1(\mathbf{r}), p_\perp(\mathbf{r}), p_\parallel(\mathbf{r})$ *is a solution of the static CGL plasma equilibrium system (1.24), where*

$$\begin{aligned}
&\mathbf{B}_1(\mathbf{r}) = f(\mathbf{r})\mathbf{B}(\mathbf{r}),\\
&p_{\perp 1}(\mathbf{r}) = C_0 \mu P(\mathbf{r}) + C_1 + \left(C_0 - f(\mathbf{r})^2/\mu\right) \mathbf{B}(\mathbf{r})^2/2, \qquad (3.4)\\
&p_{\parallel 1}(\mathbf{r}) = C_0 \mu P(\mathbf{r}) + C_1 - \left(C_0 - f(\mathbf{r})^2/\mu\right) \mathbf{B}(\mathbf{r})^2/2.
\end{aligned}$$

Remark 4. The above Corollary can be used directly to construct a wide variety of anisotropic plasma equilibrium solutions of different topologies. Indeed, starting with any harmonic function $h(\mathbf{r}) : \; \triangle h(\mathbf{r}) = 0$ and using a corresponding vacuum magnetic field $\mathbf{B} = \operatorname{grad} h(\mathbf{r})$, one can build non-degenerate CGL plasma equilibria.

In Section 3.3 below, we present several analytical examples of anisotropic CGL plasma equilibria obtained with the help of the above corollary.

3.2.3 Physical conditions and stability of new solutions

To model real phenomena, any isotropic and anisotropic MHD equilibrium solution has to satisfy natural physical conditions.

For solutions in a domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$, bounded and unbounded respectively, one should demand that the conditions (1.35), (1.36) described in Chapter 1, sec. 1.3.6, are satisfied.

If the free functions $f(\mathbf{r}), g(\mathbf{r})$ in the transformations (3.1) are separated from zero, then the transformed anisotropic solutions retain the boundedness of the original solution. The functions $f(\mathbf{r}), g(\mathbf{r})$ in every particular model must be selected so that the new anisotropic solution has proper asymptotics at $|\mathbf{r}| \to \infty$.

Now we address the question of stability of the new equilibrium solutions (3.1). Under the assumption of double-adiabatic behaviour of the plasma (1.12), it is known that the criterium for the fire-hose instability is [72] (1.39)

$$p_{\|} - p_{\perp} > \frac{\mathbf{B}^2}{\mu},$$

(or, equivalently, $\tau > 1/\mu$), and for the mirror instability – (1.40)

$$p_{\perp}\left(\frac{p_{\perp}}{6p_{\|}} - 1\right) > \frac{\mathbf{B}^2}{2\mu}.$$

Now we explicitly check these conditions for the transformed CGL equilibria $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), p_{\perp 1}(\mathbf{r}), p_{\|1}(\mathbf{r}), \rho_1(\mathbf{r})\}$ (3.1), supposing that the original isotropic MHD equilibrium configuration $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ satisfies physical conditions (1.35) or (1.36).

From (3.1), for the new solutions

$$p_{\|1} - p_{\perp 1} = \left(\frac{1}{\mu} - \frac{C_0}{f^2}\right)\ \mathbf{B_1}^2 = \frac{\mathbf{B}^2 f^2}{\mu} - C_0\mathbf{B}^2.$$

Hence the fire-hose instability is not present when

$$\frac{\mathbf{B}^2 f^2}{\mu} - C_0\mathbf{B}^2 \leq \frac{\mathbf{B_1}^2}{\mu} = \frac{\mathbf{B}^2 f^2}{\mu}.$$

Thus any choice of $C_0 \geq 0$ prevents the new solutions from having the fire-hose instability.

Now we consider the sufficient condition of the mirror instability (1.40). We define $Q = C_0\mu P(\mathbf{r}) + C_1$, and for stability demand

$$p_{\perp 1}\left(\frac{p_{\perp 1}}{6p_{\|1}} - 1\right) \leq \frac{\mathbf{B_1}^2}{2\mu},$$

which can be rewritten as

$$-\left(5Q + \frac{7}{2}\left(\frac{f^2}{\mu} - C_0\right)\mathbf{B}^2\right)\left(Q - \frac{1}{2}\left(\frac{f^2}{\mu} - C_0\right)\mathbf{B}^2\right) \leq \frac{3f^2\mathbf{B}^2}{2\mu}\left(2Q + \mathbf{B}^2\left(\frac{f^2}{\mu} - C_0\right)\right).$$

This is a square inequality with respect to an unknown function $z = f^2(\mathbf{r})$ constant on magnetic field lines and plasma streamlines:

$$\frac{\mathbf{B}^4}{2\mu}z^2 - 4\mathbf{B}^2(2Q + C_0\mathbf{B}^2)z - \frac{1}{2}\mu(10Q - 7C_0\mathbf{B}^2)(2Q + C_0\mathbf{B}^2) \leq 0. \tag{3.5}$$

From this inequality we determine the possible range of $f^2(\mathbf{r})$. If we take $C_1 \geq 0$ (and thus $Q \geq 0$ for $P \geq 0$) and assume $\mathbf{B}^2 \geq 0$ in the plasma domain, then the discriminant $D = 3\mathbf{B}^4(2Q + C_0\mathbf{B}^2)(14Q + 3C_0\mathbf{B}^2)$ is non-negative, and the roots are

$$z_{1,2} = \frac{4\mu}{\mathbf{B}^2}(2Q + C_0\mathbf{B}^2) \mp \frac{\mu\sqrt{D}}{\mathbf{B}^4}. \tag{3.6}$$

If the original plasma equilibrium is static, then on every magnetic surface S: $P|_S = \text{const} \geq 0$, hence $Q|_S = \text{const} \geq 0$, and it is easy to check that $z_1|_S(|\mathbf{B}|)$ is always concave down, while $z_2|_S(|\mathbf{B}|)$ is concave up. Therefore under the physical assumptions of non-negativity and boundedness of P and $\mathbf{B}^2$, on any magnetic surface S $\max_S z_1 < \min_S z_2$.

If the original equilibrium is not static, then the function Q is not constant on magnetic surfaces, and it should be explicitly checked that on every surface the inequality $\max_S z_1 < \min_S z_2$ holds.

The values of $f^2(\mathbf{r})$ on magnetic surfaces must be selected within the interval $\max_S z_1 \leq f^2(\mathbf{r})|_S \leq \min_S z_2$, and thus the new CGL solution will not have the mirror instability. This is the only limitation on the choice of the function $f^2(\mathbf{r})$.

In sec. 3.3 below, we discuss particular examples and explicitly verify the fire-hose and mirror instability conditions.

Conclusion. For any MHD equilibrium that satisfies natural physical conditions, by using the transformations (3.1) one can construct infinitely many anisotropic CGL equilibria that are free from the fire-hose instability. Every *static* MHD equilibrium can be transformed into an infinite family of anisotropic equilibria free from the mirror instability.

3.3 Examples of anisotropic (CGL) plasma equilibria

3.3.1 A closed flux tube with no geometrical symmetries

The transformations between isotropic and anisotropic motionless plasmas (3.4) can indeed be applied to vacuum magnetic field configurations

$$\mathbf{B} = \text{grad } f(\mathbf{r}), \quad \text{div } \mathbf{B} = 0, \tag{3.7}$$

which are equivalent to solutions of the Laplace equation $\nabla^2 f(\mathbf{r}) = 0$. Magnetic fields produced by linear electric currents represent a part of this family; they have a critical line coinciding with the line of current and decrease at infinity, according to Bio-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu\ I}{4\pi} \int_L \frac{\mathbf{dl} \times (\mathbf{r} - \mathbf{r_1})}{(\mathbf{r} - \mathbf{r_1})^3}. \tag{3.8}$$

Such magnetic fields can have different topologies, depending on the shape of current circuit. For instance, if the current circuit is flat, one readily shows that the magnetic field lines are closed, and therefore lie on magnetic surfaces (which in this case are not defined uniquely).

Such fields themselves represent degenerate plasma equilibria (1.20) with no pressure and currents, and thus are not valuable as models, but they can be used for construction of non-trivial CGL plasma equilibria.

In this example we apply the Corollary 3.1 to a magnetic field produced by a non-symmetric closed line of current having the parametrization

$$x(t) = 10.0\cos(2\pi t), y(t) = 7.7\sin(2\pi t), z(t) = 10.0(t^2 - t)\sin^2(16\pi t). \qquad (3.9)$$

For a magnetic field from such a circuit, there is no analytical representation simpler than the integral (3.8). For several starting points, we numerically traced magnetic field lines parameterized by $\mathbf{r}(t)$:

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{B}(\mathbf{r}(t)), \qquad (3.10)$$

using the Runge-Kutta method of degree 4 ($\mu/4\pi = 1, \quad I = 1$).

For the current conductor configuration (3.9), calculations show that the magnetic field around the conductor lies on 2-dimensional families of nested tori, which were reconstructed using Delaunay triangulation algorithm implemented in `tcocone` software (Tamal K. Dey *et al*, Ohio State University).

The shape of three such nested tori is shown on Fig. 3-1, whereas the positions of several families of these tori with respect to the circuit is presented on Fig. 3-2.

In the similar manner, for every initial condition that was attempted, the corresponding magnetic field line was always dense on some torus stringed on the conductor. Therefore we hypothesize that the families of tori are separated by two-dimensional surfaces.

Figure 3-3 shows the Poincare section of the dynamical system (3.10) for the initial data lying on the three tori from the picture shown on Fig. 3-1.

In a given family of nested tori, one can choose a particular torus T_0, and a transverse variable ψ continuously enumerating all the family members inside it, $0 \le \psi < \infty$. For example, one can choose $\psi|_{T_0} = 0$ to correspond to the outmost torus, and $\psi \to \infty$ near the axis of the family. Then by Corollary 3.1 one has an

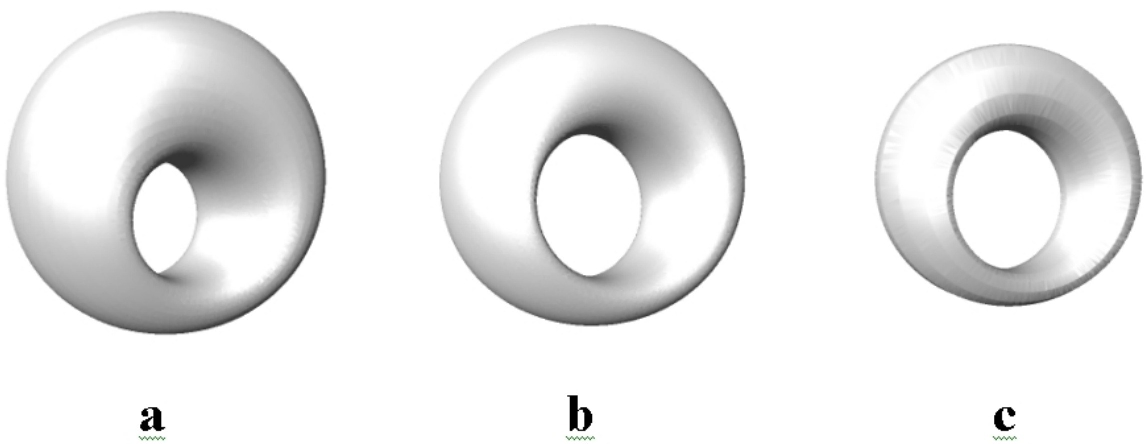

Figure 3-1: Non-symmetric toroidal magnetic surfaces of an anisotropic equilibrium.

Three sample tori of one family; the smaller ones are situated inside the bigger. This vacuum configuration is used for producing a nontrivial anisotropic plasma equilibria by the transformations 3.1 (the example from sec. 3.3.1).

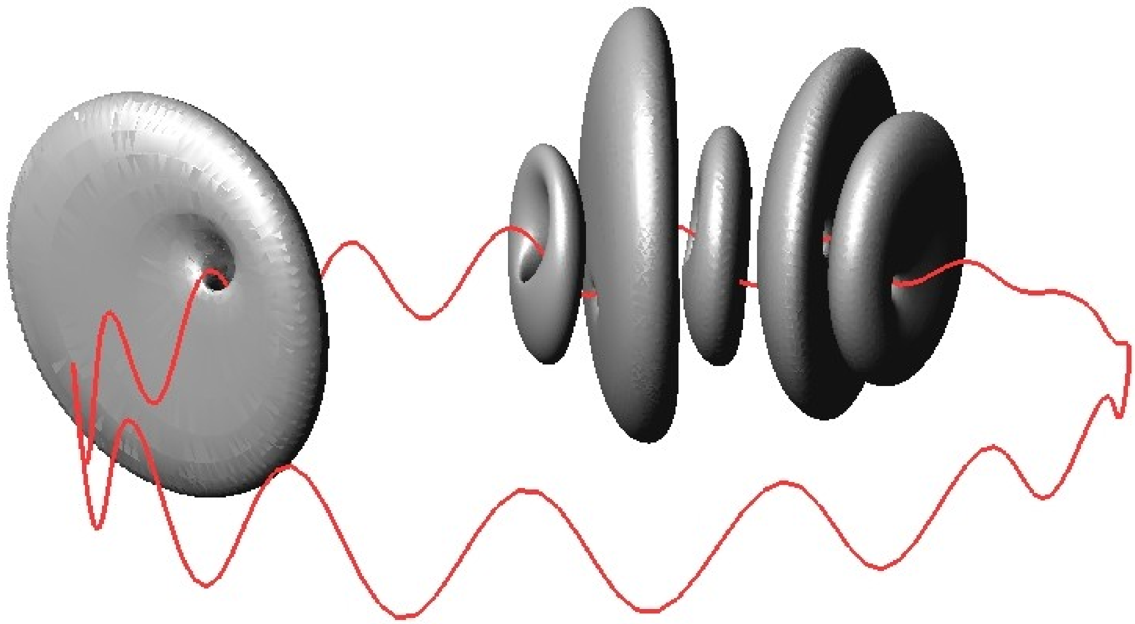

Figure 3-2: Non-symmetric magnetic flux tubes around a current conductor.

The mutual position of the current conductor and the anisotropic flux tubes around it (the solution from sec. 3.3.1).

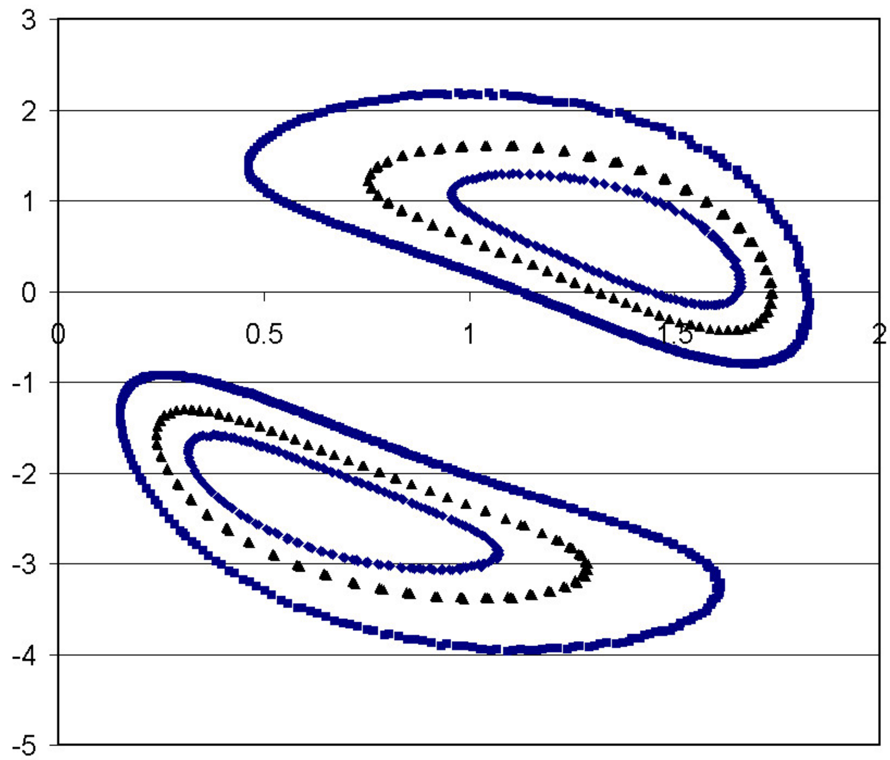

Figure 3-3: Poincare section of the non-symmetric magnetic field tangent to tori.

The Poincare section of the magnetic field lines (dynamical system (3.10)) lying on three nested magnetic surfaces of the anisotropic flux tube configuration (sec. 3.3.1).

infinite-dimensional set of CGL plasma equilibrium configurations

$$\mathbf{B}_1(\mathbf{r}) = f(\psi)\mathbf{B}(\mathbf{r}),$$

$$p_{\perp 1}(\mathbf{r}) = C_1 + (C_0 - f(\psi)^2/\mu)\ \mathbf{B}(\mathbf{r})^2/2, \tag{3.11}$$

$$p_{\parallel 1}(\mathbf{r}) = C_1 - (C_0 - f(\psi)^2/\mu)\ \mathbf{B}(\mathbf{r})^2/2.$$

$$\tau_1(\psi) = 1/\mu - C_0/f(\psi)^2.$$

We select the torus on Fig. 3-3a to be the boundary of the plasma domain $\mathcal{D}$. Estimates give $0.14 \lesssim \mathbf{B}^2|_{\mathcal{D}} \lesssim 6.82$.

We choose now $C_0 = 5, C_1 = 2$. $f(\psi)$ is an arbitrary function defined on the range of ψ; the range of $f(\psi)$ must be chosen so that within the whole plasma domain the mirror instability condition is satisfied (3.6):

$$\frac{4}{\mathbf{B}^2}(2Q + C_0\mathbf{B}^2) - \frac{\sqrt{D}}{\mathbf{B}^4} \leq \frac{f(\mathbf{r})^2}{\mu} \leq \frac{4}{\mathbf{B}^2}(2Q + C_0\mathbf{B}^2) + \frac{\sqrt{D}}{\mathbf{B}^4}.$$

Using the parameters listed above, this gives

$$4.46 \leq \frac{f(\mathbf{r})^2}{\mu} \leq 40.24. \tag{3.12}$$

The requirement for pressure positive-definiteness $p_\parallel \geq 0,\ p_\perp \geq 0$ is expressed from (3.11) and gives an additional condition on $f(\mathbf{r})^2$:

$$C_0 - \frac{2C_1}{\mathbf{B}^2} \leq \frac{f(\mathbf{r})^2}{\mu} \leq C_0 + \frac{2C_1}{\mathbf{B}^2},$$

which is numerically represented as

$$4.42 \leq \frac{f(\mathbf{r})^2}{\mu} \leq 5.58. \tag{3.13}$$

Finding the intersection of the two ranges (3.12) and (3.13), we conclude that all solutions having the range

$$4.46 \leq \frac{f(\mathbf{r})^2}{\mu} \leq 5.58 \tag{3.14}$$

satisfy physical conditions for pressure and are not subject to mirror or fire-hose instabilities.

The domain in which the solution is defined is bounded by the torus $\psi = 0$, which is a magnetic surface, therefore everywhere on the boundary $\mathbf{B}_1(\mathbf{r})$ is tangent to it. Hence we may define $\mathbf{B}_1(\mathbf{r}) \equiv 0$ outside of the domain. The discontinuity in tangent component of the magnetic field corresponds to a surface current on the bounding torus

$$\mathbf{i}_b(\mathbf{r}) = \mu^{-1}\mathbf{B}_1(\mathbf{r}) \times \mathbf{n}_1(\mathbf{r}),$$

where $\mathbf{n}_1(\mathbf{r})$ is an outward normal.

The presented exact solutions model a closed flux tube with no geometrical symmetries. The notion of toroidal flux tubes has been extensively used in theoretical MHD analysis (see, e.g., [32]) and in applications (e.g. a model of a ball lightning as a knotted system of closed force-free flux tubes presented in [22]), but appropriate exact solutions were not available.

3.3.2 An anisotropic plasma equilibrium with magnetic field dense in a 3D region

We now construct a CGL plasma equilibrium from a magnetic field produced by the same closed current circuit (3.9) and an additional straight current $I_2 = 3$ in the positive direction of z-axis.

Fig. 3-4 shows the Poincare section of the dynamical system (3.10) describing a magnetic field line starting from the point $x = 1.1$; $y = 10.0$; $z = 1.2$. The calculation thus suggests that the magnetic filed line does not lie on any compact magnetic surface, but is dense in some 3D region.

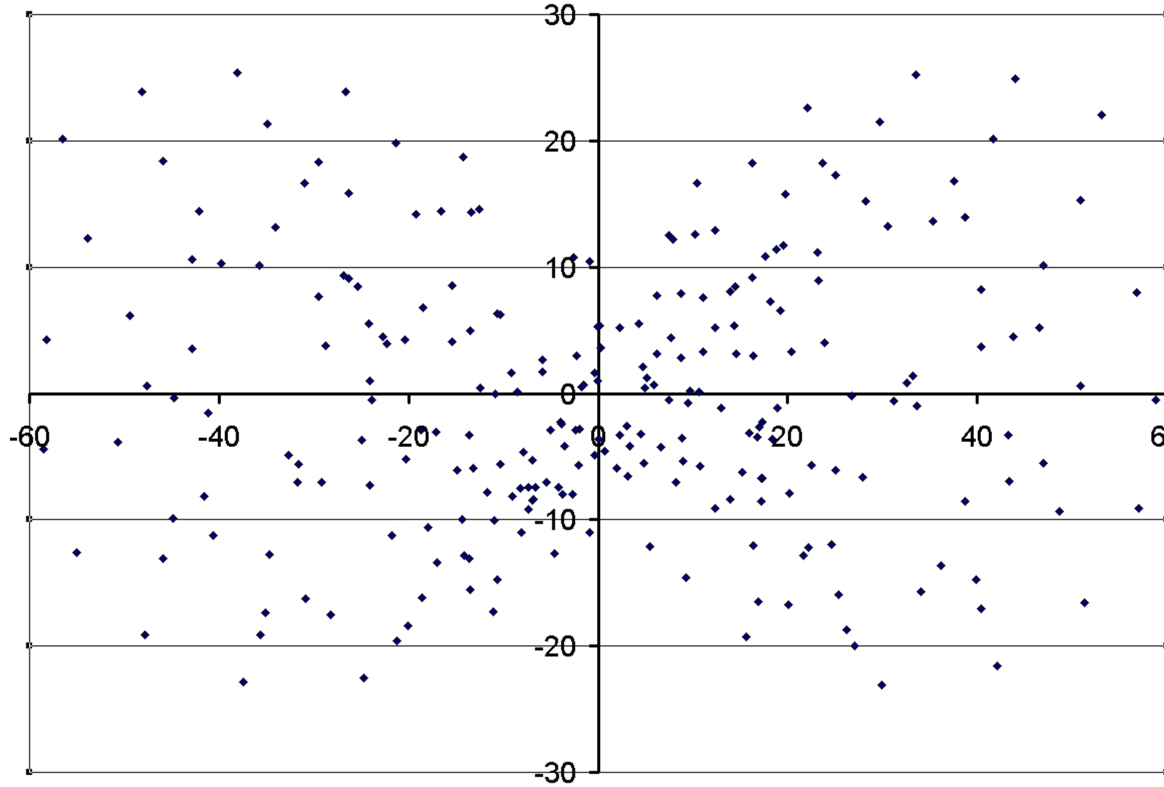

Figure 3-4: A magnetic field line dense in a 3D region.

The Poincare section of a magnetic field line (dynamical system (3.10)) for another anisotropic plasma configuration (sec. 3.3.2). The figure suggests that this magnetic field line does not lie on any 2D surface.

To get an anisotropic plasma equilibrium from this pure magnetic field configuration, we apply Corollary 3.1. If the magnetic field is dense in some 3D region, then the topology requirement on the function $f(\mathbf{r})$ is that it must be constant in the whole plasma domain.

Numerical computations cannot precisely specify the domain shape, but they suggest the boundedness of the magnetic field magnitude from above (the magnetic field lines do not tend to approach the conductors very closely). In that case, the proper choice of the constants $C_0 \geq 0, C_1 \geq 0$, will make the positive pressure requirement satisfied.

3.3.3 An anisotropic model of helically-symmetric astrophysical jets

Below we present an anisotropic helically-symmetric model of astrophysical jets. It is obtained by application of the Theorem 3.1 to certain isotropic helically symmetric MHD equilibria.

We start with the following helically symmetric [38] magnetic fields:

$$\mathbf{B}_h = \frac{\psi_u}{r}\mathbf{e}_r + B_1\mathbf{e}_z + B_2\mathbf{e}_\phi, \quad B_1 = \frac{\alpha\gamma\psi - r\psi_r}{r^2+\gamma^2}, \quad B_2 = \frac{\alpha r\psi + \gamma\psi_r}{r^2+\gamma^2}, \tag{3.15}$$

where $\mathbf{e}_r$, $\mathbf{e}_z$, $\mathbf{e}_\phi$ are the unit orts in the cylindrical coordinates r, z, ϕ and $\psi = \psi(r, u)$ is the flux function, $u = z - \gamma\phi$, $\alpha = \text{const}$, $\gamma = \text{const}$. In [8], the exact plasma equilibria (3.15), $\operatorname{curl}\mathbf{B}\times\mathbf{B} = \mu \operatorname{grad} P$, $\operatorname{div}\mathbf{B} = 0$ were obtained, that correspond to the flux functions

$$\psi_{Nmn} = e^{-\beta r^2}\left(a_N B_{0N}(y) + r^m B_{mn}(y)(a_{mn}\cos(mu/\gamma) + b_{mn}\sin(mu/\gamma))\right), \tag{3.16}$$

where N, m, n are arbitrary integers ≥ 0 satisfying the inequality $2N > 2n + m$, and $y = 2\beta r^2$. The plasma pressure is $P_h = p_0 - 2\beta^2\psi^2/\mu$, and the plasma velocity $\mathbf{V} = 0$.

The functions $B_{mn}(y)$ are polynomials [8].

The simplest exact solution (3.16) is defined for $N = 1, m = 1, n = 0$ and has the form

$$\psi_{110}(r, z, \phi) = e^{-\beta r^2}(1 - 4\beta r^2 + a_1 r \cos(z/\gamma - \phi)). \tag{3.17}$$

Fig. 3-5 shows the section $z = 0$ of the surfaces of its constant level: $\psi_{110}(r, z, \phi) =$ const for $a_1 = -1$, $\beta = 0.1$, $\gamma = \sqrt{5/2}$, $\alpha = 3/(2\gamma)$.

We now apply the "anisotropizing" transformations (Corollary 3.1) to the static exact isotropic solutions (3.15)-(3.16), and obtain new static anisotropic equilibria

$$\begin{aligned} &\mathbf{B}_a = f(\mathbf{r})\mathbf{B}_h, \\ &p_{\perp a} = C_0\mu P_h + C_1 + (C_0 - f(\mathbf{r})^2/\mu)\ \mathbf{B}_h^2/2, \\ &p_{\|a} = C_0\mu P_h + C_1 - (C_0 - f(\mathbf{r})^2/\mu)\ \mathbf{B}_h^2/2, \end{aligned} \tag{3.18}$$

and $f(\mathbf{r})$ is an arbitrary function constant on the magnetic field lines; $C_0 > 0,\ C_1$ are arbitrary constants.

Let us consider particular solutions from the family (3.18) in greater detail.

I. Anisotropic helically symmetric jets. We take the flux function in the simplest form $\psi = \psi_{110}(r, z, \phi)$, and choose a helically symmetric arbitrary function

$$f(\mathbf{r}) = (C_0 + 1/\cosh(\psi^2))^{1/2}. \tag{3.19}$$

The magnetic field and pressure functions are given by the expressions (3.18).

Fig. 3-6a represents the profiles of pressure along the x-axis (original isotropic pressure P_h shown with a thin solid line, anisotropic $p_{||a}$ with a thick dash line, and $p_{\perp a}$ with a thick solid line). The positive-pressure requirement is evidently satisfied.

On Fig. 3-6b, the original isotropic and the transformed anisotropic magnetic field magnitudes $\mathbf{B}_h^2$ and $\mathbf{B}_a{}^2$ along the x-axis are shown (isotropic with a thin solid line, and anisotropic with a thick solid line). The magnetic field is evidently bounded from above, therefore, in accordance with stability considerations presented in subsection 3.2.3, the presented sample solution is free from fire-hose and mirror instabilities.

Both figures use values $C_0 = 1.0, C_1 = 0.01$.

II. Astrophysical jet model with no symmetries. The arbitrary function $f(\mathbf{r})$ has only to be constant on magnetic field lines, not necessarily on magnetic surfaces $\psi = \text{const}$ (cf. Corollary 3.1). In the family of solutions (3.18), the magnetic field lines all go to infinity in the variable z [8]. Therefore the function $f(\mathbf{r})$ in this anisotropic solution depends on two transversal variables, and the generic exact solutions (3.18) are *non-symmetric.*

As an example the flux function $\psi = \psi_{110}(r, z, \phi)$ (3.17) and constants $a_1 = 0$, $\beta = 0.1$, $\gamma = \sqrt{5/2}$, $\alpha = 3/(2\gamma)$ (this choice indeed yields a cylindrically symmetric flux function.) A simple computation shows that the general function of two variables

$$f(\mathbf{r}) = F\left(r, \phi - \frac{2\sqrt{10}z}{2r^2 - 15}\right) \tag{3.20}$$

is constant on the lines of the magnetic field (3.15). Every magnetic field line winding on a cylindrical surface $\psi = \text{const}$ goes to infinity and is helically symmetric, but the helical constant changes from line to line; therefore the general anisotropic static plasma equilibrium solution (3.18), (3.20) has no geometrical symmetries.

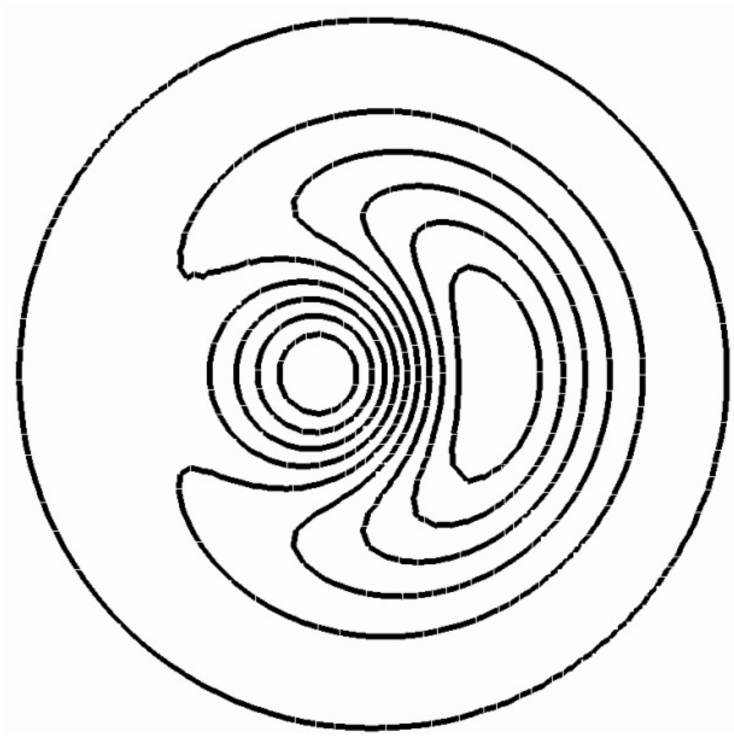

Figure 3-5: The section of helically-symmetric magnetic surfaces.

The section $z = 0$ of the magnetic surfaces $\psi(r, \phi) = \text{const}$ for an anisotropic helically-symmetric astrophysical jet model (sec. 3.3.3). (The parameter values are: $a_1 = -1$, $\beta = 0.1$, $\gamma = \sqrt{5/2}$, $\alpha = 3/(2\gamma)$).

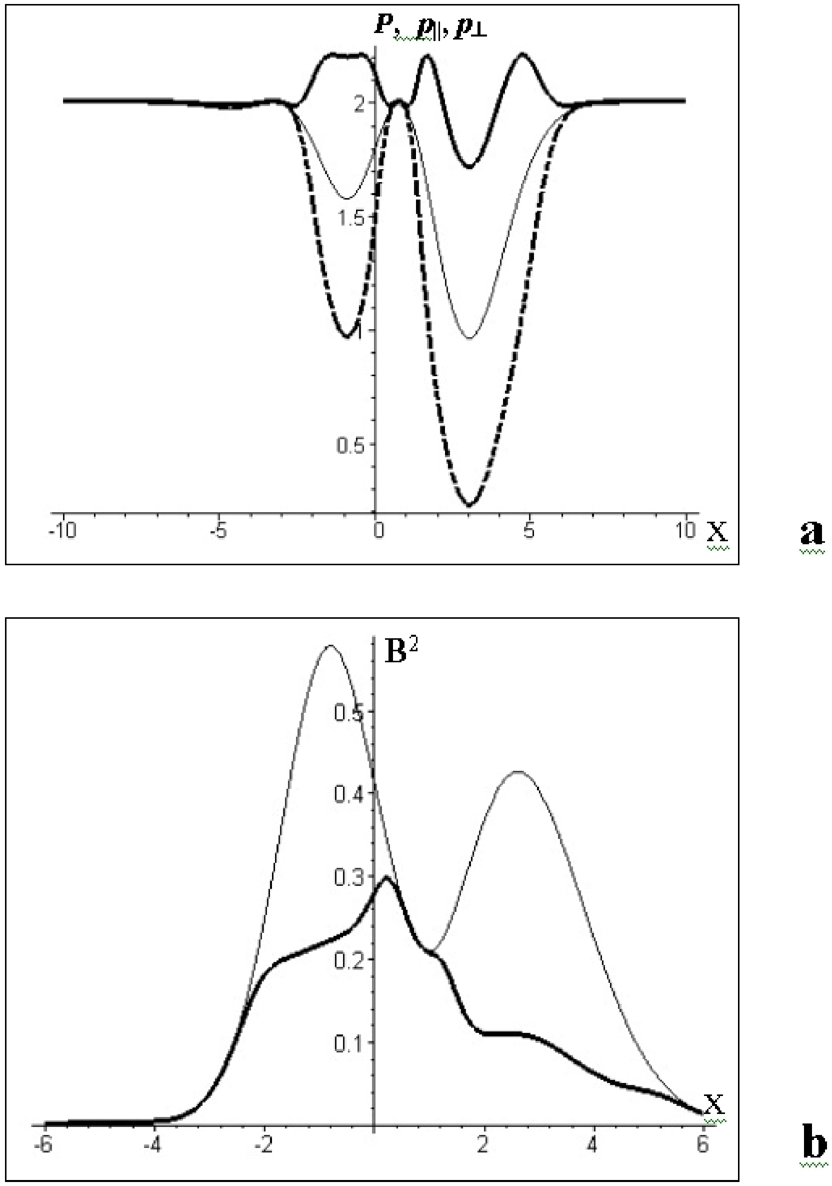

Figure 3-6: Comparison of pressure and magnetic field profiles in isotropic and anisotropic helically-symmetric equilibria.

(a) The profiles of pressure along the x-axis for a helically-symmetric astrophysical jet model ($a_1 = -1$, $\beta = 0.1$, $\gamma = \sqrt{5/2}$, $\alpha = 3/(2\gamma)$). Original isotropic pressure P_h: thin solid line, anisotropic $p_{||a}$: thick dash line; $p_{\perp a}$: thick solid line. Positive-pressure requirement is satisfied.

(b) The magnetic field magnitudes ${\mathbf{B}_h}^2$ and ${\mathbf{B}_a}^2$ for isotropic (thin line) and anisotropic (thick line) helically-symmetric astrophysical jet model (the profile along the x-axis). Same parameters as in part **(a)**.

3.4 Infinite-dimensional symmetries for anisotropic (CGL) plasma equilibria

3.4.1 The form of the symmetries

Recently Bogoyavlenskij [1, 2] found that the isotropic MHD equilibrium equations (1.16)-(1.18) possess the symmetries (1.27)-(1.28): If $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ is an MHD equilibrium, where the density $\rho(\mathbf{r})$ is constant on both magnetic field lines and streamlines, then $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), P_1(\mathbf{r}), \rho_1(\mathbf{r})\}$ is also an equilibrium solution, where

$$\mathbf{V}_1 = \frac{b(\mathbf{r})}{m(\mathbf{r})\sqrt{\mu\rho}}\mathbf{B} + \frac{a(\mathbf{r})}{m(\mathbf{r})}\mathbf{V},$$

$$\mathbf{B}_1 = a(\mathbf{r})\mathbf{B} + b(\mathbf{r})\sqrt{\mu\rho}\mathbf{V},$$

$$\rho_1 = m^2(\mathbf{r})\rho, \quad P_1 = CP + (C\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu).$$

Here $a^2(\mathbf{r}) - b^2(\mathbf{r}) = C = \text{const}$, and $a(\mathbf{r}), b(\mathbf{r}), c(\mathbf{r})$ are functions constant on both magnetic field lines and streamlines (i.e. on magnetic surfaces $\Psi = \text{const}$, when they exist).

These symmetries form an infinite-dimensional Abelian group (1.28) [2]

$$G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

where R^+ is a multiplicative group of positive numbers, and A_m is an additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on magnetic surfaces. The group G_m has eight connected components.

Below we present the symmetries of ideal *anisotropic* (CGL) plasma equilibria (1.22)-(1.23), which naturally generalize the above Bogoyavlenskij symmetries for the isotropic case.

From now on we consider only plasma configurations free of the fire-hose instability, i.e for which $1/\mu > \tau$ in the whole domain (see sec. 3.2.3 above.)

Theorem 3.2 *Let* $\{\mathbf{V}(\mathbf{r}),\ \mathbf{B}(\mathbf{r}),\ p_{\perp}(\mathbf{r}),\ p_{\parallel}(\mathbf{r}),\ \rho(\mathbf{r})\}$ *be a solution of the CGL equilibrium system (1.22)-(1.23), where the density* $\rho(\mathbf{r})$ *and the anisotropy factor* $\tau(\mathbf{r})$ *(1.21) are constant on both magnetic field lines and streamlines. Then* $\{\mathbf{V_1}(\mathbf{r}),\ \mathbf{B_1}(\mathbf{r}),\ p_{\perp 1}(\mathbf{r}),\ p_{\parallel 1}(\mathbf{r}),\ \rho_1(\mathbf{r})\}$ *is also a solution, where*

$$\begin{aligned}
&\rho_1 = m^2(\mathbf{r})\rho,\\
&\mathbf{V}_1 = \frac{b(\mathbf{r})\sqrt{1/\mu-\tau}}{m(\mathbf{r})\sqrt{\rho}}\mathbf{B} + \frac{a(\mathbf{r})}{m(\mathbf{r})}\mathbf{V},\\
&\mathbf{B}_1 = \frac{a(\mathbf{r})}{n(\mathbf{r})}\mathbf{B} + \frac{b(\mathbf{r})\sqrt{\rho}}{n(\mathbf{r})\sqrt{1/\mu-\tau}}\mathbf{V},\\
&p_{\perp 1} = Cp_{\perp} + \frac{(C\mathbf{B}^2-\mathbf{B}_1^2)}{2\mu},\\
&p_{\parallel 1} = p_{\parallel} n^2(\mathbf{r})\frac{\mathbf{B}_1^2}{\mathbf{B}^2} + p_{\perp}\left(C - n^2(\mathbf{r})\frac{\mathbf{B}_1^2}{\mathbf{B}^2}\right) + \frac{(C\mathbf{B}^2+\mathbf{B_1}^2(1-2n^2(\mathbf{r})))}{2\mu}.
\end{aligned} \tag{3.21}$$

Here

$$a^2(\mathbf{r}) - b^2(\mathbf{r}) = C = \text{const},$$

and $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ *are functions constant on both magnetic field lines and streamlines.*

Under the conditions of the theorem, the anisotropy factor $\tau(\mathbf{r})$ is transformed as follows:

$$\tau_1 \equiv \frac{p_{\parallel 1} - p_{\perp 1}}{\mathbf{B_1}^{\mathbf{2}}} = \frac{1}{\mu} - n^2(\mathbf{r})\left(\frac{1}{\mu} - \tau\right).$$

The proof of this theorem is given in the Appendix F.

3.4.2 Properties of the symmetries

The above symmetry transforms are applicable to any dynamic or static anisotropic CGL plasma configuration with density $\rho(\mathbf{r})$ and the anisotropy factor $\tau(\mathbf{r})$ constant on magnetic field lines and streamlines. For example, it can be directly applied to *static* anisotropic configurations that were obtained in section 3.3, and produce families of dynamic solutions.

Like Bogoyavlenskij symmetries (1.27), the transformations (3.21) are invertible for $C \neq 0$:

$$C\mathbf{V} = \frac{a(\mathbf{r})}{m_1(\mathbf{r})}\mathbf{V}_1 - \frac{b(\mathbf{r})\sqrt{1/\mu - \tau_1(\mathbf{r})}}{m_1(\mathbf{r})\sqrt{\rho_1(\mathbf{r})}}\mathbf{B}_1, \quad C\mathbf{B} = \frac{a(\mathbf{r})}{n_1(\mathbf{r})}\mathbf{B}_1 - \frac{b(\mathbf{r})\sqrt{\rho_1(\mathbf{r})}}{n_1(\mathbf{r})\sqrt{1/\mu - \tau_1(\mathbf{r})}}\mathbf{V}_1.$$

The structure of the arbitrary functions The arbitrary functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ must be constant on magnetic field lines and plasma streamlines, and therefore their structure depends on the topology of the original anisotropic MHD equilibrium configuration $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), \tau(\mathbf{r}), p_\perp(\mathbf{r}), \rho(\mathbf{r})\}$.

In the following topologies the structure of the unknown functions is evident:

(i). If the magnetic field $\mathbf{B}$ and the velocity $\mathbf{V}$ of the original anisotropic MHD equilibrium configuration are in every point tangent to magnetic surfaces, then the functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ must be constant on each of these surfaces.

(ii). If magnetic field and velocity are collinear, $\mathbf{B} = k(\mathbf{r})\mathbf{V}$ ($k(\mathbf{r})$ is some smooth function in $\mathbb{R}^3$), and each field line is dense on a 2-dimensional magnetic surface, then $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ have to be constant on every such surface.

(iii). Magnetic field and velocity are collinear, and field lines are closed loops or go to infinity. Then the functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ have to be constant on the plasma streamlines.

(iv). Magnetic field and velocity are collinear, and their field lines are dense in some 3D domain $\mathcal{D}$. Then the functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ are constant in $\mathcal{D}$.

The group structure of the new transformations. Consider the set G of all transformations (3.21) with $C \neq 0$ with smooth $a(\mathbf{x})$, $b(\mathbf{x})$, $m(\mathbf{x})$, and $n(\mathbf{x})$ constant on magnetic field lines and plasma streamlines, for a given anisotropic MHD equilibrium. Each transformation is prescribed by a quadruple of functions (a, b, m, n) that satisfy the conditions

$$a^2(\mathbf{r}) - b^2(\mathbf{r}) \equiv \text{const} = C \neq 0, \quad m(\mathbf{r}) \neq 0, \quad n(\mathbf{r}) \neq 0.$$

The domain E for these transformations consists of all divergence-free incompressible MHD equilibria that have the same topology of magnetic surfaces.

Consider a function $h(\mathbf{r})$ constant on the lines of $\mathbf{V}$ and $\mathbf{B}$ of an initial equilibrium configuration. This implies

$$\mathbf{B} \cdot \operatorname{grad} h(\mathbf{r}) = 0, \quad \mathbf{V} \cdot \operatorname{grad} h(\mathbf{r}) = 0.$$

For the transformed "mixed" vector fields $\mathbf{V}_1$ and $\mathbf{B}_1$ (3.21) one also has

$$\mathbf{B}_1 \cdot \operatorname{grad} h(\mathbf{r}) = 0, \quad \mathbf{V}_1 \cdot \operatorname{grad} h(\mathbf{r}) = 0,$$

therefore the function $h(\mathbf{r})$ is also constant on the lines magnetic field lines and plasma streamlines of the new plasma equilibrium configuration. This fact, together with the invertibility of the transformations (3.21) for $C \neq 0$ proves that the range of these transformations is the same as their domain. Hence the composition of the transformations is well defined.

We now show that the composition assigns on the set G the structure of an Abelian group. Indeed, the composition of the transformations (3.21) is equivalent to the 4×4

matrix multiplication

$$\begin{pmatrix} m_2 & 0 & 0 & 0 \\ 0 & n_2 & 0 & 0 \\ 0 & 0 & a_2\sqrt{\frac{\rho_1}{\rho_2}} & b_2\frac{\sqrt{1/\mu-\tau_1}}{\sqrt{\rho_2}} \\ 0 & 0 & b_2\frac{\sqrt{\rho_1}}{\sqrt{1/\mu-\tau_2}} & a_2\frac{\sqrt{1/\mu-\tau_1}}{\sqrt{1/\mu-\tau_2}} \end{pmatrix} \times \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & n_1 & 0 & 0 \\ 0 & 0 & a_1\sqrt{\frac{\rho}{\rho_1}} & b_1\frac{\sqrt{1/\mu-\tau}}{\sqrt{\rho_1}} \\ 0 & 0 & b_1\frac{\sqrt{\rho}}{\sqrt{1/\mu-\tau_1}} & a_1\frac{\sqrt{1/\mu-\tau}}{\sqrt{1/\mu-\tau_1}} \end{pmatrix}$$

$$= \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & a\sqrt{\frac{\rho}{\rho_2}} & b\frac{\sqrt{1/\mu-\tau}}{\sqrt{\rho_2}} \\ 0 & 0 & b\frac{\sqrt{\rho}}{\sqrt{1/\mu-\tau_2}} & a\frac{\sqrt{1/\mu-\tau}}{\sqrt{1/\mu-\tau_2}} \end{pmatrix},$$

where $m = m_2m_1$, $n = n_2n_1$, $a = a_2a_1 + b_2b_1$, $b = b_2a_1 + a_2b_1$. In other words,

$$(m_2, n_2, a_2, b_2) \cdot (m_1, n_1, a_1, b_1) = (m_2m_1, n_2n_1, a_2a_1 + b_2b_1, b_2a_1 + a_2b_1), \quad (3.22)$$

that implies $C = a^2 - b^2 = C_2C_1 \neq 0$. The unit quadruple is $(1, 1, 1, 0)$, the inverse transform corresponds to the quadruple $(m, n, a, b)^{-1} = (m^{-1}, n^{-1}, C^{-1}a, -C^{-1}b)$. It is evident that the above multiplication is commutative and associative. Hence the symmetries (3.21) form an abelian group G.

Let us describe the structure of the group G. We introduce the parametrization $m(\mathbf{x}) = \tau \exp \alpha(\mathbf{r})$, $n(\mathbf{x}) = \lambda \exp \beta(\mathbf{r})$, where $\alpha(\mathbf{r})$, $\beta(\mathbf{r})$ are smooth functions constant on the magnetic field lines and plasma streamlines; $\tau, \lambda = \pm 1$. For $C = \sigma k^2$, $\sigma = \pm 1$, $k > 0$, the equation $a^2(\mathbf{r}) - b^2(\mathbf{r}) = C$ is resolved in the form: $\sigma = 1$: $a(\mathbf{r}) = \eta k \operatorname{ch} \delta(\mathbf{r})$, $b(\mathbf{r}) = \eta k \operatorname{sh} \delta(\mathbf{r})$; $\sigma = -1$: $a(\mathbf{r}) = \eta k \operatorname{sh} \delta(\mathbf{r})$, $b(\mathbf{x}) = \eta k \operatorname{ch} \delta(\mathbf{r})$, where $\eta = \pm 1$ and $\delta(\mathbf{x})$ is an arbitrary smooth function constant on the magnetic field lines and plasma streamlines. Hence each transformation (3.21) corresponds to a octuple $(\alpha(\mathbf{r}), \beta(\mathbf{r}), \delta(\mathbf{r}), k, \tau, \lambda, \sigma, \eta)$. The group multiplication law then can be written in the form

$$(\alpha_1(\mathbf{r}), \beta_1(\mathbf{r}), \delta_1(\mathbf{r}), k_1, \tau_1, \lambda_1, \sigma_1, \eta_1) \cdot (\alpha_2(\mathbf{x}), \beta_2(\mathbf{x}), \delta_2(\mathbf{r}), k_2, \tau_2, \lambda_2, \sigma_2, \eta_2) =$$

$$(\alpha_1(\mathbf{r}) + \alpha_2(\mathbf{r}), \beta_1(\mathbf{r}) + \beta_2(\mathbf{r}), \delta_1(\mathbf{r}) + \delta_2(\mathbf{r}), k_1k_2, \tau_1\tau_2, \lambda_1\lambda_2, \sigma_1\sigma_2, \eta_1\eta_2).$$

Hence the group G is the direct sum

$$G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \tag{3.23}$$

Here R^+ is the multiplicative group of positive numbers $k > 0$. The A_m is the additive abelian group of smooth functions in $\mathbb{R}^3$ that are constant on the magnetic field lines and plasma streamlines for a given anisotropic MHD equilibrium. The group A_m is a linear space and an associative algebra with respect to the multiplication of functions. The group G evidently has 16 connected components.

The group G_m of Bogoyavlenskij transformations (1.27), (1.28) constitutes an infinite - dimensional subgroup of G.

The conservation of the Lagrangian. It is known that under the action of Bogoyavlenskij symmetries (1.27), the Lagrangian density function

$$L(\mathbf{r}, t) = \frac{\rho \mathbf{V}^2}{2} - \frac{\mathbf{B}^2}{2\mu} \tag{3.24}$$

of an isotropic incompressible MHD system is transformed as

$$L_1(\mathbf{r}, t) = \frac{\rho_1 \mathbf{V}_1^2}{2} - \frac{\mathbf{B_1}^2}{2\mu} = CL(\mathbf{r}, t), \tag{3.25}$$

where $C = \text{const} = a^2(\mathbf{r}) - b^2(\mathbf{r})$.

The same property holds for the generalization of Bogoyavlenskij symmetries on the anisotropic case - the symmetries (3.21) of CGL equilibria.

Direct substitution and a short computation show that the anisotropic Lagrangian density

$$L_a(\mathbf{r}, t) = \frac{\rho \mathbf{V}^2}{2} + \left(\frac{1}{\mu} - \tau \right) \frac{\mathbf{B}^2}{2} \tag{3.26}$$

is transformed by the same rule:

$$L_{a1}(\mathbf{r},t) = \frac{\rho_1 \mathbf{V}_1^2}{2} - \left(\frac{1}{\mu} - \tau_1\right)\frac{\mathbf{B_1}^2}{2} = C\left(\frac{\rho \mathbf{V}^2}{2} + \left(\frac{1}{\mu} - \tau\right)\frac{\mathbf{B}^2}{2}\right) \equiv C L_a(\mathbf{r},t). \quad (3.27)$$

In particular, this relation implies that the symmetries (3.21) with $C = 1$ preserve the Lagrangian.

Stability considerations. If the original plasma configuration possesses the inequality $1/\mu - \tau > 0$, and thus is free of the fire-hose instability, then the transformed configuration is also fire-hose-stable when $1/\mu - \tau_1 = n^2(\mathbf{r})(1/\mu - \tau) > 0$. The latter is always true if $n(\mathbf{r}) \neq 0$ in the plasma domain. Therefore the symmetries (3.21) do not produce the fire-hose instability.

The possibility for the transformed solution to have the mirror and other instabilities needs to be studied separately in each case.

3.4.3 Connection with Lie point transformations

As it is shown in Chapter 2 of this work (theorems 2.1, 2.2), the original Bogoyavlenskij symmetries (1.27) are equivalent to certain Lie point transformations of the isotropic MHD equilibrium equations, and can be obtained independently by Lie group analysis of that system, provided that the general solution topology (the existence of magnetic surfaces to which vector fields $\mathbf{B}$ and $\mathbf{V}$ are tangent) and the incompressibility condition are explicitly taken into account in the form of additional constraints:

$$\rho(\mathbf{r}) = \rho(\Psi(\mathbf{r})), \quad \mathrm{grad}(\Psi(\mathbf{r})) \cdot \mathbf{B} = 0, \quad \mathrm{grad}(\Psi(\mathbf{r})) \cdot \mathbf{V} = 0.$$

Here $\Psi(\mathbf{r})$ is a magnetic surface function (or, more generally, a function constant on magnetic field lines and plasma streamlines.)

Statements similar to theorems 2.1, 2.2 of Chapter 2 are true for the symmetries (3.21) of anisotropic plasma equilibria [73].

Theorem 3.3 *Consider the incompressible anisotropic CGL equilibrium system of equations (1.22)-(1.23), where the density* $\rho(\mathbf{r})$ *and anisotropy factor* $\tau(\mathbf{r})$ *(1.21) are constant on both magnetic field lines and streamlines. This system admits the infinitesimal operators*

$$X^{(1)} = M(\mathbf{r})\left(\sum_{k=1}^{3} B_k \frac{1/\mu - \tau}{\rho}\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k \frac{\partial}{\partial B_k} - \frac{1}{\mu}(\mathbf{V}\cdot\mathbf{B})\frac{\partial}{\partial p_\perp}\right), \tag{3.28}$$

$$X^{(2)} = \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2p_\perp \frac{\partial}{\partial p_\perp}, \tag{3.29}$$

$$X^{(3)} = N(\mathbf{r})\left(2\rho\frac{\partial}{\partial \rho} - \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k}\right), \tag{3.30}$$

$$X^{(4)} = L(\mathbf{r})\left(2\left(\frac{1}{\mu} - \tau\right)\frac{\partial}{\partial \tau} - \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + \frac{B^2}{\mu}\frac{\partial}{\partial p_\perp}\right), \tag{3.31}$$

$$X^{(5)} = \frac{\partial}{\partial p_\perp}. \tag{3.32}$$

These operators form a basis of the Lie algebra of infinitesimal operators in the class of Lie point transformations $\{\mathbf{x}' = \mathbf{x},\ \mathbf{u}' = \mathbf{g}(\mathbf{u}, a)\}$. *Here* $L(\mathbf{r}), M(\mathbf{r}), N(\mathbf{r})$ *are arbitrary smooth functions constant on both magnetic field lines and streamlines.*

Theorem 3.4 *Lie point transformations defined by (3.28)-(3.32) are equivalent to the group* G *of transformations (3.21), (3.23).*

These theorems are proven in exactly the same way as the corresponding theorems of Chapter 2 for the isotropic case (cf. [73].)

3.5 Conclusion

In this chapter we present two methods of constructing new anisotropic plasma equilibrium configurations as solutions to the Chew-Goldberger-Low (CGL) system of partial differential equations (1.22)-(1.23) [26].

The CGL system is a continuum approximation used to describe plasmas in which the mean free path for particle collisions is long compared to Larmor radius, for instance, this is the case in strongly magnetized or rarified plasmas. Unlike isotropic magnetohydrodynamics (MHD) where plasma pressure is a scalar, in the CGL approximation pressure is a tensor with two different components: along the magnetic field $p_{\parallel}$ and in the transverse direction $p_{\perp}$. The Chew-Goldberger-Low system is used to model and study anisotropic plasmas in different areas of physics, such as Earth ionosphere studies [27], plasma confinement [28], and others.

In this chapter we considered equilibrium plasma flows and static configurations, which are particularly important in many applications.

In section 3.2 we present new infinite-dimensional transformations (3.1) between isotropic (MHD) and anisotropic (CGL) plasma equilibria. These transformations can be applied to any static plasma equilibrium and to a wide class of dynamic equilibria (those with density ρ constant on plasma streamlines and magnetic field lines) and yield physically interesting anisotropic equilibrium solutions. The result is formulated in Theorem 3.1, which contains the explicit form of the transformations.

The new anisotropic solutions obtained from these transformations retain the topology of the original isotropic plasma equilibrium solution.

In subsection 3.2.1 we separately discuss the form of the new transformations when they are applied to static MHD equilibria ($\mathbf{V} = \mathbf{0}$). It appears that the transformations can be applied even to degenerate plasma equilibria - pure magnetic fields in vacuum - to produce non-degenerate CGL plasma equilibria.

For the solutions of the equations to model the physical reality, they must satisfy natural boundary conditions for a phenomenon under consideration. These constraints are discussed in subsection 3.2.3, along with another important issue for equilibrium solutions - their stability.

It is shown that if the free functions $f(\mathbf{r}), g(\mathbf{r})$ in the transformations (3.1) are finite and separated from zero, then the transformed anisotropic solutions retain the boundedness of the original solution.

No common procedure for proving the stability of an MHD or a CGL plasma equilibrium is available; however, there are explicit instability criteria. In subsection 3.2.3, we show that the new anisotropic solutions obtained with the help of transformations (3.1) can be made free of the fire-hose instability (and, in the static case, of the mirror instability) by the proper choice of the transformation parameters.

The examples of using the transformations to construct new anisotropic plasma equilibrium solutions are given in Section 3.3. The first example is a closed non-symmetric anisotropic plasma tube spanned by nested toroidal flux surfaces. It is obtained by applying the transformations (3.1) to a pure ("vacuum") magnetic field of a closed thin current conductor.

The second example (subsection 3.3.2) suggests the existence of static anisotropic non-symmetric plasma equilibria with magnetic field lines dense in a 3D domain. The exact form of the solution is known (in the form of definite integrals), but the shape of the magnetic field lines was reconstructed from the dynamical system $d\mathbf{r}/dt = \mathbf{B}(\mathbf{r})$ numerically. The computations suggest the topology mentioned above.

The third example (subsection 3.3.3) is a model of anisotropic helically-symmetric astrophysical jets. It is based on the family of solutions for isotropic MHD equations obtained in [8].

In this chapter (sec. 3.4) we also present a new family of topology-dependent infinite-dimensional symmetries (3.21) of anisotropic incompressible CGL plasma equilibrium equations. These symmetries can be used to produce new solutions in an explicit algebraic form. They depend on three arbitrary functions that are constant on magnetic field lines and plasma streamlines of the original anisotropic equilibrium. The new transformations constitute an abelian group $G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ with sixteen connected components.

The presented symmetries generalize the known Bogoyavlenskij symmetries (1.27) [1, 2] for isotropic incompressible plasma equilibria. The group $G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2$ of Bogoyavlenskij symmetries is indeed a subgroup of G.

The new symmetries (3.21) depend on all three spatial variables $\mathbf{r} = (x, y, z)$; as Bogoyavlenskij symmetries, they are capable of breaking geometrical symmetries, if the original equilibrium is field-aligned.

Using these symmetries, one can construct dynamic CGL plasma equilibria from static ones. The symmetries create only solutions free from fire-hose instability. With the choice $C = 1$, the Lagrangian density is preserved.

It is shown that the new symmetries are equivalent to the isotropic-case Bogoyavlenskij symmetries under the action of the "anisotropizing" transformations (3.1): given an isotropic equilibrium, the consecutive application of Bogoyavlenskij symmetries (1.27) and the "anisotropizing" transformations gives the same result as first transforming it into an anisotropic configuration by (3.1) and then applying the generalized Bogoyavlenskij symmetries (3.21).

In the way similar to that for Bogoyavlenskij symmetries of isotropic plasma equilibria, the anisotropic equilibrium symmetries (3.21) are equivalent to certain Lie point transformations of the CGL equilibrium system. Theorems 3.3 and 3.4 in section 3.4.3 are parallel to theorems 2.1, 2.2 of Chapter 2.

The corresponding Lie point symmetries can be found by the general Lie group analysis of the CGL equilibrium system *only* if the general solution topology (the existence of magnetic surfaces) and the incompressibility condition are explicitly taken into account in the form of additional constraints.

Chapter 4

Plasma equilibrium equations in coordinates connected with magnetic surfaces

4.1 Introduction

As noted in Chapter 1 (sec. 1.4), due to the essential non-linearity and complexity of the systems of isotropic and anisotropic plasma equilibrium equations (1.16)-(1.17) and (1.22)-(1.23), only several classes of exact solutions to these systems have been found; the majority for the static isotropic case (1.20). Among the known solutions, many have very restricted applicability as physical models, because they do not satisfy necessary physical requirements (see Chapter 1, sec. 1.3.6.) The only known general methods that produce families of static isotropic solutions from known ones are dimension reduction methods (e.g. Grad-Shafranov and JFKO equations, Chapter 1, sec. 1.3.5) and Bogoyavlenskij symmetries (1.27).

In this chapter, we develop a method of building exact isotropic and anisotropic plasma equilibria with and without dynamics, and often without geometrical symmetry, in different geometries and with physically relevant properties.

We start from representing the system of *static classical plasma equilibrium* equations in coordinates connected with magnetic surfaces (sec. 4.2.) In many important cases, for a given set of magnetic surfaces, an *orthogonal coordinate system* can be constructed, with one of the coordinates constant on the magnetic surfaces. In such coordinates, the static plasma equilibrium system is reduced to two partial differential equations for two unknown functions. One of the equations of the system is a "truncated" Laplace equation, and the second has an energy-connected interpretation.

The suggested representation of the plasma equilibrium system is used for producing particular exact solutions in different geometries.

A set of coordinates is defined by its metric tensor coefficients; we establish sufficient conditions for the metric coefficients under which exact solutions to plasma equilibrium equations can be found. We also prove that in coordinates where the Laplace equation admits 2-dimensional solutions, non-trivial exact plasma equilibria of a certain type can be built.

The ways of finding new systems of coordinates where plasma equilibrium equations have exact solutions are discussed; the examples are given.

In many systems of coordinates, classical and non-classical, non-trivial *gradient* vector fields can be built, tangent to prescribed sets of magnetic surfaces (sec. 4.3.) Though gradient fields by themselves represent only degenerate plasma equilibria with constant pressure and no electric currents, and cannot model physical phenomena, they can serve as initial solutions in infinite-parameter transformations (such as (1.27), (3.1)) that produce non-trivial dynamic and static, isotropic and anisotropic plasma equilibrium configurations.

In section 4.4, we explicitly construct families of exact isotropic and anisotropic plasma equilibria. We start from static gradient and non-gradient solutions with magnetic surfaces being nested spheres, ellipsoids, non-circular cylinders, and surfaces of other types. These solutions, by virtue of transformations (1.27), (3.1), give rise to *families* of more complicated dynamic and static, isotropic and anisotropic equilibrium configurations. In the majority of constructed equilibria, all plasma parameters and the magnetic energy have finite values in the plasma domain.

The value of some of such solutions as models of astrophysical phenomena is discussed. It is shown that some essential features of the models and the relations between macroscopic parameters are in the agreement with astrophysical observations. Unlike the majority of existing models, the presented solutions are exact and generally non-symmetric.

4.2 Isotropic MHD equilibrium equations in coordinates connected with magnetic surfaces

In this section, a theorem is stated and proven that shows that the static plasma equilibrium system (1.20) with a particular restriction on the type of magnetic surfaces can be rewritten in a compact form of two equations (as opposed to four equations for a general static plasma equilibrium).

In the following sections, the properties of the new representation and its application for construction of exact solutions of different kinds of plasma equilibria are discussed.

The question whether it is possible, given a family of smooth surfaces $A(x, y, z) =$ const in $\mathbb{R}^3$, to construct (at least locally) two other families of surfaces so that the three families form a triply orthogonal system, was answered by Darboux [74].

It is known (result due to Darboux) that for two orthogonal families of surfaces to admit the third one orthogonal to both, the two families must intersect in the lines of curvature.

If $\mathbf{n}(\mathbf{r}) = \operatorname{grad} A(\mathbf{r})/|\operatorname{grad} A(\mathbf{r})|$ is a unit normal to the family of surfaces $A(x, y, z) =$ const, then a unit vector $\mathbf{k}(\mathbf{r})$ tangent to the lines of curvature of the family is found from the eigenvalue problem

$$(\mathbf{k}(\mathbf{r}) \cdot \operatorname{grad})\mathbf{n}(\mathbf{r}) = \lambda \mathbf{k}(\mathbf{r}),$$

where λ is also an unknown function.

A condition that there exists a second family of surfaces orthogonal to the family $A(x, y, z) = \text{const}$ and intersecting it in the lines of curvature is that the vector $\mathbf{k}(\mathbf{r})$ satisfies the equation [75]

$$\mathbf{k}(\mathbf{r}) \cdot \operatorname{curl} \mathbf{k}(\mathbf{r}) = 0,$$

which is a PDE of the second order with respect to the components of $\mathbf{n}(\mathbf{r})$, and thus of the third order w.r.t. the determining function $A(\mathbf{r})$ of the original family of surfaces.

Finally, *a family of surfaces* $A(x, y, z) = \text{const}$ *is a part of a triply orthogonal system if anf only if the function* $A(\mathbf{r})$ *satisfies a certain nonlinear partial differential equation of order 3* [74, 75].

Such families of surfaces were called by Darboux *the families of Lamé*.

Thus, for a given family of Lamé, one can construct a system of orthogonal coordinates with one of the coordinates constant on surfaces $A(x, y, z) = \text{const}$.

There exist many examples of families of Lamé; they include sets of parallel surfaces; sets of surfaces of revolution; Ribaucour surfaces, and other families [75].

In orthogonal coordinates, the metric tensor is

$$g_{ij} = {H_i}^2 \delta_{ij}, \quad {H_i}^2 = \left(\frac{\partial x}{\partial u^i}\right)^2 + \left(\frac{\partial y}{\partial u^i}\right)^2 + \left(\frac{\partial z}{\partial u^i}\right)^2, \quad i, j = 1, 2, 3, \tag{4.1}$$

where H_i are the scaling (Lamé) coefficients.

Further on we may use any of two notations

$$\{H_1, H_2, H_3\} \equiv \{H_u, H_v, H_w\}; \quad \{u^1, u^2, u^3\} \equiv \{u, v, w\}. \tag{4.2}$$

The subscript in each scaling coefficient indicates the coordinate it relates to; all other subscripts used below mean corresponding partial derivatives.

Theorem 4.1 *To every solution of the system*

$$\frac{\partial}{\partial u}\left(\frac{H_v H_w}{H_u}\Phi_u\right) + \frac{\partial}{\partial v}\left(\frac{H_u H_w}{H_v}\Phi_v\right) = 0, \tag{4.3}$$

$$\frac{1}{H_u^2}\Phi_u \Phi_{uw} + \frac{1}{H_v^2}\Phi_v \Phi_{vw} = -P_w \tag{4.4}$$

in some orthogonal coordinates (u, v, w) *with scaling coefficients* $\{H_u, H_v, H_w\}$ *there corresponds a solution to the isotropic static Plasma Equilibrium system (1.20) with magnetic surfaces* $w = \text{const}$ *forming a family of Lamé, the pressure* $P = P(w)$*, and the magnetic field*

$$\mathbf{B} = \frac{\Phi_u}{H_u}\mathbf{e_u} + \frac{\Phi_v}{H_v}\mathbf{e_v}. \tag{4.5}$$

Proof. The usual differential operators in orthogonal coordinates (u, v, w) have the form

$$\operatorname{grad} f = \mathbf{e}_u \frac{1}{H_u}\frac{\partial f}{\partial u} + \mathbf{e}_v \frac{1}{H_v}\frac{\partial f}{\partial v} + \mathbf{e}_w \frac{1}{H_w}\frac{\partial f}{\partial w}; \tag{4.6}$$

$$\operatorname{div} \mathbf{A} = \frac{1}{H_u H_v H_w}\left(\frac{\partial}{\partial u} H_v H_w A_1 + \frac{\partial}{\partial v} H_u H_w A_2 + \frac{\partial}{\partial w} H_u H_v A_3\right); \tag{4.7}$$

$$\operatorname{curl}\mathbf{A} = \frac{1}{H_u H_v H_w}\begin{vmatrix} H_u\mathbf{e_u} & H_v\mathbf{e_v} & H_w\mathbf{e_w} \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ H_u A_1 & H_v A_2 & H_w A_3 \end{vmatrix} \tag{4.8}$$

for any differentiable function $f(u, v, w)$ and vector field $\mathbf{A}(u, v, w){=}A_1(u, v, w)\mathbf{e}_u+$ $A_2(u, v, w)\mathbf{e}_v+$ $A_3(u, v, w)\mathbf{e}_w$. From now on we will assume the dependence of all functions on (u, v, w) and will not write it explicitly.

Given the conditions of the theorem (4.3), (4.4), we prove that the magnetic field $\mathbf{B}$ (4.5) and the pressure $P(w)$ satisfy the plasma equilibrium system (1.20)

$$\operatorname{curl}\mathbf{B}\times\mathbf{B} = \mu\ \operatorname{grad}\ P, \quad \operatorname{div}\mathbf{B} = 0.$$

First, we verify that the field $\mathbf{B}$ (4.5) is solenoidal. Indeed, after the substitution of (4.5) into the expression (4.7) for the divergence, one gets the equation (4.3), which is true by assumptions of the theorem.

Second, we check the remaining equation of the static plasma equilibrium system: $\operatorname{curl}\mathbf{B}\times\mathbf{B} = \mu\operatorname{grad}P$. Substituting the magnetic field $\mathbf{B}$ (4.5) into the expression (4.8) for the curl, we get

$$\operatorname{curl}\mathbf{B} = -\frac{1}{H_v H_w}\frac{\partial^2\Phi}{\partial v\partial w}\mathbf{e_u} + \frac{1}{H_u H_w}\frac{\partial^2\Phi}{\partial u\partial w}\mathbf{e_v}.$$

This vector field is generally not collinear with $\mathbf{B}$. A simple computation shows that the u- and v- components of (curl $\mathbf{B}$) $\times\,\mathbf{B}$ are identically zero, and the w-component (in the compact notation, with partial derivatives of $\Phi(u, v, w)$ denoted by subscripts) is

$$-\frac{1}{H_u^2 H_w}\Phi_u\Phi_{uw} - \frac{1}{H_v^2 H_w}\Phi_v\Phi_{vw}.$$

But this is equal, by the condition (4.4) of the theorem, to P_w/H_w, which is exactly the w-component of the pressure gradient $\operatorname{grad}P(w)$ in the coordinates (u, v, w).

Thus the magnetic field $\mathbf{B}$ (4.5) and the pressure $P(w)$ describe a valid plasma equilibrium configuration (1.20).

Finally we remark that the levels of constant pressure $P(w) = \text{const}$, and thus the coordinate surfaces $w = \text{const}$, are the magnetic surfaces of the plasma equilibrium, to which (curl $\mathbf{B}$) and $\mathbf{B}$ are tangent. By the assumption of the theorem, (u, v, w) is an orthogonal system, therefore the family of surfaces $w = \text{const}$ must be a *family of Lamé*.

The theorem is proven. □

Remark 1. It is evident that converse is also true: given a static MHD equilibrium (1.20) with magnetic surfaces being surfaces of Lamé, using the same argument we show that it satisfies the system (4.3), (4.4).

The expression (4.3) is the (u, v)-part of the Laplace's equation in the coordinates (u, v, w). Therefore the system of static MHD equilibrium equations (1.20) with magnetic surfaces being surfaces of Lamé is (at least locally) equivalent to the system

$$\triangle_{(u,v)}\Phi = 0, \tag{4.9}$$

$$\text{grad}_{(u,v)}\Phi \cdot \text{grad}_{(u,v)}\Phi_w = -P_w, \tag{4.10}$$

where the subscript (u, v) means that only u- and v- parts of the corresponding differential operators are used.

Remark 2. In coordinate systems where $H_u = H_u(u, v)$, $H_v = H_v(u, v)$, the second equation of the system, (4.10), has a simple energy-connected interpretation. Indeed, the equation can be rewritten as

$$\frac{1}{H_w}\frac{\partial}{\partial w}\frac{1}{2}\left(\text{grad}_{(u,v)}\Phi\right)^2 = -\frac{1}{H_w}P_w,$$

which is, by (4.5), equivalent to the relation

$$\frac{1}{H_w}\frac{\partial}{\partial w}\left(\frac{\mathbf{B}^2}{2}+P\right)=0. \tag{4.11}$$

For incompressible plasma equilibria, the latter means that *the component of the gradient of total energy density in the direction normal to the magnetic surfaces vanishes.* Therefore for any MHD equilibrium configuration in which magnetic surfaces $w = \text{const}$ form a family of Lamé, and where $H_u = H_u(u,v)$, $H_v = H_v(u,v)$, *the total energy can be finite only if the plasma domain is bounded in the direction transverse to magnetic surfaces.*

An example where the plasma domain is *not* bounded in the direction perpendicular to magnetic surfaces is the family of axially symmetric incompressible solutions described in Chapter 1, sec. 1.4 (example 4). In every layer $c_1 < z < c_2$, the total energy is infinite (though the magnetic energy is indeed finite).

Remark 3. The electric current density $\mathbf{J} = \frac{1}{\mu}\operatorname{curl}\mathbf{B}$ is written in terms of Φ as follows:

$$\mathbf{J} = \frac{1}{\mu}\left(-\frac{1}{H_v H_w}\frac{\partial^2\Phi}{\partial v\partial w}\mathbf{e_u} + \frac{1}{H_u H_w}\frac{\partial^2\Phi}{\partial u\partial w}\mathbf{e_v}\right). \tag{4.12}$$

Remark 4. As noted by Lundquist [76], the static MHD equilibrium equations (1.20) are equivalent to the time-independent incompressible Euler equations that describe ideal fluid equilibria. Therefore static Euler equations may also be presented in the form (4.3), (4.4).

Remark 5. As will be shown in the sections below, in many cases appropriate orthogonal coordinates (u, v, w) required by the above theorem may be introduced *globally* in the plasma domain $\mathcal{D}$.

4.3 Applications and properties of the coordinate representation

In the current section, we prove statements that provide machinery for constructing exact solutions to different types of plasma equilibrium systems (not only the static isotropic case).

In the **first subsection** of this section, a theorem is proven that gives sufficient conditions on the scaling coefficients H_u, H_v, H_w, such that in the corresponding coordinate system (u, v, w) a force-free solution of the plasma equilibrium system (4.3), (4.4) of certain form exists. (The solutions independent on one of the variables are not included in the formulation of this theorem - they are treated in the first lemma of the second subsection).

This theorem may be used to look for solutions of a particular type in any prescribed geometry, i.e. when a coordinate system (u, v, w) (or at least a Lamé family of magnetic surfaces $w(x, y, z) = \text{const}$) is specified. The examples are discussed in Section 4.4.

Another way of building solutions with the help of this theorem is finding metric coefficients that satisfy both the conditions of the theorem and null the Riemann tensor R_{ijkl}, thus corresponding to some coordinates in the flat space $\mathbb{R}^3$. The examples of the use of this approach are also given in Section 4.4.2. They contain families of plasma equilibria with magnetic surfaces being cylinders of non-symmetric cross-sections, well-defined in all 3-dimensional space or its subdomains, and having physical boundary/asymptotic conditions.

The procedure of reconstruction of the explicit connection of curvilinear coordinates (u, v, w) with cartesian coordinates (x, y, z) from scaling coefficients H_u, H_v, H_w

is well-defined and described in the Appendix G.

In the **second subsection** of this section, a lemma is proven that shows how to construct trivial "vacuum" (curl-free) magnetic fields in any coordinate system which affords 2-dimensional solutions to the Laplace equation.

Another lemma presents a transformation from such a "vacuum" magnetic vector field depending only on two variables into an extended magnetic vector field which is *not* force-free or gradient, and thus gives rise to a new *non-trivial* solution to plasma equilibrium equations (1.20). This transformation will be used to generalize examples in subsequent sections.

The third lemma works in coordinate systems whose metric coefficients depend on variables (u, v) only. It generalizes "vacuum" curl-free solutions tangent to surfaces $w = \text{const}$ to "vacuum" fields that have non-zero w-components. The use of this lemma is illustrated in the example section 4.4.1.

The use of trivial "vacuum" magnetic fields for the construction of *non-trivial* isotropic and anisotropic, static and dynamic plasma equilibria, where electric current density $\mathbf{J}$, plasma pressure P and velocity $\mathbf{V}$ are generally non-zero, is discussed in the **third subsection**.

Though "vacuum" magnetic fields are indeed gradient fields, their geometry may not be simple, as Lemma 4.1 and its applications show. For example, such fields can have non-symmetric field lines on spheres and ellipsoids, and surfaces of other types.

Gradient "vacuum" fields as plasma equilibria are degenerate - such equilibria have constant pressure and no electric currents, and thus cannot model physical phenomena. But they can serve as initial solutions in the infinite-parameter transformations like (1.27) and (3.1), and give rise to non-trivial dynamic and static isotropic and anisotropic plasma equilibrium configurations.

The examples of new exact isotropic and anisotropic plasma equilibrium solutions obtained using the above statements are found in section 4.4.

4.3.1 The construction of exact Isotropic Plasma Equilibria in a prescribed geometry

First, we discuss the methods of construction of exact analytical solutions to *static* plasma equilibria (PE) that arise from the representation (4.9), (4.10).

In a prescribed geometry (i.e. given an orthogonal coordinate system (u, v, w) or a Lamé family of magnetic surfaces $w(x, y, z) = \text{const}$), one can directly look for a solution $\{(\Phi(u, v, w), P(w)\}$ to the system (4.3), (4.4). For $\mathbb{R}^3$, many coordinate systems were found and studied. The need for new systems originates both from applications (search for natural descriptions of configurations of specific geometry), and from applied mathematical motivation, such as the need in systems where certain equations of mathematical physics (Laplace, Helmholtz, etc.) are separable.

A vast literature is devoted to this subject. In the book [77] by Moon and Spencer, 40 coordinate systems are described in detail, which allow separation and R-separation of Laplace and Helmholtz equations. The book contains a comprehensive description of the coordinate systems, including scaling coefficients, Stäckel matrices, general particular solutions to important equations and illustrations. This book also includes all classical systems as spherical, ellipsoidal and toroidal coordinates, as well as many more esoteric and complicated ones.

In the other book by the same authors [78], several methods of construction of new systems of coordinates with desired properties and symmetries are described, and necessary and sufficient constraints on such systems are given that allow the separability and R-separability of the Laplace, Helmholtz and vector Helmholtz equations.

Here we specify conditions for the metric tensor components of curvilinear coordinates sufficient for a certain class of Force-Free plasma equilibria solutions to exist. Examples in the section to follow illustrate the use of these sufficient conditions for the known sets of coordinates and the ways of finding new suitable coordinates explicitly.

Theorem 4.2 ***(i)*** *If the scaling coefficients of some coordinates* (u, v, w) *satisfy*

$$H_u/H_v = a(u)b(v)c(w), \quad H_w = \mathcal{F}\left(w, \lambda(v) - \mu(u)\frac{C_2(w)}{c^2(w)C_1(w)}\right), \tag{4.13}$$

then the function

$$\Phi(u, v, w) = C_1(w)\mu(u) + C_2(w)\lambda(v) \tag{4.14}$$

defines a solution to the system of isotropic plasma equilibrium system (4.3), (4.4) with pressure $P(w) = const$*, and thus a force-free plasma equilibrium (1.15), provided that* $C_1(w)$ *and* $C_2(w)$ *are chosen so that* $\dfrac{dC_1^2(w)}{dw} + c^2(w)\dfrac{dC_2^2(w)}{dw} = 0$. *Here* $\mu(u) = \int a(u)du$, $\lambda(v) = \int 1/b(u)du$.

The force-free proportionality coefficient (cf. (1.15)) is

$$\alpha(\mathbf{r}) = \alpha(w) = \frac{1}{H_w C_2(w)}\frac{dC_1(w)}{dw}.$$

(ii) *Moreover, if the scaling coefficients satisfy*

$$H_u/H_v = a(u)b(v), \quad H_w = \mathcal{F}(w), \tag{4.15}$$

then

$$\Phi_1(u, v, w) = \int t(k)\left[C_1(w)e^{\tau n(k)\mu}\cos(n(k)\lambda) + C_2(w)e^{\tau n(k)\mu}\sin(n(k)\lambda)\right]dk, \tag{4.16}$$

$$\Phi_2(u, v, w) = \int t(k)\left[C_1(w)e^{\tau n(k)\lambda}\cos(n(k)\mu) + C_2(w)e^{\tau n(k)\lambda}\sin(n(k)\mu)\right]dk, \tag{4.17}$$

define solutions to the system of isotropic plasma equilibrium system (4.3), (4.4), for any $C_1(w)$ *and* $C_2(w)$ *satisfying* $\dfrac{d}{dw}(C_1^2(w) + C_2^2(w)) = 0$. *Here again* $\mu(u) =$

$\int a(u)du$, $\lambda(v) = \int 1/b(u)du$, $\tau = \pm 1$, $n(k)$ is an arbitrary function, and $t(k)$ is an arbitrary generalized function (for each solution, $n(k), t(k)$ must be chosen so that the integral converges).

These solutions correspond to force-free plasma equilibria (1.15) with the coefficient

$$\alpha(\mathbf{r}) = \alpha(w) = \frac{1}{\mathcal{F}(w)C_2(w)} \frac{dC_1(w)}{dw}.$$

Proof. (i) Substituting the relations (4.13) into the first plasma equilibrium equation (4.3), we get

$$\frac{1}{a(u)} \frac{\partial}{\partial u} \left(H_w \frac{1}{a(u)} \frac{\partial \Phi}{\partial u} \right) + c^2(w) b(v) \frac{\partial}{\partial v} \left(H_w b(v) \frac{\partial \Phi}{\partial v} \right) = 0.$$

By performing the change of variables $\mu(u) = \int a(u)du$, $\lambda(v) = \int 1/b(u)du$, the equation is brought into a simple form

$$\frac{\partial}{\partial \mu} \left(H_w \frac{\partial \Phi}{\partial \mu} \right) + c^2(w) \frac{\partial}{\partial \lambda} \left(H_w \frac{\partial \Phi}{\partial \lambda} \right) = 0.$$

The second equation of the system, (4.4), is rewritten under the force-free field assumption $P(w) = \text{const}$ as

$$\frac{\partial \Phi}{\partial \mu} \frac{\partial^2 \Phi}{\partial \mu \partial w} + c^2(w) \frac{\partial \Phi}{\partial \lambda} \frac{\partial^2 \Phi}{\partial \lambda \partial w} = 0.$$

Under the assumption about the form of $\Phi(u, v, w)$ (4.14) these equations become

$$C_1(w) \frac{\partial H_w}{\partial \mu} + c^2(w) C_2(w) \frac{\partial H_w}{\partial \lambda} = 0, \quad \frac{dC_1^2(w)}{dw} + c^2(w) \frac{dC_2^2(w)}{dw} = 0.$$

The first equation resolves for H_w to coincide with (4.13), and the second one is the necessary connection between the coefficients $C_1(w)$, $C_2(w)$, $c(w)$ specified in the conditions of part (i) of the theorem.

Using the formula for the magnetic field from Theorem 4.1, the expression for the magnetic field can be found:

$$\mathbf{B} = \frac{\Phi_u}{H_u} \mathbf{e_u} + \frac{\Phi_v}{H_v} \mathbf{e_v} = \frac{C_1(w) a(u)}{H_u} \mathbf{e_u} + \frac{C_2(w)}{b(v) H_v} \mathbf{e_v}.$$

This is a force-free magnetic field, because it occurs as a solution of the Plasma Equilibrium equations with $P(w) = \text{const}$. The expression for $\operatorname{curl}\mathbf{B}$ (from (4.12)) is:

$$\operatorname{curl}\mathbf{B} = -\frac{1}{H_v H_w}\frac{\partial^2\Phi}{\partial v\partial w}\mathbf{e_u} + \frac{1}{H_u H_w}\frac{\partial^2\Phi}{\partial u\partial w}\mathbf{e_v} = -\frac{1}{H_v H_w b(v)}\frac{dC_2(w)}{dw}\mathbf{e_u} + \frac{a(u)}{H_u H_w}\frac{dC_1(w)}{dw}\mathbf{e_v}.$$

Therefore the force-free proportionality coefficient $\alpha(\mathbf{r})$ is

$$\alpha(\mathbf{r}) = \frac{(\operatorname{curl}\mathbf{B})_1}{(\mathbf{B})_\mathbf{1}} = \frac{(\operatorname{curl}\mathbf{B})_2}{(\mathbf{B})_\mathbf{2}} = \frac{1}{H_w(w)C_2(w)}\frac{dC_1(w)}{dw}.$$

Thus part (i) is proven.

(ii) First we remark that as $H_w = \mathcal{F}(w)$, a change of variables can be performed: $w_1 = w_1(w) = \int \mathcal{F}(w)dw$ such that the corresponding scaling coefficient $H_{w_1} = 1$.

In coordinates (u, v, w_1), we substitute (4.15) and the force-free requirement $P(w) = \text{const}$ into the system (4.3), (4.4) under consideration, to get, in the way parallel to the proof of part (i),

$$\Phi_{\mu\mu} + \Phi_{\lambda\lambda} = 0, \quad \Phi_\mu \Phi_{\mu w_1} + \Phi_\lambda \Phi_{\lambda w_1} = 0.$$

The first equation of the above couple is independent on w, and evidently integrands of both $\Phi_1(u, v, w)$ and $\Phi_2(u, v, w)$ (4.16), (4.17) satisfy it, because they are harmonic in $(\mu,\ \lambda)$.

Concerning the second equation, direct substitution and the use of the condition $dC_1^2(w)/dw + dC_2^2(w)/dw = 0$ shows that functions $\Phi_1(u, v, w)$, $\Phi_2(u, v, w)$ satisfy it.

For both $\Phi_1(u, v, w)$ and $\Phi_2(u, v, w)$, the corresponding $\mathbf{B}$ is a force-free magnetic field, because it occurs as a solution of the plasma equilibrium equations with $P(w) = \text{const}$.

The force-free proportionality coefficient $\alpha(\mathbf{r})$ is

$$\alpha(\mathbf{r}) = \frac{(\operatorname{curl}\mathbf{B})_1}{(\mathbf{B})_\mathbf{1}} = \frac{(\operatorname{curl}\mathbf{B})_2}{(\mathbf{B})_\mathbf{2}} = \frac{1}{\mathcal{F}(w)C_2(w)}\frac{dC_1(w)}{dw}.$$

This concludes the proof of part (ii) and of the theorem. □

Remark 1. In $\mathbb{R}^3$, for a general orthogonal coordinate system, the Riemann tensor has six independent nonzero components, that are expressed through the metric tensor components g_{ii} (see Appendix G.)

If metric coefficients g_{ii} (or, equivalently, scaling coefficients H_u, H_v, H_w) are found for which all Riemann equations are satisfied, then the connection of the coordinates (u, v, w) and the cartesian coordinates (x, y, z) can be reconstructed explicitly (Appendix G.)

Remark 2. Relations of the type (4.13) and (4.15) between coordinates' scaling coefficients are not unnatural. The simplest example is coordinates obtained by a conformal transformation of the complex plane. This and other examples are discussed in sec. 4.4 below.

Remark 3. The integrals (4.16), (4.17) contain a generalized function $t(k)$, and therefore, with different choices of $t(k)$, can become, for instance, finite or infinite sums, or combinations of sums and continuous integrals.

4.3.2 Construction and generalization of "vacuum" magnetic fields

Lemma 4.1 *In any coordinate system where the 3D Laplace equation* $\triangle_{(u,v,w)}\phi(u, v, w) = 0$ *admits a solution independent of one of the variables* (w)*, there exists a trivial ("vacuum") magnetic field configuration*

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{B} = 0 \tag{4.18}$$

corresponding to this solution, and this magnetic field is tangent to surfaces $w =$ const.

Proof. If a solution of the Laplace equation $\phi(u,v)$ independent of w is given,

$$\frac{\partial}{\partial u}\frac{H_v H_w}{H_u}\frac{\partial\phi(u,v)}{\partial u}+\frac{\partial}{\partial v}\frac{H_u H_w}{H_v}\frac{\partial\phi(u,v)}{\partial v}+\frac{\partial}{\partial w}\frac{H_u H_v}{H_w}\frac{\partial\phi(u,v)}{\partial w}=0,$$

then it is indeed at the same time a solution to the "truncated" Laplace equation (4.9).

It also is a solution to the second plasma equilibrium equation (4.10) when $P(w) = \text{const}$, because it nulls the left-hand side identically. Thus $\phi(u,v)$ defines a force-free plasma magnetic field (4.5).

From the fact that $\phi(u,v)$ satisfies the system (4.9), (4.10) it follows that the magnetic field (4.5) is tangent to the coordinate surfaces $w = \text{const}$, by derivation of the equations.

For every such solution the electric current $\mathbf{J}$ (4.12) vanishes, thus making such plasma equilibrium a *vacuum magnetic field* configuration.

The lemma is proven.□

The second Lemma presents a transformation of a "vacuum" curl-free magnetic vector field depending only on two variables into an extended magnetic vector field (4.20) which is *not* force-free or gradient, and thus gives rise to a new *non-degenerate* solution to plasma equilibrium equations (1.20). This transformation will be used to build more general examples in subsequent sections.

Lemma 4.2 *If $\phi(u,v)$ is a solution to the system (4.9),(4.10) in the coordinates (u,v,w) with properties*

$$H_u = H_u(u,v) = H_v, \quad H_w = H_w(w), \tag{4.19}$$

then not only the magnetic field (4.5) with pressure $P(w) = \text{const}$ solves the Plasma Equilibrium equations (1.20), but so does the extended magnetic field

$$\mathbf{B} = \frac{1}{H_u}\frac{\partial\phi}{\partial u}\mathbf{e_u} + \frac{1}{H_v}\frac{\partial\phi}{\partial v}\mathbf{e_v} + K(u,v)\mathbf{e_w} \tag{4.20}$$

with pressure

$$P = C - K^2(u,v)/2, \tag{4.21}$$

where $K(u,v)$ satisfies

$$\frac{\partial^2 K(u,v)}{\partial u^2} + \frac{\partial^2 K(u,v)}{\partial v^2} = 0, \qquad \operatorname{grad}\phi(u,v)\cdot\operatorname{grad}K(u,v) = 0.$$

Proof. Without loss of generality we assume $H_w(w) = 1$, which can always be done by scaling the variable w.

Under the assumptions (4.19) about the coordinate system, the equation (4.9) can be rewritten as

$$\frac{\partial^2 \phi(u,v)}{\partial u^2} + \frac{\partial^2 \phi(u,v)}{\partial v^2} = 0,$$

therefore $\phi(u,v)$ is harmonic, and there exists its harmonic conjugate satisfying the Cauchy-Riemann equations

$$\frac{\partial \phi(u,v)}{\partial u} = \frac{\partial K(u,v)}{\partial v}, \qquad \frac{\partial \phi(u,v)}{\partial v} = -\frac{\partial K(u,v)}{\partial u}.$$

We substitute the extended form magnetic field (4.20) into plasma equilibrium equations (1.20). The equation $\operatorname{div}\mathbf{B} = 0$ evidently holds, as the term $H_u H_v K(u,v)$ is independent of w (see (4.7)). For $\operatorname{curl}\mathbf{B}$ we get, using (4.8) and the Cauchy-Riemann equations:

$$\operatorname{curl}\mathbf{B} = \frac{1}{H_v H_w}\frac{\partial K(u,v)}{\partial v}\mathbf{e_u} - \frac{1}{H_u H_w}\frac{\partial K(u,v)}{\partial u}\mathbf{e_v} = \frac{1}{H_u}\frac{\partial \phi(u,v)}{\partial u}\mathbf{e_u} + \frac{1}{H_u}\frac{\partial \phi(u,v)}{\partial u}\mathbf{e_v},$$

therefore

$$\operatorname{curl}\mathbf{B}\times\mathbf{B} = \frac{K(u,v)}{H_u}\frac{\partial \phi(u,v)}{\partial v}\mathbf{e_u} - \frac{K(u,v)}{H_u}\frac{\partial \phi(u,v)}{\partial u}\mathbf{e_v} = -\operatorname{grad}\frac{K^2(u,v)}{2}.$$

Introducing the pressure $P = C - K^2(u,v)/2$, we have the full Plasma Equilibrium system (1.20) satisfied. The lemma is proven.□

The following lemma uses assumptions about the scaling coefficients different from those in the previous one, and extends a "vacuum" curl-free magnetic vector field depending only on two variables and tangent to surfaces $w = \text{const}$ to "vacuum" fields that have non-zero w-components.

Lemma 4.3 *If $\phi(u, v)$ is a 2-dimensional solution to the plasma equilibrium system (4.9),(4.10) in the coordinates (u, v, w) with properties*

$$H_u = H_u(u, v), \quad H_v = H_v(u, v), \quad H_w = a(w)\mathcal{F}(u, v), \tag{4.22}$$

then not only the magnetic field (4.5) with pressure $P(w) = \text{const}$ solves the Plasma Equilibrium equations (1.20), but so does the extended magnetic field

$$\mathbf{B} = \frac{1}{H_u}\frac{\partial\phi}{\partial u}\mathbf{e_u} + \frac{1}{H_v}\frac{\partial\phi}{\partial v}\mathbf{e_v} + \frac{D}{H_w}\mathbf{e_w}; \quad D = \text{const}. \tag{4.23}$$

This magnetic filed is a vacuum magnetic field: $\operatorname{div}\mathbf{B} = 0, \ \operatorname{curl}\mathbf{B} = 0.$

Proof. Without loss of generality we assume $a(w) = 1$, which can be done by scaling the variable w.

The magnetic field (4.5) produced by a 2-dimensional solution to the plasma equilibrium system (4.9),(4.10) $\phi(u, v)$ is a gradient field, by Lemma 4.1.

If the w-component of the type $\dfrac{D}{H_w}\mathbf{e_w}$, $D = \text{const}$, is added, then the divergence (4.7) and the curl (4.8) of the field do not change, therefore the field (4.23) is a "vacuum" gradient magnetic field.

The lemma is proven.□

4.3.3 The availability of "vacuum" magnetic fields and their use for building exact non-trivial plasma equilibrium configurations

(i) Use of "vacuum" magnetic fields. As plasma equilibria, vacuum magnetic fields are *trivial* and can not be used for direct modelling of real physical equilibrium phenomena, where electric current $\mathbf{J}$, plasma pressure P and velocity $\mathbf{V}$ are generally non-zero. However, they can be used as an initial solution to construct new *non-trivial* solutions to Plasma Equilibrium equations in static and dynamic cases, for isotropic and anisotropic plasmas.

Indeed, the application of Bogoyavlenskij symmetries (1.27) to such a configuration results in non-trivial *field-aligned isotropic MHD solutions* with $P \neq \text{const}$, $\mathbf{J} \neq 0$, and density being an arbitrary function of magnetic surface variable.

By the application of the transforms (3.1) to a static vacuum magnetic field configuration, a static anisotropic CGL plasma equilibria are obtained, also non-trivial in the sense $p_{\|}, p_{\perp} \neq \text{const}$, $\mathbf{J} \neq 0$ and having the same topology as the original vacuum field. These static anisotropic equilibria can be extended further on the dynamic case with the help of the analog of Bogoyavlenskij symmetries for CGL plasmas (Chapter 3 sec. 3.4).

Appropriate examples are given in Section 4.4.

(ii) Availability of vacuum magnetic fields. By Lemma 4.1, the solutions to the "truncated" Laplace's equation (4.9) are available in any coordinate system that allows simple separability of Laplace's equation or where two-dimensional solutions exist. Many classical and esoteric coordinate systems *do* admit two-dimensional solutions of the Laplace equation, as can be found in literature, for example, [77].

New systems of coordinates may be constructed where the Laplace's equation will be separable or have two-dimensional solutions. The list of necessary and sufficient conditions on the metric coefficients is available in [78].

Another method of construction of gradient fields tangent to nested non-symmetric compact surfaces and dense on them was suggested in [79] and is described in Chapter 3, sec. 3.3.

4.4 The construction of exact Plasma Equilibria

In this section, we provide examples that illustrate the use of the system (4.9), (4.10) connected with magnetic surfaces for building new dynamic and static, isotropic and anisotropic plasma equilibria.

In the **first subsection**, we use the approach formulated above in Theorems 4.1, 4.2 and Lemmas 4.1 – 4.3 to produce various examples of non-degenerate plasma equilibrium configurations.

The first example combines the results of Theorems 4.1 and 4.2 to produce a set of non-Beltrami force-free plasma equilibria (1.15) in a prescribed geometry - with spherical magnetic surfaces. The force-free coefficient $\alpha(\mathbf{r})$ (1.15) in this example is a function of the *spherical radius*. Due to this fact, a linear combination of such solutions with different axis orientations can be taken, to produce force-free plasma equilibrium configurations *with no geometrical symmetries.*

The presented force-free solutions have a singularity on the z-axis $x = y = 0$; it makes the magnetic energy infinite.

With the help of the coordinate representation (4.9), (4.10), we prove that the infiniteness of energy is not the feature of the particular exact solutions on spheres chosen here, but of all force-free (and most of non-force-free) fields with spherical magnetic surfaces.

As a second example, we build exact dynamic isotropic and anisotropic plasma equilibrium configurations that model *solar flares* in the coronal plasma near an active region. The resulting model is essentially non-symmetric and presented in an exact form; it reproduces important features of solar flares known from observations.

In the third example, the coordinate representation is used to build a particular "vacuum" magnetic field in prolate spheroidal coordinates. From that field, by the symmetries (1.27) and the transformations (3.1), families of non-degenerate (and generally non-symmetric) isotropic and anisotropic plasma equilibria with dynamics are constructed, which model the quasi-stationary phase of mass exchange between two spheroidal objects. Plasma domains of different geometry and topology can be chosen; the total magnetic energy of the configurations is finite.

In the **second subsection**, we discuss the ways of finding new coordinate systems in flat 3D space (with the Riemannian symbol $R_{ijkl} = 0$), in which exact solutions the system (4.3), (4.4) can be found.

Two such ways are suggested here: (a) Search for the metric coefficients satisfying the conditions of Theorem 4.2 that also satisfy the differential equations $R_{ijkl} = 0$; and (b) use orthogonal coordinate transformations $(x, y, z) \rightarrow (u, v, w)$ from flat coordinates (x, y, z), such that the new metric coefficients conform with the conditions of Theorem 4.2, so that solutions described in this theorem exist.

Both approaches are illustrated with examples. In particular, using (b) and Lemmas 4.1, 4.2, plasma equilibria with non-circular cylindrical magnetic surfaces are be obtained.

4.4.1 Examples of exact plasma equilibria

Example 1: Force-free plasma equilibria with spherical magnetic surfaces

In the first example, we build a set of solutions to isotropic force-free plasma equilibria with magnetic surfaces being *spheres*.

The spherical coordinates (r, θ, ϕ) will be denoted respectively (w, u, v) to agree with the notation of previous sections. The scaling coefficients then are

$$H_u = w, \quad H_v = w \sin u, \quad H_w = 1.$$

These scaling coefficients conform with the formulation of both parts of Theorem 4.2, with $a(u) = 1/\sin u$, $b(v) = c(w) = 1$. Therefore after the substitution

$$\mu(u) = \int a(u)du = \ln\left(\frac{1 - \cos u}{\sin u}\right), \quad P(\mathbf{r}) = \text{const},$$

the equations of plasma equilibrium (4.3), (4.4) simplify to

$$\frac{\partial^2 \Phi}{\partial \mu^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0 \tag{4.24}$$

and

$$\frac{\partial \Phi}{\partial \mu}\frac{\partial^2 \Phi}{\partial \mu \partial w} + \frac{\partial \Phi}{\partial v}\frac{\partial^2 \Phi}{\partial v \partial w} = 0,$$

or

$$\left(\frac{\partial \Phi}{\partial \mu}\right)^2 + \left(\frac{\partial \Phi}{\partial v}\right)^2 = a(\mu, v), \tag{4.25}$$

where $a(\mu, v)$ is an arbitrary function. From part (i) of Theorem 4.2, the function

$$\Phi = C_1(w)\mu + C_2(w)v + C_3, \tag{4.26}$$

is a solution of this system. It corresponds to the magnetic field (4.5) in spherical coordinates (r, θ, ϕ):

$$\mathbf{B} = \frac{C_1(r)}{r \sin \theta}\mathbf{e}_\theta + \frac{C_2(r)}{r \sin \theta}\mathbf{e}_\phi. \tag{4.27}$$

Here $C_1(r)$ and $C_2(r)$ are any smooth functions of the spherical radius r that satisfy the relation $\partial/\partial w(C_1^2(w)+C_2^2(w))=0$. (4.27) is a force-free field with

$$\operatorname{curl}\mathbf{B}=\mu\mathbf{J}=-\frac{dC_1^2(r)/dr}{r\sin\theta\sqrt{C_2(r)^2}}\mathbf{e}_\theta-\frac{dC_1(r)/dr}{r\sin\theta}\mathbf{e}_\phi \tag{4.28}$$

and the force-free field proportionality coefficient $\alpha(r)$ (cf. (1.15)):

$$\alpha(r)=\frac{dC_1(r)/dr}{C_2(r)}. \tag{4.29}$$

Discussion. An example of a solution of the type (4.26)-(4.29) with a choice

$$C_1(r)=\sin(-\kappa^2r^2+d_1r+d_2),\quad \kappa=2,\ d_1=2.5,\ d_2=6.25$$

is presented on Fig. 4-1, where magnetic field lines tangent to a spherical magnetic surface are shown.

This is a force-free configuration with spherical magnetic surfaces ($\alpha=\alpha(r)$). It has a pole-type singularity at $r=0$ and at $\theta=\pm\pi$, i.e. on the z-axis. The magnetic energy in the volume occupied by plasma $\int_V \frac{B^2}{2}dV$ is infinite.

The magnetic field lines enter the singularity without infinitely rotating around it: $(B_\theta/B_\phi)|_{\theta=0}=C_1(r)/C_2(r)$, where r is the spherical radius.

Remark 1. Using the representation (4.24), (4.25), it is possible to show that the infiniteness of energy is not the feature of the particular type of the solution chosen (4.26), but of all force-free (and most of non-force-free) fields with spherical magnetic surfaces.

From the equation (4.4) it follows that for the spherical case

$$\mathbf{B}^2=\frac{1}{w^2}\left(\left(\frac{\partial\Phi}{\partial u}\right)^2+\frac{1}{\sin^2u}\left(\frac{\partial\Phi}{\partial v}\right)^2\right)=-\frac{1}{w^2}\int_0^w h^2\frac{dP(h)}{dh}dh+\frac{1}{w^2}a_1(u,v), \tag{4.30}$$

where $a_1(u,v)$ is generally not identically zero, and is *never* identically zero for force-free plasmas (because $\mathbf{B}^2$ is not identically zero in the plasma domain). Therefore

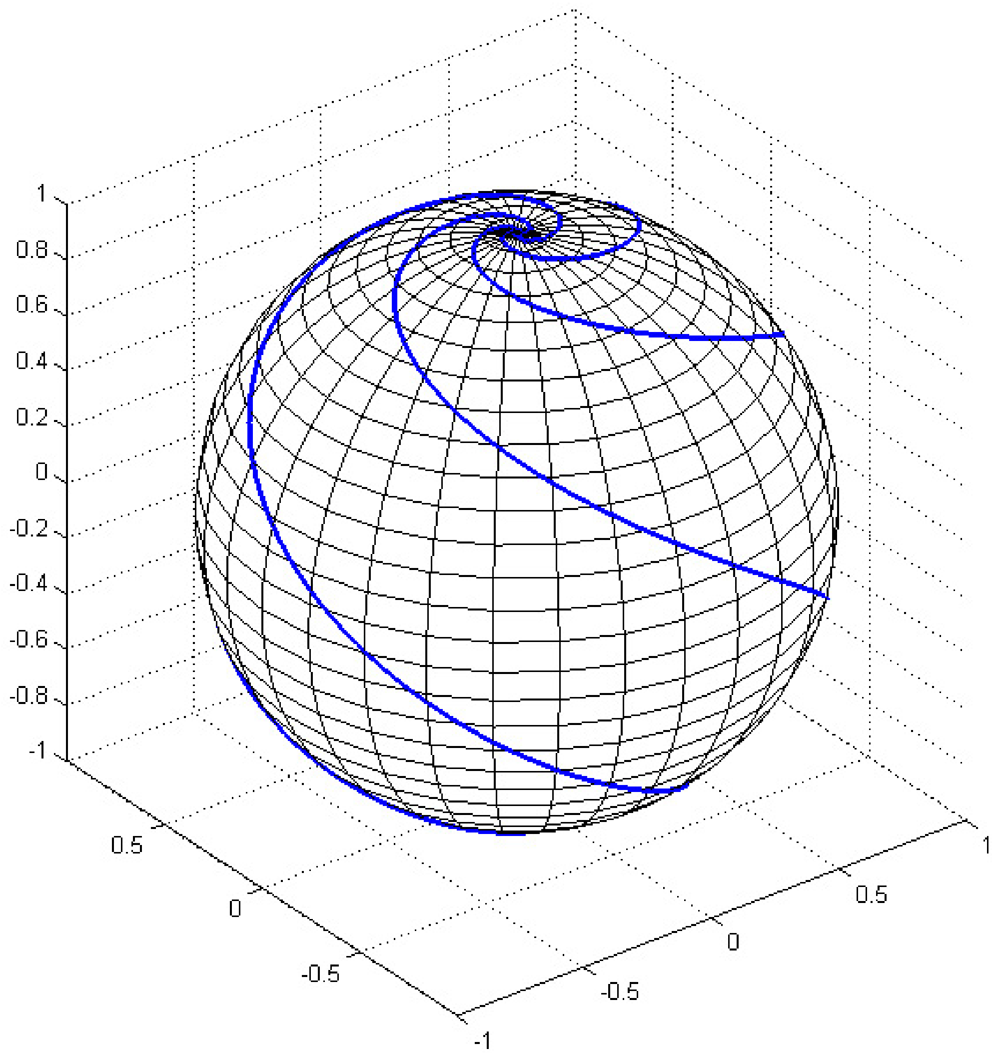

Figure 4-1: A force-free field tangent to spheres.

Force-free field magnetic field lines of the solution (4.28) tangent to a spherical magnetic surface.

any force-free and a general non-force-free plasma equilibrium configuration has a pole-type singularity at the origin $r = 0$, and infinite magnetic energy if the plasma region includes the origin.

Remark 2. Non-symmetric force-free equilibria with spherical magnetic surfaces. The force-free equilibrium equations (1.15) admit the addition of solutions that have the same magnetic surfaces and the same parameter $\alpha(\mathbf{r})$. In the above example, the magnetic surfaces are spheres; the parameter $\alpha(\mathbf{r}) = \alpha(r)$ depends only on the spherical radius and is constant on any given sphere. Thus, adding several solutions of the type (4.26)-(4.29) with differently directed axes of symmetry z, but with the same $\alpha(r)$, one obtains exact non-degenerate and *non-symmetric* force-free plasma equilibria.

Several magnetic lines of a solution that is a sum of two equilibrium force-free configurations 4.27 with different mutual orientation of the axes are presented on Fig. 4-2.

Remark 3. Other Force-Free fields tangent to spheres. Analytical force-free magnetic equilibria with spherical magnetic surfaces other than (4.27) are readily found. First, the part (ii) of Theorem 4.2 is applicable to spherical coordinates, hence (4.16) is a family of force-free equilibria on spheres. (Solutions (4.17) are not ϕ-periodic and cannot be used.)

Also, in accordance with Lemma 4.1 of Section 4.3, any harmonic function of the type $\Phi = \Phi(\mu, v)$ (4.24) results in a vacuum magnetic field configuration on nested spheres: $\operatorname{div} \mathbf{B} = 0$, $\operatorname{curl} \mathbf{B} = 0$. As a plasma configuration, it is trivial, but it may be used to building non-trivial isotropic and anisotropic plasma equilibrium configurations, by the same methods as shown below for magnetic fields tangent to ellipsoids.

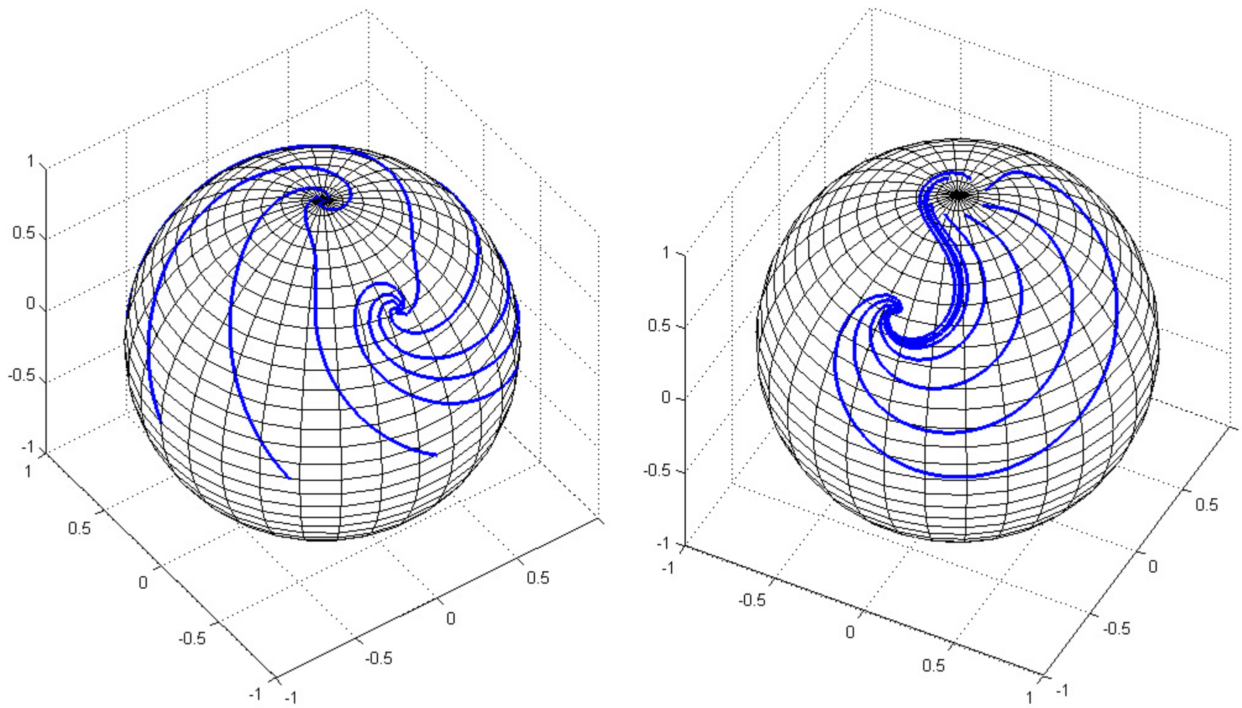

Figure 4-2: A non-symmetric force-free field tangent to spheres.

Magnetic lines of a solution that is a sum of two equilibrium force-free configurations (4.27) with different mutual orientation of the axes.

Left: both configurations have upward-directed axes;
Right: one configuration has an upward-directed axis, another - a downward-directed axis.

Remark 4. The same constructions can be applied to Euler's equation describing the equilibrium motion of an incompressible fluid.

Example 2: Isotropic and anisotropic plasma equilibria tangent to nested ellipsoids

In this example, we construct a family of generally non-symmetric plasma equilibria with ellipsoidal magnetic surfaces.

We start from the construction of vacuum magnetic fields tangent to ellipsoids, using the magnetic-surface-connected representation of plasma equilibria equations (Theorem 4.1) and Lemma 4.1. Then transformations are applied to this trivial solution to produce non-trivial isotropic and anisotropic plasma equilibria.

An application of the resulting solutions to modelling the solar photosphere plasma near active regions is discussed.

(i). A vacuum magnetic field tangent to ellipsoids. The ellipsoidal coordinates are [77]:

$$u = \theta, \quad b^2 < \theta^2 < c^2,$$
$$v = \lambda, \quad 0 \le \lambda^2 < b^2,$$
$$w = \eta, \quad c^2 < \eta^2 < +\infty.$$

The coordinate surfaces are

$$\frac{x^2}{\eta^2} + \frac{y^2}{\eta^2 - b^2} + \frac{z^2}{\eta^2 - c^2} = 1, \qquad (\text{ellipsoids}, \eta = \text{const}),$$
$$\frac{x^2}{\theta^2} + \frac{y^2}{\theta^2 - b^2} - \frac{z^2}{c^2 - \theta^2} = 1, \qquad (\text{one-sheet hyperboloids}, \theta = \text{const}),$$
$$\frac{x^2}{\lambda^2} - \frac{y^2}{b^2 - \lambda^2} - \frac{z^2}{c^2 - \lambda^2} = 1, \qquad (\text{two-sheet hyperboloids}, \lambda = \text{const}).$$

Laplace's equation is separable in ellipsoidal coordinates, and we take a solution depending only on (θ, λ), so that its gradient has zero η-projection transverse to ellipsoids, but is tangent to them:

$$\Phi_1(\theta,\lambda) = \left(A_1 + B_1\mathrm{sn}^{-1}\left(\sqrt{\frac{c^2-\theta^2}{c^2-b^2}}, \sqrt{\frac{c^2-b^2}{c^2}}\right)\right)\left(A_2 + B_2\mathrm{sn}^{-1}\left(\frac{\lambda}{b}, \frac{b}{c}\right)\right).$$

Here $\mathrm{sn}(x,k)$ is the Jacobi elliptic sine function. The inverse of it is an incomplete elliptic integral

$$F_{ell}(z,k) = \int_0^z \frac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,dt.$$

$\Phi_1(\theta,\lambda)$ does not depend on w, and therefore evidently satisfies both equations (4.3), (4.4). The resulting magnetic field (4.5) is tangent to ellipsoids $\eta = \mathrm{const}$, and has a singularity at $\theta = \lambda$, i.e. on the plane $y = 0$.

However one may verify that for a plasma region $c < \eta_1 < \eta < \eta_2$ the total magnetic energy $\int_V B^2/2dv$ is finite. Also, if one restricts to a half-space $y > 0$ or $y < 0$, then the magnetic field is well-defined in a continuous and differentiable way.

If the magnetic field is tangent to the boundary of a domain, one can safely assume that outside of it $\mathbf{B} = 0$ identically. This is achieved, as usual, by introducing a boundary surface current

$$\mathbf{i}_b(\mathbf{r}_1) = \mu^{-1}\mathbf{B}(\mathbf{r}_1) \times \mathbf{n}_{out}(\mathbf{r}_1), \tag{4.31}$$

where $\mathbf{r}_1$ is a point on the boundary of the domain, and $\mathbf{n}_{out}$ is an outward normal.

Fig. 4-3 shows several magnetic field lines for the case $(b = 7, c = 10, A_1 = A_2 = 0, B_1 = 1/100, B_2 = 1/30)$ on the ellipsoid $\eta = 12$. For this set of constants, the vector of the magnetic field has the form is

$$\mathbf{B}_0 = \frac{F_{ell}\left(\frac{\lambda}{7}, \frac{7}{10}\right)}{\sqrt{(\theta^2-\lambda^2)(\eta^2-\theta^2)}}\mathbf{e}_\theta - \frac{F_{ell}\left(\sqrt{\frac{100-\theta^2}{51}}, \sqrt{\frac{51}{100}}\right)}{\sqrt{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}}\mathbf{e}_\lambda \tag{4.32}$$

This "vacuum" (gradient) magnetic field is used to produce non-trivial dynamic isotropic and anisotropic plasma equilibria, as shown below.

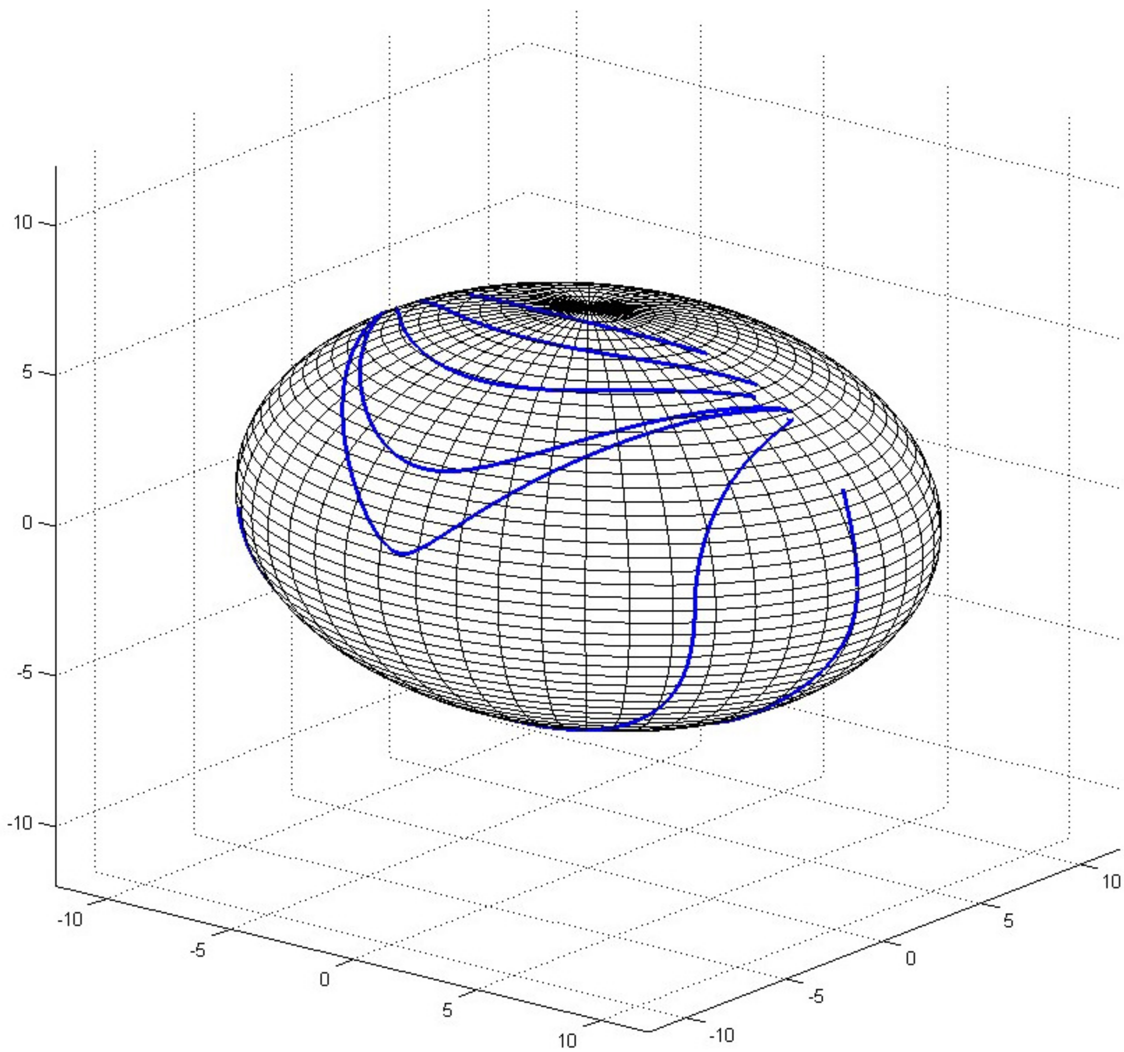

Figure 4-3: A magnetic field tangent to ellipsoids.

Lines of the magnetic field (4.32) tangent to the ellipsoid $\eta = 12$. The shown ellipsoid is a magnetic surface from the family of nested ellipsoids η = const in classical ellipsoidal coordinates.
This configuration is smoothly defined in the half-space $y > 0$.

We remark that though the magnetic field lines of the field (4.32) have a plane of symmetry $x = 0$, a non-zero choice of constants (A_1, A_2, B_1, B_2) would produce a *completely non-symmetric* magnetic field tangent to a family of ellipsoids.

(ii). Isotropic dynamic plasma equilibrium with ellipsoidal magnetic surfaces. The above vacuum magnetic field $\mathbf{B_0}$ is indeed a trivial solution to the general isotropic plasma equilibrium system (1.16)-(1.17) with $\mathbf{V} = 0,\ P = P_0 = \text{const}$ and an arbitrary density function $\rho = \rho_0(\mathbf{r})$.

If we choose $\rho_0(\mathbf{r})$ to be constant on magnetic field lines (plasma streamlines do not exist as there is no flow), then the infinite-parameter transformations (1.27) become applicable to such configuration. Applying them formally, we obtain a family of isotropic plasma equilibria

$$\begin{aligned}
&\mathbf{B}_1 = m(\mathbf{r})\mathbf{B}_0, \quad \mathbf{V}_1 = \frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\mathbf{B}_0, \\
&\rho_1 = a^2(\mathbf{r})\rho_0(\mathbf{r}), \quad P_1 = CP_0 - n^2(\mathbf{r})\mathbf{B}_0^2/(2\mu), \\
&m^2(\mathbf{r}) - n^2(\mathbf{r}) = C = \text{const},
\end{aligned} \tag{4.33}$$

where $a(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r}), \rho_0(\mathbf{r})$ are functions constant on magnetic field lines and streamlines (which coincide in this case, as $\mathbf{V}_1$ and $\mathbf{B}_1$ are collinear).

We consider a plasma configuration in a region $\mathcal{D}$ in the half-space $y > 0$ between two ellipsoid shells $\eta_1,\ \eta_2\ :\ c < \eta_1 < \eta < \eta_2$, (we take $c = 10$; c is one of the parameters of the elliptic coordinate systems used for this solution; cf. sec. 4.4.1.)

Outside of the region, we assume $\mathbf{B}_1 = 0$, by introducing a corresponding surface current (4.31). We also assume $\mathbf{V}_1 = 0$, which can be done because the streamlines are tangent to the boundary of the plasma domain $\mathcal{D}$.

The magnetic field lines in the chosen region are not dense on any 2D surface or in any 3D domain, therefore the arbitrary functions $a(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r}), \rho_0(\mathbf{r})$ can be chosen (in a smooth way) to have a constant value on each magnetic field line, thus being in fact functions of *two variables* enumerating all magnetic field lines in the region of interest (for example, η and λ, which specify the beginning of every magnetic field line).

We remark that unlike the initial field $\mathbf{B}_0$, the vector fields $\mathbf{B}_1$ and $\mathbf{V}_1$ are neither potential nor force-free: for example, $\operatorname{curl}\mathbf{B}_1 = \operatorname{grad} m(\mathbf{r}) \times \mathbf{B}_0 \nparallel \mathbf{B}_1$. But both $\mathbf{B}_1$ and $\mathbf{V}_1$ satisfy the solenoidality requirement.

Direct verification shows that, with a non-singular choice of the arbitrary functions, the total magnetic energy $E_m = 1/2 \int_V B_1^2 \, dv$ and the kinetic energy $E_k = 1/2 \int_V \rho \, V_1^2 \, dv$ are finite. The magnetic field, velocity, pressure and density $(\mathbf{B}_1, \ \mathbf{V}_1, \ \rho_1, \ P_1)$ are defined in a continuous and differentiable way.

The presented model is not unstable according to the known sufficient instability condition for incompressible plasma equilibria with flows proven in [47] (see Chapter 1, sec. 1.5.) The latter states that if $\mathbf{V} \nparallel \mathbf{B}$, then a plasma equilibrium with constat density is unstable. In the presented example, $\mathbf{V} \parallel \mathbf{B}$ (the density ρ_1 can be chosen constant).

(iii). Anisotropic plasma equilibrium with ellipsoidal magnetic surfaces. When the mean free path for particle collisions is long compared to Larmor radius, (e.g. in strongly magnetized plasmas), the tensor-pressure CGL approximation should be used. The model suggested here describes a rarefied plasma behaviour in a strong magnetic field looping out of the star surface.

To construct an anisotropic CGL extension of the above isotropic model, we use the transformations (3.1) (Chapter 3) from MHD to CGL equilibrium configurations. Given $\mathbf{B}_1, \mathbf{V}_1, P_1, \rho_1$ determined by (4.33) with some choice of the arbitrary functions $a(\mathbf{r})$, $m(\mathbf{r})$, $n(\mathbf{r})$, $\rho_0(\mathbf{r})$, we obtain an anisotropic equilibrium $\mathbf{B}_2, \mathbf{V}_2, p_{\|2}, p_{\perp 2}, \rho_2$ defined as

$$\begin{aligned}
&\mathbf{B}_2 = f(\mathbf{r})\mathbf{B}_1, \quad \mathbf{V}_2 = g(\mathbf{r})\mathbf{V}_1, \quad \rho_2 = C_0\rho_1\mu/g(\mathbf{r})^2, \\
&p_{\perp 2} = C_0\mu P_1 + C_1 + \left(C_0 - f(\mathbf{r})^2/\mu\right)\, \mathbf{B}_1^2/2, \\
&p_{\|2} = C_0\mu P_1 + C_1 - \left(C_0 - f(\mathbf{r})^2/\mu\right)\, \mathbf{B}_1^2/2,
\end{aligned} \tag{4.34}$$

$f(\mathbf{r})$, $g(\mathbf{r})$ are arbitrary functions constant on the magnetic field lines and streamlines, i.e. again on constant on every plasma magnetic field line, and C_0, C_1 are arbitrary constants.

Setting $P_0 = 0$ in (4.33) and making an explicit substitution, we get

$$\begin{aligned}
&\mathbf{B}_2 = f(\mathbf{r})m(\mathbf{r})\mathbf{B}_0, \quad \mathbf{V}_2 = g(\mathbf{r})\frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\mathbf{B}_0, \quad \rho_2 = C_0 a^2(\mathbf{r})\rho_0(\mathbf{r})\mu/g(\mathbf{r})^2, \\
&p_{\perp 2} = C_1 + \frac{\mathbf{B}_0^2}{2\mu}\left(C_0 C\mu - f^2(\mathbf{r})m^2(\mathbf{r})\right), \\
&p_{\|2} = C_1 + \frac{\mathbf{B}_0^2}{2\mu}\left(f^2(\mathbf{r})m^2(\mathbf{r}) - C_0 C\mu - 2C_0 n^2(\mathbf{r})\right).
\end{aligned} \tag{4.35}$$

It is known (Chapter 3, sec. 3.2.3; also [79]) that for the new equilibrium to be free from a fire-hose instability, the transformations (3.1) must have $C_0 > 0$.

$p_\perp$ is the pressure component perpendicular to magnetic field lines. It is due to the rotation of particles in the magnetic field. Therefore in strongly magnetized or rarified plasmas, where the CGL equilibrium model is applicable, the behaviour of $p_\perp$ should reflect that of $\mathbf{B}^2$.

In the studies of the solar wind flow in the Earth magnetosheath, the relation

$$p_\perp/p_\| = 1 + 0.847(B^2/(2p_\|)) \tag{4.36}$$

was proposed [27]. We denote $k(\mathbf{r}) = C_0 C\mu - f^2(\mathbf{r})m^2(\mathbf{r})$ and select the constants and functions $C_0, C, f(\mathbf{r}), m(\mathbf{r})$ so that $k(\mathbf{r}) \geq 0$ in the space region under consideration.

From (4.35), we have:

$$p_{\perp 2} - p_{\| 2} = \frac{\mathbf{B}_0^2}{2\mu}(2k(\mathbf{r}) + 2C_0 n^2(\mathbf{r})),$$

or

$$\frac{p_{\perp 2}}{p_{\| 2}} = 1 + \frac{2k(\mathbf{r}) + 2C_0 n^2(\mathbf{r})}{\mu f^2(\mathbf{r}) m^2(\mathbf{r})} \frac{\mathbf{B}_2^2}{2p_{\| 2}},$$

which generalizes and includes the experimental result (4.36).

(iv). A model of plasma behaviour in arcade solar flares.

Solar flares are phenomena that take place in the photospheric region of the solar atmosphere and are connected with a sudden release of huge energies (typically $10^{22} - 10^{25}$ J) ([80], pp. 331-348). Particle velocities connected with this phenomenon (about 10^3 m/s) are rather small compared to typical coronal velocities ($\sim 5 \cdot 10^5$ m/s), therefore equilibrium models are applicable.

Morphologically two types of solar flares are distinguished: loop arcades (magnetic flux tubes) and two-ribbon flares. Flares themselves and post-flare loops are grounded in from active photospheric regions.

As noted in [80], p. 332, *"rigorous theoretical modelling has mainly been restricted to symmetric configurations, cylindrical models of coronal loops and two-dimensional arcades."*

The configurations described in (ii) and (iii) can serve as *non-symmetric* 3D isotropic and anisotropic models of quasi-equilibrium plasma in flare and post-flare loops, where magnetic field and inertia terms prevail upon the gravitation potential term in the plasma equilibrium equations:

$$\mathbf{V} \times \operatorname{curl} \mathbf{V} \gg \operatorname{grad} \varphi, \quad \frac{1}{\mu}\mathbf{B} \times \operatorname{curl} \mathbf{B} \gg \rho \operatorname{grad} \varphi.$$

where φ is the star gravitation field potential.

The relative position and form of the magnetic field lines in the model, with respect to the star surface, are shown on Fig. 4-4. The characteristic shape of the magnetic field energy density $\mathbf{B}$ and the pressure P along a particular magnetic field line, for the isotropic case (ii), are given on Fig. 4-5.

Magnetic field lines in the model are not closed; therefore by introducing a surface current of the type (4.31), a plasma domain $\mathcal{D}$ can indeed be selected to have any tubular loop shape, and the magnetic field can be assumed zero outside (together with the velocity in models (ii), (iii).) The current sheet introduction is not artificial – as argued in [81], in a general 3D coronal configurations the current sheets between flux tubes are formed (see also: [80], p. 343.)

This isotropic MHD model (ii) is valid when the mean free path of plasma particles is much less than the typical scale of the problem, so that the picture is maintained nearly isotropic via frequent collisions.

However, the CGL framework must be adopted when plasma is rarefied or strongly magnetized. For such plasmas, we propose the anisotropic model (iii), for which the requirement of plasma being rarefied can be satisfied by choosing $a(\mathbf{r})$ sufficiently small.

Example 3: Isotropic and anisotropic plasma equilibria in prolate spheroidal coordinates. A model of mass exchange between two distant spheroidal objects

In this example, families of non-symmetric exact plasma equilibria is prolate spheroidal coordinates with finite magnetic energy are obtained, in isotropic and anisotropic frameworks. On the basis of these solutions, a model of the quasi-equilibrium stage of mass exchange by a plasma jet between two distant spheroidal objects is suggested.

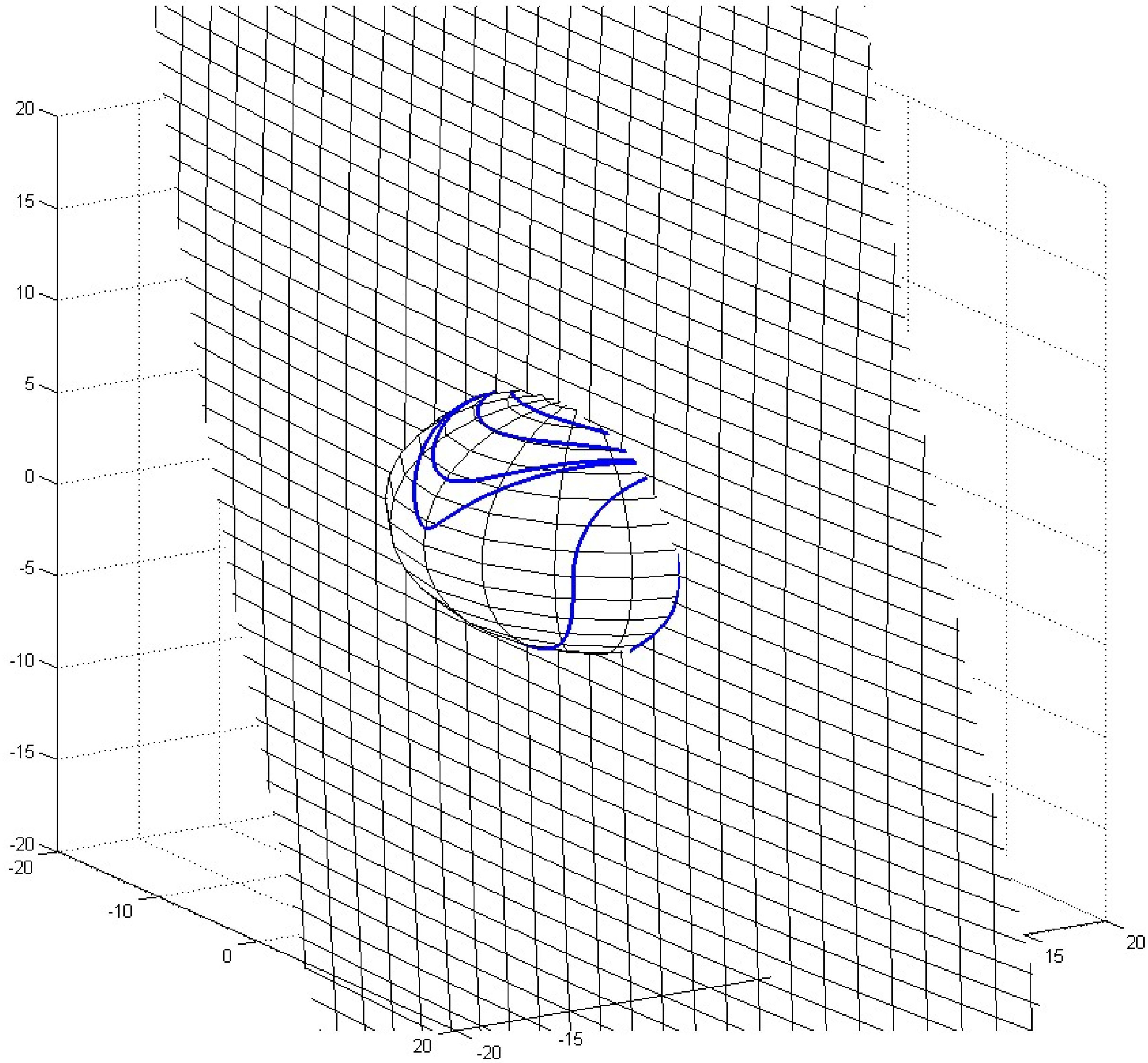

Figure 4-4: A solar flare model – magnetic field lines.

The model of a solar flare as a coronal plasma loop near an active photospheric region. The position and shape of several magnetic field lines are shown with respect to the star surface; magnetic field is tangent to nested half-ellipsoids.

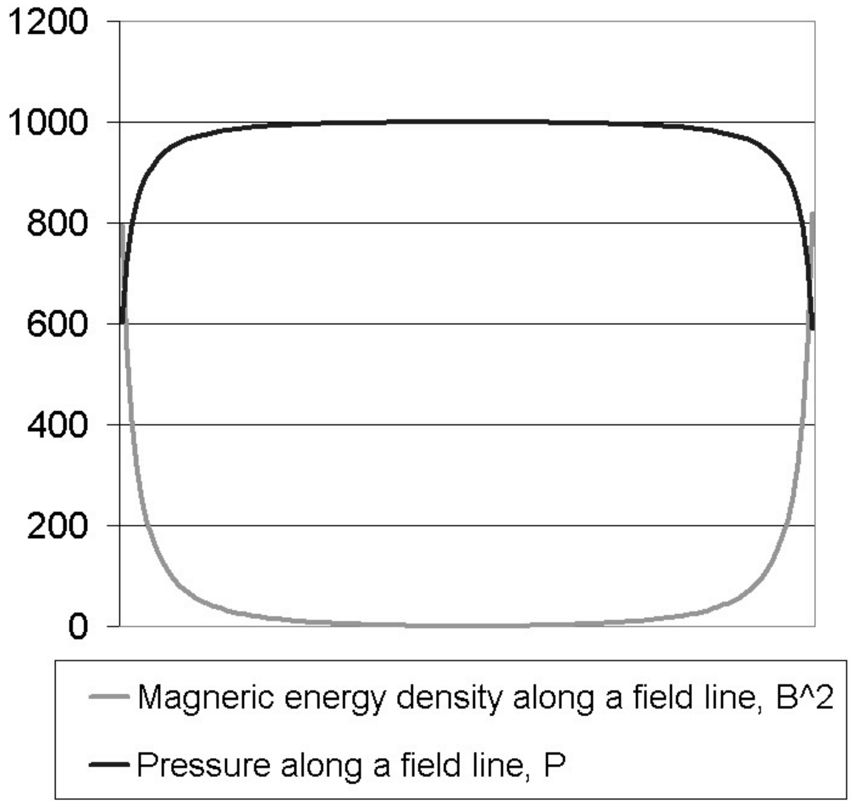

Figure 4-5: A solar flare model – plasma parameter profiles.

A model of a solar flare – a coronal plasma loop near an active photospheric region. The figure shows the characteristic shape of the magnetic field energy density $\mathbf{B}$ and the pressure P curves along a particular magnetic field line. (The isotropic case.)

(i). Vacuum magnetic configuration in prolate spheroidal coordinates. Consider the prolate spheroidal system of orthogonal coordinates:

$$u = \theta, \quad 0 \leq \theta \leq \pi,$$

$$v = \phi, \quad 0 \leq \phi < 2\pi,$$

$$w = \eta, \quad 0 \leq \eta < +\infty.$$

The coordinate surfaces are

$$\begin{aligned}
&\frac{x^2}{a^2 \sinh^2 \eta} + \frac{y^2}{a^2 \sinh^2 \eta} + \frac{z^2}{a^2 \cosh^2 \eta} = 1, && \text{(prolate spheroids}, \eta = \text{const}),\\
&-\frac{x^2}{a^2 \sin^2 \theta} - \frac{y^2}{a^2 \sin^2 \theta} + \frac{z^2}{a^2 \cos^2 \theta} = 1, && \text{(two-sheet hyperboloids}, \theta = \text{const}),\\
&\tan(\phi) = \frac{y}{x}, && \text{(half planes}, \phi = \text{const}),
\end{aligned}$$

and the metric coefficients

$$g_{\eta\eta} = g_{\theta\theta} = a^2(\sinh^2 \eta + \sin^2 \theta), \quad g_{\phi\phi} = a^2 \sinh^2 \eta \sin^2 \theta.$$

It is known that the 3-dimensional Laplace equation is separable in this system, and it admits the axially-symmetric family of solutions of this equation [77]:

$$\Phi = H(\eta)T(\theta),$$

$$H(\eta) = A_1 \mathcal{P}_p(\cosh \eta) + B_1 \mathcal{L}_p(\cosh \eta), \quad T(\theta) = A_2 \mathcal{P}_p(\cos \theta) + B_2 \mathcal{L}_p(\cos \theta),$$

where $\mathcal{L}_p(z) \equiv \mathcal{L}_p^0(z)$, $\mathcal{L}_p(z) \equiv \mathcal{L}_p^0(z)$ are the *Legendre wave functions* of first and second kind respectively.

This solution evidently satisfies both plasma equilibrium equations in magnetic-surface-related coordinates (4.9), (4.10): the first one because the usual and truncated Laplace equations coincide when Φ is a function of two variables, and the second - identically due to the independence of Φ on w.

In the case of integer p, the above 2-dimensional solution expresses in ordinary Legendre functions of the first and second kind:

$$H(\eta) = A_1 P_p(\cosh \eta) + B_1 Q_p(\cosh \eta), \quad T(\theta) = A_2 P_p(\cos \theta) + B_2 Q_p(\cos \theta). \quad (4.37)$$

From the above family, we select a particular axially symmetric function $\Phi(\eta,\theta)$ with an asymptotic condition

$$\lim_{|\mathbf{r}|\to\infty} \Phi(\eta,\theta) = M_0 z, \tag{4.38}$$

i.e. such that its gradient is asymptotically a constant vector field in the cartesian z-direction: $\operatorname{grad}\Phi(\eta,\theta) = M_0\mathbf{e}_z$.

Another requirement is that the potential $\Phi(\eta,\theta)$ is constant on a given prolate spheroid $\eta = \eta_0$.

We search for $\Phi(\eta,\theta)$ as a combination of particular solutions (4.37). The boundary conditions are

$$\begin{aligned} &\Phi(\eta,\theta) = M_0 z = M_0 a \sinh\eta \sin\theta, && \eta \gg \eta_0, \\ &\Phi(\eta,\theta) = 0, && \eta = \eta_0. \end{aligned} \tag{4.39}$$

The term with $Q_p(\cos\theta)$ can not be present in the solution, because this function is infinite on the z axis. Hence the solution can be sought as a combination of terms

$$\Phi_p = A_p P_p(\cosh\eta) P_p(\cos\theta) + B_p Q_p(\cosh\eta) P_p(\cos\theta).$$

When $\eta \gg 1$, $Q_p(\cosh\eta) \to 0$ for any $p \geq 0$, so, using just one p, from the first boundary condition we get

$$M_0 a \sinh\eta \sin\theta = A_p P_p(\cosh\eta) P_p(\cos\theta),$$

which resolves as

$$p = 1, \quad A_1 = M_0 a.$$

The second boundary condition suggests

$$0 = A_1 P_1(\cosh\eta_0) P_1(\cos\theta) + B_1 Q_1(\cosh\eta_0) P_1(\cos\theta),$$

yielding the expression for B_1.

We note the the function $Q_1(z)$ has the explicit form

$$Q_1(z) = \frac{1}{2} z \ln \frac{z+1}{z-1} - 1.$$

Finally, the required solution takes the form

$$\Phi_0(\eta, \theta) = M_0 a \cos\theta \left\{ n \cosh\eta - \cosh\eta_0 \frac{Q_1(\cosh\eta)}{Q_1(\cosh\eta_0)} \right\},$$

and the corresponding magnetic field $\mathbf{B} = \operatorname{grad}\Phi(\eta, \theta)$

$$\mathbf{B}_0 = \operatorname{grad}\Phi_0(\eta, \theta) = \frac{1}{\sqrt{g_{\eta\eta}}} \frac{\partial \Phi_0(\eta, \theta)}{\partial \eta} \mathbf{e}_\eta + \frac{1}{\sqrt{g_{\theta\theta}}} \frac{\partial \Phi_0(\eta, \theta)}{\partial \theta} \mathbf{e}_\theta. \tag{4.40}$$

The magnetic surfaces this field is tangent to are nested widening circular tubes along z-axis, perpendicular to the spheroid $\eta = \eta_0$ and asymptotically approaching circular cylinders $x^2 + y^2 = \text{const}$. A graph for the choice $\{a = 2,\ M_0 = 1,\ \eta_0 = 0.3\}$ with a single magnetic field line shown is presented on Fig. 4-6.

Each tube is uniquely defined by the value of θ of its intersection with the base spheroid. The two shown on Fig. 4-6 correspond to $\theta_1 = 0.07$ and $\theta_2 = 0.12$.

If a "winding" polar component

$$\frac{D}{H_\phi}\mathbf{e}_\phi, \quad H_\phi = \sqrt{g_{\phi\phi}} = a \sinh\eta \sin\theta, \quad D = \text{const},$$

is added to the field (4.40), which can be done by Lemma 4.3 of section 4.3 above, a new vacuum magnetic field is obtained:

$$\mathbf{B}_w = \operatorname{grad}\Phi_0(\eta, \theta) = \frac{1}{\sqrt{g_{\eta\eta}}} \frac{\partial \Phi_0(\eta, \theta)}{\partial \eta} \mathbf{e}_\eta + \frac{1}{\sqrt{g_{\theta\theta}}} \frac{\partial \Phi_0(\eta, \theta)}{\partial \theta} \mathbf{e}_\theta + \frac{D}{H_\phi}\mathbf{e}_\phi. \tag{4.41}$$

A graph for $D = 2.13$ showing two opposite field lines winding around a magnetic surface is presented on Fig. 4-7.

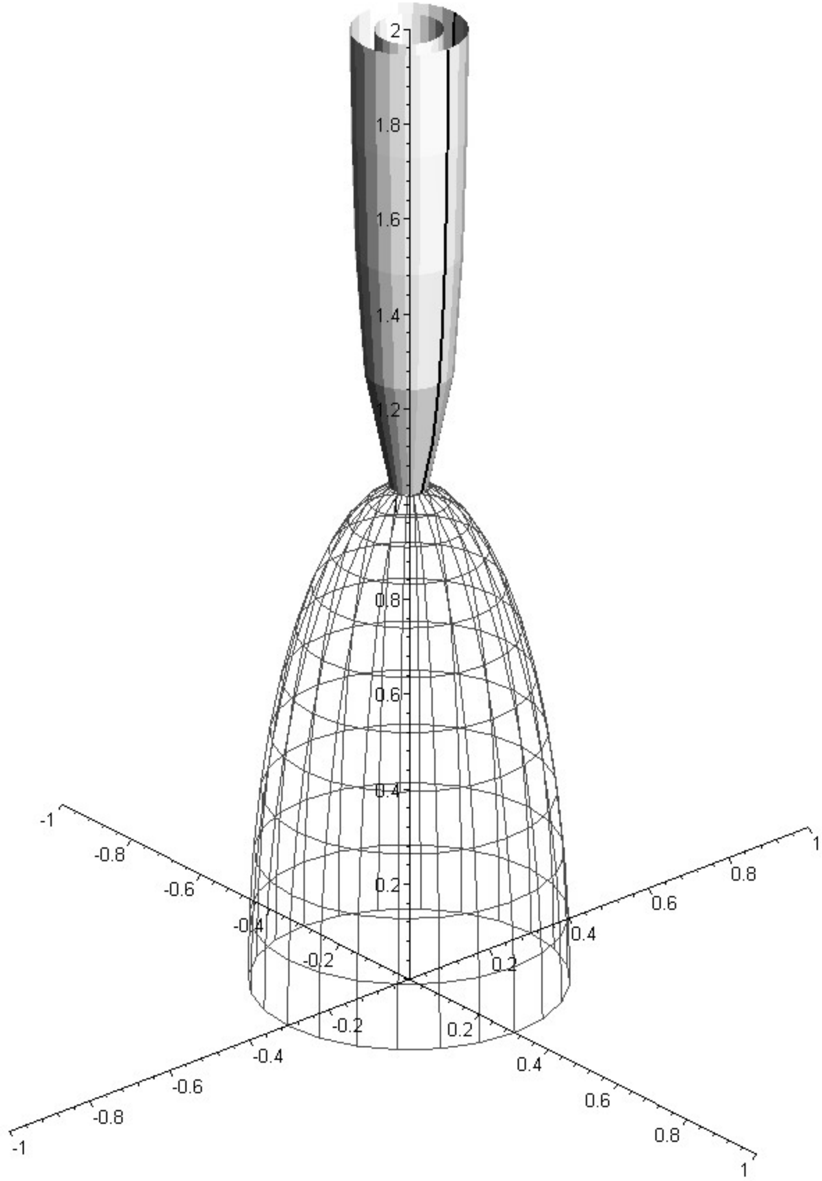

Figure 4-6: A magnetic field flux tube normal to a prolate spheroid.

The magnetic field (4.40) and magnetic surfaces in prolate spheroidal coordinates. The magnetic surfaces are nested widening circular tubes along the z-axis, perpendicular to the spheroid $\eta = \eta_0$ and asymptotically approaching circular cylinders $x^2 + y^2 = \text{const.}$

The graph is built for the choice $\{a = 2,\ M_0 = 1,\ \eta_0 = 0.3\}$. The two magnetic surfaces shown here correspond to $\theta_1 = 0.07$ and $\theta_2 = 0.12$.

A sample magnetic field line on the outer surface ($\theta_2 = 0.12$) is plotted.

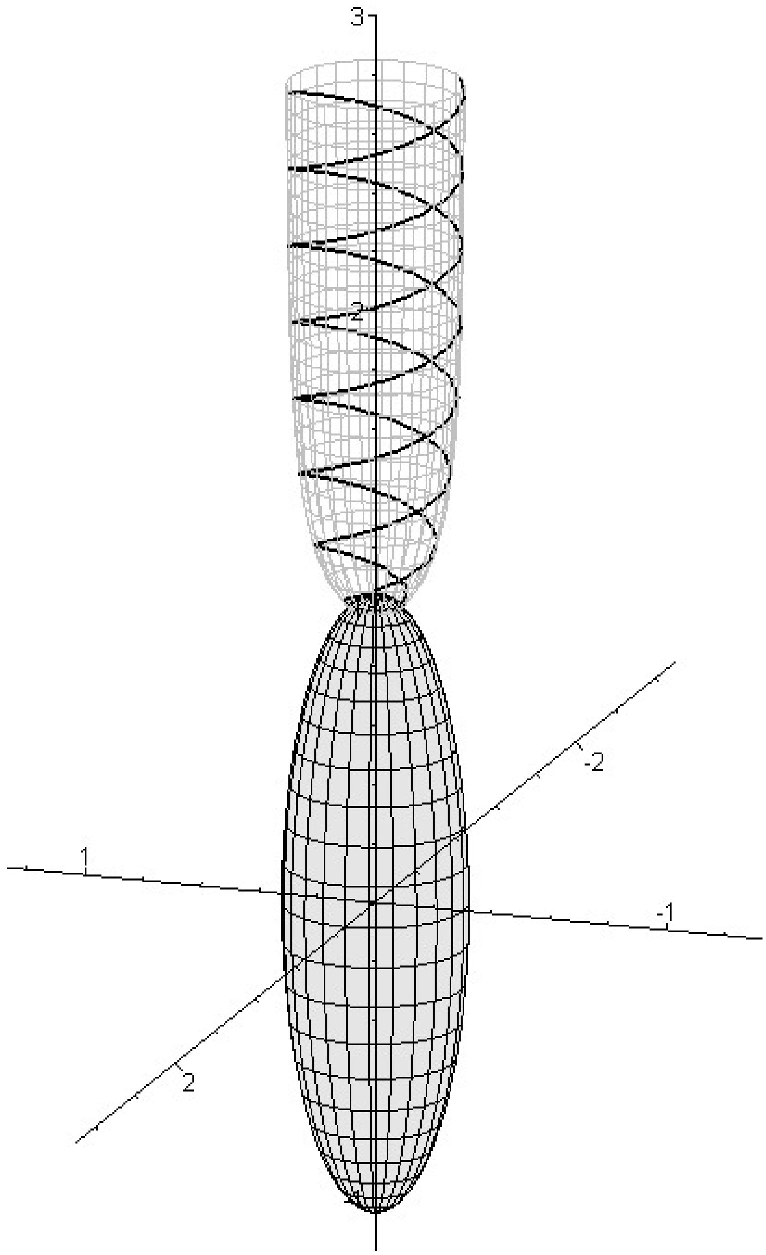

Figure 4-7: A winding magnetic field in prolate spheroidal coordinates.

Two opposite field lines of the magnetic field (4.41) winding around a magnetic surface are shown; a solution is constructed in prolate spheroidal coordinates.
The magnetic surfaces are nested widening circular tubes along the z-axis, perpendicular to the spheroid $\eta = \eta_0$ and asymptotically approaching circular cylinders $x^2 + y^2 = \text{const}$. The magnetic surface shown here is defined by its intersection with the spheroid at $\theta_0 = 0.3$.
The graph is built for the choice $\{a = 2,\ M_0 = 1,\ \eta_0 = 0.3\}$, $D = 2.13$.

The physical model. We use the vacuum solutions with and without the polar component, (4.40) and (4.41), to model a quasi-equilibrium process of *mass exchange by a plasma jet between two distant spheroidal objects.*

The useful property of the solutions is that their magnetic surfaces tend to cylinders by construction. Also, if $\mathbf{B}$ is a vacuum magnetic field, then $(-\mathbf{B})$ is a vacuum magnetic field, too.

Hence one may effectively glue one copy of such solution with another copy, the latter being rotated on the angle π with respect to an axis orthogonal to the axis of the magnetic surface, translated on the distance much longer than the size of the initial spheroid (Fig. 4-8), and taken with the opposite sign.

It is possible to show that for a given solution ((4.40) or (4.41)), the rate of growth of the tube radius B_r/B_z has the leading term z^{-3} at $z \to \infty$, hence the magnetic field lines of the "glued" solution will not have significant "cusps" - discontinuities of derivatives.

The resulting force-free vacuum magnetic field can be used to construct isotropic plasma equilibria with flow by virtue of the Bogoyavlenskij symmetries (1.27), or anisotropic plasma equilibria with and without flow, with the help of the transformations (3.1) obtained in Chapter 3 of this work.

For example, after the application of the symmetries (1.27) to the field $\mathbf{B}_w$, one gets an *isotropic dynamic configuration*

$$\mathbf{B}_1 = m(\mathbf{r})\mathbf{B}_w, \quad \mathbf{V}_1 = \frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\mathbf{B}_w,$$

$$\rho_1 = a^2(\mathbf{r})\rho_0(\mathbf{r}), \quad P_1 = CP_0 - n^2(\mathbf{r})\mathbf{B}_w^2/(2\mu). \tag{4.42}$$

$$m^2(\mathbf{r}) - n^2(\mathbf{r}) = C = \text{const},$$

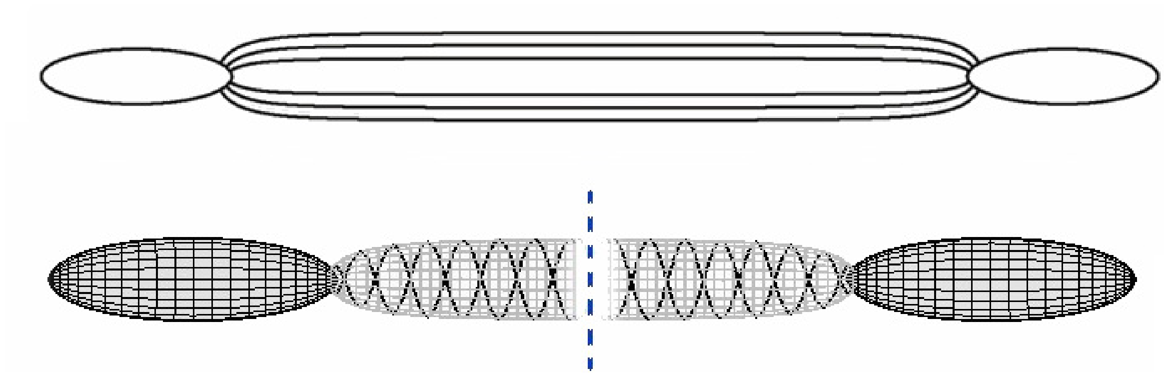

Figure 4-8: A model of mass exchange between two distant spheroidal objects by a plasma jet.

The magnetic surfaces are nested widening circular tubes along the z-axis, perpendicular to the spheroid $\eta = \eta_0$ and asymptotically approaching circular cylinders $x^2 + y^2 = \text{const}$: $B_r/B_z = O(z^{-3})$ at $z \to \infty$.
Shown here is the procedure of gluing one copy of a solution ((4.41) or (4.40)) with another copy, rotated on the angle π with respect to an axis orthogonal to the axis of the magnetic surface, translated on the distance much longer than the size of the initial spheroid, and taken with the opposite sign.

where $a(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r}), \rho_0(\mathbf{r})$ are functions constant on magnetic field lines and streamlines (which coincide, as $\mathbf{V}_1$ and $\mathbf{B}_1$ are collinear).

To construct an anisotropic CGL extension of the above isotropic model, we again use the transformations (3.1). The resulting anisotropic equilibrium $\mathbf{B}_2, \mathbf{V}_2, p_{\|2}, p_{\perp 2}, \rho_2$ is then defined by (P_0 was set to 0):

$$\mathbf{B}_2 = f(\mathbf{r})m(\mathbf{r})\mathbf{B}_w, \quad \mathbf{V}_2 = g(\mathbf{r})\frac{n(\mathbf{r})}{a(\mathbf{r})\sqrt{\mu\rho_0(\mathbf{r})}}\mathbf{B}_w, \quad \rho_2 = C_0 a^2(\mathbf{r})\rho_0(\mathbf{r})\mu/g(\mathbf{r})^2,$$

$$p_{\perp 2} = C_1 + \frac{\mathbf{B}_w^2}{2\mu}\left(C_0 C\mu - f^2(\mathbf{r})m^2(\mathbf{r})\right), \tag{4.43}$$

$$p_{\|2} = C_1 + \frac{\mathbf{B}_w^2}{2\mu}\left(f^2(\mathbf{r})m^2(\mathbf{r}) - C_0 C\mu - 2C_0 n^2(\mathbf{r})\right).$$

The physical requirements and applicability bounds are the same as described in the previous model (see sec. 4.4.1). The *relation between the pressure components* of anisotropic pressure tensor, is also the same:

$$\frac{p_{\perp 2}}{p_{\|2}} = 1 + \frac{2k(\mathbf{r}) + 2C_0 n^2(\mathbf{r})}{\mu f^2(\mathbf{r})m^2(\mathbf{r})}\frac{\mathbf{B}_2^2}{2p_{\|2}}, \quad k(\mathbf{r}) = C_0 C\mu - f^2(\mathbf{r})m^2(\mathbf{r}),$$

which is in the agreement with the observation-based empiric formula (4.36).

We remark that the same way as in the previous model, the values of all the arbitrary functions of the transformations (1.27), (3.1) can be chosen separately not on every magnetic surface, but on *every magnetic field line*. Thus these free functions are actually functions of two independent variables specifying the origin of every magnetic line on the starting spheroid, and the resulting exact solution *has no geometrical symmetries*.

If the constant D in the initial field $\mathbf{B}_w$ (4.41) is different from zero, then the w - component of this field has a singularity on the z-axis, and the **plasma domain** $\mathcal{D}$ must be restricted to a volume between two nested magnetic surfaces so that the

z-axis is excluded (see Fig. 4-9). However, the families of transformed isotropic (4.42) and anisotropic (4.43) magnetic fields $\mathbf{B}_1$, $\mathbf{B}_2$ *are smooth everywhere*, if the non-singular field $\mathbf{B}_0$ (4.40) is used instead of $\mathbf{B}_w$. Then the plasma domain $\mathcal{D}$ can be chosen to be a region inside any flux tube or between two nested ones (Fig. 4-9.) The domain on Fig. 4-9a is not simply connected; the one on Fig. 4-9b is simply connected. The axis of symmetry of the magnetic surfaces coincides with the big axis of both spheroids.

In dynamic isotropic and anisotropic cases, by the properties of solutions constructed from static configurations $\mathbf{V} = 0$ by the transformations (1.27) or (3.1), the plasma velocity has the same direction as the magnetic field, $\mathbf{V} \parallel \mathbf{B}$, so the configuration is interpreted as a magnetically driven matter flow from one spheroid to another.

The presented solution models the quasi-equilibrium stage during the time interval T, with the requirement

$$T \cdot S_{max} \cdot \max_{\mathcal{D}} |\rho \mathbf{V}| \ll M_0,$$

where M_0 is the mass of the spheroid objects, and S_{max} is the area of the maximal section of the plasma domain transverse to the flow lines.

4.4.2 Coordinates in which exact plasma equilibrium solutions can be constructed

In this subsection, we consider methods of finding new coordinate systems in flat 3D space, in which exact solutions to the Plasma Equilibrium equations (1.20) can be built.

Figure 4-9: A model of mass exchange between two distant spheroidal objects by a plasma jet: possible plasma domains.

Examples of possible plasma domains $\mathcal{D}$ for isotropic and anisotropic equilibria models (sec. 4.4.1.)

By Theorem 4.2, to every set of functions H_u, H_v, H_w, Φ, P that satisfy the equations (4.3), (4.4), such that all components of the Riemann tensor $R_{ijkl}(H_u, H_v, H_w) = 0$, there exists an orthogonal coordinate system (u, v, w) in $\mathbb{R}^3$, where an isotropic plasma equilibrium (1.20) is realized.

Therefore one of the ways to look for new plasma equilibria is to search for the metric coefficients as solutions to the differential equations $R_{ijkl} = 0$, which also satisfy the conditions of Theorem 4.2. In such coordinate systems, the exact force-free plasma equilibria (4.14) exist. (An example is given in subsec. 4.4.2.)

Once the metric coefficients are found, the reconstruction of the explicit connection of coordinates (u, v, w) with cartesian coordinates (x, y, z) from the scaling coefficients H_u, H_v, H_w is well-defined and described in Appendix G.

In $\mathbb{R}^3$, for a general orthogonal coordinate system, the Riemann tensor has six independent nonzero components, that are expressed through the metric tensor components g_{ii} as follows ([82], p.119):

$$\begin{aligned}
&R_{hijk} = 0, && (h,i,j,k \neq) \\
&R_{hiik} = \sqrt{g_{ii}} \left(\frac{\partial^2 \sqrt{g_{ii}}}{\partial u^h \partial u^k} - \frac{\partial \sqrt{g_{ii}}}{\partial u^h} \frac{\partial \ln \sqrt{g_{hh}}}{\partial u^k} - \frac{\partial \sqrt{g_{ii}}}{\partial u^k} \frac{\partial \ln \sqrt{g_{kk}}}{\partial u^h} \right), && (h,i,k \neq) \\
&R_{hiih} = \sqrt{g_{ii}} \sqrt{g_{hh}} \left[\frac{\partial}{\partial u^h} \left(\frac{1}{\sqrt{g_{hh}}} \frac{\partial \sqrt{g_{ii}}}{\partial u^h} \right) + \frac{\partial}{\partial u^i} \left(\frac{1}{\sqrt{g_{ii}}} \frac{\partial \sqrt{g_{hh}}}{\partial u^i} \right) \right. \\
&\qquad\qquad \left. + \sum_{m \neq h,i} \frac{1}{g_{mm}} \frac{\partial \sqrt{g_{ii}}}{\partial u^m} \frac{\partial \sqrt{g_{hh}}}{\partial u^m} \right], && (h \neq i)
\end{aligned} \tag{4.44}$$

Here $h, i, j, k, m = 1, 2, 3$, and the notation (4.1) - (4.2) is assumed.

These six equations are non-linear partial differential equations. Using the conditions of Theorem 4.2 as constraints, simplest particular solutions can be found, which in most cases are reduced to metrics describing spherical, cylindrical or cartesian coordinates. As an example, a particular set of non-circular cylindrical coordinates is constructed in subsection 4.4.2.

A common way of construction of new solutions of non-linear partial differential equations is starting from a known particular solution and using symmetries, for example, Lie point symmetries.

As known particular solutions, spherical or cylindrical coordinates can serve. As for Lie point transformations, it can be shown that the Riemann equations (4.44) themselves admit only the local symmetries generated by

$$X_R = \sum_{i=1}^{3} \left(g_{ii} \left(c_1 - 2\frac{dF_i(u^i)}{du^i} \right) \frac{\partial}{\partial g_{ii}} \quad + \quad F_i(u^i) \frac{\partial}{\partial u^i} \right), \tag{4.45}$$

which corresponds to simple independent coordinate scalings

$$u^i \to M_i(u^i), \quad g_{ii} \to \frac{g_{ii}}{M_i(u^i)^2}, \quad i = 1..3, \tag{4.46}$$

and the global homogeneous scalings $g_{ii} \to k^2 g_{ii}$. Therefore the Lie symmetry method can not be applied for generating new suitable sets of coordinates.

A more common and efficient approach of finding new triply orthogonal coordinate systems is using general coordinate transformations that preserve orthogonality, do not form Lie groups and are non-trivial as opposed to (4.46). An example of such transformations considered in this work is conformal transformations of the complex plane, which generate sets of non-circular cylindrical coordinates. The latter by construction have the form of metric coefficients acceptable by Theorem 4.2. The examples of the corresponding plasma equilibria are given in subsection 4.4.2 below.

Example 1. Coordinate systems found from Riemann's equations

In this example, an orthogonal coordinate system that satisfies the requirements of the Theorem 4.2 is found as a particular solution to the Riemann equations (4.44).

We assume the scaling coefficients have the form

$$H_u^2 = f_1(u) f_2(v) h(w), \quad H_v^2 = g_1(u) g_2(v) h(w), \quad H_w^2 = k_1(w),$$

which satisfies the Theorem, and find a sample solution

$$f_1(u) = u^3, \;\; f_2(v) = \sin^2 v, \;\; g_1(u) = 1, \;\; g_2(v) = 4, \;\; h(w) = 1+w^2, \;\; k_1(w) = 4w^2/h(w)$$

Using the method of the reconstruction of coordinates from the scaling coefficients as described in the Appendix G, we find the expression for cartesian coordinates

$$x = \frac{2}{5}e^{u+2v}\cos(v-2u)+\frac{1}{5}e^{u+2v}\sin(v-2u), \;\; y = \frac{1}{5}e^{u+2v}\cos(v-2u)-\frac{2}{5}e^{u+2v}\sin(v-2u), \quad z = w^2.$$

The magnetic surfaces $w = C$ are thus planes $z = C^2$, and the surfaces $u = \text{const}, v = \text{const}$ are orthogonal families of vertical spiral cylinders, shown on the Fig. 4-10.

In this coordinate system, by Theorems 4.1 and 4.2, exact force-free magnetic field of the form (4.5), (4.14) exist.

Example 2. Generation of orthogonal coordinate systems by coordinate transformations

Given a coordinate system (x^1, x^2, x^3) in $\mathbb{R}^3$ that satisfies $R_{ijkl} = 0$, one can use an arbitrary coordinate transformation

$$u^i = u^i(x^1, x^2, x^3), \quad i = 1, 2, 3,$$

and, by tensor transformation rules, the Riemann tensor of the resulting coordinates will also be identically zero:

$$R'_{ijkl} = R'_{abcd}\frac{\partial x^a}{\partial u^i}\frac{\partial x^b}{\partial u^j}\frac{\partial x^c}{\partial u^k}\frac{\partial x^d}{\partial u^l} = 0, \quad i, j, k, l, a, b, c, d = 1, 2, 3.$$

Here we give an example of transformations that produce orthogonal coordinates and have metric coefficients satisfying the sufficient condition for a force-free plasma equilibrium of the type (4.14) to exist.

Consider the plane transformations

$$x = \xi_1(u, v), \quad y = \xi_2(u, v) \tag{4.47}$$

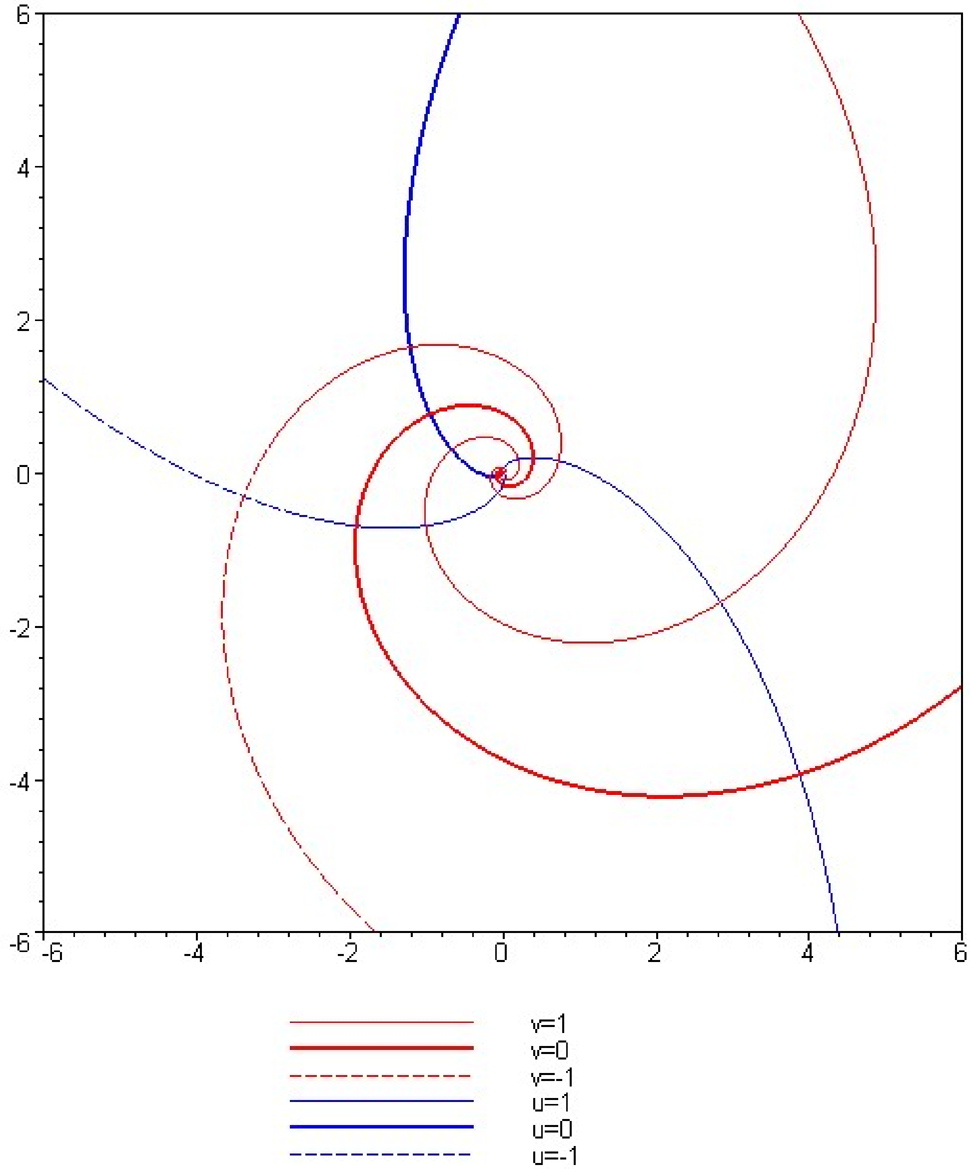

Figure 4-10: An example of cylindrical orthogonal coordinates where exact plasma equilibria can be built.

A section of orthogonal families of vertical spiral cylinders $u = \text{const}, v = \text{const}$, that together with planes $z = \text{const}$ form an orthogonal coordinate system satisfying the conditions of Theorem 4.2.

satisfying the Cauchy-Riemann conditions

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}. \tag{4.48}$$

(Here x, y are cartesian and u, v curvilinear coordinates). The property of conformal mappings is that it preserves angles, hence the families of curves $u = \text{const}$, $v = \text{const}$ in the plane are mutually orthogonal.

If we consider the corresponding 3D cylindrical mapping

$$x = \xi_1(u, v), \quad y = \xi_2(u, v), \quad z = w, \tag{4.49}$$

it defines an orthogonal coordinate system with metric coefficients

$$g_{11} = g_{22} = \left(\frac{\partial \xi_1(u, v)}{\partial u}\right)^2 + \left(\frac{\partial \xi_2(u, v)}{\partial u}\right)^2, \quad g_{33} = 1. \tag{4.50}$$

First type of solutions. These metric coefficients exactly satisfy the conditions of the above Theorem 4.2, and in the coordinates (u, v, w) a force-free magnetic field

$$\mathbf{B} = \left(\frac{C_1(w)}{\sqrt{g_{11}}}, \frac{C_2(w)}{\sqrt{g_{11}}}, 0\right), \quad \frac{\partial}{\partial w}(C_1^2(w) + C_2^2(w)) = 0. \tag{4.51}$$

Many examples of such cylindrical transformations can be suggested. The simplest ones include power, logarithmic, exponential, hyperbolic, elliptic and other types of conformal complex plane mappings. We do not consider solutions of this type in detail here.

Remark. The field lines of force-free magnetic fields (4.51) lie in planes $z = \text{const}$. A constant z- component can be added to the fields of this type as follows:

$$\mathbf{B} = \left(\frac{C_1(w)}{\sqrt{g_{11}}}, \frac{C_2(w)}{\sqrt{g_{11}}}, D\right), \quad \frac{\partial}{\partial w}(C_1^2(w) + C_2^2(w)) = 0, \quad D = \text{const}. \tag{4.52}$$

Then the current $\mathbf{J} = \operatorname{curl} \mathbf{B}/\mu$ does not change, and the equilibrium remains force-free.

Second type of solutions. The solutions presented below can be built in *any* coordinate system obtained from the cartesian coordinates (x, y, z) by a conformal plane transformation (4.49).

In the transformed coordinates (4.49), the metric coefficients are (4.50), and the complete Laplace equation evidently can have 2-dimensional solutions $\Phi(u, v)$:

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0.$$

Therefore, by Lemma 4.1, in these coordinates there exists a vacuum magnetic field (4.5) in the $z = $ const-plane defined by

$$\mathbf{B} = \left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\Phi}{\partial u}, \quad \frac{1}{\sqrt{g_{11}}}\frac{\partial\Phi}{\partial v}, \quad 0 \right),$$

which can be given a non-trivial z-component using Lemma 4.2 in Section 4.3:

$$\mathbf{B} = \left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\Phi}{\partial u}, \quad \frac{1}{\sqrt{g_{11}}}\frac{\partial\Phi}{\partial v}, \quad K(u,v) \right), \ \ \triangle K(u,v) = 0, \ \ \operatorname{grad}\Phi(u,v)\cdot\operatorname{grad}K(u,v) = 0.$$

By Lemma 4.2, such field and the pressure $P(u, v) = C - K^2(u, v)/2$ $(C = \text{const})$ satisfy the full plasma equilibrium system (1.20)

$$\operatorname{curl}\mathbf{B}\times\mathbf{B} = \mu \operatorname{grad} P, \quad \operatorname{div}\mathbf{B} = 0.$$

Example. We now give a particular example in elliptic cylindrical coordinate system defined by a conformal transformation that acts on the complex plane as $Z' = a\cosh Z$:

$$x = a\cosh u \cos v, \quad y = a\sinh u \sin v, \quad z = w.$$

Then we choose a function satisfying $\triangle_{(u,v)}\Phi(u, v) = 0$:

$$\Phi(u, v) = \sinh u \cos v + 0.1 \sinh 2u \cos 2v - 3v + C_1.$$

A conjugate harmonic function for it is

$$K(u,v) = \cosh u \sin v + 0.1 \cosh 2u \sin 2v + 3u.$$

The level curves $K(u,v) = \text{const}$ are presented on Fig. 4-11 and coincide with the projections of the magnetic field lines on the (x,y)-plane.

The corresponding plasma equilibrium solution on the cylinders $K(u,v) = \text{const}$ has a simple representation

$$\mathbf{B} = \frac{\cosh u \cos v + 0.2 \cosh 2u \cos 2v}{\sqrt{\cosh^2 u - \cos^2 v}} \mathbf{e_u} + \frac{-\sinh u \sin v - 0.2 \sinh 2u \sin 2v - 3}{\sqrt{\cosh^2 u - \cos^2 v}} \mathbf{e_v} + (3u + \cosh u \sin v + 0.1 \cosh 2u \sin 2v) \mathbf{e_z} \tag{4.53}$$

The u- and v-components of this magnetic field evidently have a singularity at $u = v = 0$ of the order ρ^{-1}, where $\rho = \sqrt{u^2 + v^2}$ is the "distance" to singularity. In cartesian coordinates, the singularity is located at $(x = \pm 1, y = 0)$.

The levels of the magnetic surface function $K(u,v) = \text{const}$ encircle the singularities, and degenerate into a line segment as $K(u,v) \to 0$, as seen from Fig. 4-11

If the magnetic field (4.53) with pressure $P(u,v) = C - K^2(u,v)/2$ (4.21) is to be used in any **isotropic** model, the plasma domain $\mathcal{D}$ is to be restricted to the cylindrical volume between any two magnetic surfaces: $\mathcal{D} = \{(u,v) : 0 < K_1 \leq K(u,v) \leq K_2$. Using the fact that the magnetic field is tangent to the surfaces $K(u,v) = \text{const}$, this can be done by introducing a boundary surface current (4.31). Then outside of $\mathcal{D}$ magnetic field is zero: $\mathbf{B} \equiv 0$.

However, if one is to use the described configuration as an initial solution for building an **anisotropic static equilibrium** by virtue of the transformations (3.1),

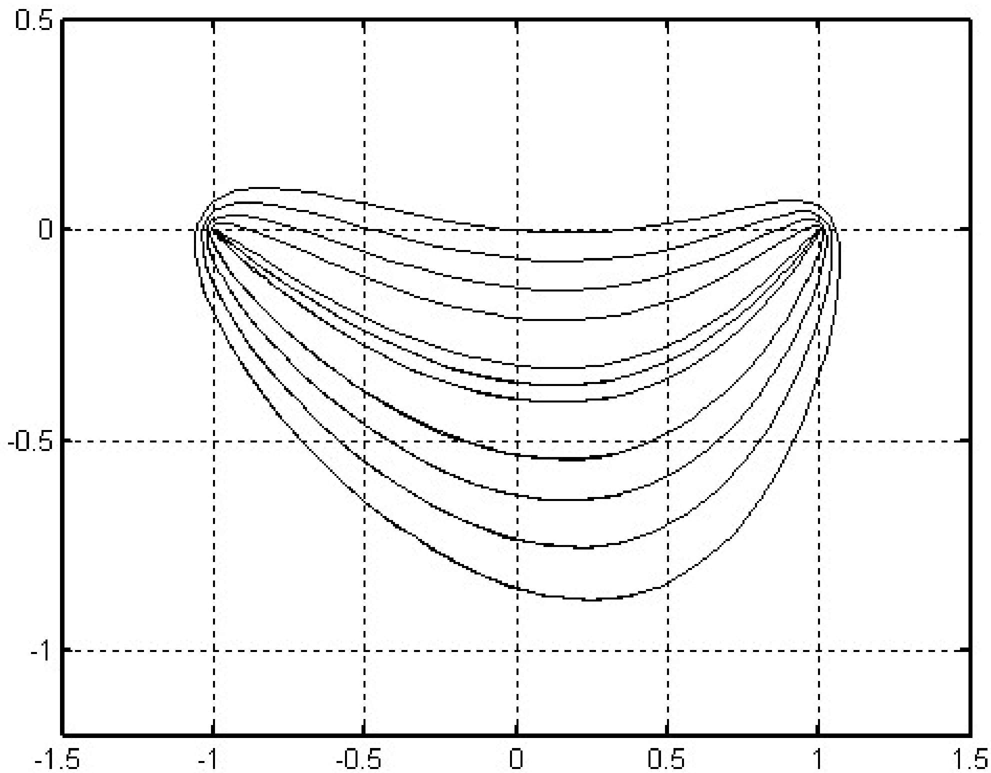

Figure 4-11: An example of non-circular cylindrical magnetic surfaces.

A family of cylinders with non-symmetric closed cross-sections that are the magnetic surfaces of sample isotropic and anisotropic plasma equilibria found in sec. 4.4.2.

then he should select the function $f(\mathbf{r})$ (that must be constant on magnetic surfaces, $f(\mathbf{r}) = f(K)$) as follows:

$$f(K) = \frac{1}{\max\limits_{K(u,v)=K} \{|B_1|, |B_2|\}} a(K),$$

where $a(K)$ is some function with compact support and the property $\frac{da(K)}{dK}|_{K=0} = 0$, and $|B_1|, |B_2|$ are respectively the u- and v-components of the field (4.53).

This function is evidently finite in any domain $\mathcal{D}$ bounded by a level $K(u,v) \leq K$. On the degenerate surface $K(u,v) = 0$ by continuity we have $\mathbf{B} = 0$. On the outer boundary of the domain, a surface current (4.31) must be introduced, to ensure so that $\mathbf{B} \equiv 0$ outside of $\mathcal{D}$.

Vacuum magnetic fields in rotational coordinate systems

Among the classical and esoteric coordinate systems where the Laplace equation is separable or R-separable, many are rotationally symmetric systems, with metric coefficients independent of the polar angle ϕ.

In all such systems, the Laplace equation has solutions independent of ϕ. Examples are toroidal coordinates, usual and inverse prolate and oblate spheroidal coordinates, cap-cyclide, disk-cyclide, cardioid coordinates and several others (see [77]).

By Lemma 4.1, "vacuum" gradient magnetic fields can be built in such coordinates, tangent to the magnetic surfaces, which are in this case vertical half-planes $\phi = \text{const}$.

By Lemma 4.3, a non-zero ϕ-component can be added to these vacuum magnetic fields, to make them non-planar.

Examples of such vacuum magnetic fields and the corresponding isotropic and anisotropic plasma configurations obtained from them by transformations (1.27), (3.1) will be built in consequent papers.

Particular solutions obtained in different coordinates can have simple algebraic representation only in the corresponding coordinates, therefore different rotational coordinate systems may not be considered equivalent, from the computational point of view.

We also remark that the magnetic fields constructed this way *can not* be found from Grad-Shafranov equation, which describes plasma equilibria with magnetic surfaces $\Psi(r, z) = \text{const}$, whereas in the above case magnetic surfaces are $\Psi = \Psi(\psi) = \text{const}$.

4.5 Conclusion

In this chapter, a method of construction of exact plasma equilibria is presented. Using it, new dynamic and static equilibria in different geometries are constructed, in both the isotropic (classical magnetohydrodynamics, (1.16)-(1.17)) and anisotropic tensor-pressure (Chew-Goldberger-Low, (1.22)-(1.23)) frameworks.

The method is based on representing the system of static isotropic plasma equilibrium equations (1.20) in coordinates (u, v, w), such that magnetic surfaces are the coordinate level surfaces $w = \text{const}$. As shown in Theorem 4.1 of section 4.2, it is possible if the family of magnetic surfaces is a family of Lamé; in such coordinates the system is reduced to two partial differential equations for two unknown functions. The first of the equations of the system is a "truncated" Laplace equation (4.3), and the second one, (4.4), has an energy-connected interpretation (see Remark 2 in sec. 4.2).

Instead of four unknown functions of the static MHD equilibrium system, $\mathbf{B}(\mathbf{r})$ and $P(\mathbf{r})$, that depend on three spatial variables, the new system of equations employs only two functions - $\Phi(u, v, w)$ and $P(w)$, and the magnetic field $\mathbf{B}(\mathbf{r})$ is reconstructed from the relation (4.5).

In section 4.3, sufficient conditions on the metric coefficients are established under which exact solutions of particular types can be found in corresponding coordinates.

Theorem 4.2 states that if the metric coefficients satisfy the relations (4.13) or (4.15), then respectively the families of force-free solutions $\Phi(u, v, w)$ (4.14) and (4.16), (4.17) with $P(w) = \text{const}$ exist in such coordinates.

Lemma 4.1 of the same section shows that in coordinates where the general 3D Laplace equation admits 2-dimensional solutions, "vacuum" magnetic fields $\operatorname{div} \mathbf{B} = 0$, $\operatorname{curl} \mathbf{B} = 0$ with non-trivial geometry (and tangent to magnetic surfaces $w = \text{const}$) can be built. Lemmas 4.2 and 4.3 extend this class of solutions in some coordinate systems.

In subsection 4.3.3, the value of "vacuum" gradient fields for plasma equilibrium modelling is discussed. By themselves, gradient fields represent trivial plasma equilibria with constant pressure and no electric currents, and cannot model real plasma phenomena, where the electric current density $\mathbf{J}$, pressure P and velocity $\mathbf{V}$ are generally non-zero. However, they can serve as initial solutions in the infinite-parameter transformations (1.27), (3.1) that produce complex dynamic and static isotropic and anisotropic plasma equilibrium configurations with non-constant pressure(s) and currents.

In the examples section 4.4, we use previously developed machinery for the construction of particular examples of plasma equilibria.

Subsection 4.4.1 contains various examples of non-degenerate plasma equilibrium configurations generated by using Theorems 4.1, 4.2 and Lemmas 4.1 – 4.3.

The first example (subsection 4.4.1) is a set of non-Beltrami Force-Free plasma equilibria (1.15) in a prescribed geometry - with spherical magnetic surfaces and the force-free coefficient $\alpha(\mathbf{r})$ (1.15) being a function of the spherical radius. When used in a linear combination, these solutions give rise to force-free fields with *no geometrical symmetries* tangent to spheres.

In the second example, we build exact dynamic isotropic and anisotropic plasma equilibrium configurations with magnetic fields tangent to *ellipsoids*. It was done by employing Lemma 4.1 in ellipsoidal coordinates and producing non-trivial "vacuum" magnetic fields tangent to ellipsoids. The latter were transformed into families of dynamic isotropic and anisotropic plasma equilibria by virtue of Bogoyavlenskij symmetries (1.27) and "anisotropizing" transformations (3.1).

The solutions are well-defined and have a finite magnetic energy in half-space. They model solar coronal flares near the active regions of the Sun photosphere. The resulting anisotropic model is an essentially non-symmetric exact solution, unlike other available models (see [80].) The solution reproduces important features of solar flares known from observations.

In the third example, the coordinate representation is used to build a particular trivial "vacuum" magnetic field in prolate spheroidal coordinates. From that field, also by the symmetries (1.27) and the transformations (3.1), we construct families of non-degenerate (and generally non-symmetric) isotropic and anisotropic plasma equilibria with dynamics, which model the quasi-stationary phase of mass exchange between two spheroidal objects. Plasma domains of different geometry and topology may be chosen for the model.

In the subsection 4.4.2 of the example section, ways of construction of new coordinate systems in flat 3D space ($R_{ijkl} = 0$) in which exact equilibria (4.3), (4.4) can be found are discussed.

Two ways are suggested: to search for the metric coefficients satisfying the conditions of Theorem 4.2 or Lemma 4.1, that also satisfy the differential equations $R_{ijkl} = 0$; or to use orthogonal coordinate transformations $(x, y, z) \to (u, v, w)$ from flat coordinates (x, y, z), such that the new metric coefficients conform with the conditions of Theorem 4.2 or Lemma 4.1.

Both approaches are illustrated with examples. In particular, using the second approach and Lemmas 4.1, 4.2, plasma equilibria with non-circular cylindrical magnetic surfaces and realistic values of plasma parameters are obtained.

Summary

In this thesis, an analytical study of symmetries and other properties of isotropic and anisotropic, dynamic and static plasma equilibrium systems is performed, and new families of exact equilibrium solutions are constructed.

In **Chapter 2**, we study the possibility of finding complex intrinsic symmetries of systems of partial differential equations, such as Bogoyavlenskij symmetries for isotropic plasma equilibria, by applying a general method. More precisely, we answer the question about the possibility of obtaining the Bogoyavlenskij symmetries (1.27) and (1.29) of the MHD equilibrium equations using the Lie group formalism. This question was raised soon after the discovery of the symmetries.

The Lie symmetry method [55] is applicable to any system of PDEs; it is generally capable of detecting both simple geometric symmetries (e.g., rotations, scaling transforms and translations) and more complicated ones. They can be used to build particular solutions of the system under consideration, to reduce the order and to obtain invariants. Self-similar solutions constructed from Lie symmetries often have a transparent physical meaning.

We prove that certain Lie groups of point transformations of isotropic MHD equilibrium equations (1.16)-(1.17), which were found independently using the classical Lie approach, are *equivalent* to Bogoyavlenskij symmetries G_m, G_{0m} (1.28), (1.31) for

these equations.

It is important to remark that the Bogoyavlenskij symmetries can be found as Lie point transformations of the MHD equilibrium system *only* if the general solution topology, the existence of magnetic surfaces to which vector fields **B** and **V** are tangent, is explicitly taken into account in the form of two additional constraints:

$$\rho(\mathbf{r}) = \rho(\Psi(\mathbf{r})), \quad \mathrm{grad}(\Psi(\mathbf{r})) \cdot \mathbf{B} = 0, \quad \mathrm{grad}(\Psi(\mathbf{r})) \cdot \mathbf{V} = 0,$$

where $\Psi(\mathbf{r})$ is a magnetic surface function (or, more generally, a function constant on magnetic field lines and plasma streamlines of an equilibrium configuration).

The potential symmetry analysis [70, 71] of the static isotropic plasma equilibrium system (1.20) is performed, and the conclusion is drawn that the invertible linearization of this system is not possible.

In **Chapter 3**, an infinite-dimensional set of transformations between solutions of the isotropic Magnetohydrodynamic (MHD) equilibrium equations and solutions of the anisotropic equilibrium equations in the Chew-Goldberger-Low (CGL) approximation are presented. These transformations depend on the topology of the original solution and allow the building of a wide class of anisotropic plasma equilibrium solutions with a variety of physical properties and topologies, including 3D solutions with no geometrical symmetries [79].

Anisotropic plasma equilibrium configurations with and without magnetic surfaces can be built using these transformations. Several examples are presented in Chapters 3 and 4.

It is also shown that anisotropic CGL plasma equilibria possess topology-dependent infinite-dimensional symmetries similar to Bogoyavlenskij symmetries (1.27) for isotropic MHD equilibria. These symmetries can be used to construct new anisotropic plasma equilibria from known ones, in particular, dynamic equilibrium solutions from a given static equilibrium. As the original Bogoyavlenskij symmetries, these symmetries can be found from the Lie group analysis procedure applied to the CGL equilibrium system with appropriate additional constraints [73].

In **Chapter 4**, a method of the construction of analytical solutions to classical isotropic and anisotropic plasma equilibrium equations (in static and dynamic cases) is suggested. The method is based on the general existence of magnetic surfaces to which the plasma magnetic field is tangent.

For many important cases, given a set of magnetic surfaces, an orthogonal coordinate system can be constructed, with one of the coordinates constant on the magnetic surfaces. The system of static classical plasma equilibrium equations rewritten in such coordinates is reduced to two partial differential equations for two unknown functions, one of the equations being a "truncated" form of the Laplace equation.

We establish sufficient conditions for coordinate systems, under which exact solutions to plasma equilibrium equations can be found. It is shown that in coordinates where the Laplace equation admits 2-dimensional solutions, a certain type of non-trivial exact plasma equilibria can be built.

Using the coordinate representation described above, in different systems of curvilinear coordinates, classical and non-classical, non-trivial gradient fields can be built, tangent to the prescribed sets of magnetic surfaces. By using infinite-parameter transformations (1.27), (3.1), and (3.21), families of complex dynamic and static isotropic and anisotropic plasma equilibrium configurations can be constructed from such gradient fields.

As examples, we build exact plasma equilibrium solutions with magnetic surfaces being nested spheres, ellipsoids, non-circular cylinders, and surfaces of other types.

The value of some of these solutions as models of astrophysical phenomena is discussed. It is shown that characteristic features and relations between macroscopic parameters in the models agree with astrophysical observations.

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Appendices

A Proof of Theorem 2.1

Proof.

(i) First, we prove that the operators $X^{(1)} - X^{(4)}$ are admissible for incompressible isotropic MHD equilibrium system with density $\rho(\mathbf{r})$ constant on magnetic field lines and streamlines. The complete system of equations under consideration can be written as follows:

$$\rho\mathbf{V} \times (\text{curl } \mathbf{V}) - \frac{1}{\mu}\,\mathbf{B} \times (\text{curl } \mathbf{B}) - \text{grad } P - \rho \text{ grad}\frac{\mathbf{V}^2}{2} = 0, \tag{A.1}$$

$$\text{div } \mathbf{B} = 0, \ \text{div } \mathbf{V} = 0, \ \text{curl } (\mathbf{V} \times \mathbf{B}) = 0, \tag{A.2}$$

$$\rho(\mathbf{r}) = \rho(\Psi(\mathbf{r})), \quad \text{grad}(\Psi(\mathbf{r})) \cdot \mathbf{B} = 0, \quad \text{grad}(\Psi(\mathbf{r})) \cdot \mathbf{V} = 0. \tag{A.3}$$

Here $\Psi(\mathbf{r})$ is an arbitrary function constant on magnetic field lines and streamlines (hence on magnetic surfaces, when they exist).

The system (A.1)-(A.3) consists of $l = 10$ equations. It has $n = 3$ independent and $m = 8$ dependent variables:

$$\mathbf{x} = (x, y, z), \quad \mathbf{u} = (V_1, V_2, V_3, \ B_1, B_2, B_3, \ \Psi, \ P). \tag{A.4}$$

Let us apply the Lie procedure described in Section 2.2, assuming that the transformations do not depend on spatial variables, and that the spatial variables themselves are not transformed. Thus we are looking for the transformations of the type

$$(x')^i = f^i(\mathbf{x}, \mathbf{u}, a) \equiv x^i; \quad (u')^k = g^k(\mathbf{x}, \mathbf{u}, a) \equiv g^k(\mathbf{u}, a); \tag{A.5}$$

$$i = 1, \ldots, 3, \quad k = 1, \ldots, 8.$$

They form a subgroup of the general Lie group of transformations (2.4).

Remark. Without the assumptions (A.5), handling the computations takes significantly longer time and puts much higher demands on computer resources. The problem of performing the complete group analysis of the MHD equilibrium system with density constant on magnetic filed field lines and streamlines (A.1)-(A.3) is therefore out of the scope of this work and will be addressed in a subsequent paper.

The unknown quantities to be found are the tangent vector field coordinates

$$\eta^k(\mathbf{u}) = \frac{\partial g^k(\mathbf{u}, a)}{\partial a}|_{a=0}, \quad k = 1, \ldots, 8. \tag{A.6}$$

(We have $\xi^i(\mathbf{x}, \mathbf{u}) = \frac{\partial f^i(\mathbf{x},\mathbf{u},a)}{\partial a}|_{a=0} = 0 \ , i = 1, \ldots, 3.$)

Applying the corresponding prolonged $\mathbf{h}$ (2.5) to every equation of the system (A.1)-(A.3), we get the following system of equations:

$$\begin{aligned}
&\xi_2^4 B_2 + \xi_3^4 B_3 - \xi_1^8 + \eta^5(-\partial B_2/\partial x + \partial B_1/\partial y) + \eta^6(\partial B_1/\partial z - \partial B_3/\partial x) - \\
&\xi_1^5 B_2 - \xi_1^6 B_3 - \eta^2 \rho(\Psi)\partial V_1/\partial y - \eta^3 \rho(\Psi)\partial V_1/\partial z - \xi_1^1 \rho(\Psi) V_1 \\
&-\xi_2^1 \rho(\Psi) V_2 - \xi_3^1 \rho(\Psi) V_3 - \eta^7(V_1 \partial V_1/\partial x + V_2 \partial V_1/\partial y + V_3 \partial V_1/\partial z) d\rho(\Psi)/d\Psi \\
&-\eta^1 \rho(\Psi)\partial V_1/\partial x = 0,
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
&\xi_1^5 B_1 + \xi_3^5 B_3 - \xi_2^8 + \eta^4(\partial B_2/\partial x - \partial B_1/\partial y) + \eta^6(-\partial B_3/\partial y + \partial B_2/\partial z) - \\
&\eta^1 \rho(\Psi)\partial V_2/\partial x - \eta^3 \rho(\Psi)\partial V_2/\partial z - \eta^7(V_1 \partial V_2/\partial x + V_2 \partial V_2/\partial y \\
&+V_3 \partial V_2/\partial z) d\rho(\Psi)/d\Psi - \xi_1^2 \rho(\Psi) V_1 - \xi_2^2 \rho(\Psi) V_2 - \xi_3^2 \rho(\Psi) V_3 - \xi_2^4 B_1 \\
&-\xi_2^6 B_3 - \eta^2 \rho(\Psi)\partial V_2/\partial y = 0,
\end{aligned} \tag{A.8}$$

$$\xi_1^6 B_1 + \xi_2^6 B_2 - \xi_3^8 - \eta^1 \rho(\Psi)\partial V_3/\partial x - \eta^2 \rho(\Psi)\partial V_3/\partial y - \eta^3 \rho(\Psi)\partial V_3/\partial z - \xi_1^3 \rho(\Psi) V_1$$
$$-\eta^7 (V_1 \partial V_3/\partial x + V_2 \partial V_3/\partial y + V_3 \partial V_3/\partial z) d\rho(\Psi)/d\Psi$$
$$-\xi_2^3 \rho(\Psi) V_2 - \xi_3^3 \rho(\Psi) V_3 + \eta^4(-\partial B_1/\partial z + \partial B_3/\partial x) + \eta^5(\partial B_3/\partial y$$
$$-\partial B_2/\partial z) - \xi_3^4 B_1 - \xi_3^5 B_2 = 0, \tag{A.9}$$

$$\xi_1^4 + \xi_2^5 + \xi_3^6 = 0, \tag{A.10}$$

$$\xi_1^1 + \xi_2^2 + \xi_3^3 = 0, \tag{A.11}$$

$$\begin{aligned} &\eta^4(-\partial V_2/\partial y - \partial V_3/\partial z) + \eta^5 \partial V_1/\partial y + \eta^6 \partial V_1/\partial z \\ &+\eta^1(\partial B_2/\partial y + \partial B_3/\partial z) - \eta^2 \partial B_1/\partial y - \eta^3 \partial B_1/\partial z \\ &+\xi_2^5 V_1 + \xi_3^6 V_1 + \xi_2^1 B_2 + \xi_3^1 B_3 - \xi_2^4 V_2 - \xi_2^2 B_1 - \xi_3^3 B_1 - \xi_3^4 V_3 = 0, \end{aligned} \tag{A.12}$$

$$\begin{aligned} &\eta^5(-\partial V_3/\partial z - \partial V_1/\partial x) + \eta^6 \partial V_2/\partial z + \eta^4 \partial V_2/\partial x \\ &+\eta^2(\partial B_3/\partial z + \partial B_1/\partial x) - \eta^1 \partial B_2/\partial x - \eta^3 \partial B_2/\partial z \\ &+\xi_1^4 V_2 + \xi_3^6 V_2 + \xi_1^2 B_1 + \xi_3^2 B_3 - \xi_3^5 V_3 - \xi_1^1 B_2 - \xi_3^3 B_2 - \xi_1^5 V_1 = 0, \end{aligned} \tag{A.13}$$

$$\begin{aligned} &\eta^6(-\partial V_2/\partial y - \partial V_1/\partial x) + \eta^4 \partial V_3/\partial x + \eta^5 \partial V_3/\partial y \\ &+\eta^3(\partial B_1/\partial x + \partial B_2/\partial y) - \eta^1 \partial B_3/\partial x - \eta^2 \partial B_3/\partial y \\ &+\xi_1^4 V_3 + \xi_2^5 V_3 + \xi_1^3 B_1 + \xi_2^3 B_2 - \xi_1^6 V_1 - \xi_2^6 V_2 - \xi_1^1 B_3 - \xi_2^2 B_3 = 0, \end{aligned} \tag{A.14}$$

$$\eta^4 \partial\Psi/\partial x + \eta^5 \partial\Psi/\partial y + \eta^6 \partial\Psi/\partial z + \xi_1^7 B_1 + \xi_2^7 B_2 + \xi_3^7 B_3 = 0, \tag{A.15}$$

$$\eta^1 \partial\Psi/\partial x + \eta^2 \partial\Psi/\partial y + \eta^3 \partial\Psi/\partial z + \xi_1^7 V_1 + \xi_2^7 V_2 + \xi_3^7 V_3 = 0. \tag{A.16}$$

According to (2.10), we need to solve the above ten determining equations under the condition that the original equations are also satisfied. For this purpose, we express ten derivatives

$$\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}, \frac{\partial B_3}{\partial z}, \frac{\partial V_1}{\partial x}, \frac{\partial V_2}{\partial x}, \frac{\partial V_3}{\partial x}, \frac{\partial V_3}{\partial z}, \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}.$$

from the system (A.1)-(A.3) and substitute them, together with explicitly written prolonged vector field coordinates (2.7), into (A.7)-(A.16).

To solve the resulting system, and obtain the tangent vector field coordinates (A.6), one should use the fact that the latter do not depend on derivatives u_i^k. Setting in all ten determining equations the coefficients at different derivatives to zero, we get 141 dependent partial differential equations on 8 unknown functions $\eta^1, .., \eta^8$.

Using the `Rif` package in Waterloo Maple software to reduce this system, we obtain the following equations:

$$\begin{aligned}
&\eta^1 = \eta^1(V_1, B_1, \Psi),\ \eta^2 = \eta^2(V_2, B_2, \Psi),\ \eta^3 = \eta^3(V_3, B_3, \Psi),\\
&\eta^4 = \eta^4(V_1, B_1, \Psi),\ \eta^5 = \eta^5(V_2, B_2, \Psi),\ \eta^6 = \eta^6(V_3, B_3, \Psi),\\
&\eta^7 = \eta^7(\Psi).
\end{aligned} \tag{A.17}$$

When these equations are substituted into the original system, it reduces to as few as 21 independent equations, which are integrated by hand to give the infinitesimal operator

$$\begin{aligned}
X = &\sum_{k=1}^{3} \left(M(\Psi)\frac{B_k}{\mu\rho} + (C_1 - N(\Psi))V_k \right) \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} (M(\Psi)V_k + C_1 B_k) \frac{\partial}{\partial B_k}\\
&+ \left(-\frac{1}{\mu} (\mathbf{V} \cdot \mathbf{B})\, M(\Psi) + 2C_1 P + C_2 \right) \frac{\partial}{\partial P} + 2\rho N(\Psi) \frac{\partial}{\partial \rho}.
\end{aligned} \tag{A.18}$$

Here $M(\Psi) = M(\Psi(\mathbf{r}))$, $N(\Psi) = N(\Psi(\mathbf{r}))$ are arbitrary functions constant on magnetic field lines and streamlines, and C_1, C_2 are free constants.

The operator (A.18) is evidently a general linear combination of infinitesimal operators (2.11)-(2.14).

We now verify that they form a Lie algebra basis. Indeed, their commutators are

$$\left[X^{(1)}, X^{(2)}\right] = \left[X^{(1)}, X^{(4)}\right] = \left[X^{(2)}, X^{(3)}\right] = \left[X^{(3)}, X^{(4)}\right] = 0,$$

$$\left[X^{(2)}, X^{(4)}\right] = -2X^{(4)},$$

$$\left[X^{(1)}, X^{(3)}\right] = Q(\Psi)X^{(1)},$$

where

$$Q(\Psi) = N(r) - 2\rho(\Psi)\frac{\partial M(\Psi)}{\partial \Psi}/M(\Psi)\frac{\partial \rho(\Psi)}{\partial \Psi}.$$

Thus the part (i) of the theorem is proven.

(ii) Now we show that the operators $X^{(5)}, X^{(6)}$ (2.15) - (2.16) are admissible for compressible MHD equilibria with the ideal gas equation of state and entropy constant along the streamlines. The complete system of equations under consideration in this case is

$$\rho\mathbf{V} \times (\text{curl } \mathbf{V}) - \frac{1}{\mu}\ \mathbf{B} \times (\text{curl } \mathbf{B}) - \text{grad } P - \rho \text{ grad}\frac{\mathbf{V}^2}{2} = 0, \tag{A.19}$$

$$\text{div } \mathbf{B} = 0,\ \text{div } \rho\mathbf{V} = 0,\ \text{curl } (\mathbf{V} \times \mathbf{B}) = 0, \tag{A.20}$$

$$\text{grad } \Psi(\mathbf{r}) \cdot \mathbf{B} = 0, \quad \text{grad } \Psi(\mathbf{r}) \cdot \mathbf{V} = 0, \tag{A.21}$$

$$P = \rho^{\gamma} \exp(S/c_v),\ \text{grad } S \cdot \mathbf{V} = 0. \tag{A.22}$$

Here $\Psi(\mathbf{r})$ is an arbitrary function constant on magnetic field lines and streamlines. It is needed because the function $N(\mathbf{r})$ with the same properties enters into $X^{(2)}$.

The system (A.19)-(A.22) consists of $l = 12$ equations. It has $n = 3$ independent and $m = 10$ dependent variables:

$$\mathbf{x} = (x, y, z), \quad \mathbf{u} = (V_1, V_2, V_3, B_1, B_2, B_3, \Psi, P, \rho, S). \tag{A.23}$$

In a manner parallel to that in (i), we look for a subgroup of Lie point transformations of the type (A.5). Applying the operator $\mathbf{h}$ (2.5) to every equation of the system under consideration, we get the following system of equations:

$$\xi_2^4 B_2 + \xi_3^4 B_3 - \xi_1^8 + \eta^5(-\partial B_2/\partial x + \partial B_1/\partial y) + \eta^6(\partial B_1/\partial z - \partial B_3/\partial x)-$$
$$\xi_1^5 B_2 - \xi_1^6 B_3 - \eta^2\rho(\Psi)\partial V_1/\partial y - \eta^3\rho(\Psi)\partial V_1/\partial z - \xi_1^1\rho(\Psi)V_1 - \xi_2^1\rho(\Psi)V_2 - \xi_3^1\rho(\Psi)V_3$$
$$-\eta^7(V_1\partial V_1/\partial x + V_2\partial V_1/\partial y + V_3\partial V_1/\partial z)d\rho(\Psi)/d\Psi - \eta^1\rho(\Psi)\partial V_1/\partial x = 0, \tag{A.24}$$

$$\xi_1^5 B_1 + \xi_3^5 B_3 - \xi_2^8 + \eta^4(\partial B_2/\partial x - \partial B_1/\partial y) + \eta^6(-\partial B_3/\partial y + \partial B_2/\partial z)-$$
$$\eta^1\rho(\Psi)\partial V_2/\partial x - \eta^3\rho(\Psi)\partial V_2/\partial z - \eta^7(V_1\partial V_2/\partial x + V_2\partial V_2/\partial y + V_3\partial V_2/\partial z)d\rho(\Psi)/d\Psi$$
$$-\xi_1^2\rho(\Psi)V_1 - \xi_2^2\rho(\Psi)V_2 - \xi_3^2\rho(\Psi)V_3 - \xi_2^4 B_1 - \xi_2^6 B_3 - \eta^2\rho(\Psi)\partial V_2/\partial y = 0, \tag{A.25}$$

$$\xi_1^6 B_1 + \xi_2^6 B_2 - \xi_3^8 - \eta^1\rho(\Psi)\partial V_3/\partial x - \eta^2\rho(\Psi)\partial V_3/\partial y - \eta^3\rho(\Psi)\partial V_3/\partial z - \xi_1^3\rho(\Psi)V_1$$
$$-\eta^7(V_1\partial V_3/\partial x + V_2\partial V_3/\partial y + V_3\partial V_3/\partial z)d\rho(\Psi)/d\Psi - \xi_2^3\rho(\Psi)V_2 - \xi_3^3\rho(\Psi)V_3$$
$$+\eta^4(-\partial B_1/\partial z + \partial B_3/\partial x) + \eta^5(\partial B_3/\partial y - \partial B_2/\partial z) - \xi_3^4 B_1 - \xi_3^5 B_2 = 0, \tag{A.26}$$

$$\xi_1^4 + \xi_2^5 + \xi_3^6 = 0, \tag{A.27}$$

$$\rho(\xi_1^1 + \xi_2^2 + \xi_3^3) + V_1\xi_1^9 + V_2\xi_2^9 + V_3\xi_3^9 = 0, \tag{A.28}$$

$$\begin{aligned}&\eta^4(-\partial V_2/\partial y - \partial V_3/\partial z) + \eta^5\partial V_1/\partial y + \eta^6\partial V_1/\partial z\\ &+\eta^1(\partial B_2/\partial y + \partial B_3/\partial z) - \eta^2\partial B_1/\partial y - \eta^3\partial B_1/\partial z\\ &+\xi_2^5 V_1 + \xi_3^6 V_1 + \xi_2^1 B_2 + \xi_3^1 B_3 - \xi_2^4 V_2 - \xi_2^2 B_1 - \xi_3^3 B_1 - \xi_3^4 V_3 = 0,\end{aligned} \tag{A.29}$$

$$\begin{aligned}&\eta^5(-\partial V_3/\partial z - \partial V_1/\partial x) + \eta^6\partial V_2/\partial z + \eta^4\partial V_2/\partial x\\ &+\eta^2(\partial B_3/\partial z + \partial B_1/\partial x) - \eta^1\partial B_2/\partial x - \eta^3\partial B_2/\partial z\\ &+\xi_1^4 V_2 + \xi_3^6 V_2 + \xi_1^2 B_1 + \xi_3^2 B_3 - \xi_3^5 V_3 - \xi_1^1 B_2 - \xi_3^3 B_2 - \xi_1^5 V_1 = 0,\end{aligned} \tag{A.30}$$

$$\begin{aligned}
&\eta^6(-\partial V_2/\partial y - \partial V_1/\partial x) + \eta^4 \partial V_3/\partial x + \eta^5 \partial V_3/\partial y \\
&+\eta^3(\partial B_1/\partial x + \partial B_2/\partial y) - \eta^1 \partial B_3/\partial x - \eta^2 \partial B_3/\partial y \qquad (A.31)\\
&+\xi_1^4 V_3 + \xi_2^5 V_3 + \xi_1^3 B_1 + \xi_2^3 B_2 - \xi_1^6 V_1 - \xi_2^6 V_2 - \xi_1^1 B_3 - \xi_2^2 B_3 = 0,
\end{aligned}$$

$$\eta^4 \partial \Psi/\partial x + \eta^5 \partial \Psi/\partial y + \eta^6 \partial \Psi/\partial z + \xi_1^7 B_1 + \xi_2^7 B_2 + \xi_3^7 B_3 = 0, \qquad (A.32)$$

$$\eta^1 \partial \Psi/\partial x + \eta^2 \partial \Psi/\partial y + \eta^3 \partial \Psi/\partial z + \xi_1^7 V_1 + \xi_2^7 V_2 + \xi_3^7 V_3 = 0, \qquad (A.33)$$

$$\eta^8 - \left(\gamma \rho^{\gamma-1} \eta^9 + \rho^\gamma \eta^{10}/c_v\right) \exp(S/c_v) = 0, \qquad (A.34)$$

$$\eta^1 \partial S/\partial x + \eta^2 \partial S/\partial y + \eta^3 \partial S/\partial z + \xi_1^{10} V_1 + \xi_2^{10} V_2 + \xi_3^{10} V_3 = 0. \qquad (A.35)$$

This system becomes the system of determining equations after using the fact that the original equations (A.19)-(A.22) are satisfied, and upon the explicit substitution of prolonged vector field coordinates (2.7).

The unknown quantities to be found are the tangent vector field coordinates

$$\eta^k(\mathbf{u}) = \frac{\partial g^k(\mathbf{u}, a)}{\partial a}\Big|_{a=0}, \quad k = 1, \dots, 10. \qquad (A.36)$$

To ensure that the original equations (A.19)-(A.22) are satisfied, we use them to express the quantities

$$\frac{\partial \Psi}{\partial x}, \ \frac{\partial \Psi}{\partial y}, \ \frac{\partial B_3}{\partial x}, \ \frac{\partial B_3}{\partial z}, \ \frac{\partial V_1}{\partial x}, \ \frac{\partial V_2}{\partial x}, \ \frac{\partial V_3}{\partial x}, \ \frac{\partial V_3}{\partial z}, \ \frac{\partial S}{\partial x}, \ \frac{\partial S}{\partial y}, \ \frac{\partial S}{\partial z}, \ P,$$

and substitute them into (A.24)-(A.35).

Maple software shows that after setting the coefficients at different derivatives to zero in all twelve determining equations, one gets 187 dependent partial differential equations.

To be able to perform computations in reasonable time, we make a simplifying assumption, supposing that the tangent vector field coordinates (A.36) depend not on all variables $\mathbf{u}$, but only on some of them, as follows:

$$\begin{aligned}
&\eta^1 = \eta^1(V_1, \Psi),\ \eta^2 = \eta^2(V_2, \Psi),\ \eta^3 = \eta^3(V_3, \Psi),\\
&\eta^4 = \eta^4(B_1),\ \eta^5 = \eta^5(B_2),\ \eta^6 = \eta^6(B_3),\\
&\eta^7 = 0,\ \eta^8 = \eta^8(P),\ \eta^9 = \eta^9(\rho, \Psi),\ \eta^{10} = \eta^{10}(\Psi).
\end{aligned} \tag{A.37}$$

We remark that other choices of simplifying assumptions, when the functions η^k depend no more than two variables, did not give more general results than the choice above.

Under the assumptions (A.37), the 187 equations mentioned above can be reduced (using Waterloo Maple with `Rif` package) to only ten independent equations, from which only two are partial differential equations, and the other eight are algebraic:

$$\begin{aligned}
&\eta^1 = \frac{V_1(\eta^{10}\rho + (\gamma-1)c_v\eta^9)}{2c_v\rho},\quad \eta^2 = \frac{V_2(\eta^{10}\rho + (\gamma-1)c_v\eta^9)}{2c_v\rho},\\
&\eta^3 = \frac{V_3(\eta^{10}\rho + (\gamma-1)c_v\eta^9)}{2c_v\rho},\quad \eta^4 = \frac{B_1(\eta^{10}\rho + \gamma c_v\eta^9)}{2c_v\rho},\\
&\eta^5 = \frac{B_2(\eta^{10}\rho + \gamma c_v\eta^9)}{2c_v\rho},\quad \eta^6 = \frac{B_3(\eta^{10}\rho + \gamma c_v\eta^9)}{2c_v\rho},\quad \eta^7 = 0,\\
&\eta^8 = \frac{P(\eta^{10}\rho + \gamma c_v\eta^9)}{c_v\rho},\quad \frac{\partial\eta^9}{\partial\Psi} = -\frac{\partial\eta^{10}}{\partial\Psi}\frac{\rho}{\gamma c_v},\quad \frac{\partial\eta^9}{\partial\rho} = \frac{\eta^9}{\rho}.
\end{aligned} \tag{A.38}$$

The solution of the system (A.38) directly yields the infinitesimal operators (2.15), (2.16). This proves the part (ii) and so completes the proof of the theorem. □

B Alternative proof of Theorem 2.1

(i) First, we prove that the operator $X^{(1)}$ is admissible for the system (1.16)-(1.17) in the case of incompressible plasma.

For the case $C = 1$, we can write $b(\mathbf{r})$, $c(\mathbf{r})$ in the Bogoyavlenskij symmetries

(1.27) as

$$b(\mathbf{r}) = \eta \cosh(\beta(\mathbf{r})), \;\; c(\mathbf{r}) = \eta \sinh(\beta(\mathbf{r})). \tag{B.39}$$

Then for $\eta = 1, a(\mathbf{r}) = 1$ the symmetries (1.27) become

$$\begin{aligned}
&\mathbf{B}_1 = \cosh(\beta(\mathbf{r}))\mathbf{B} + \sinh(\beta(\mathbf{r}))\sqrt{\mu\rho}\mathbf{V},\\
&\mathbf{V}_1 = \frac{\sinh(\beta(\mathbf{r}))}{\sqrt{\mu\rho}}\mathbf{B} + \cosh(\beta(\mathbf{r}))\mathbf{V},\\
&\rho_1 = \rho, \;\; P_1 = P + (\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu);
\end{aligned} \tag{B.40}$$

these transformations have additive Lie group structure [2]. Writing $\beta(\mathbf{r}) = \tau M(\mathbf{r})/\sqrt{\mu\rho}$, and treating τ as a group parameter, we find according to (2.6):

$$\begin{aligned}
&\xi^i(\mathbf{V}, \mathbf{B}, p, \rho) = 0, \quad i = 1, 2, 3;\\
&(\eta^1, \eta^2, \eta^3) = \frac{\partial \mathbf{V}_1}{\partial \tau}|_{\tau=0} = \mathbf{B}\frac{M(\mathbf{r})}{\mu\rho};\\
&(\eta^4, \eta^5, \eta^6) = \frac{\partial \mathbf{B}_1}{\partial \tau}|_{\tau=0} = \mathbf{V}M(\mathbf{r});\\
&\eta^7 = 0, \quad \eta^8 = \frac{\partial P_1}{\partial \tau}|_{\tau=0} = -\frac{M(\mathbf{r})}{\mu}(\mathbf{V}\cdot\mathbf{B}).
\end{aligned}$$

This set of tangent vector field coordinates corresponds exactly to the infinitesimal operator (2.11).

To get the infinitesimal operators (2.12) and (2.13) from Bogoyavlenskij symmetries (1.27), we take $b(\mathbf{r}) = const = \exp(\tau\delta)$, $c(\mathbf{r}) = 0$, $a(\mathbf{r}) = \exp(N(\mathbf{r})\tau)$, where $N(\mathbf{r})$ is constant on magnetic field lines and streamlines and $\delta = const$. Then $C = \exp(2\tau\delta)$, and the symmetries become

$$\begin{aligned}
&\mathbf{B}_1 = \exp(\tau\delta)\mathbf{B},\\
&\mathbf{V}_1 = \exp(\tau(\delta - N(\mathbf{r}))\mathbf{V},\\
&\rho_1 = \exp(2\tau N(\mathbf{r}))\rho, \quad P_1 = \exp(2\tau\delta)P,
\end{aligned}$$

Treating τ as a group parameter, we find the corresponding infinitesimal operator

$$X = (\delta - N(\mathbf{r}))\sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \delta \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2\delta P \frac{\partial}{\partial P} + 2N(\mathbf{r})\rho \frac{\partial}{\partial \rho},$$

which is a superposition of $X^{(2)}$ (2.12) and $X^{(3)}$ (2.13).

Finally, the operator (2.14) represents the shifts $P_1 = P + C_0,\ C_0 = const$, and thus is evidently admissible by the MHD equilibrium equations, which depend only on the derivatives of pressure.

The commutator relations are given in Appendix A.

This proves part (i).

(ii) To prove that the operators (2.15), (2.16) are admitted by the compressible isotropic MHD equilibrium equations (1.16),(1.17),(1.19) for any density function, we take the transformations (1.29)-(1.30) with the following choice of parameters:

$$a(\mathbf{r}) = \exp(N(\mathbf{r})\tau),\ b = \exp(\delta\tau)$$

where $N(\mathbf{r})$ is a function constant on both magnetic field lines and streamlines. Then the formulas (1.29)-(1.30) become a Lie group with respect to addition in parameter τ:

$$\rho_1 = \exp(2N(\mathbf{r})\tau)\rho, \quad \mathbf{B}_1 = \exp(\delta\tau)\mathbf{B}, \quad \mathbf{V}_1 = \exp((\delta - N(\mathbf{r}))\tau)\mathbf{V}, \tag{B.41}$$

$$P_1 = \exp(2\delta\tau)P, \quad S_1 = S + 2c_v\left(\delta - \gamma N(\mathbf{r})\right)\tau.$$

The tangent vector field coordinates corresponding to this Lie group of transformations are found according to (2.6):

$$\xi^1 = \xi^2 = \xi^3 = 0,$$

$$\eta^i = V_i(\delta - N(\mathbf{r})), \quad \eta^{i+3} = B_i\delta, \quad i = 1, 2, 3,$$

$$\eta^7 = 2N(\mathbf{r})\rho, \quad \eta^8 = 2\delta P, \quad \eta^9 = 2c_v(\delta - \gamma N(\mathbf{r})).$$

These tangent vector field coordinates give rise to the infinitesimal operators (2.15) (set $N(\mathbf{r}) = 0$) and (2.16) (set $\delta = 0$). This completes the proof of part (ii) of the theorem. □

C Proof of Theorem 2.2

The following lemma is necessary in the proof of Theorem 2.2.

Lemma C.1 *The incompressible isotropic MHD equilibrium system of equations (1.16)-(1.18) admits the discrete symmetries*

$$\mathbf{B}_1 = \pm\mathbf{B}, \quad \mathbf{V}_1 = \pm\mathbf{V}, \quad P_1 = P, \quad \rho_1 = \rho \tag{C.42}$$

and

$$\mathbf{B}_1 = \mathbf{V}\sqrt{\mu\rho}, \quad \mathbf{V}_1 = \mathbf{B}/\sqrt{\mu\rho}, \quad P_1 = -P - (\mathbf{B}^2 + \mathbf{V}^2\mu\rho)/(2\mu), \quad \rho_1 = \rho. \tag{C.43}$$

Compressible isotropic MHD equilibrium equations (1.16)-(1.17) with ideal gas state equation (1.19), for arbitrary density, admit the discrete symmetries (C.42).

Proof of the Lemma.

The proof is based on the complexification of parameters of known continuous point symmetries of the systems under consideration.

First we consider the incompressible MHD equilibrium system (1.16)-(1.18). By Theorem 2.2, it admits the infinitesimal operators $X^{(1)}, X^{(2)}, X^{(3)}$ (2.11)-(2.13) and therefore the continuous Lie point transformations (2.18)-(2.20).

If we take an equilibrium configuration $\{\mathbf{V}, \mathbf{B}, P, \rho\}$ and apply the transformation (2.19) with $\tau = \tau_1$ and then (2.20) with $\tau = \tau_2$, we get a new solution

$$\rho_1 = \exp(2N(\mathbf{r})\tau_2)\rho, \quad \mathbf{B}_1 = \exp(\tau_1)\mathbf{B}, \quad \mathbf{V}_1 = \exp(\tau_1 - N(\mathbf{r})\tau_2)\mathbf{V}, \quad P_1 = P\ .$$

Now, using the combinations $\{\tau_1 = \pi i,\ N(\mathbf{r})\tau_2 = 0\}$, $\{\tau_1 = \pi i,\ N(\mathbf{r})\tau_2 = \pi i\}$, $\{\tau_1 = 0,\ N(\mathbf{r})\tau_2 = -\pi i\}$, we get all transformations (C.42), with $\rho_1 = \rho$.

To prove that the discrete symmetry (C.43) is admissible, we take an equilibrium configuration $\{\mathbf{V}, \mathbf{B}, P, \rho\}$ and apply first the transformation (2.18) with $M(\mathbf{r})\tau = \sqrt{\mu\rho}\pi i/2$, and then the transformation (2.19) with $\tau = -\pi i/2$. The final result is real and coincides with the required formula (C.43). The density ρ is not transformed.

The existence of the refection symmetry (C.42) for the compressible case is proven the same way as for incompressible, using the operators (2.15)-(2.16) and the corresponding transformations (2.22)-(2.23). The lemma is proven. □

Proof of Theorem 2.2.

(i). In the case $C > 0$, we denote $C = q^2,\ \eta = \pm 1,\ \sigma = \pm 1,\ \lambda = \pm 1$, and write $a(\mathbf{r}),\ b(\mathbf{r}),\ c(\mathbf{r})$ in Bogoyavlenskij symmetries (1.27) as

$$a(\mathbf{r}) = \eta \exp(N(\mathbf{r})),\ b(\mathbf{r}) = \sigma q \cosh(\beta(\mathbf{r})),\ c(\mathbf{r}) = \lambda q \sinh(\beta(\mathbf{r})).$$

Therefore the transformations (1.27) become

$$\begin{aligned} &\mathbf{B}_1 = q\sigma \cosh(\beta(\mathbf{r}))\mathbf{B} + q\lambda \sinh(\beta(\mathbf{r}))\sqrt{\mu\rho}\mathbf{V}, \\ &\mathbf{V}_1 = q\lambda \frac{\sinh(\beta(\mathbf{r}))}{\eta \exp(N(\mathbf{r}))\sqrt{\mu\rho}}\mathbf{B} + q\sigma \frac{\cosh(\beta(\mathbf{r}))}{\eta \exp(N(\mathbf{r}))}\mathbf{V}, \\ &\rho_1 = \exp(2N(\mathbf{r}))\rho,\ P_1 = q^2 P + (q^2\mathbf{B}^2 - \mathbf{B}_1^2)/(2\mu), \end{aligned} \qquad \text{(C.44)}$$

From the original solution $\{\mathbf{V}, \mathbf{B}, P, \rho\}$, using appropriately the reflections (C.42) and the mixing transformations (2.18), we can obtain a solution

$$\tilde{\mathbf{B}}_1 = \sigma \cosh(\beta(\mathbf{r}))\mathbf{B} + \lambda \sinh(\beta(\mathbf{r}))\sqrt{\mu\rho}\mathbf{V},$$

$$\tilde{\mathbf{V}}_1 = \lambda\frac{\sinh(\beta(\mathbf{r}))}{\eta\sqrt{\mu\rho}}\mathbf{B} + \sigma\frac{\cosh(\beta(\mathbf{r}))}{\eta}\mathbf{V}, \tag{C.45}$$
$$\tilde{\rho}_1 = \rho,\ \tilde{P}_1 = P + (\mathbf{B}^2 - \tilde{\mathbf{B}}_1^2)/(2\mu).$$

This intermediate solution can be scaled by applying (2.19) with $\tau = \ln q$ and (2.20) with $\tau = 1$ to obtain (C.44).

In the case $C < 0$, we denote $C = -q^2,\ \eta = \pm 1,\ \sigma = \pm 1,\ \lambda = \pm 1$. Then

$$a(\mathbf{r}) = \eta\exp(N(\mathbf{r})),\ b(\mathbf{r}) = \sigma q\sinh(\beta(\mathbf{r})),\ c(\mathbf{r}) = \lambda q\cosh(\beta(\mathbf{r})).$$

Therefore the transformations (1.27) can be written as

$$\mathbf{B}_1 = q\sigma\sinh(\beta(\mathbf{r}))\mathbf{B} + q\lambda\cosh(\beta(\mathbf{r}))\sqrt{\mu\rho}\mathbf{V},$$
$$\mathbf{V}_1 = q\lambda\frac{\cosh(\beta(\mathbf{r}))}{\eta\exp(N(\mathbf{r}))\sqrt{\mu\rho}}\mathbf{B} + q\sigma\frac{\sinh(\beta(\mathbf{r}))}{\eta\exp(N(\mathbf{r}))}\mathbf{V}, \tag{C.46}$$
$$\rho_1 = \exp(2N(\mathbf{r}))\rho,\ P_1 = -q^2P - (q^2\mathbf{B}^2 + \mathbf{B}_1^2)/(2\mu)$$

We will show that this transform can be found from an original solution by combining (2.18)-(2.20), (C.42), (C.43).

First, to the original solution $\{\mathbf{V}, \mathbf{B}, P, \rho\}$ we apply (C.43) to obtain

$$\mathbf{B}_2 = \mathbf{V}\sqrt{\mu\rho},\quad \mathbf{V}_2 = \mathbf{B}/\sqrt{\mu\rho},\quad P_2 = -P - (\mathbf{B}^2 + \mathbf{V}^2\mu\rho)/(2\mu),\quad \rho_2 = \rho. \tag{C.47}$$

In the way described above in this proof and using (2.18)-(2.20) and (C.42), the solution $\{\mathbf{V}_2, \mathbf{B}_2, P_2, \rho_2\}$ can be transformed to another solution

$$\mathbf{B}_3 = q\lambda\cosh(\beta(\mathbf{r}))\mathbf{B}_2 + q\sigma\sinh(\beta(\mathbf{r}))\sqrt{\mu\rho_2}\mathbf{V}_2,$$
$$\mathbf{V}_3 = q\sigma\frac{\sinh(\beta(\mathbf{r}))}{\eta\exp(N(\mathbf{r}))\sqrt{\mu\rho_2}}\mathbf{B}_2 + q\lambda\frac{\cosh(\beta(\mathbf{r}))}{\eta\exp(N(\mathbf{r}))}\mathbf{V}_2, \tag{C.48}$$
$$\rho_3 = \exp(2N(\mathbf{r}))\rho_2,\ P_3 = q^2P_2 + (q^2\mathbf{B}_2^2 - \mathbf{B}_3^2)/(2\mu),$$

After the substitution of $\{\mathbf{V}_2, \mathbf{B}_2, P_2, \rho_2\}$, the new solution (C.48) coincides with the desired form (C.46).

The fact that the operators (2.11)-(2.13), and thus the corresponding transformations (2.18)-(2.20), can be obtained from Bogoyavlenskij symmetries (1.27) is proven in Appendix B This proves part (i).

(ii). Consider the Bogoyavlenskij symmetries (1.29)-(1.30) for compressible MHD equilibria with $b \geq 0,\ a(\mathbf{r}) > 0$. To an equilibrium $\{\mathbf{V}, \mathbf{B}, P, \rho, S\}$ we apply the Lie point transformations (2.22) with $\tau = \ln b$ and (2.23) with $\{\tau = 1,\ N(\mathbf{r}) = \ln a(\mathbf{r})\}$. This converts the original solution exactly into the form (1.29)-(1.30).

Suppose now that $b \leq 0, a(\mathbf{r}) > 0$. Let $\{\mathbf{V}, \mathbf{B}, P, \rho, S\}$ be an MHD equilibrium. Then by applying to it first (C.42) in the form $\mathbf{B} \to -\mathbf{B},\quad \mathbf{V} \to -\mathbf{V}$, and then the Lie symmetries (2.22) with $\tau = \ln|b|$ and (2.23) with $\{\tau = 1,\ N(\mathbf{r}) = \ln a(\mathbf{r})\}$ we obtain

$$\rho_1 = a^2(\mathbf{r})\rho, \quad \mathbf{B}_1 = b\mathbf{B}, \quad \mathbf{V}_1 = \frac{b}{a(\mathbf{r})}\mathbf{V}, \quad P_1 = b^2 P,$$

$$S_1 = S + 2c_v\left(\ln|b| - \gamma \ln|a(\mathbf{r})|\right).$$

This form coincides with the transform (1.29)-(1.30).

The cases when $a(\mathbf{r})$ can be negative are treated in the same way. In the points where $a(\mathbf{r}) < 0$, an additional reflection transformation (C.42) in the form $\mathbf{B} \to \mathbf{B},\quad \mathbf{V} \to -\mathbf{V}$ needs to be applied to the original solution.

Thus the composition of Lie symmetries (2.22) and (2.23) yields the transformation (1.29)-(1.30) of compressible MHD equilibria.

Conversely, the operators (2.15)-(2.16), and so the transformations (2.22) and (2.23), the are implied by Bogoyavlenskij symmetries (1.29)-(1.30), as shown in Appendix B.

This proves part (ii) and completes the proof of Theorem 2.2. □

D Potential symmetry analysis and non-linearizability of static isotropic MHD equilibrium equations

Potential symmetries of static isotropic MHD equilibria

In this subsection, the notion and basic properties of potential symmetries of differential equations are discussed, and the algorithm of potential symmetry analysis is applied to the static isotropic MHD equilibrium equations (1.20).

The classical Lie and Lie-Bäcklund symmetry methods [55] are applicable to any system of partial differential equations (with sufficiently smooth coefficients) $R\{\mathbf{x}, \mathbf{u}\}$:

$$\begin{aligned} &\mathbf{E}(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, ..., \underset{p}{\mathbf{u}}) = 0, \\ &\mathbf{E} = (E^1, \ldots, E^l),\ \mathbf{x} = (x^1, \ldots, x^n),\ \mathbf{u} = (u^1, \ldots, u^m). \end{aligned} \tag{D.49}$$

Here $\mathbf{x}$ is a vector of independent variables, $\mathbf{u}$ – a vector of dependent variables, and $\underset{k}{\mathbf{u}}$ are partial derivatives of order k.

These methods let one find one-parametric Lie groups of point transformations from a solution $(\mathbf{x}, \mathbf{u})$ to a solution $(\mathbf{x}', \mathbf{u}')$ of the system $R\{\mathbf{x}, \mathbf{u}\}$:

$$\begin{aligned} &(x')^i = x^i + a\xi^i(\mathbf{x}, \mathbf{u}) + O(a^2), \quad (i = 1, \ldots, n), \\ &(u')^j = u^j + a\eta^j(\mathbf{x}, \mathbf{u}) + O(a^2), \quad (j = 1, \ldots, m) \end{aligned} \tag{D.50}$$

for the classical Lie symmetries, and

$$\begin{aligned} &(x')^i = x^i + a\xi^i(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, ..., \underset{k}{\mathbf{u}}) + O(a^2), \quad (i = 1, \ldots, n), \\ &(u')^j = u^j + a\eta^j(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, ..., \underset{k}{\mathbf{u}}) + O(a^2), \quad (j = 1, \ldots, m), \end{aligned} \tag{D.51}$$

$(1 \leq k \leq p)$ for the Lie-Bäcklund symmetries.

The above symmetries are often referred to as *point*, or *local* symmetries, because in every point they depend on the value of the solution in the same point. However, for some systems, *non-local symmetries* can be built – solution-to-solution transformations that depend on the values of functions in more than one point.

If some equations of the system $R\{\mathbf{x},\mathbf{u}\}$ can be put into a *conserved* form,

$$\begin{aligned} &D_i f^i(\mathbf{x},\mathbf{u},\underset{1}{\mathbf{u}},...,\underset{p-1}{\mathbf{u}})=0,\\ &D_i=\frac{\partial}{\partial x^i}+\sum_{k=1}^{m}\left[u^k_i\frac{\partial}{\partial u^k}+...+u^k_{ii_1i_2...i_{p-1}}\frac{\partial}{\partial u^k_{i_1i_2...i_{p-1}}}\right], \end{aligned} \quad \text{(D.52)}$$

then every such equation can be equivalently replaced by n equations [70]:

$$f^i(\mathbf{x},\mathbf{u},\underset{1}{\mathbf{u}},...,\underset{p-1}{\mathbf{u}})=\sum_{i<j}(-1)^j\frac{\partial}{\partial x^j}\Psi^{ij}+\sum_{i<j}(-1)^{i-1}\frac{\partial}{\partial x^j}\Psi^{ji},\quad i,j=1,...,n. \quad \text{(D.53)}$$

With every replacement of a conserved-form equation (D.52) with a set (D.53), $\frac{1}{2}n(n-1)$ additional dependent variables Ψ^{ij} is introduced. The antisymmetric tensor Ψ^{ij} plays a role of the vector potential of solenoidal vector fields, the conserved form (D.52) being the analog of the solenoidality condition.

Definition. The *auxiliary system* $S\{\mathbf{x},\mathbf{u},\mathbf{v}\}$ of a PDE system $R\{\mathbf{x},\mathbf{u}\}$ is a system of equations obtained by replacing all equations of $R\{\mathbf{x},\mathbf{u}\}$ that have the conserved form (D.52) by n equations (D.53).

If additional constraints on Ψ^{ij} (i.e. the *gauge choice*) are not imposed upon such substitution, then the system becomes underdetermined [70]. Indeed, if C is the number of equations of the original system (D.49) that have a conserved form, then from these equations one gets $N=nC$ equations (D.53), so the number of equations is raised by $(n-1)C$. Therefore the total number of introduced independent functions (potentials) must also be $(n-1)C$, i.e. appropriate number of constraints must be chosen so that every conserved-form equation gives rise to $n-1$ independently defined *potentials* $\{v^h\}$.

The total set of independent potentials in the auxiliary system is $\mathbf{v}=(v^1,...,v^N)$, where $N=(n-1)C$.

Now assume that the auxiliary system $S\{\mathbf{x},\mathbf{u},\mathbf{v}\}$ admits a one-parameter Lie group of point symmetries [70]:

$$\begin{aligned} (x')^i &= x^i + a\xi^i(\mathbf{x},\mathbf{u},\mathbf{v}) + O(a^2), \quad (i=1,\ldots,n),\\ (u')^j &= u^j + a\eta^j(\mathbf{x},\mathbf{u},\mathbf{v}) + O(a^2), \quad (j=1,\ldots,m),\\ (v')^k &= v^k + a\zeta^k(\mathbf{x},\mathbf{u},\mathbf{v}) + O(a^2), \quad (k=1,\ldots,N). \end{aligned} \tag{D.54}$$

Definition. The point symmetry (D.54) defines a *potential symmetry* of the system (D.49) if and only if some of the infinitesimals $(\xi^i(\mathbf{x},\mathbf{u},\mathbf{v}),\eta^j(\mathbf{x},\mathbf{u},\mathbf{v}))$ explicitly depend on the potentials $\mathbf{v}$ [70].

Theorem D.1 (Bluman, Kumei) *[70, 71]. A potential symmetry of $R\{\mathbf{x},\mathbf{u}\}$ ia a nonlocal symmetry of $R\{\mathbf{x},\mathbf{u}\}$.*

Examples of non-trivial potential symmetries are contained in the book [70] and other works by Bluman *et al.* The examples include potential symmetries of nonlinear heat and wave equations, Maxwell equations and other systems.

Remark. The potential symmetry method is a nonlocal extension of the classical Lie method. It is applicable to ordinary and partial differential equations and to ODE/PDE systems. The set of symmetries of the auxiliary system *includes* the usual Lie point symmetries of the original system $R\{\mathbf{x},\mathbf{u}\}$.

The potential symmetries can be used to construct exact invariant solutions and to study analytical properties of equations, e.g. the possibility of their linearization (see the next subsection).

Here we perform the potential symmetry analysis of the classical system of static isotropic plasma equilibrium equations (1.20), that, as shown below, can be completely written in the conserved form, prefectly suitable for potential symmetry analysis.

Below we assume the notation $\mathbf{B} = (B_1, B_2, B_3)$ for the magnetic field in the cartesian coordinates (x, y, z).

Statement. *The system of static isotropic plasma equilibrium equations (1.20) is equivalent to the following system of four equations that have conserved form:*

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{1}{2}\left(B_1^2 - B_2^2 - B_3^2\right) - P\right) + \frac{\partial}{\partial y}(B_1 B_2) + \frac{\partial}{\partial z}(B_1 B_3) = 0;\\
&\frac{\partial}{\partial x}(B_1 B_2) + \frac{\partial}{\partial y}\left(\frac{1}{2}\left(B_2^2 - B_3^2 - B_1^2\right) - P\right) + \frac{\partial}{\partial z}(B_2 B_3) = 0;\\
&\frac{\partial}{\partial x}(B_1 B_3) + \frac{\partial}{\partial y}(B_2 B_3) + \frac{\partial}{\partial z}\left(\frac{1}{2}\left(B_3^2 - B_1^2 - B_2^2\right) - P\right) = 0,\\
&\frac{\partial}{\partial x}B_1 + \frac{\partial}{\partial y}B_2 + \frac{\partial}{\partial z}B_3 = 0.
\end{aligned} \tag{D.55}$$

The proof is simple; it is done directly by considering the four equations of the system (1.20) and using usual vector calculus relation.

This conserved-form representation of the plasma equilibrium system allows one to perform the complete potential symmetry analysis, following the algorithm described above. The auxiliary system $S\{\mathbf{x}, \mathbf{u}, \mathbf{v}\}$ includes $N = (n-1)S = (3-1) \times 4 = 8$ independent potentials and has the form

$$\begin{aligned}
&\left(\frac{1}{2}\left(B_1^2 - B_2^2 - B_3^2\right) - P\right) = v_y^3 - v_z^2, \quad B_1 B_2 = v_z^1 - v_x^3, \quad B_1 B_3 = v_x^2 - v_y^1,\\
&B_1 B_2 = v_y^6 - v_z^5, \quad \frac{1}{2}\left(B_2^2 - B_3^2 - B_1^2\right) - P = v_z^4 - v_x^6, \quad B_2 B_3 = v_x^5 - v_y^4,\\
&B_1 B_3 = v_y^9 - v_z^8, \quad B_2 B_3 = v_z^7 - v_x^9, \quad \frac{1}{2}\left(B_3^2 - B_1^2 - B_2^2\right) - P = v_x^8 - v_y^7,\\
&B_1 = v_y^{12} - v_z^{11}, \quad B_2 = v_z^{10} - v_x^{12}, \quad B_3 = v_x^{11} - v_y^{10}.
\end{aligned} \tag{D.56}$$

Here $\mathbf{v} = (v^1, ..., v^{12})$ are the potentials, and by lower subscripts x, y, z we denoted the corresponding partial derivatives, for the shortness of notation. Only 8 potentials are independent, because the following Lorentz constraints are added:

$$\begin{aligned}
&v_x^1 + v_y^2 + v_z^3 = 0, \quad v_x^4 + v_y^5 + v_z^6 = 0,\\
&v_x^7 + v_y^8 + v_z^9 = 0, \quad v_x^{10} + v_y^{11} + v_z^{12} = 0.
\end{aligned} \tag{D.57}$$

The result can be formulated as a theorem:

Theorem D.2 *The projection of the complete set of symmetries of the auxiliary system (D.56) of the static isotropic plasma equilibrium equations (1.20) onto the space of variables* $(B_1, B_2, B_3, P,\ x, y, z)$ *is independent of the potentials* v^k *and is given by the operator*

$$\begin{aligned} X^s = \; & (K_2 x - c_7 y - c_3 z + c_9)\frac{\partial}{\partial x} + (c_7 x + K_2 y - c_6 z + c_8)\frac{\partial}{\partial y} + (c_3 x + c_6 y + K_2 z + c_5)\frac{\partial}{\partial z} \\ & + \left(\frac{1}{2}K_1 B_1 - B_3 c_3 - B_2 c_7\right)\frac{\partial}{\partial B_1} + \left(-c_6 B_3 + \frac{1}{2}K_1 B_2 + c_7 B_1\right)\frac{\partial}{\partial B_2} \\ & + \left(B_2 c_6 + B_1 c_3 + \frac{1}{2}K_1 B_3\right)\frac{\partial}{\partial B_3} + (K_1 P + c_2)\frac{\partial}{\partial P}. \end{aligned} \tag{D.58}$$

Thus the static isotropic MHD equilibrium system of equations (1.20) admits no potential symmetries.

Here K_1, K_2 *and* c_i *are arbitrary constants.*

Proof. The system under consideration is (D.56)-(D.57); it consists of $l = 16$ equations and employs 3 independent and 4 dependent variables, and 12 potentials:

$$\mathbf{x} = (x, y, z), \quad \mathbf{u} = (\ B_1, B_2, B_3,\ P), \quad \mathbf{v} = (v^1, ..., v^{12}).$$

The potentials are written separately from the independent variables to underline that the current system of equations is an auxiliary system.

To this system we apply the standard procedure of Lie group analysis described in sec. 2.2 of the present chapter. The unknown quantities to be found are the tangent vector field coordinates for $\mathbf{x},\ \mathbf{u},\ \mathbf{v}$ respectively:

$$\xi^i(\mathbf{x}, \mathbf{u}, \mathbf{v}),\ \ \eta^k(\mathbf{x}, \mathbf{u}, \mathbf{v}),\ \ \zeta^p(\mathbf{x}, \mathbf{u}, \mathbf{v}), \quad i = 1, 2, 3;\ k = 1, 2, 3, 4;\ p = 1, ..., 12. \tag{D.59}$$

Applying the appropriate prolonged operator $\mathbf{h}$ (2.5) to every equation of the system (D.56)-(D.57), we get the system of 16 determining equations, which needs to be solved under the condition (2.10) that the original equations are also satisfied.

For this purpose, we express ten derivatives

$$v_z^2, v_z^1, v_y^1, v_z^5, v_z^4, v_y^4, v_z^8, v_z^7, v_y^7, v_z^{11}, v_z^{10}, v_y^{10}, v_x^1, v_x^4, v_x^7, v_x^{10},$$

in the specified order, from the system (D.56)-(D.57) and substitute them, together with explicitly written prolonged vector field coordinates (2.7), into the determining equations.

To solve the resulting system, and obtain the tangent vector field coordinates (D.59), one should use the fact that the latter do not depend on derivatives u_i^k, v_i^p. Setting in all determining equations the coefficients at such derivatives to zero, one gets 564 dependent partial differential equations on 16 unknown functions $\xi^1, ..., \xi^3$, $\eta^1, .., \eta^4$, $\zeta^1, .., \zeta^{12}$ (D.59).

Again the `Rif` package for the Waterloo Maple software is used for the reduction of this system, and one observes that the tangent vector field components corresponding to $(x, y, z), (B_1, B_2, B_3, P)$, i.e. the functions $\xi^1, ..., \xi^3$, $\eta^1, .., \eta^4$, and do not depend on the potentials:

$$\frac{\partial}{\partial v^p}\xi^i = 0, \quad \frac{\partial}{\partial v^p}\eta^k = 0, \quad i = 1,2,3; \; k = 1,2,3,4; \; p = 1, ..., 12.$$

The rest of the equations on $\xi^1, ..., \xi^3$, $\eta^1, .., \eta^4$ has the form

$$\frac{\partial^2}{\partial z^2}\xi^2 = 0, \; \frac{\partial^2}{\partial z^2}\xi^3 = 0;$$
$$\frac{\partial}{\partial B_k}\xi^i = 0, \; \frac{\partial}{\partial B_k}\eta^4 = 0, \quad i, k = 1,2,3;$$
$$\frac{\partial}{\partial q}\eta^k = 0, \quad k = 1,2,3,4; \; q = x, y, z;$$
$$\frac{\partial}{\partial P}\eta^k = 0, \quad k = 1,2,3;$$
$$\frac{\partial}{\partial B_1}\eta^1 = \frac{\eta^3 B_3 + \eta^1 B_1 + \eta^2 B_2}{{B_2}^2 + {B_3}^2 + {B_1}^2};$$
$$\frac{\partial}{\partial B_1}\eta^2 = \frac{-{B_1}^2 B_3 \frac{\partial}{\partial z}\xi^2 + {B_1}^2\eta^2 - B_3\frac{\partial}{\partial z}\xi^2 {B_2}^2 - {B_3}^3\frac{\partial}{\partial z}\xi^2 + {B_3}^2\eta^2 - B_3\eta^3 B_2 - B_1\eta^1 B_2}{B_1({B_2}^2 + {B_3}^2 + {B_1}^2)};$$
$$\frac{\partial}{\partial B_1}\eta^3 = \frac{\frac{\partial}{\partial z}\xi^2 {B_2}^3 + {B_1}^2\eta^3 + B_2{B_3}^2\frac{\partial}{\partial z}\xi^2 - B_2 B_3\eta^2 + \eta^3{B_2}^2 - B_1\eta^1 B_3 + {B_1}^2 B_2\frac{\partial}{\partial z}\xi^2}{B_1({B_2}^2 + {B_3}^2 + {B_1}^2)};$$

$$\frac{\partial}{\partial B_2}\eta^1 = \frac{B_1^2 B_3 \frac{\partial}{\partial z}\xi^2 - B_1^2\eta^2 + B_3\frac{\partial}{\partial z}\xi^2 B_2^2 + B_3^3\frac{\partial}{\partial z}\xi^2 - B_3^2\eta^2 + B_3\eta^3 B_2 + B_1\eta^1 B_2}{B_1(B_2^2 + B_3^2 + B_1^2)};$$
$$\frac{\partial}{\partial B_2}\eta^2 = \frac{\eta^3 B_3 + \eta^1 B_1 + \eta^2 B_2}{B_2^2 + B_3^2 + B_1^2}; \quad \frac{\partial}{\partial B_2}\eta^3 = -\frac{\partial}{\partial z}\xi^2;$$
$$\frac{\partial}{\partial B_3}\eta^1 = \frac{-\frac{\partial}{\partial z}\xi^2 B_2^3 - B_1^2\eta^3 - B_2 B_3^2\frac{\partial}{\partial z}\xi^2 + B_2 B_3\eta^2 - \eta^3 B_2^2 + B_1\eta^1 B_3 - B_1^2 B_2\frac{\partial}{\partial z}\xi^2}{B_1(B_2^2 + B_3^2 + B_1^2)};$$
$$\frac{\partial}{\partial B_3}\eta^2 = \frac{\partial}{\partial z}\xi^2;$$

$$\frac{\partial}{\partial B_3}\eta^3 = \frac{\eta^3 B_3 + \eta^1 B_1 + \eta^2 B_2}{B_2^2 + B_3^2 + B_1^2}; \quad \frac{\partial}{\partial P}\eta^4 = \frac{2\eta^1 B_1 + 2\eta^2 B_2 + 2\eta^3 B_3}{B_2^2 + B_3^2 + B_1^2};$$
$$\frac{\partial}{\partial x}\xi^1 = \frac{\partial}{\partial y}\xi^2 = \frac{\partial}{\partial z}\xi^3; \quad \frac{\partial}{\partial y}\xi^3 = -\frac{\partial}{\partial z}\xi^2,$$
$$\frac{\partial}{\partial x}\xi^2 = \frac{-B_1^2 B_3\frac{\partial}{\partial z}\xi^2 + B_1^2\eta^2 - B_3\frac{\partial}{\partial z}\xi^2 B_2^2 - B_3^3\frac{\partial}{\partial z}\xi^2 + B_3^2\eta^2 - B_3\eta^3 B_2 - B_1\eta^1 B_2}{B_1(B_2^2 + B_3^2 + B_1^2)};$$

$$\frac{\partial}{\partial x}\xi^3 = \frac{\frac{\partial}{\partial z}\xi^2 B_2^3 + B_1^2\eta^3 + B_2 B_3^2\frac{\partial}{\partial z}\xi^2 - B_2 B_3\eta^2 + \eta^3 B_2^2 - B_1\eta^1 B_3 + B_1^2 B_2\frac{\partial}{\partial z}\xi^2}{B_1(B_2^2 + B_3^2 + B_1^2)};$$
$$\frac{\partial}{\partial y}\xi^1 = \frac{B_1^2 B_3\frac{\partial}{\partial z}\xi^2 - B_1^2\eta^2 + B_3\frac{\partial}{\partial z}\xi^2 B_2^2 + B_3^3\frac{\partial}{\partial z}\xi^2 - B_3^2\eta^2 + B_3\eta^3 B_2 + B_1\eta^1 B_2}{B_1(B_2^2 + B_3^2 + B_1^2)};$$
$$\frac{\partial}{\partial z}\xi^1 = \frac{-\frac{\partial}{\partial z}\xi^2 B_2^3 - B_1^2\eta^3 - B_2 B_3^2\frac{\partial}{\partial z}\xi^2 + B_2 B_3\eta^2 - \eta^3 B_2^2 + B_1\eta^1 B_3 - B_1^2 B_2\frac{\partial}{\partial z}\xi^2}{B_1(B_2^2 + B_3^2 + B_1^2)};$$

and do not depend on the potentials. Hence the base system of equations (1.20) has no potential symmetries, according to the definition given in the beginning of this Appendix.

Integrating the above equations, we obtain the complete list of Lie point symmetry generators (2.8) of the base system (1.20), which exactly coincides with (D.58). The theorem is proven. □

Discussion. The above theorem shows that the point symmetries of the full auxiliary system $S\{\mathbf{x}, \mathbf{u}, \mathbf{v}\}$ (D.56) do not include symmetries for which the tangent vector field coordinates $(\xi^i(\mathbf{x}, \mathbf{u}, \mathbf{v}), \eta^j(\mathbf{x}, \mathbf{u}, \mathbf{v}))$ corresponding to $(B_1, B_2, B_3, P,\ x, y, z)$ depend explicitly on the potentials.

The symmetries (D.58) are rather trivial and could be found without the potential symmetry framework. They correspond to the orthogonal rotations $SO(3)$, shifts and uniform scalings of the space, and the scalings of the dependent variables of the type

$$\mathbf{B} \to \exp(aR)\mathbf{B}, \quad P \to \exp(2aR)P, \quad R = \text{const.}$$

(For the reconstruction see Olver [55].)

In the subsection below, the results of the Theorem D.2 are used for the study of possibility of the linearization of the system of static isotropic plasma equilibrium equations (1.20).

Non-linearizability of static isotropic MHD equilibrium equations

If all infinitesimal generators of point symmetries admitted by a system $R\{\mathbf{x}, \mathbf{u}\}$ are known, then it is possible to determine whether or not this system can be *linearized by an invertible mapping*, and construct such invertible mapping, if it exists [70, 71].

The necessary and sufficient conditions of the existence of the linearizing mapping for a PDE system are given, for example, in [71].

In particular, the necessary condition reads:

Theorem D.3 (Bluman) *[70, 71]. If there is an invertible mapping of* $R\{\mathbf{x}, \mathbf{u}\}$ *with the number of independent variables* $m \geq 2$ *into a system with independent variables* $\mathbf{z} = (z^1, ..., z^n)$ *and dependent variables* $\mathbf{w} = (w^1, ..., w^m)$*, then*

(i) The mapping is a point transformation

$$z^j = \phi^j(\mathbf{x}, \mathbf{u}), \quad w^k = \psi_k(\mathbf{x}, \mathbf{u}); \quad j = 1, ..., n, \ k = 1, ..., m.$$

(ii) The original system $R\{\mathbf{x},\mathbf{u}\}$ *must admit infinitesimal generators of the form (2.8) with*

$$\xi^i(\mathbf{x},\mathbf{u}) = \sum_{\sigma=1}^{m} \alpha^i_\sigma(\mathbf{x},\mathbf{u}) F_\sigma(\mathbf{x},\mathbf{u}), \quad i = 1,...,n,$$
$$\eta^k(\mathbf{x},\mathbf{u}) = \sum_{\sigma=1}^{m} \beta^k_\sigma(\mathbf{x},\mathbf{u}) F_\sigma(\mathbf{x},\mathbf{u}), \quad k = 1,...,m,$$

where $\alpha^i_\sigma(\mathbf{x},\mathbf{u})$, $\beta^k_\sigma(\mathbf{x},\mathbf{u})$ *are specific functions of* $(\mathbf{x},\mathbf{u})$, *and* $\mathbf{F} = (F_1,...,F_m)$ *is an arbitrary solution of some linear system* $L[\mathbf{X}]\mathbf{F} = 0$ *with* $L[\mathbf{X}]$ *a linear operator depending on independent variables*

$$\mathbf{X} = (X^1(\mathbf{x},\mathbf{u}),...,X^n(\mathbf{x},\mathbf{u})).$$

The potential symmetry method described in [70, 71] and reviewed in the previous subsection can be a useful tool in the problem of linearization of a given system $R\{\mathbf{x},\mathbf{u}\}$.

If the system itself does not satisfy the theorem, i.e. does not possess necessary point symmetries, its auxiliary system sometimes can be linearized [71]. However, the result for the static plasma equilibrium system is negative, even in the framework of the theory of potential symmetries.

Theorem D.4 *The system (1.20) of static plasma equilibrium equations can not be linearized – neither in its usual form*

$$\text{curl }\mathbf{B}\times\mathbf{B} = \mu \ \text{grad} \ P, \quad \text{div }\mathbf{B} = 0,$$

nor in its auxiliary representation $S\{\mathbf{x},\mathbf{u},\mathbf{v}\}$ *(D.56).*

Proof.

(i) According to Theorem D.2, the projection of the symmetries of the auxiliary system on the space of variables $(B_1, B_2, B_3, P, \ x, y, z)$ is (D.55); it represents exactly the set of the classical Lie point symmetries of static plasma equilibrium system (1.20).

These symmetries evidently do not contain free functions $F_\sigma(\mathbf{x},\mathbf{u})$ that are arbitrary solutions of some linear system $L[\mathbf{X}]\mathbf{F}=0$ and satisfy the conditions of Theorem D.3. Thus the latter does not hold, and the system (1.20) cannot be invertibly transformed into a linear system.

(ii) By Theorem D.2, the set of potential symmetries of the system under consideration is empty. Therefore the symmetries of the auxiliary system (D.56) also do not satisfy the conditions of Theorem D.3, and $S\{\mathbf{x},\mathbf{u},\mathbf{v}\}$ can not be invertibly transformed to a linear system.

The theorem is proven. □

E Proof of Theorem 3.1

Proof.

Let us insert the quantities (3.1) into the incompressible system of CGL plasma equilibrium equations (1.22)-(1.23), div $\mathbf{V}=0$, assuming that $\{\mathbf{V}(\mathbf{r}),\mathbf{B}(\mathbf{r}),P(\mathbf{r}),\rho(\mathbf{r})\}$ is an anisotropic MHD equilibrium and satisfies (1.16)-(1.18).

To simplify the notation, we do not write the dependence of functions on $\mathbf{r}$ explicitly.

The functions $f(\mathbf{r})$, $g(\mathbf{r})$ are constant on the magnetic field lines and streamlines, therefore

$$\begin{aligned}\text{div }\mathbf{B}_1 &= f\ \text{div }\mathbf{B} + \mathbf{B}\ \text{grad }f = 0,\\ \text{div }\mathbf{V}_1 &= g\ \text{div }\mathbf{V} + \mathbf{V}\ \text{grad }g = 0;\end{aligned} \tag{E.60}$$

Also, using a vector calculus identity

$$\text{curl}(s\mathbf{q}) = s \text{ curl } \mathbf{q} + \text{grad}(s) \ \times \mathbf{q}, \tag{E.61}$$

we conclude that

$$\text{curl}(\mathbf{V_1} \times \mathbf{B_1}) = 0, \tag{E.62}$$

therefore equations (1.23) are satisfied.

To prove that (1.22) holds, we first observe that

$$\rho_1 \mathbf{V_1} \times \text{curl } \mathbf{V_1} - \left(\tfrac{1}{\mu} - \tau_1\right) \mathbf{B_1} \times \text{curl } \mathbf{B_1} =$$
$$\rho_1 g^2 \ \mathbf{V} \times \text{curl } \mathbf{V} - \left(\tfrac{1}{\mu} - \tau_1\right) f^2 \ \mathbf{B} \times \text{curl } \mathbf{B} + \mathbf{V}^2 \rho_1 g \ \text{grad}(g) - \mathbf{B}^2 \left(\tfrac{1}{\mu} - \tau_1\right) f \ \text{grad}(f) =$$
$$C_0\mu \left(\rho \ \mathbf{V} \times \text{curl } \mathbf{V} - \tfrac{1}{\mu}\mathbf{B} \times \text{curl } \mathbf{B}\right) + \mathbf{V}^2 \rho_1 g \ \text{grad}(g) - \mathbf{B}^2 \left(\tfrac{1}{\mu} - \tau_1\right) f \ \text{grad}(f) =$$
$$C_0\mu \left(\text{grad } P + \rho \ \text{grad}\tfrac{\mathbf{V}^2}{2}\right) + \mathbf{V}^2 \rho_1 g \ \text{grad}(g) - \mathbf{B}^2 \left(\tfrac{1}{\mu} - \tau_1\right) f \ \text{grad}(f) =$$
$$C_0\mu \text{grad } P + C_0\rho\mu \text{grad } \mathbf{V}^2/2 + \tfrac{C_0\rho\mu\mathbf{V}^2}{2g^2}\text{grad } g^2 - \tfrac{\mathbf{B_1}^2}{2} \ \text{grad}(\tau_1).$$

According to the remark (3.2), τ_1 is constant on both magnetic field lines and streamlines, therefore

$$\mathbf{B_1} \cdot \text{grad } \tau_1 = 0,$$

The right-hand side of (1.22) is

$$\text{grad } p_{\perp 1} + \rho_1 \ \text{grad}\tfrac{\mathbf{V_1}^2}{2} + \tau_1 \ \text{grad}\tfrac{\mathbf{B_1}^2}{2} =$$
$$\text{grad}\left(p_{\perp 1} + \rho_1 \ \tfrac{\mathbf{V_1}^2}{2} + \tau_1 \ \tfrac{\mathbf{B_1}^2}{2}\right) - \tfrac{\mathbf{B_1}^2}{2} \ \text{grad}(\tau_1) - \tfrac{\mathbf{V_1}^2}{2} \ \text{grad}(\rho_1) =$$
$$C_0\mu \text{grad } P + C_0\rho\mu \text{grad } \mathbf{V}^2/2 + \tfrac{C_0\rho\mu\mathbf{V}^2}{2g^2}\text{grad } g^2 - \tfrac{\mathbf{B_1}^2}{2} \ \text{grad}(\tau_1)$$

and is identically equal to the left hand side. The theorem is proven. □

F Proof of Theorem 3.2

Proof.

First, we remark that from pressure transformation formulas (3.21) it follows that

$$\tau_1 \equiv \frac{p_{\parallel 1} - p_{\perp 1}}{\mathbf{B_1}^{\mathbf{2}}} = \frac{1}{\mu} - n^2(\mathbf{r}) \left(\frac{1}{\mu} - \tau \right).$$

Now let $w(\mathbf{x})$ be any function that is constant on magnetic field lines and plasma streamlines. Then

$$\mathbf{B} \cdot \operatorname{grad} w(\mathbf{x}) = 0, \quad \mathbf{V} \cdot \operatorname{grad} w(\mathbf{x}) = 0. \tag{F.63}$$

For any smooth vector field $\mathbf{A}$, a vector calculus identity holds

$$\mathbf{A} \times \operatorname{curl} \mathbf{A} = -(\mathbf{A} \cdot \operatorname{grad})\mathbf{A} + \operatorname{grad}(\mathbf{A}^2/2). \tag{F.64}$$

Using it, we rewrite equation (1.22) under consideration, assuming density $\rho(\mathbf{r})$ and the anisotropy factor $\tau(\mathbf{r})$ (1.21) constant on both magnetic field lines and streamlines, as

$$\rho(\mathbf{V} \cdot \operatorname{grad})\mathbf{V} - \left(\frac{1}{\mu} - \tau \right) (\mathbf{B} \cdot \operatorname{grad})\mathbf{B} = -\operatorname{grad}(p_\perp + \frac{\mathbf{B}^2}{2\mu}). \tag{F.65}$$

In (3.21), the coefficients at $\mathbf{B}$, $\mathbf{V}$ in the formulas defining $\mathbf{B_1}$, $\mathbf{V_1}$ are evidently constant on magnetic field lines and plasma streamlines.

Using formulae (F.63), (F.64), we get

$$\begin{gathered} \rho(\mathbf{V} \cdot \operatorname{grad})\mathbf{V} - \left(\tfrac{1}{\mu} - \tau\right) (\mathbf{B} \cdot \operatorname{grad})\mathbf{B} + \operatorname{grad}(p_\perp + \tfrac{\mathbf{B}^2}{2\mu}) = \\ (a^2(\mathbf{r}) - b^2(\mathbf{r})) \left(\rho(\mathbf{V_1} \cdot \operatorname{grad})\mathbf{V_1} - \left(\tfrac{1}{\mu} - \tau_1\right) (\mathbf{B_1} \cdot \operatorname{grad})\mathbf{B_1} + \operatorname{grad}(p_{\perp 1} + \tfrac{\mathbf{B_1}^2}{2\mu}) \right) \end{gathered} \tag{F.66}$$

Thus the functions ρ_1, $\mathbf{B}_1$, $\mathbf{V}_1$, $p_{\perp 1}$, $p_{\parallel 1}$ satisfy equation (F.65) and therefore the CGL equilibrium equation (1.22).

The equations div $\mathbf{V_1} = 0$, div $\mathbf{B_1} = 0$ are evidently satisfied due to (F.63).

Now consider the quantity

$$\mathbf{V_1} \times \mathbf{B_1} = \frac{a^2(\mathbf{r}) - b^2(\mathbf{r})}{m(\mathbf{r})n(\mathbf{r})}(\mathbf{V} \times \mathbf{B}).$$

The scalar factor on the left hand side is constant on magnetic field lines and plasma streamlines, therefore (F.63) applies. Hence

$$\mathrm{curl}(\mathbf{V_1} \times \mathbf{B_1}) = \frac{a^2(\mathbf{r}) - b^2(\mathbf{r})}{m(\mathbf{r})n(\mathbf{r})}\,\mathrm{curl}(\mathbf{V} \times \mathbf{B}) + \mathrm{grad}\left(\frac{a^2(\mathbf{r}) - b^2(\mathbf{r})}{m(\mathbf{r})n(\mathbf{r})}\right) \cdot (\mathbf{V} \times \mathbf{B}) = 0.$$

Thus all anisotropic CGL equilibrium equations (1.22)-(1.23) are satisfied by the transformed quantities (3.21), and the theorem is proven. $\square$

G Explicit reconstruction of coordinates from metric tensor components.

Here we describe the procedure of the reconstruction of the explicit form of coordinates $(u^1, u^2, u^3) = (u, v, w)$ from known metric tensor components that eliminate the Riemann tensor and have the property $g_{ij} = \delta_{ij} H_i^2$, ($\delta_{ij}$ is the Kronecker symbol). We follow Eisenhart [82].

In $\mathbb{R}^3$, for a general orthogonal coordinate system, the Riemann tensor has six independent nonzero components (4.44).

If metric coefficients g_{ii} (or, equivalently, scaling coefficients H_u, H_v, H_w) are found for which the above Riemann equations are satisfied, then the coordinates (u, v, w) can be reconstructed explicitly (see the next subsection).

First, the symbol β_{ij} is evaluated:

$$\beta_{ij} = \frac{1}{H_i}\frac{\partial H_j}{\partial u^i} \quad (i \neq j; \quad i, j = 1..3).$$

Then one solves the linear system

$$\frac{\partial Y_j^i}{\partial u^k} = \beta_{jk} Y_k^i, \qquad \frac{\partial Y_j^i}{\partial u^j} = -\sum_l \beta_{lj} Y_l^i \qquad (k \neq j; \quad i,j,k,l = 1..3)$$

for the quantities Y_j^i $(i,j = 1..3)$.

Finally, the expression for the cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$ is obtained in terms of coordinates (u, v, w) from another linear system:

$$\frac{\partial x^i}{\partial u^j} = H_j Y_j^i \qquad (i,j = 1..3).$$

From the general solution to this system, one that defines an orthogonal coordinate system can always be selected [82]. Then the functions

$$x^i = f^i(u, v, w) \qquad (i = 1..3)$$

define a triply orthogonal family of hypersurfaces $u = \text{const}, v = \text{const}, w = \text{const}$.