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Exact spherical vortex-type equilibrium flows in fluids and plasmas

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ABSTRACT

The famous Hill's solution describing a spherical vortex with nested toroidal pressure surfaces, bounded by a sphere, propelling itself in an ideal Eulerian fluid, is re-derived using Galilei symmetry and the Bragg–Hawthorne equations in spherical coordinates. The correspondence between equilibrium Euler equations of fluid dynamics and static magnetohydrodynamic equations is used to derive a generalized vortex type solution that corresponds to dynamic fluid equilibria and static plasma equilibria with a nonzero azimuthal vector field component, satisfying physical boundary conditions. Separation of variables in Bragg–Hawthorne equation in spherical coordinates is used to construct further new fluid and plasma equilibria with nested toroidal flux surfaces, featuring respectively boundary vorticity sheets and current sheets. Finally, the instability of the original Hill's vortex with respect to certain radial perturbations of the spherical flux surface is proven analytically and illustrated numerically.

1. Introduction

In the general context of fluid dynamics, physically meaningful exact solutions to the fundamental Euler and Navier–Stokes equations, even those possessing some symmetries, are extremely challenging to find. At the same time, when known, and even when rather simple, such solutions represent a valuable resource, possibly on their own as physical models, and/or as a basis to test numerical methods for direct numerical simulations of more complex setups. The same is true about solutions to ideal magnetohydrodynamics (MHD) equations and related models. In recent years, some progress has been made in these directions. For example, new conservation laws of Euler and Navier–Stokes equations have been found in helical symmetry and in a new time-dependent helical coordinate system [1,2]; new exact solutions of Navier–Stokes equations were derived in [3] through similarity reductions; symmetry transformations mapping solutions of MHD equations into families of new solutions were found [4,5]; families of new exact plasma equilibria with axial and helical symmetry were derived in [6–8], and so on. The geometrical feature of equilibrium fluid and plasma models, the existence of flux (magnetic) surfaces to which the velocity and/or magnetic field is tangent, is an essential component of modern analysis of plasma phenomena (e.g., [9,10]).

In 1894 Micaiah John Muller Hill [11] published an article describing a sphere moving symmetrically with regards to an axis through a stationary fluid. Using cylindrical coordinates and assuming that the azimuthal velocity component is zero, Hill was able to find an exact solution of equilibrium dynamic Euler equations that describes a vortex represented by a set of toroidal flux surfaces inside the sphere and an outside solution featuring velocity that decays towards infinity. In the plasma physics context, in the magnetohydrodynamics approximation, a static equilibrium of a plasma is described by the same equations as a dynamic equilibrium fluid flow. A vortex-type solution in the MHD framework was put forth by Morikawa [12] (see also [13–15]), and then, more generally, by Bobnev [16] who considered a spherical vortex in an ideally conducting fluid. In this work, several small mistakes were made, in particular, the plasma vortex described by Bobnev in fact corresponded to a lower internal pressure on the magnetic axis, and a higher outside pressure, and thus could not represent a fireball in a vacuum as claimed by the author. Interestingly enough, in 1995, R. Kaiser and D. Lortz [17] again considered the problem of a spherical vortex in MHD equilibrium to model ball lighting, essentially re-deriving the vortex solution Bobnev found earlier.

Axially symmetric equilibrium solutions of Euler incompressible flows and static MHD equations satisfy the Bragg–Hawthorne equation.[1] Broad literature is devoted to this equation, its variants, and applications, for different forms of arbitrary functions it involves. Some important works, including Refs. [12–15,17,18], considered the Bragg–Hawthorne equation in spherical coordinates, which led to families of particular exact solutions associated with spherical boundaries or separatrices in fluid or plasma domains.

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[1] The Bragg–Hawthorne equation, called Grad-Shafranov equation in plasma physics, was first derived in 1898 by William Mitchinson Hicks, and only gained popularity after being re-derived in 1950 by William Hawthorne and Stephen Bragg.

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In the current paper, first, we are interested in writing down a modern and much simpler derivation of Hill's spherical vortex, using the Galilei symmetry admitted by fluid dynamics equations, going to a moving frame of reference, and finding an equilibrium solution there. The equilibrium solution of Euler's equations in the moving frame is found using axial symmetry, that is, with the help of the corresponding reduction described by the Bragg–Hawthorne equation. written in spherical coordinates (Section 2). In Section 3, we review the problem of a stationary plasma vortex, and the corresponding exact solution of static magnetohydrodynamics equations. This solution family has a nonzero azimuthal magnetic field component. Using the correspondence between static plasma equilibria and dynamic equilibrium fluid flow, the Hill's vortex solution can be generalized to include nonzero azimuthal velocity component (Section 4). Section 5 considers spherical coordinate separation of variables for the Grad-Shafranov equation to produce new exact MHD vortex-type solutions spanned by toroidal magnetic surfaces and bounded by a current sheet. Finally, in Section 6, we revisit the question of stability of Hill's spherical vortex itself. The linear instability of the Hill's vortex with respect to certain axisymmetric perturbations that preserve circulation was shown in Ref. [19]. A nonlinear stability study was undertaken in the paper [20] which was based on the perturbation of magnetic surfaces, but contains few details, and results we were unable to reproduce. In our study, the stability of the Hill's vortex is examined by performing an axisymmetric perturbation of the spherical boundary surface of the vortex. The problem for such a perturbation is shown to yield a third-order ODE eigenvalue problem, which has solutions corresponding to unbounded exponential growth of certain perturbations, thus proving the Hill's vortex nonlinear instability. The paper is concluded with a discussion in Section 7.

2. Hill's spherical vortex: a modern derivation

A sphere of radius R moving through a stationary fluid directed along the z axis can be modeled with the incompressible Euler equations. Starting with the equations of motion for an incompressible fluid

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\chi}\,\text{grad}\;P, \tag{2.1a}$$

$$\text{div}\;\mathbf{V} = 0, \tag{2.1b}$$

where $\mathbf{V}$ denotes the fluid velocity, p the pressure, and χ the density, the well known result that the incompressible Euler equations are invariant under a general Galilean transformations motivates the change of variables

$$\mathbf{V}(\mathbf{r},t) = \tilde{\mathbf{v}}(\mathbf{r} - Z(t)\mathbf{e}_z) + Z'(t)\mathbf{e}_z, \quad P(\mathbf{r},t) = \tilde{P}(\mathbf{r} - Z(t)\mathbf{e}_z). \tag{2.2}$$

Here $Z(t)$ is an arbitrary function of time and $\tilde{\mathbf{v}}$, $\tilde{P}$ denote fluid parameters measured in the corresponding moving frame of reference.

Assuming that the moving frame of reference is moving at the same speed as the spherical vortex, and the density is constant, after omitting the tilde on the new variables, the Euler equations can be written as

$$\text{curl}\;\mathbf{v}\times\mathbf{v} = \text{grad}\;H, \tag{2.3a}$$

$$\text{div}\;\mathbf{v} = 0, \tag{2.3b}$$

where

$$H = -\left(\frac{P}{\chi} + \frac{\mathbf{v}^2}{2}\right) \tag{2.4}$$

is a modified pressure term. In the rest of this section H will simply be referred to as the pressure. Assuming that the motion is axially symmetric, it is natural to use cylindrical coordinates (r,ϕ,z) and set $\mathbf{v}$ and H independent of the azimuthal angle ϕ. In doing so, one can reduce (2.3) to the well known Bragg–Hawthorne equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + F(\psi)F'(\psi) = r^2 H'(\psi). \tag{2.5}$$

In (2.5), ψ is the stream function that arises from the incompressibility condition (2.3b). Indeed, in the axial symmetry, (2.3b) has the form of a two-dimensional divergence

$$\text{div}\;\mathbf{v} \equiv \frac{1}{r}(rv_r)_r + (v_z)_z,$$

which yields $(rv^r)_r + (rv^z)_z = 0$, and is the integrability condition of the PDEs

$$rv_r = \psi_z, \quad rv_z = -\psi_r, \tag{2.6}$$

defining the stream function. In terms of ψ, the velocity is given by

$$\mathbf{v} = \frac{\psi_z}{r}\mathbf{e}_r + \frac{F(\psi)}{r}\mathbf{e}_\phi + \frac{-\psi_r}{r}\mathbf{e}_z, \tag{2.7}$$

and F, H are arbitrary functions of ψ. Following Hill's assumption who considered a two-component axially symmetric flow, the azimuthal component of the velocity is set to zero, giving the condition

$$F(\psi) = 0. \tag{2.8}$$

When (2.8) holds, one has the simplified Bragg–Hawthorne equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = -r^2 H'(\psi), \tag{2.9}$$

and the velocity given by

$$\mathbf{v} = \frac{\psi_z}{r}\mathbf{e}_r + \frac{-\psi_r}{r}\mathbf{e}_z. \tag{2.10}$$

In the case (2.8), from (2.9) and (2.10), it follows that the fluid vorticity $\boldsymbol{\omega} = \text{curl}\;\mathbf{v}$ is given by

$$\boldsymbol{\omega} = r^2 H'(\psi)\mathbf{e}_\phi. \tag{2.11}$$

Hence when the pressure is constant, $H = H_0$, one has $\boldsymbol{\omega} = 0$, which corresponds to an irrotational flow.

The arbitrary function $H(\psi)$ can be chosen such that (2.9) becomes separable in spherical coordinates, and the asymptotics of the pressure $H(\psi)$ is physically relevant. As far as separability of (2.9) goes, $H'(\psi)$ generally cannot be of higher degree than linear in ψ.[2] In regards to the asymptotics, the pressure far away from the sphere must not change and needs to be the ambient pressure H_0. In particular, one may employ a piecewise linear form of $H(\psi)$ with appropriate matching at the boundary:

$$H(\psi) = \begin{cases} H_0 - 10\delta\psi, & \rho < R, \\ H_0, & \rho > R. \end{cases} \tag{2.12}$$

Here the coefficient 10δ is chosen to make the calculation cleaner. The problem is now decomposed into two pieces: the rotational flow inside the sphere, satisfying

$$H(\psi) = H_0 - 10\delta\psi, \qquad \frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = 10\delta r^2, \tag{2.13}$$

and the irrotational flow outside of the sphere, defined by

$$H(\tilde{\psi}) = H_0, \qquad \frac{\partial^2\tilde{\psi}}{\partial r^2} + \frac{\partial^2\tilde{\psi}}{\partial z^2} - \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r} = 0. \tag{2.14}$$

Along with these two equations, there is the condition that both pieces must have matching pressure and velocity components at the boundary of the sphere ($r^2 + z^2 = R^2$). For matching pressure, this implies that for the inside solution, $\psi(r,z) = 0$ when $r^2 + z^2 = R^2$. It turns out that one can effectively seek solutions to (2.13) and (2.14) in spherical coordinates (ρ,θ,ϕ), in the separated form $\psi(\rho,\theta) = R(\rho)\Theta(\theta)$. Here standard spherical coordinates are related to cylindrical coordinates by $r = \rho\sin\theta$, $z = \rho\cos\theta$. Converting the above problem into spherical coordinates gives

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\psi = 10\delta\rho^2\sin^2\theta \tag{2.15}$$

[2] Multiple exact solutions of the Bragg–Hawthorne equation and its generalizations have been obtained in cases when it becomes linear; see, for example, Refs. [8,21–24].

inside the sphere, and

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\psi=0 \tag{2.16}$$

outside the sphere. The velocity components inside and outside are given by

$$\mathbf{v}_{in}=\frac{1}{\rho^2\sin\theta}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\rho-\frac{1}{\rho\sin\theta}\frac{\partial\psi}{\partial\rho}\mathbf{e}_\theta,\qquad \mathbf{v}_{out}=\frac{1}{\rho^2\sin\theta}\frac{\partial\tilde\psi}{\partial\theta}\mathbf{e}_\rho-\frac{1}{\rho\sin\theta}\frac{\partial\tilde\psi}{\partial\rho}\mathbf{e}_\theta. \tag{2.17}$$

Along with this, the matching conditions and the need for $\psi(\rho,\theta)$ to be regular at $\rho=0$ give the following four boundary conditions

$$\psi(R,\theta)=0,\quad |\psi(0,\theta)|<\infty,\quad \left.\frac{\partial\psi}{\partial\theta}\right|_{\rho=R}=\left.\frac{\partial\tilde\psi}{\partial\theta}\right|_{\rho=R},\quad \left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=R}=\left.\frac{\partial\tilde\psi}{\partial\rho}\right|_{\rho=R}. \tag{2.18}$$

A general solution for the inhomogeneous inside Eq. (2.15) is sought in the form of $\psi(\rho,\theta)=\psi(\rho,\theta)_{gen}+\psi(\rho,\theta)_{part}$ where $\psi(\rho,\theta)_{gen}$ is a general solution to the homogeneous version of (2.15) given by

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\psi=0, \tag{2.19}$$

and $\psi(\rho,\theta)_{part}$ is a particular solution to (2.15). One particular solution is easily found to be

$$\psi(\rho,\theta)_{part}=\delta\rho^4\sin^2\theta. \tag{2.20}$$

The general solution to (2.19) is obtained by separation of variables, $\psi(\rho,\theta)=R(\rho)\Theta(\theta)$. Upon substitution the separated form into (2.19) one arrives at two ODEs

$$\rho^2R''-CR=0, \tag{2.21}$$

$$\left((-\csc\theta)\Theta'\right)'=C\Theta\csc\theta, \tag{2.22}$$

where C is a separation constant to be determined. Using the change of variables

$$t=\cos\theta,\quad \Theta(\theta)=T(t),$$

Eq. (2.22) becomes

$$(1-t^2)T''(t)+CT(t)=0. \tag{2.23}$$

This ODE can be connected to the associated Legendre ODE with the transformation $T(t)=\sqrt{1-t^2}P(t)$, leading to

$$(1-t^2)P''(t)-2tP'(t)+\left(C-\frac{1}{1-t^2}\right)P(t)=0. \tag{2.24}$$

The Eq. (2.24) is related to the associated Legendre ODE [17] given by

$$(1-x^2)\tilde P''(x)-2x\tilde P'(x)+\left(l(l+1)-\frac{m^2}{1-x^2}\right)\tilde P(x)=0. \tag{2.25}$$

In particular, (2.24) matches (2.25) when $m=1$ and $C=\ell(\ell+1)$. The Eq. (2.25) has nonsingular solutions on the interval $[-1,1]$ only when ℓ and m are integer values [25]. For $m=1$, the associated Legendre polynomials have the form

$$P_\ell(x)=-\sqrt{1-x^2}\frac{d}{dx}\mathcal{P}_\ell(x), \tag{2.26}$$

where $\mathcal{P}_\ell$ refers to the ℓ^{th} order Legendre polynomial. One then arrives at the regular solutions to (2.23) given by

$$T_\ell(t)=-(1-t^2)\frac{d}{dt}\mathcal{P}_\ell,$$

that can be written as

$$T_\ell(t)=(\ell+1)\mathcal{P}_{\ell+1}(t)-(\ell+1)t\mathcal{P}_\ell(t). \tag{2.27}$$

This gives $\Theta(\theta)$ as

$$\Theta_\ell(\theta)=(\ell+1)\mathcal{P}_{\ell+1}(\cos\theta)-(\ell+1)\cos\theta\ \mathcal{P}_\ell(\cos\theta). \tag{2.28}$$

The value $C=\ell(\ell+1)$ can now be substituted into (2.21) and yields

$$\rho^2R''(\rho)-\ell(\ell+1)R(\rho)=0.$$

The latter ODE has the solution

$$R_\ell(\rho)=a_\ell\rho^{\ell+1}+b_\ell\rho^{-\ell}.$$

As the solution is required to be regular at $\rho=0$, b_ℓ will be set to zero. Separated solutions to the homogeneous PDE (2.19) are therefore given by

$$\psi_\ell(\rho,\theta)=a_\ell\rho^{\ell+1}\Theta_\ell(\theta),\qquad \ell=0,1,2,\dots, \tag{2.29}$$

giving the general solution for ψ inside the sphere in terms of the particular solution (2.20) and a Fourier series:

$$\psi(\rho,\theta)=\delta\rho^4\sin^2\theta+\sum_{\ell=0}^{\infty}a_\ell\rho^{\ell+1}\Theta_\ell(\theta). \tag{2.30}$$

The condition that the pressure be continuous across the spherical boundary reduces to the condition that $\psi(R,\theta)=0$, as specified in (2.18). One consequently has

$$\sum_{\ell=0}^{\infty}a_\ell R^{\ell+1}\Theta_\ell(\theta)=-\delta R^4\sin^2\theta. \tag{2.31}$$

The solutions $\Theta_\ell(\theta)$ form a complete orthogonal basis, since (2.22) is a classical Sturm–Liouville second-order linear ODE with weight $w(\theta)=\csc\theta$. Using the observation that $\Theta_1(\theta)=-\sin^2\theta$ Eq. (2.31) can be written as

$$\sum_{\ell=0}^{\infty}a_\ell R^{\ell+1}\Theta_\ell(\theta)=\delta R^4\Theta_1(\theta). \tag{2.32}$$

By multiplying the above equation by $-\csc\theta\,\Theta_\ell(\theta)$ and integrating from $0<\theta<\pi$ one arrives at

$$a_\ell R^{\ell+1}=\frac{-\int_0^\pi\delta R^4\csc\theta\,\Theta_1(\theta)\Theta_\ell(\theta)\,d\theta}{-\int_0^\pi\csc\theta\,\Theta_\ell^2(\theta)\,d\theta}.$$

The right hand side is zero due to the orthogonality of $\Theta_\ell(\theta)$ for all ℓ except when $\ell=1$. In this case one obtains the condition that $a_1=\delta R^2$. Therefore the solution inside the sphere can be written as

$$\psi(\rho,\theta)=\delta\rho^2\sin^2\theta(\rho^2-R^2). \tag{2.33}$$

For the outside of the sphere, the solution is the same as the homogeneous solution to (2.15), and is given by the series

$$\tilde\psi(\rho,\theta)=\sum_{\ell=0}^{\infty}\left(c_\ell\rho^{\ell+1}+\frac{d_1}{\rho}\right)\Theta_\ell(\theta). \tag{2.34}$$

The fourth condition in (2.18) yields

$$\sum_{\ell=0}^{\infty}\left(c_\ell(\ell+1)R^\ell-\frac{d_1}{R^2}\right)\Theta_\ell(\theta)=3\delta R^3\sin^2\theta. \tag{2.35}$$

Using the orthogonality of $\Theta_\ell(\theta)$ as discussed before, one again has $\ell=1$. Lastly, the third condition in (2.18) gives

$$\left(c_\ell R^2+\frac{d_1}{R}\right)=0. \tag{2.36}$$

which leads to $c_1=-d_1/R^3$. Substituting this back into (2.35) with $\ell=1$, one finally obtains the complete solution for the stream function in the whole space. It is given by

$$\psi(\rho,\theta)=\begin{cases}\delta\rho^2\sin^2\theta(\rho^2-R^2), & \rho<R,\\ \frac{2}{3}\delta R^2\sin^2\theta\left(\frac{\rho^3-R^3}{\rho}\right), & \rho>R.\end{cases} \tag{2.37}$$

This can be written in cylindrical coordinates as

$$\psi(r,z)=\begin{cases}\delta\left((r^2z^2+r^4)-R^2r^2\right), & r^2+z^2<R^2,\\ \frac{2}{3}\delta R^2r^2\left(1-\frac{R^3}{(r^2+z^2)^{3/2}}\right), & r^2+z^2>R^2.\end{cases} \tag{2.38}$$

The velocity components are consequently computed from (2.10) and are given by

$$v_r = \begin{cases} 2\delta rz, & r^2+z^2<R^2, \\ \dfrac{2\delta R^5 rz}{(r^2+z^2)^{5/2}}, & r^2+z^2>R^2, \end{cases} \tag{2.39a}$$

$$v_z = \begin{cases} 2\delta\left(R^2-r^2-z^2\right), & r^2+z^2<R^2, \\ \dfrac{4}{3}\delta R^2 + \dfrac{2\delta R^5}{3}\dfrac{(r^2-2z^2)}{(r^2+z^2)^{5/2}}, & r^2+z^2>R^2. \end{cases} \tag{2.39b}$$

Moving back into the lab frame using the Galilei transformation given by (2.2), one arrives at the following time-dependent fluid velocity component expressions:

$$V_r = \begin{cases} 2\delta r(z-Z(t)), & r^2+(z-Z(t))^2<R^2 \\ \dfrac{2\delta R^5 r(z-Z(t))}{(r^2+(z-Z(t))^2)^{5/2}}, & r^2+(z-Z(t))^2>R^2, \end{cases} \tag{2.40a}$$

and

$$V_z = \begin{cases} Z'(t)+2\delta\left(R^2-r^2-(z-Z(t))^2\right), & r^2+(z-Z(t))^2<R^2, \\ Z'(t)+\dfrac{4}{3}\delta R^2+\dfrac{2\delta R^5}{3}\dfrac{(r^2-2(z-Z(t))^2)}{(r^2+(z-Z(t))^2)^{5/2}}, & r^2+(z-Z(t))^2>R^2. \end{cases} \tag{2.40b}$$

In particular, the center of the vortex moves along the z-axis according to the position function $Z(t)$. The pressure in the stationary frame of reference is given by

$$H(r,z) = \begin{cases} H_0 - 10\delta^2\left(r^2\left((z-Z(t))^2+r^2-R^2\right)\right), & r^2+(z-Z(t))^2<R^2, \\ H_0, & r^2+(z-Z(t))^2>R^2. \end{cases} \tag{2.41}$$

One additional boundary condition that can be considered is the behavior of the velocity far away from the spherical vortex. In particular, if the fluid that the sphere is moving through is stationary, it is natural to demand $V_r, V_z \to 0$ as $r^2+z^2\to\infty$. The first limit for V_r is trivially satisfied, $\lim_{r^2+z^2\to\infty} v_r = 0$, however, for the z component of velocity, v_z one gets

$$\lim_{r^2+z^2\to\infty} v_z = Z'(t)+\frac{4}{3}\delta R^2 = 0. \tag{2.42}$$

This gives the additional condition that $Z'(t) = -\frac{4}{3}\delta R^2$, and implies that for the ambient velocity to vanish at infinity, the group velocity of the moving spherical vortex must be constant, with a speed that is proportional to the square of the radius. :

$$Z(t) = Z_0 - \frac{4}{3}\delta R^2 t. \tag{2.43}$$

An illustration of the Hill's vortex cross-section is given in Fig. 1. The vortex is comprised of nested toroidal flux surfaces on the inside of the sphere, and unbounded flux surfaces in the outside region. The spherical surface constitutes a separatrix between the two families of the flux surfaces, with X-points where it meets the z-axis.

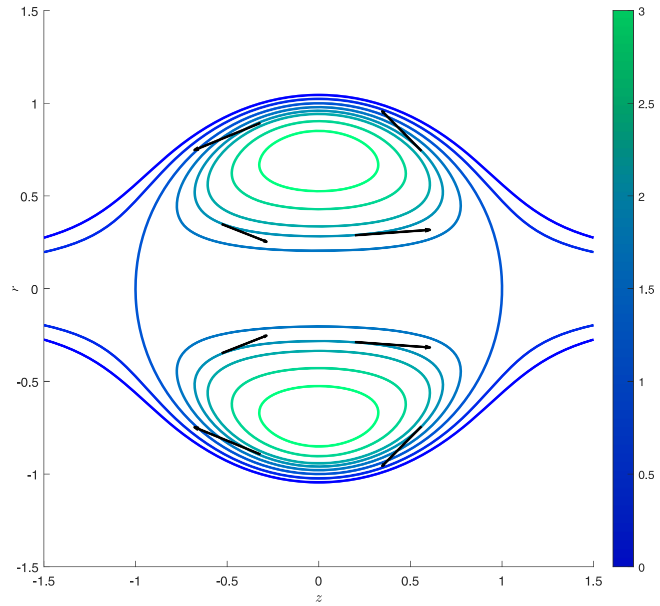


Fig. 1. A cross-section of surfaces $H(\psi)=\text{const}$ in the lab frame given by (2.41). Here $R=1$, $H_0=1$, $\delta=1$ and $t=0$. The black arrows correspond to the velocity vectors on a given surface. Via the first equation of (2.3), both **v** and curl **v** are tangent to this surface.

3. A stationary spherical plasma vortex

A similar problem to Hill's spherical vortex is the concept of a spherical vortex moving through an ideally conducting fluid. With this problem, negligibly small fluid motion ($\mathbf{V}=0$) is assumed which gives the starting point as the static equilibrium magnetohydrodynamic (MHD) equations

$$\text{curl}\,\mathbf{B}\times\mathbf{B} = \mu\,\text{grad}\,P, \tag{3.1a}$$

$$\text{div}\,\mathbf{B} = 0, \tag{3.1b}$$

where μ is the magnetic permeability coefficient. Here, the main differences between Hill's spherical vortex and this stationary conducting spherical vortex is: the search for **v** inside and outside the sphere is replaced with the search for the magnetic field **B**, and the azimuthal component of this magnetic field is *not* assumed to be zero. Two assumptions of this conducting spherical vortex are that the pressure goes to a constant value taken to be zero at the boundary of the sphere (similar to Hill's spherical vortex), and every magnetic field component goes to zero at the boundary. The last condition here regarding the magnetic field is chosen in this way because asymptotically, the magnetic field must decay at least as quickly as a dipole moment. It was shown in [17] that the only solution outside of the sphere consistent with proper asymptotic behavior of the internal pressure and magnetic field is $\mathbf{B}=0$.

The spherical magnetic vortex is assumed to have inherent axial symmetry which allows the reduction of (3.1) to a single PDE

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+I(\psi)I'(\psi) = -r^2P'(\psi). \tag{3.2}$$

equivalent to the Bragg–Hawthorne equation (2.5) and called in the MHD context the *Grad-Shafranov equation*. The magnetic field components are given by

$$\mathbf{B} = \frac{\psi_z}{r}\mathbf{e}_r + \frac{I(\psi)}{r}\mathbf{e}_\phi - \frac{\psi_r}{r}\mathbf{e}_z. \tag{3.3}$$

Inside of the sphere, the pressure $P(\psi)$ and the arbitrary function related to the toroidal magnetic field $I(\psi)$ are taken to be linear (as any higher power series expansion of $P(\psi)$ and $I(\psi)$ makes (3.2) not separable in spherical coordinates). Therefore, these arbitrary functions are written as (see, e.g., [15])

$$P(\psi) = P_0 - \gamma\psi, \quad I(\psi) = \lambda\psi. \tag{3.4}$$

The Grad-Shafranov equation now becomes a second order linear homogeneous PDE. This equation can be converted to spherical coordinates to yield

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\lambda^2\right]\psi = \gamma\rho^2\sin^2\theta, \tag{3.5}$$

where the magnetic field is given by

$$\mathbf{B} = \frac{1}{\rho^2 \sin\theta}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\rho + \frac{I(\psi)}{\rho\sin\theta}\mathbf{e}_\phi - \frac{1}{\rho\sin\theta}\frac{\partial\psi}{\partial\rho}\mathbf{e}_\theta. \tag{3.6}$$

Following a procedure similar to that of the previous section, $\psi(\rho,\theta) = \psi(\rho,\theta)_{gen} + \psi(\rho,\theta)_{part}$ where $\psi(\rho,\theta)_{gen}$ is a general solution to the homogeneous version of (3.5) given by

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \lambda^2\right]\psi = 0, \tag{3.7}$$

and

$$\psi(\rho,\theta)_{part} = \frac{\delta}{\lambda^2}\rho^2\sin^2\theta. \tag{3.8}$$

The general solution to (3.7) is obtained by a separated solution $\psi(\rho,\theta) = R(\rho)\Theta(\theta)$. Upon substituting the separated form into (3.7) one arrives at the two ODEs

$$\rho^2 R''(\rho) - (c+\lambda^2)R(\rho) = 0, \tag{3.9}$$

$$\left((-\csc\theta)\Theta'\right)' = C(\csc\theta)\Theta. \tag{3.10}$$

One can notice that (3.10) is the exact same as in the previous section given by (2.22). Therefore, due to the $\sin^2\theta$ dependence in (3.8) and the orthogonality of $\Theta_\ell(\theta)$ given by (2.28), one can conclude in a similar fashion to the previous section that the only value of ℓ which satisfies the condition of the pressure P going to the constant ambient pressure P_0 on the boundary is $\ell = 1$. This gives the following separated ansatz to use:

$$\psi(\rho,\theta) = G(\rho)\rho^2\sin^2\theta. \tag{3.11}$$

Upon substituting the above into Eq. (3.5), the second order linear ODE on $G(\rho)$ is obtained:

$$G''(\rho) + \frac{4}{\rho}G'(\rho) + G(\rho)\lambda^2 = \gamma. \tag{3.12}$$

This ODE equation, along with the following physical conditions gives a well posed eigenvalue problem [16].

1. To guarantee a finite magnetic energy inside the sphere, $\lim_{\rho\to 0}|G(\rho)| < \infty$.
2. The magnetic field components given by (3.6) must vanish at the domain boundary, as discussed in [16,17], hence $G'(R) = G(R) = 0$.
3. The pressure must go to a constant ambient pressure P_0 at the boundary, which also implies $G(R) = 0$.

One can show that the general solution to (3.12) is given by

$$G(\rho) = C_1\frac{\rho\lambda\sin(\rho\lambda)+\cos(\rho\lambda)}{\rho^3} + C_2\frac{\rho\lambda\cos(\rho\lambda)-\sin(\rho\lambda)}{\rho^3} + \frac{\gamma}{\lambda^2}. \tag{3.13}$$

From the first condition above, $C_1 = 0$. The second condition gives a countable number of normalized eigenvalues $\lambda_n = \lambda R$ corresponding to the nth root of the transcendental equation

$$x^2\tan x - 3\tan x + 3x = 0. \tag{3.14}$$

Lastly, the third condition gives a value for γ depending on the value of λ_n,

$$\gamma_n = -C_2\lambda_n^2\frac{\lambda_n\cos\lambda_n - \sin\lambda_n}{R^5}. \tag{3.15}$$

This gives the flux function inside of the sphere as

$$\psi(\rho,\theta) = \left(C_2\frac{(\rho\lambda_n/R)\cos(\rho\lambda_n/R) - \sin(\rho\lambda_n/R)}{\rho} + \frac{\rho^2R^2\gamma_n}{\lambda_n^2}\right)\sin^2\theta. \tag{3.16}$$

which can be written in terms of a first order spherical Bessel function of the first kind, j_1 as

$$\psi(\rho,\theta) = \left(\tilde{C}_2\frac{\rho\lambda_n}{R}j_1\left(\frac{\rho\lambda_n}{R}\right) + \frac{\rho^2R^2\gamma_n}{\lambda_n^2}\right)\sin^2\theta. \tag{3.17}$$

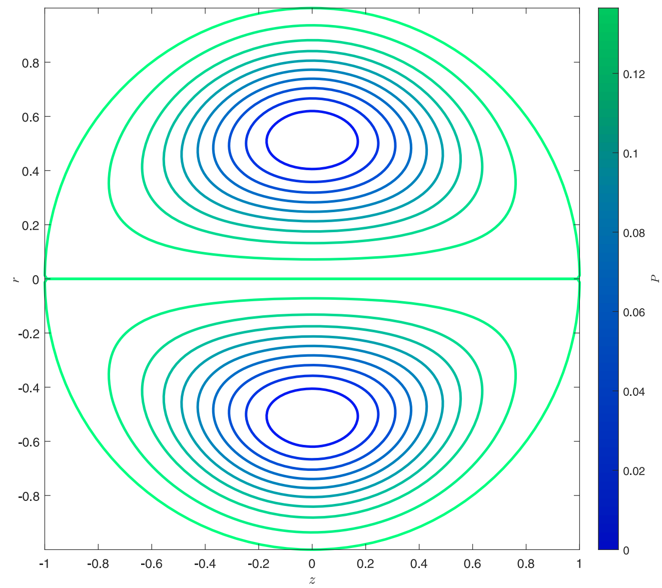


Fig. 2. Pressure profile of static spherical vortex in ideally conducting fluid given by $P(\psi_n) = P_0 - \gamma_n\psi_n$ where ψ_n is given by (3.16) for $R = 1$, $n = 1$ and $C_2 = 1$.

Outside of the sphere $\rho > R$ all of the magnetic field components are zero and the pressure is equal to the ambient pressure P_0. An example of this solution for $n = 1$ has its pressure shown in Fig. 2.

A few other solutions are shown for higher values of n. In Fig. 3 pressure profiles $P(\psi_n) = P_0 - \gamma_n\psi_n$ for ψ_n given by (3.16) with $n = 2$ and $n = 3$ can be seen. In Fig. 4, $n = 4$ and $n = 5$.

4. A generalized version of Hill's spherical vortex

In this section, we use the results of Section 3 to generalize the Hill's vortex in an ideal fluid (Section 2) onto the case of nonzero toroidal velocity component.

In a moving frame of reference, assuming axial invariance, the Euler equations (2.1) can be reduced to the Bragg–Hawthorne equation (2.5). We start from said equation in spherical coordinates, now without the assumption that the ϕ-component of velocity (2.7) vanishes, which meant $F(\psi) = 0$ (2.8). We have

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\psi + F(\psi)F'(\psi) = -H'(\psi)\rho^2\sin^2\theta. \tag{4.1}$$

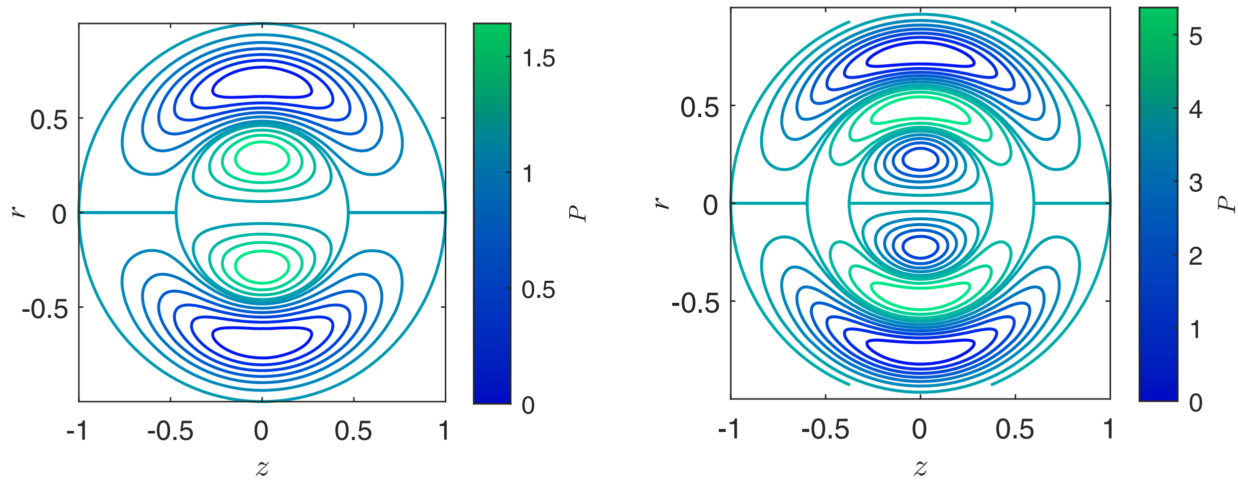
The arbitrary functions are again chosen as the highest power series expansion in ψ such that the PDE (4.1) becomes separable, and the asymptotics of the pressure $H(\psi)$ and the toroidal velocity component function $F(\psi)$ behave physically. As far as separability of (4.1) goes, both functions cannot be of higher degree then linear in ψ. In regards to the asymptotics, the pressure far away from the sphere is chosen to not change, and thus needs to be the ambient pressure H_0:

$$H(\psi) = \begin{cases} H_0 - \gamma\psi, & \rho < R, \\ H_0, & \rho > R. \end{cases} \tag{4.2}$$

Similarly, $F(\psi)$ must also not change far away from the sphere. however, $F(\psi) = F_0$ where $F_0 = $ const is not allowed as it corresponds to a singular V^ϕ (cf. (2.7)). This gives the free function form

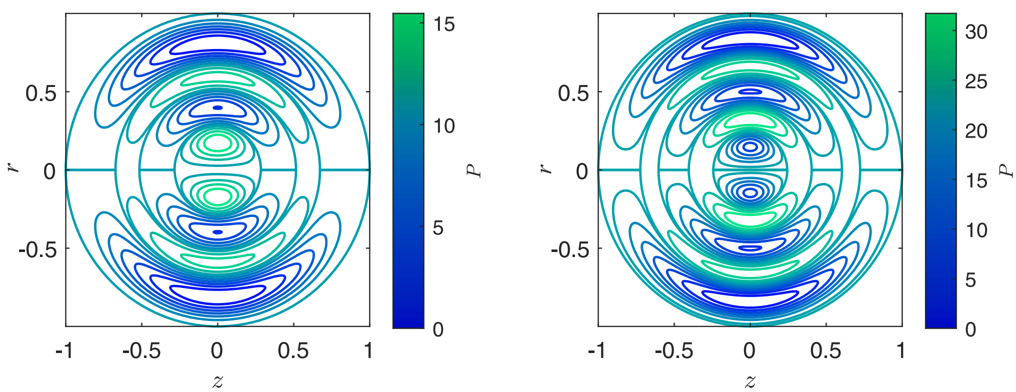
$$F(\psi) = \begin{cases} \lambda\psi, & \rho < R, \\ 0, & \rho > R. \end{cases} \tag{4.3}$$

The above ansatz allows one to decompose the spherical Bragg-Hawthorne equation into two problems as before, one inside and one outside of the sphere, namely:



(a) (b)

Fig. 3. Pressure profile of static spherical vortex in ideally conducting fluid given by $P(\psi_n) = P_0 - \gamma_n \psi_n$ where ψ_n is given by (3.16) for $C_2 = 1$, $R = 1$, $n = 2$ on the left, and $n = 3$ on the right.



(a) (b)

Fig. 4. Pressure profile of static spherical vortex in ideally conducting fluid given by $P(\psi_n) = P_0 - \gamma_n \psi_n$ where ψ_n is given by (3.16) for $C_2 = 1$, $R = 1$, $n = 4$ on the left, and $n = 5$ on the right.

1. Rotational flow inside the sphere $\rho < R$

$$H(\psi) = H_0 - \gamma\psi, \tag{4.4a}$$

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \lambda^2\right]\psi = \gamma\rho^2\sin^2\theta. \tag{4.4b}$$

2. Irrotational, force-free flow outside the sphere $\rho > R$

$$H(\tilde\psi) = H_0, \tag{4.5a}$$

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\tilde\psi = 0. \tag{4.5b}$$

The velocity components inside and outside are given by

$$\mathbf{v}_{in} = \frac{1}{\rho^2\sin\theta}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\rho + \frac{F(\psi)}{\rho\sin\theta}\mathbf{e}_\phi - \frac{1}{\rho\sin\theta}\frac{\partial\psi}{\partial\rho}\mathbf{e}_\theta, \tag{4.6}$$

and

$$\mathbf{v}_{out} = \frac{1}{\rho^2\sin\theta}\frac{\partial\tilde\psi}{\partial\theta}\mathbf{e}_\rho - \frac{1}{\rho\sin\theta}\frac{\partial\tilde\psi}{\partial\rho}\mathbf{e}_\theta, \tag{4.7}$$

respectively. The matching condition for the pressure and velocity and the regularity of $\psi(\rho,\theta)$ at $\rho = 0$ yield the conditions

$$\psi(R,\theta) = 0, \quad |\psi(0,\theta)| < \infty, \quad \left.\frac{\partial\psi}{\partial\theta}\right|_{\rho=R} = \left.\frac{\partial\tilde\psi}{\partial\theta}\right|_{\rho=R}, \quad \left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=R} = \left.\frac{\partial\tilde\psi}{\partial\rho}\right|_{\rho=R}. \tag{4.8}$$

From the last section, a solution inside the sphere that is bounded at the origin was found to be

$$\psi(\rho,\theta) = \left(C\frac{\rho\lambda\cos(\rho\lambda) - \sin(\rho\lambda)}{\rho} + \frac{\gamma}{\lambda^2}\rho^2\right)\sin^2\theta, \tag{4.9}$$

and from the first section, the solution outside of the sphere is given by

$$\tilde\psi(\rho,\theta) = \rho^2\sin^2\theta\left(A + \frac{B}{\rho^3}\right). \tag{4.10}$$

After applying the matching pressure boundary condition given by the first equation in (4.8) one obtains the transcendental equation between λ and γ

$$C\lambda^2 R\cos(R\lambda) - C\lambda\sin(R\lambda) + R^3\gamma = 0. \tag{4.11}$$

Using the third boundary condition in (4.8) one obtains

$$A = -\frac{B}{R^3}, \tag{4.12}$$

which gives the outside solution

$$\tilde\psi(\rho,\theta) = B\rho^2\sin^2\theta\left(\frac{1}{\rho^3} - \frac{1}{R^3}\right). \tag{4.13}$$

Lastly, the final boundary condition in (4.8) allows one to solve for B in terms of the other constants, giving

$$B = \frac{CR\lambda^3\cos(R\lambda) + C\lambda^4R^2\sin(R\lambda) - C\lambda^2\sin(R\lambda) - 2\gamma R^3}{3\lambda^2}. \tag{4.14}$$

The three conditions on the constants given by (4.11), (4.12) and (4.14) yield $\psi(\rho,\theta)$ in the whole space:

$$\psi(\rho,\theta) = \begin{cases} \left(C\dfrac{\rho\lambda\cos(\rho\lambda)-\sin(\rho\lambda)}{\rho} + \dfrac{\gamma}{\lambda^2}\rho^2\right)\sin^2\theta, & \rho < R, \\ \dfrac{CR\lambda^3\cos(R\lambda)+C\lambda^4R^2\sin(R\lambda)-C\lambda^2\sin(R\lambda)-2\gamma R^3}{3\lambda^2}\rho^2\sin^2\theta\left(\dfrac{1}{R^3}-\dfrac{1}{\rho^3}\right), & \rho > R. \end{cases} \tag{4.15}$$

This solution (4.15) of the spherical Grad-Shafranov Eqs. (4.4) and (4.5) is a more general version of the original Hill's spherical vortex stream function (2.37) because:

- The ϕ component of the velocity is non-zero inside of the sphere, whereas Hill's original vortex solution had $V_\phi = 0$.
- There is the choice of freedom for three constants, C, λ or γ, and R, whereas Hill's original solution only has a choice of freedom for R and one constant δ.

The asymptotics of the velocity field outside of the sphere behaves in a suitable manner as this is the same outside solution of Hill's spherical vortex, given in cylindrical coordinates by (2.39a) and (2.39b) which has correct asymptotics as discussed in [11].

One interesting remark in relation to the MHD vortex of Section 3 is that if the outside magnetic field in the vortex must vanish outside, which corresponds in this case to the coefficient of the outside solution given in (4.15) as B, then this problem reduces to the problem in the previous section, and the Eqs. (4.11) and (4.14) reduce to the transcendental equations given by (3.14) and (3.15) as they should. This requirement was briefly discussed in [17].

5. Spherical coordinate separation of variables for the Grad-Shafranov equation in the context of the MHD vortex

In Section 3 above, a separated solution in spherical coordinates to the Grad-Shafranov Eq. (5.1) was obtained to satisfy boundary conditions that correspond to a spherical vortex moving through a stationary fluid. During this, the behavior of $\Theta_l(\theta)$ given by (2.28) was restricted to $\ell = 1$ to satisfy the boundary conditions. In this section, a fully separated solution is considered in its own right.

Proceeding as in Section 3 up until (3.10), the linear Grad-Shafranov equation in spherical coordinates

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \lambda^2\right]\psi = \gamma\rho^2\sin^2\theta. \tag{5.1}$$

which corresponds to the free functions from Section 3 given by $I(\psi) = \lambda\psi$ and $P(\psi) = P_0 - \gamma\psi$. A solution in the form of $\psi(\rho,\theta) = \psi(\rho,\theta)_{gen} + \psi(\rho,\theta)_{part}$ is sought with

$$\psi(\rho,\theta)_{part} = \frac{\gamma\rho^2\sin^2\theta}{\lambda^2}.$$

Separated solutions for the homogeneous version of (5.1) are sought in the form

$$\psi(\rho,\theta) = R(\rho)\Theta(\theta). \tag{5.2}$$

The homogeneous version of Eq. (5.1) then reduces to the two ODEs

$$\rho^2R''(\rho) - (C + \lambda^2)R(\rho) = 0, \tag{5.3}$$

$$\Theta''(\theta) - \frac{\cos\theta}{\sin\theta}\Theta'(\theta) + c\Theta(\theta) = 0, \tag{5.4}$$

where C is a separation constant to be determined. In Section 2, the separation constant was found to be $C = \ell(\ell+1)$ for $\ell \in \mathbb{N}$ with a

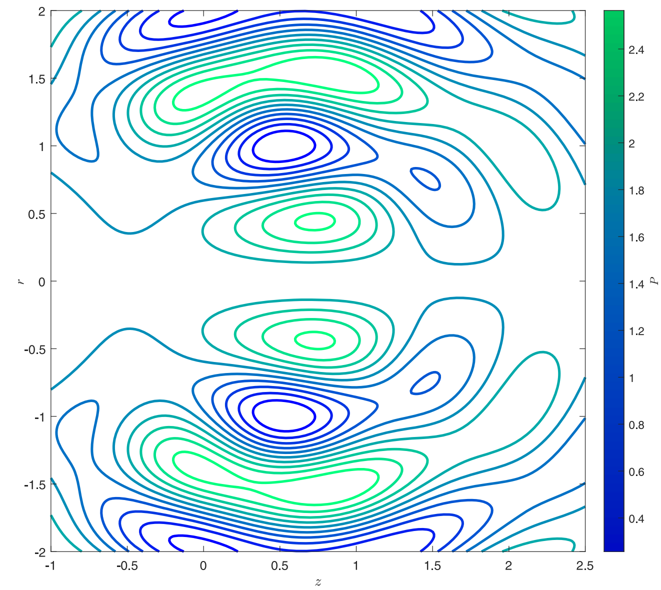


Fig. 5. A cross-section of magnetic surfaces where the magnetic surfaces are shown by $P(\psi) = \text{const}$ for $P = P_0 - \gamma\psi$ where ψ is given by (5.9). Here $\gamma = 1$, $\lambda = 1$, $n = 5$, $a_l = \ell$, $\ell = 1,2,3,4,5$. Any toroidal surface can be considered a truncated solution with some outer surface described by a current sheet (see, e.g., [8] for details).

solution to (5.4) given by

$$\Theta_\ell(\theta) = (\ell+1)\mathcal{P}_{\ell+1}(\cos\theta) - (\ell+1)\cos\theta\,\mathcal{P}_l(\cos\theta). \tag{5.5}$$

The ODE (5.3) yields

$$\rho^2R''(\rho) - (\ell(\ell+1) + \lambda^2)R(\rho) = 0. \tag{5.6}$$

This has a solution in terms of the Bessel function of the first kind

$$R_\ell(\rho) = \sqrt{\rho}\,\mathcal{J}\left(\frac{2\ell+1}{2}, \rho\lambda\right). \tag{5.7}$$

A separated solution to the homogeneous version of (5.1) is thus given by (5.2) with (5.5) and (5.7):

$$\psi_\ell(\rho,\theta) = \sqrt{\rho}\,\mathcal{J}\left(\frac{2\ell+1}{2}, \rho\lambda\right)\Big((\ell+1)\mathcal{P}_{\ell+1}(\cos\theta) - (\ell+1)\cos\theta\,\mathcal{P}_\ell(\cos\theta)\Big). \tag{5.8}$$

Because the PDE (5.1) is linear, any linear combination of the separated solutions (5.8) with the addition of the particular solution will also be a solution. This can be written in a general way as

$$\psi(\rho,\theta) = \frac{\gamma\rho^2\sin^2\theta}{\lambda^2} + \sum_{\ell=0}^{n} a_\ell\psi_\ell(\rho,\theta) \tag{5.9}$$

(see also [15]). Clearly this solution is no longer related to the spherical vortex but is an MHD equilibrium solution that can be considered in its own. A sample profile $P = P_0 - \gamma\psi$ of constant pressure surfaces, or equivalently, magnetic surfaces of this static MHD configuration with ψ given by (5.9) can be seen in Fig. 5.

Recall that natural physical requirements on MHD solutions are regularity and sufficient smoothness of the dependent variable (the components of **B** in (3.1), (3.3), and the corresponding behavior of the flux function ψ). Additionally, such conditions include the requirement of finite magnetic energy in the plasma domain $\mathcal{V}$,

$$\int_{\mathcal{V}} \frac{\mathbf{B}^2}{2\mu}\,dV \quad < \quad +\infty, \tag{5.10}$$

and the pressure asymptotics

$$P \to P_0 = \text{const} \quad \text{as} \quad |\mathbf{x}| \to +\infty. \tag{5.11}$$

Assuming nonnegative pressure values, one requires $P_0 = 0$ for vacuum configurations, whereas $P_0 > 0$ corresponds to a plasma supported by the pressure of an ambient medium. If the plasma domain is unbounded in one direction z, the finite energy requirement (5.10) may be restricted to a slice $z_1 \le z \le z_2$.

On the boundary surface $\partial\mathcal{V}$ of the plasma domain, if the magnetic field does not vanish there, it is commonly tangent to the boundary, $\mathbf{B}\cdot\mathbf{n} = 0$, on the "inside" side of $\partial\mathcal{V}$, and zero outside $\mathcal{V}$. In this case, the plasma domain boundary $\partial\mathcal{V}$ is a magnetic surface that simultaneously is a current sheet. Indeed, recall that the volumetric current density is related to the magnetic field by the equation

$$\mathbf{J} = \frac{1}{\mu}\operatorname{curl}\mathbf{B}. \tag{5.12}$$

Integration of the PDE (5.12) in a domain transverse to the boundary yields the surface electric current density given by

$$\mathbf{K} = \frac{\mathbf{B}}{\mu} \times \mathbf{n}, \tag{5.13}$$

that is, the current sheet coinciding with $\partial\mathcal{V}$.

Axially symmetric configurations like the one depicted in Fig. 5, i.e., solutions following from the separated ansatz (5.2) and their linear combinations (5.9), will often have magnetic surfaces given by closed curves in the (x, z) cross-section planes, that is, have the shape of tori in three dimensions. Such magnetic surfaces, when endowed with a current sheet as described above, may serve as the boundaries of plasma domain, with plasma confined within such a surface and magnetic field vanishing outside. Such configurations yield valid MHD equilibrium solutions satisfying all necessary physical conditions.

In the context of fluid dynamics, solutions described above, with $\mathbf{B} \equiv \mathbf{v}$ now denoting the fluid velocity, may be similarly restricted to a domain bounded by a given flux surface, with that surface being a *vorticity sheet* instead of the current sheet for the MHD model.

6. Instability of the Hill's spherical vortex

Consider the surface of the sphere $\rho = R$ corresponding to the boundary of the Hill's spherical vortex (Section 2). The dynamic equation satisfied by the stream function ψ, found in Hill's paper [11], is given by

$$\left(\frac{\partial}{\partial t} + \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\right)\left[\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right] = 0. \tag{6.1}$$

The main step in the current analysis is the introduction of an axisymmetric perturbation

$$\rho \mapsto \rho\cdot(1 + \epsilon h(\theta, t)) \tag{6.2}$$

of the Hill's solution (2.37) inside the sphere. The substitution yields the time-dependent stream function

$$\psi(\rho,\theta) = \delta\rho^2(1+\epsilon h(\theta,t))^2\sin^2\theta(\rho^2(1+\epsilon h(\theta,t))^2 - R^2). \tag{6.3}$$

This perturbed solution is now substituted into the spherical version of the dynamic ψ-Eq. (6.1), in order to analyze the dynamics of the perturbation $h(\theta, t)$. Specifically, we substitute $\rho = R$ to analyze the perturbation of the spherical boundary surface of the vortex itself. Discarding terms beyond the first order in ϵ, the following third order PDE for $h(\theta, t)$ (*the h-equation*) is obtained:

$$\begin{aligned} 2R\delta\sin\theta\frac{\partial^3 h}{\partial\theta^3} + \frac{\partial^3 h}{\partial t\partial\theta^2} + 6R\delta\cos\theta\frac{\partial^2 h}{\partial\theta^2} + 3\frac{\cos\theta}{\sin\theta}\frac{\partial^2 h}{\partial t\partial\theta} \\ - \frac{40R\delta}{\sin\theta}\left(\cos^2\theta - \frac{17}{20}\right)\frac{\partial h}{\partial\theta} + 20\frac{\partial h}{\partial t} = 0. \end{aligned} \tag{6.4}$$

The h-equation is a linear homogeneous PDE and thus is separable: one can seek its solutions as $h(\theta, t) = \Theta(\theta)T(t)$ where $\Theta(\theta)$ and $T(t)$ satisfy the separated ODEs

$$\begin{aligned} \frac{d^3\Theta}{d\theta^3} = -3\left(\frac{\cos\theta}{\sin\theta} + \frac{\lambda}{6R\delta\sin\theta}\right)\frac{d^2\Theta}{d\theta^2} \\ + \left(20\frac{\cos^2\theta}{\sin^2\theta} - 3\frac{\cos\theta}{2R\delta\sin^2\theta}\lambda - \frac{17}{\sin^2\theta}\right)\frac{d\Theta}{d\theta} - 10\frac{\lambda}{R\delta\sin\theta}\Theta, \end{aligned} \tag{6.5}$$

$$\frac{dT}{dt} = \lambda T. \tag{6.6}$$

where λ is the eigenvalue.[3]

The T-equation above has the exponential solution $T(t) = Ae^{\lambda t}$. The Θ Eq. (6.5) can be converted into a simpler equation with the transformation $z = \cos\theta$ with $\Theta(\theta) = Z(z)$. This gives

$$(1-z^2)\frac{d^3Z}{dz^3} - (2K+6z)\frac{d^2Z}{dz^2} + 8\left(2 + \frac{Kz}{1-z^2}\right)\frac{dZ}{dz} - \frac{40K}{1-z^2}Z = 0, \tag{6.7}$$

where $\lambda = 4KR\delta$. Solutions to (6.7) can be expressed as a linear combination of the following functions written in terms of the hypergeometric functions

$$Z_1 = \mathcal{H}\left(\left[\frac{3}{4} + \frac{\sqrt{89}}{4}, \frac{3}{4} - \frac{\sqrt{89}}{4}\right], \frac{1}{2}, z^2\right), \tag{6.8a}$$

$$Z_2 = z\mathcal{H}\left(\left[\frac{5}{4} + \frac{\sqrt{89}}{4}, \frac{5}{4} - \frac{\sqrt{89}}{4}\right], \frac{3}{2}, z^2\right), \tag{6.8b}$$

$$Z_3 = -Z_1\int_{z_0}^{z} z(z+1)^{1+\frac{\lambda}{4R\delta}}(z-1)^{1-\frac{\lambda}{4R\delta}} Z_2\,dz + Z_2\int_{z_0}^{z}(z+1)^{1+\frac{\lambda}{4R\delta}}(z-1)^{1-\frac{\lambda}{4R\delta}} Z_1\,dz. \tag{6.8c}$$

Here z_0 is an arbitrary constant satisfying $z_0 < z$. One should notice that both the first and second solution of (6.8) do not depend on the separation constant λ. This is because (6.7) can be written as

$$\mathcal{L} = \left(\frac{d}{dz} - \frac{2K}{1-z^2}\right)\mathcal{G}, \tag{6.9}$$

where

$$\mathcal{G} \equiv (1-z^2)\frac{d^2Z}{dz^2} - 4z\frac{dZ}{dz} + 20Z = 0. \tag{6.10}$$

Here (6.10) has the general solution

$$Z = C_1Z_1 + C_2Z_2 \tag{6.11}$$

where Z_1 and Z_2 are given in (6.8).

Because λ appears neither in Z_1 nor in Z_2, there will exist $h(\theta, t)$ which grows exponentially in time because λ may be positive. However, one must check and make sure that these $h(\theta, t)$ that grow in time correspond to regular surfaces ψ = const. One such $h(\theta, t)$ that gives regular surfaces utilizes Z_1 given above by (6.8a). This yields $h(\theta, t)$ as

$$h(\theta,t) = Ae^{\lambda t}\mathcal{H}\left(\left[\frac{3}{4} + \frac{\sqrt{89}}{4}, \frac{3}{4} - \frac{\sqrt{89}}{4}\right], \frac{1}{2}, \cos^2\theta\right) \tag{6.12}$$

This is now substituted into (6.3). After expanding out, and converting back to cylindrical coordinates, one arrives at the stream function form

$$\begin{aligned} \psi(r,z,t) = \delta r^2(r^2+z^2-R^2) \\ -2A\epsilon^2\delta r^2 e^{\lambda t}(R^2 - 2r^2 - 2z^2)\mathcal{H}\left(\left[\frac{3}{4} + \frac{\sqrt{89}}{4}, \frac{3}{4} - \frac{\sqrt{89}}{4}\right], \frac{1}{2}, \frac{r^2}{r^2+z^2}\right). \end{aligned} \tag{6.13}$$

[3] It is interesting to note that because (6.5) is third-order, there is no analog of Sturm–Liouville theory, and no guarantee of the possibility to obtain a general solution as a linear combination of modes described by (6.5), (6.6).

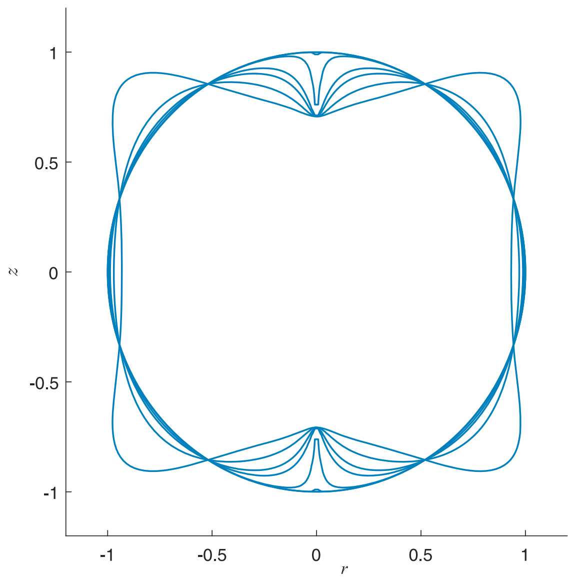


Fig. 6. Circle: the cross-section of the unperturbed spherical surface $\psi(r,z,t)=0$ (6.13) at $t\to-\infty$ or $\epsilon=0$. Curves: the evolution of the cross-section of the axisymmetric perturbed surface $\psi(r,z,t)=0$ (6.13) for $\epsilon=0.0001$, $\delta=1$, $R=1$, $A=1$, $\lambda=1$, at several different times $0<t<16$.

Several plots of the evolution of the surface $\psi=0$ are shown in Fig. 6. All instances of the surfaces shown in Fig. 6 are regular smooth axially symmetric surfaces. Despite the irregular look of these surfaces, the implicit derivative dz/dr of (6.13) when $\psi(r,z,t)=0$ can be shown to be zero on the cylindrical axis $r=0$. The spherical surface in Fig. 6 corresponds to $t\to-\infty$ or $\epsilon=0$, that is, a vanishing perturbation.

The above analysis leads to the following conclusion that Hill's spherical vortex is in general not linearly stable with respect to stream function surface perturbations described by (6.3).

7. Discussion

The self-propelling Hill's vortex (2.40) moving in an ambient asymptotically stationary fluid is a remarkable exact closed-form analytic solution of the ideal inviscid fluid dynamics equations. As shown in Section 2, instead of a tedious derivation presented in the original Hill's work [11], Hill's vortex solution can be obtained systematically from the equilibrium Euler equations (2.1): through the use of Galilei transformation (2.2) to pass to the moving frame of reference (Eqs. (2.3)), the use of axially symmetric velocity representation (2.7), with vanishing azimuthal component (2.8), through the stream function ψ that satisfies the Bragg–Hawthorne equation (2.5). The latter is subsequently considered in the linear case (2.12), converted to spherical coordinates, and solved by separation of variables inside and outside the spherical domain (Eqs. (2.15) and (2.16)), to yield the stream function in cylindrical coordinates (2.38). The corresponding velocity components (2.39) in the moving frame and (2.40) in the laboratory frame are continuous across the spherical boundary; velocity components in the stationary frame of reference satisfy $V_r, V_z\to 0$ as $r^2+z^2\to\infty$, and the pressure outside the vortex is the constant ambient pressure (see (2.41)).

Equilibrium fluid flow Eqs. (2.3) coincide with static magnetohydrodynamic Eqs. (3.1), with velocity in the former corresponding to the magnetic field in the latter. We show that the plasma vortex solution obtained by Bobnev [16] and Kaiser and Lortz [17] (see also Refs. [12–14]) is a generalization of the Hill's vortex that can be obtained in the same manner when one proceeds without the assumption of a vanishing azimuthal vector field component (Section 3). Indeed, through the indicated correspondence, this solution yields a generalized azimuthally rotating fluid vortex with the stream function (4.15) (Section 4).

The Bragg–Hawthorne/Grad-Shafranov equation in spherical coordinates (5.1) can be solved in the linear case by separation of variables, with the general solution (5.9) being a linear combination a particular solution and any number of separated solutions. This solution family contains multiple solutions whose flux surfaces are bent cylinders or tori, with no symmetry other than axial (Section 5; see, e.g., Fig. 5). In the context of plasma physics, spherical coordinates were used in Refs. [12–14,17,18] and other works to obtain families of exact static MHD equilibrium solutions. While such solutions do not usually satisfy physical requirements of finite energy, constant pressure, and magnetic field vanishing at infinity, in all space, special boundary conditions can be chosen corresponding to various physical situations. For example, a plasma configuration can be artificially "truncated" to restrict the plasma domain to the inside of any given, for example, toroidal magnetic surface which is simultaneously considered to be a current sheet. Outside of that surface, magnetic field is set to zero; this setup satisfies the material equations across the plasma domain boundary.[4] In the context of fluid dynamics, one is thus able to use families of separated solutions, and the general solution (5.9) in particular, as a stream function of a fluid flow confined within some chosen toroidal flux surface serving also as a vorticity sheet, with no fluid motion outside that domain.

More general linear (e.g., [8] and references therein) and nonlinear ansätze for the arbitrary functions in (2.5), (3.2) have been used in the literature. For example, Solov'ev [26] used a power series expansion ψ about the magnetic axis, and obtained cubic solutions that yield toroidal surfaces of ellipse-shaped cross-sections near the magnetic axis, and more generally, magnetic surfaces with separatrices (see formula (19) in [27]).

In Section 6, we revisited the question of stability of the original Hill's vortex. Considering a perturbation (6.2) of the spherical vortex boundary, we arrived at a cubic linear homogeneous Eq. (6.4) for the perturbation $h(\theta,t)$ in spherical coordinates. Separation of variables yielded an exponential time dependence and a third-order spatial ODE. It was shown that remarkably, some solutions of this ODE do not involve the spectral parameter, and therefore can correspond to exponential growth of $h(\theta,t)$ in time. A specific example was presented in Fig. 6 which shows a mode of decomposition of the spherical vortex boundary. In comparison, an earlier study [19] used a different approach: it considered linear stability of Hill's vortex with respect to axisymmetric perturbations that preserve circulation, which led to a linear integrodifferential perturbation equation and an eigenvalue problem arising from the consideration of the dynamics of linearized perturbations of the boundary of the Hill's vortex. Both stable and unstable eigenmodes were identified; the unstable eigenmodes were given by peaks localized at the rear stagnation point. This is in agreement with our analysis that showed growing perturbations of a similar kind located near both front ad rear stagnation points. It remains an open problem to study the stability of the generalized Hill's vortex with nonzero azimuthal component (Section 4).

In the context of plasma dynamics it is important to mention Galas-Bogoyavlenskij symmetry transformations [4,5]

$$\begin{aligned} &\mathbf{B}_1 = b(\psi)\mathbf{B} + c(\psi)\sqrt{\mu\chi}\,\mathbf{V}, \qquad \mathbf{V}_1 = \frac{c(\psi)}{a(\psi)\sqrt{\mu\chi}}\mathbf{B} + \frac{b(\psi)}{a(\psi)}\mathbf{V},\\ &P_1 = CP + \frac{C\mathbf{B}^2 - \mathbf{B}_1^2}{2\mu}, \qquad \chi_1 = a^2(\psi)\chi \end{aligned} \tag{7.1}$$

[4] We note that on other equilibrium solutions, for example, ones given in Refs. [21,22], the current density may vanish on the plasma domain boundary.

that map solutions $\mathbf{V}$, $\mathbf{B}$, P and χ (plasma density) of equilibrium MHD equations

$$\begin{aligned}
&\operatorname{div} \chi\mathbf{V} = 0,\\
&\chi\mathbf{V}\times\operatorname{curl}\mathbf{V} - \frac{1}{\mu}\mathbf{B}\times\operatorname{curl}\mathbf{B} - \operatorname{grad} P - \chi\operatorname{grad}\frac{\mathbf{V}^2}{2} = 0,\\
&\operatorname{curl}(\mathbf{V}\times\mathbf{B}) = 0,\\
&\operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{V} = 0
\end{aligned}$$

into an infinite family of solutions $\mathbf{V}_1$, $\mathbf{B}_1$, P_1 and χ_1. In (7.1), the density χ must be constant on both magnetic surfaces (or more generally, on magnetic field lines and plasma streamlines), $a(\psi)$ and $b(\psi)$ are arbitrary functions constant on both magnetic fields lines and streamlines, and $b^2(\psi) - c^2(\psi) = C =$ const is a free constant. The transformations (7.1) can be used to construct further exact explicit dynamic plasma equilibrium solutions, in particular, field-aligned solutions with nonzero plasma velocity, from static MHD configurations described in Sections 3 and 5 above. For a compact static equilibrium bounded by a magnetic surface, the new magnetic field and velocity will be tangent to that surface, satisfying closed boundary conditions. It is worth noting that Throumoulopoulos et al. [28] showed that if the plasma magnetic field has finite toroidal and poloidal components, axisymmetric equilibrium with purely poloidal flow is not possible except when the toroidal component of the magnetic field vanishes.

There exists broad literature that includes multiple models and solutions in the domain of the current contribution. In particular, field-aligned solutions, for axially symmetric cases different from those analyzed in this work and certain helically symmetric settings, were considered in [6,7,24]. Equations for field-aligned and non-field-aligned dynamic incompressible MHD equilibria in a particular ansatz were obtained in [29], and their exact solutions for sheared flows were found in [27]. A generalized Grad-Shafranov equation describing anisotropic plasmas and the corresponding toroidal analytic equilibrium solutions were obtained in [30]. Bogoyavlenskij-type transformations and the corresponding field-aligned analytical solutions for anisotropic plasmas were considered in [31]. In relation to those results and results presented in this paper, in future work, it is of interest to develop further closed-form exact solutions corresponding to physical plasma equilibria, in particular, configurations in the anisotropic plasma regime, and analyze their stability. In more general settings for Grad-Shafranov-type and similar models, when the governing equations are essentially not linearizable, it is intended to use the theory of approximate symmetries [32–34] to work towards constructing approximate solutions of such PDE models.

CRediT authorship contribution statement

Jason M. Keller: Writing – original draft, Validation, Software, Methodology, Formal analysis, Conceptualization. **Alexei F. Cheviakov:** Writing – review & editing, Validation, Supervision, Methodology, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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