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Galilei-invariant and energy-preserving extensions of Benjamin–Bona–Mahony-type equations

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ABSTRACT

The classical Benjamin–Bona–Mahony equation (BBM equation) models unidirectional propagation of long gravity surface waves of small amplitude. Unlike many other water wave models, it lacks the Galilean invariance, which is an essential property of physical systems. It is shown that by an addition of a higher asymptotic order nonlinear term, this deficiency can be corrected, giving rise to a new Galilei invariant Benjamin–Bona–Mahony equation (iBBM equation). Moreover, further additional higher-order terms can be chosen in a way that the augmented model preserves the energy conservation property along with Hamiltonian and Lagrangian structures. The resulting equation is referred to as energy-preserving Benjamin–Bona–Mahony equation (eBBM).

It is shown that both the classical BBM equation and the energy-preserving eBBM equations belong to a one-parameter (α) family that shares essentially the same local and nonlocal symmetries, conservation laws, Hamiltonian, and Lagrangian structures, with the BBM and eBBM equations corresponding to parameter values $\alpha = 0$ and $\alpha = 1$, respectively. Symmetry and conservation law classifications reveal a special case $\alpha = 1/3$, which is shown to correspond to a rescaled version of the celebrated integrable Camassa–Holm (CH) equation. Local symmetries and conservation laws are computed, and numerical solution behaviour is compared for the three BBM-type modes and the CH-equivalent $\mathrm{eBBM}_{1/3}$ model.

1. Introduction

The celebrated Benjamin–Bona–Mahony equation (BBM equation) was derived for the first time by D. Peregrine[1] in perfect agreement with Arnold's principle[a]. The BBM equation in scaled variables reads:

$$u_t + u_x + u u_x - u_{xxt} = 0, \tag{1.1}$$

where subscripts $(\cdot)_x$ and $(\cdot)_t$ denote ∂_x and ∂_t — the partial derivative with respect to the spatial x and temporal t variables respectively. The composition of these operators, *e.g.* ∂^2_{xt} applied to a (smooth) function u is denoted by u_{xt}. The paper[2] that gave the name[b] to this equation appeared six years later, along with one of the first numerical studies of its solitary wave interactions[4] by the same authors. One year later, a mathematical analysis of the BBM equation was published in Ref. 5. The numerical scheme employed in Ref. 4 was based on the analytical inversion of the elliptic operator $\mathbb{1} - \partial^2_{xx}$, where $\mathbb{1}$ is the identity operator in an appropriate functional space. These numerical experiments demonstrated the inelastic nature of the head-on collision in the BBM equation. We have to mention also another historical study[6] of the solitary wave interactions in the BBM equation.

Regarding the numerical tools applied to solve numerically the BBM equation, we already mentioned the integral formulation discretized with the classical trapezoidal rule used in Ref. 4. Later, this method has been studied and further employed in Ref. 5. A finite difference scheme has been proposed by D. H. Peregrine (1966)[1] and it was employed later by J. Hammack to study the wave generation by impulsive bottom motion[7] in the framework of the linear, weakly nonlinear (he referred to it as the Peregrine–Benjamin–Bona–Mahony equation (PBBM equation) to respect the correct historical attributions)

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[a] This principle can be summarized in two following points:
The Arnold Principle If a notion bears a personal name, then this name is not the name of the discoverer.
The Berry Principle The Arnold Principle is applicable to itself.

[b] We have to mention here that in some papers another name was adopted for the same equation — Regularized Long Wave equation (RLW equation). To complete the description of various avatars of the BBM equation, we mention also that this equation without linear advection term was called Equal Width equation (EW equation) in certain Ref. 3.

[c] They found this method to be the most accurate among all schemes considered in their studies.[10,11].

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and experimental measurements. This study had obvious applications to the understanding of the tsunami generation processes.[8,9] Another second order two-level finite difference scheme has been proposed later in Ref. 10. In the follow-up paper, Eilbeck and McGuire (1977) employed the three-level finite difference method[c] to study the interaction of solitary waves in the BBM equation. They came to a surprising conclusion (the bold face is ours):

> *[...] It was found that the solitary waves passed through a strongly nonlinear interaction region and reappeared with their original amplitudes, correct to a numerical error of $\lesssim 0.3\%$. Since this error is less than the error for the single solitary waves test program this means that the RLW solitary waves behave as true solitons to within the numerical error of the calculation. The calculation was repeated with various values of the soliton amplitude such that c in (1.4) varied in the range $0.05 \leq c \leq 1.0$ and the ratio of the two soliton amplitudes, $r = \frac{c_1}{c_2}$ was varied in the range $1.5 \leq r \leq 10.0$.* ***In all cases the typical collisional stability of a soliton-like interaction was observed.*** *[...]*

One year earlier, Abdulloev et al. (1976) already found that the interaction in the BBM equation was inelastic.[12] Moreover, in 1979, P. Olver proved that the BBM equation possesses only three *local* conservation laws.[13] Hence, it is not an S-integrable equation, in contrast to, for example, the celebrated Korteweg–de Vries (KdV) equation[14]:

$$u_t + u u_x + u_{xxx} = 0.$$

Hence, Eilbeck and McGuire attributed the inelastic effects to the numerical error. At the same time, Eilbeck and McGuire made also completely right observations about the solitary wave interactions in their paper.[11] To cite one example we can mention:

> *[...] The only obvious difference between the two solutions seems to be that the KDV collision region is smaller than the RLW collision region. This is presumably due to the fact that KDV solitons are narrower than RLW solitons with the same amplitude, as can be seen from eqs. (1.3) and (1.4). [...]*

The need for more accurate simulations pushed L. R. T. Gardner and G. A. Gardner (1990) to develop a Galerkin-type method based on cubic B-splines for local solution representation on each element.[15] Then, this method was applied to study the interaction of solitary waves in the BBM equation, and the inelastic character of this interaction was confirmed, in agreement with previous results of Abdulloev et al. (1976).[12] Moreover, L. R. T. Gardner and G. A. Gardner examined also the emergence of solitary waves from the initial Maxwellian pulse initial condition. Later, the continuous Galerkin methods for dispersive PDEs have been brought to perfection by V. Dougalis and his school.[16,17]

Motivated by the apparent existing controversy between finite difference and finite element methods predictions, the first Fourier-type pseudo-spectral exponentially accurate method has been proposed in Ref. 18. However, the final accuracy of pseudo-spectral methods for nonlinear problems is crucially dependent on the employed anti-aliasing technique (named the '*restrain operator*' in Ref. 18 or '*filtering*' in other studies such as Ref. 19).

At this point we shall stop the historical review of the BBM equation because all the references devoted to this equation are *practically* uncountable. To name a few subsequent developments in the framework of the BBM equation we shall just mention that the sharp well-posedness for this equation was studied in Ref. 20, a multi-symplectic numerical scheme was proposed in Ref. 21, the BBM equation on networks has been considered in Ref. 22, the solitonic gas in KdV–BBM equation has been studied in Ref. 23 and the optimal wave generation process in the framework of the forced BBM equation was studied in Ref. 24. One of the main (physical) drawbacks of the BBM equation is the absence of the Galilean invariance. This question has already been risen in the previous study of one of the authors of the present manuscript.[25] A remedy was proposed there. In this work we continue to discuss the same issue from the point of view of the modern symmetry and conservation law *systematic* analysis. On the other hand, we focus on a specific family of equations to make the perimeter of our study more embraced. For the first time, we extract an integrable equation based on physical reasoning and symmetry-type considerations. We hope that it will help the community to assess better the rôle played by integrable models in nonlinear waves.

The present manuscript is organized as follows. The classical BBM equation, its local symmetries, conservation laws, Hamiltonian and Lagrangian formulations, and other properties are briefly reviewed in Section 2. The new Galilei-invariant BBM-type model, asymptotically equivalent to the BBM equation (in the sense specified below), called the invariant Benjamin–Bona–Mahony equation (iBBM equation), is derived and studied in Section 3. In particular, this equation is shown to have an interesting exponential-type conserved quantity, but it lacks the common conservation of energy present for the classical BBM equation. In Section 4, by using the conservation law multiplier ideas and adding another term of a higher asymptotic order, we obtain a new PDE, referred to as the energy-preserving Benjamin–Bona–Mahony equation (eBBM) equation, which has both the Galilean invariance and energy conservation properties. Compared to the classical BBM equation (1.1), it also admits an additional conservation law with a time-dependent density. Similarly to the KdV equation and the classical BBM equation (1.1), the potential form of the eBBM equation arises from a Lagrangian variational formulation.

Both the classical BBM equation (1.1) and the eBBM equation are members of a one-parameter family of PDEs (the so-called *α-family*) considered in Section 5. All equations in this family share three common point symmetries and three local conservation laws, including a conserved quantity corresponding to the Hamiltonian density. The potential form of each PDE from the α-family yields a Lagrangian variational formulation. Local and nonlocal symmetry and local conservation law computations and classifications with respect to the constant parameter α are performed. It is shown that additional symmetries and conservation laws arise in only two cases, one corresponding to the eBBM ($\alpha = 1$), and another one to $\alpha = 1/3$. The latter case, referred to as eBBM$_{1/3}$ equation, admits local symmetries and conservation laws with characteristics of negative half-integer orders, similar to those of the well-known integrable Short Pulse equation (SP equation) and Camassa–Holm (CH) models that appeared in the context of nonlinear optics.[26,27] Indeed, it turns out that the eBBM$_{1/3}$ equation is nothing but a scaled version of the celebrated integrable Camassa–Holm (CH) equation. The four models of interest in this paper, namely, the BBM equation, the iBBM equation, the eBBM, and the eBBM$_{1/3}$, are investigated numerically in Section 6, in terms of the form and behaviour of solitary wave and transient solutions, and the numerical conservation of energy.

2. The classical BBM equation and its properties

The original BBM equation in scaled variables is given by

$$v_t + \sqrt{gd}\, v_x + \frac{3}{2}\, v v_x - \frac{d^2}{6}\, v_{xxt} = 0, \tag{2.1}$$

where d is the still water depth, g is the gravity acceleration constant and $v(x, t)$ is the horizontal velocity variable, usually associated with the depth-averaged horizontal velocity of the fluid. After a further rescaling change of variables, the PDE (2.1) can be written in the form

$$u_t + u_x + \frac{3}{2}\, \varepsilon u u_x - \frac{\delta^2}{6}\, u_{xxt} = 0, \tag{2.2}$$

which explicitly contains the small parameters ε, δ. It is also assumed that

$$\varepsilon \sim \delta^2 \ll 1,$$

which describes the well-known asymptotic Boussinesq regime.[17] Under the following rescaling

$$u := \frac{2}{3\varepsilon} u^*, \qquad x := \frac{\sqrt{6}}{\delta} x^*, \qquad t := \frac{\sqrt{6}}{\delta} t^* \tag{2.3}$$

and after omitting the asterisks, one recovers the familiar form (1.1) of the BBM equation. For the presentation of analytical results, we shall use the BBM equation in the form (1.1) since it is the simplest one. We also notice that by a translation change of variables $u \leftarrow w - 1$, or by a change of the reference frame and a subsequent rescaling, the BBM equation (1.1) can be recast in a yet simpler form:

$$w_t + w w_x - w_{xxt} = 0.$$

Up to the last term of the same order, the BBM equation coincides with the KdV and has the same travelling wave reduction ODE form, which prompts the existence of solitary wave solutions to the BBM equation even if their shape and velocity differ. Indeed, such solutions have been found in Ref. 13.

2.1. Symmetries

The point symmetries of the PDE (1.1) are given by the following infinitesimal generators:

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad \mathrm{X}_3 = t\,\partial_t - (u + 1)\,\partial_u. \tag{2.4}$$

As it can be seen, unlike the KdV, this model is not Galilean invariant as it was diagnosed in Ref. 25. The global symmetry group corresponding to X_1 in (2.4) is the time translation $t^* := t + a,\, x^* := x,\, u^* := u$, where $a \in \mathbb{R}$ is the group parameter, with $a = 0$ corresponding to an identity transformation. Similarly, the symmetry generator X_2 corresponds to the spatial translation $t^* := t,\, x^* := x + a,\, u^* = u$, and the generator X_3 to the scaling of t and $u + 1$:

$$t^* := \mathrm{e}^{-a} t, \qquad x^* := x, \qquad u^* + 1 := \mathrm{e}^{a}(u + 1). \tag{2.5}$$

We note that for the BBM equation in the form (2.2) involving the small parameters, the infinitesimal generators (2.4) are given by

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad \mathrm{X}_3 = \varepsilon t\,\partial_t - \left(\varepsilon u + \frac{2}{3}\right)\partial_u.$$

It follows that the BBM equation (1.1), (2.2) is not Galilei-invariant. However, since it contains a small parameter in the form (2.2), one can consider the so-called *approximate point symmetries*[28] with infinitesimal generators in the form

$$\mathrm{X} = \mathrm{X}^{(0)} + \varepsilon\,\mathrm{X}^{(1)} + \mathcal{O}(\varepsilon^2). \tag{2.6}$$

For the BBM equation in the form (2.2) involving small parameters, considered with a single small parameter $\varepsilon \sim \delta^2$, we note that the zeroth-order part

$$u_t + u_x + \mathcal{O}(\varepsilon) = 0$$

is linear. So, there are infinitely many approximate symmetries. In particular, an *approximate* Galilean symmetry is present. It has the form (2.6) with $\mathrm{X}^{(0)} = \partial_u$ and $\mathrm{X}^{(1)} = \frac{3}{2} t\,\partial_x$. Finally, the approximate Galilean symmetry is given by

$$\mathrm{X}_G = \partial_u + \frac{3}{2}\varepsilon t\,\partial_x, \tag{2.7}$$

with the corresponding global group

$$t^* := t, \qquad x^* := x + \frac{3}{2} a \varepsilon t, \qquad u^* := u + a,$$

involving the group parameter $a \in \mathbb{R}$.

Due to the presence of the translation symmetries $\mathrm{X}_{1,2}$ in (2.4), the BBM equation admits a travelling wave reduction $u(x,t) = q(\xi)$, with $\xi = x - ct$, with $c \in \mathbb{R}$. In particular, it has solitary travelling wave solutions[13] of the form

$$u(x,t) = \frac{3c^2}{1 - c^2}\,\mathrm{sech}^2\left[\frac{c}{2}\left(x - \frac{t}{1 - c^2} + d\right)\right],$$

where $c, d \in \mathbb{R}$ are arbitrary constants. The case $|c| > 1$ corresponds to a left-travelling wave of depression, while the case $|c| < 1$ to a right-travelling wave of elevation.

The BBM equation is not believed to be S-integrable or C-integrable. In particular, no Lax pair has been found and the space of local Conservation Law (CL) is only three-dimensional, as discussed below.

2.2. Conservation laws

The three local conservation laws holding on solutions of the BBM equation (1.1) are given by

$$\mathcal{D}_t(u - u_{xx}) + \mathcal{D}_x\left(u + \frac{u^2}{2}\right) = 0, \tag{2.8}$$

$$\mathcal{D}_t\left(\frac{1}{2}(u^2 + u_x^2)\right) + \mathcal{D}_x\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 - u u_{xt}\right) = 0, \tag{2.9}$$

$$\mathcal{D}_t\left(\frac{1}{2}u^2 + \frac{1}{6}u^3\right) + \mathcal{D}_x\left(\frac{1}{8}u^4 + \frac{1}{2}(u - u_{xt} + 1)u^2 - u u_{xt} + \frac{1}{2}(u_{xt}^2 - u_t^2)\right) = 0.$$

The above CL forms are unique up to a CL-equivalence.[29] They are readily found from the multiplier (the so-called *direct method*) approach (see *e.g.*, Ref.[29] and references therein) from the respective multipliers

$$\Lambda_1 = 1, \qquad \Lambda_2 = u, \qquad \Lambda_3 = u + \frac{1}{2}u^2 - u_{xt}. \tag{2.10}$$

The three corresponding (locally) conserved quantities in an arbitrary interval $x \in (a,b) \subseteq \mathbb{R}$ with vanishing boundary effects (either zero or periodic boundary conditions), are given by

$$\mathcal{M}(t) = \int_a^b (u - u_{xx})\,\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t} = 0, \tag{2.11}$$

$$\mathcal{E}(t) = \int_a^b \frac{1}{2}(u^2 + u_x^2)\,\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = 0, \tag{2.12}$$

$$\mathcal{H}(t) = \frac{1}{2}\int_a^b \left(u^2 + \frac{1}{3}u^3\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0. \tag{2.13}$$

The first CL corresponds to the conservation of the momentum. The second CL corresponds to the conservation of the kinetic energy. Indeed, u^2 and u_x^2 represent the squares of the horizontal and vertical[d] velocity components. The meaning of the third conservation law will become clearer below. It has been shown in Ref. 13 that the above list of CLs is complete.

2.3. Hamiltonian and Lagrangian structure

The conservation law (2.13) corresponds to a Hamiltonian of the BBM equation (1.1), *c.f.* Ref. 25 [Section §2.2]. Indeed, the Hamiltonian formulation is provided by

$$u_t = \mathbb{J}\,\frac{\delta\mathcal{H}}{\delta u}, \tag{2.14}$$

where $\delta/\delta u$ denotes the usual variational derivative (that is, the Euler operator; see *e.g.*, Refs. 29, 31). The nonlocal symplectic operator $\mathbb{J}$ and the Hamiltonian density are defined as

$$\mathbb{J} = \left(\mathbb{1} - \partial_x^2\right)^{-1}\cdot(-\partial_x), \tag{2.15}$$

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}}\left(u^2 + \frac{1}{3}u^3\right)\mathrm{d}x, \tag{2.16}$$

[d] In this footnote we clarify the connection between u_x with the vertical velocity. Let us denote by (U, V) the 2D velocity field in the fluid. If we approximate the horizontal velocity U by its depth-average value u which does not depend on the vertical coordinate y, i.e., take $U \approx u$, then from the flow incompressibility condition $U_x + V_y = 0$ and the bottom impermeability, we obtain that $V = -(y + d)u_x$. For more details, refer to Ref. 30, Section §2.1.

with $\mathbb{1}$ being the identity operator.

The BBM equation (1.1) also arises from a variational principle. The linearization of the PDE (1.1) is not self-adjoint as it stands. However, the potential equation

$$w_{xt} + w_{xx} + w_x w_{xx} - w_{xxxt} = 0 \tag{2.17}$$

written for the nonlocal variable $w(x,t)$ defined through $u := w_x$, has a self-adjoint linearization, and arises from a Lagrangian action[32]

$$\mathcal{L}[w] \overset{\text{def}}{:=} -\frac{1}{2}\int_{\mathbb{R}} \left(\frac{1}{3}w_x^3 + w_x^2 + w_x w_t + w_{xx} w_{xt}\right) \mathrm{d}x. \tag{2.18}$$

3. A Galilei-invariant BBM equation

The invariant Benjamin–Bona–Mahony equation (iBBM equation) equation was proposed in Ref. 25, Section §2.3 to recover the Galilean invariance property by adding an asymptotically negligible term to Eq. (1.1). The iBBM equation after rescaling (2.3) is given by

$$u_t + u_x + uu_x - u_{xxt} - uu_{xxx} = 0. \tag{3.1}$$

Note that in dimensionless variables involving the small parameters ε and δ, it has the form:

$$u_t + u_x + \frac{3}{2}\varepsilon uu_x - \frac{\delta^2}{6}u_{xxt} - \frac{\varepsilon\delta^2}{4}uu_{xxx} = 0. \tag{3.2}$$

From this last equation it can be seen that the last term has the asymptotic order $\mathcal{O}(\varepsilon^2) \sim \mathcal{O}(\delta^4) \ll \mathcal{O}(\varepsilon + \delta^2)$ under the usual Boussinesq regime assumption that $\varepsilon \sim \delta^2$.

We would like to note that in the iBBM equation, the second linear term u_x cannot be removed by a translation of variables as before. However, instead, one may be interested to consider also the iBBM equation without the u_x term:

$$u_t + uu_x - u_{xxt} - uu_{xxx} = 0. \tag{3.3}$$

This model (3.3) has the following properties:

- From the physical point of view, it corresponds to the regime of small gravity ($g \ll 1$) and finite depth ($d \sim 1$, $\sqrt{gd} \sim 1$).
- The scaling symmetry X_3 in (2.4) is preserved.

The travelling wave solutions to (3.3) are explicit exponentially decaying peakons. If $\xi = x - ct$, one gets the following reduction:

$$-cu' + uu' + cu''' - uu''' = (u - c)(u' - u''') = 0,$$

where by primes we denote the ordinary derivatives with respect to ξ. It is not difficult to check that *peakons*

$$u(x,t) = C_1 + C_2 \mathrm{e}^{|x - ct|} \tag{3.4}$$

$C_1, C_2 = \text{const}$, are exact solutions to (3.3). Numerical simulations suggest that isolated peakons (3.4) are stable only in the integrable cases $p \in \{0\}2, 3$. The iBBM equation (3.3) is a part of the peakon subfamily ($\kappa = 0$) of the famous b-family of PDEs

$$u_t + 2\kappa u_x + (b+1)uu_x - bu_x u_{xx} - u_{xxt} - uu_{xxx} = 0 \tag{3.5}$$

that includes the two well-known integrable models: the Degasperis–Procesi equation[33] ($b = 3$) and the Camassa–Holm equation[34,35] ($b = 2$), which admit peakon solutions when $\kappa = 0$. The iBBM equation (3.3) itself corresponds to $b = 0$.

3.1. Symmetries and conservation laws

The point symmetries of the iBBM equation (3.1) are given by

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad \mathrm{X}_3 = t\partial_x + \partial_u, \tag{3.6}$$

with Galilei invariance restored and the scaling symmetry lost, as compared to the symmetries (2.4) of the BBM equation (1.1). In particular, the generator X_3 corresponds to the global group

$$t^* := t, \qquad x^* := x + t, \qquad u^* := u + a \tag{3.7}$$

with the group parameter $a \in \mathbb{R}$. We note that for the iBBM equation in the dimensionless form (3.2) explicitly involving small scaling parameters, the exact point symmetries (3.6) become

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad \mathrm{X}_3 = \frac{3}{2}\varepsilon t\partial_x + \partial_u.$$

The latter exact symmetry generator coincides with the approximate generator X_G of the BBM equation (2.7).

No theoretical completeness result or the maximal order of local conservation laws is known for the iBBM equation. Noting that the equation has a Kovalevskaya form with respect to u_{xxx}, we employ the direct construction (multiplier) method[36,37] and restricting to the third order multipliers of the form

$$\Lambda = \Lambda(x, t, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, u_{xxt}, u_{xtt}, u_{ttt}).$$

Unlike the original BBM equation, the iBBM equation (3.1) in this ansatz has two CLs corresponding to the multipliers

$$\Lambda_1 = 1, \qquad \Lambda_2 = \mathrm{e}^{u - u_{xx}}.$$

The corresponding local CLs are given by

$$\mathcal{D}_t(u - u_{xx}) + \mathcal{D}_x\left(\frac{1}{2}(u^2 + u_x^2) + u(1 - u_{xx})\right) = 0, \tag{3.8a}$$

$$\mathcal{D}_t\left(\mathrm{e}^{u - u_{xx}}\right) + \mathcal{D}_x\left(u\mathrm{e}^{u - u_{xx}}\right) = 0. \tag{3.8b}$$

The first CL (3.8a) describes the conservation of momentum property (the same conserved density as in (2.8) for the BBM equation), while the two remaining local conservation laws do not hold for the iBBM equation. In particular, the conservation of energy (2.9) does not carry over, being replaced by the exponential CL (3.8b). The respective global conserved quantities for vanishing or periodic boundary conditions over an interval $(a, b) \subseteq \mathbb{R}$ are given by

$$\mathcal{M}(t) = \int_a^b (u - u_{xx})\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t} = 0,$$
$$\mathcal{P}(t) = \int_a^b \mathrm{e}^{u - u_{xx}}\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = 0.$$

No Lagrangian or Hamiltonian formulation is known for the iBBM equation (3.1).

4. An energy-preserving Galilei-invariant BBM model

In the previous Section we observed that compared to the BBM equation in the iBBM equation, the physically important Galilean invariance property was recovered, but the energy conservation law (2.9) was lost. The goal of this Section is to add another higher-order $\mathcal{O}(\delta^4)$ term to iBBM equation (3.2) to recover the conservation of energy as an exact conservation law. We assume that the multiplier for the energy CL is given by $\Lambda = u$ as in the usual BBM equation (1.1). We observe that the iBBM equation (3.1) taken with the same multiplier becomes

$$u(u_t + u_x + uu_x - u_{xxt} - uu_{xxx}) = \mathcal{D}_t\left(\frac{1}{2}(u^2 - uu_{xx})\right) + \mathcal{D}_x\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 + \frac{1}{2}(u_x u_t - uu_{xt})\right) - u^2 u_{xxx} \equiv 0.$$

The last non-divergence term we have:

$$u^2 u_{xxx} = \mathcal{D}_x(u^2 u_{xx}) - 2uu_x u_{xx}.$$

Consequently, we construct a new governing equation with an extra term:

$$u_t + u_x + uu_x - u_{xxt} - uu_{xxx} - 2u_x u_{xx} = 0. \tag{4.1}$$

The same equation in dimensionless variables with small parameters is given by

$$u_t + u_x + \frac{3}{2}\varepsilon uu_x - \frac{\delta^2}{6}u_{xxt} - \frac{\varepsilon\delta^2}{4}uu_{xxx} - \frac{\varepsilon\delta^2}{2}u_x u_{xx} = 0. \tag{4.2}$$

The last model has one more additional term (the last term) of the order $\mathcal{O}(\varepsilon\delta^2) = \mathcal{O}(\delta^4)$. If one uses the strictly asymptotic reasoning, these extra terms have to be neglected and one shall loose the Galilean invariance and the energy conservation properties. The model (4.1) (unscaled), (4.2) (in scaled variables) has another advantage of having the energy CL with the same multiplier as the original BBM equation. Moreover, since the new term is Galilean-invariant, it will also have the Galilean boost symmetry as did the iBBM equation. From now on, we shall refer to Eq. (4.1) as the energy-preserving Benjamin–Bona–Mahony equation (eBBM).

Remark 4.1. The eBBM equation (4.1) appeared in physical (unscaled) variables in Ref. 25, Remark 1, Equation (13) using a slightly different variational (more precisely, Hamiltonian) reasoning. So, we refer to Ref. 25, Section §2.3 for the Hamiltonian structure of the eBBM equation. Later, this equation appeared (without the linear advection term u_x, which can be easily removed by changing the frame of reference) in Ref. 38 under the name of a regularized Burgers equation (rB equation).

Remark 4.2. There is a striking similarity between the eBBM equation and the p-family of PDEs non-zero dispersion, given by

$$u_t + \kappa u_x + (p+1)uu_x - pu_xu_{xx} - u_{xxt} - uu_{xxx} = 0.$$

The case $\kappa \equiv 0$ corresponds to the peakon p-family (3.5) including the Camassa–Holm ($p = 2$)[34,35,39] and Degasperis–Procesi ($p = 3$) integrable equations. However, it is not difficult to see that Eq. (4.1) does not belong to this PDE family. A similar model equation was also derived in Ref. 39, Equation (3.17)[e] as a unidirectional reduction of the SGN system system. However, the authors recognized the lack of the Galilean invariance in this equation, despite the beauty of its derivation.

4.1. Well-posedness theory

A class of nonlinear equations including the eBBM has been studied in two papers by Z. Yin.[40,41] The following local existence result has been proven there:

Theorem 4.1. *Given an initial datum $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, then there exists a maximal time $T^\star = T^\star(\|u_0\|_s) > 0$ and a unique solution u to Eq. (4.1) such that*

$$u = u(\cdot, u_0) \in C\left([0, T^\star[; H^s(\mathbb{R})\right) \bigcap C^1\left([0, T^\star[; H^{s-1}(\mathbb{R})\right).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$H^s \longrightarrow C\left([0, T^\star[; H^s(\mathbb{R})\right) \bigcap C^1\left([0, T^\star[; H^{s-1}(\mathbb{R})\right),$$
$$u_0 \longmapsto u(\cdot, u_0)$$

is continuous. If $T^\star < +\infty$, then

$$\lim_{t\uparrow T^\star} \|u(t,\cdot)\|_{H^s} = +\infty.$$

Moreover, if $s \geq 3$ and $T^\star < +\infty$, then

$$\lim_{t\uparrow T^\star} \inf_{x\in\mathbb{R}} u_x(t,x) = -\infty.$$

A better bound on the blow-up time $T^\star$ was obtained in Ref. 38:

Theorem 4.2. *Let the initial datum $u_0 \in H^s$ be non-trivial with $s \geq 2$ and let $\forall t < T^\star$:*

$$m(t) \stackrel{\text{def}}{:=} \inf_{x\in\mathbb{R}} u_x(t,x) < 0 < \sup_{x\in\mathbb{R}} u_x(t,x) \stackrel{\text{def}}{=:} M(t).$$

- *If $|m(0)| \geq M(0)$, then*
$$-\frac{1}{\inf_{x\in\mathbb{R}} u_{0,x}(x)} \leq T^\star \leq -\frac{2}{\inf_{x\in\mathbb{R}} u_{0,x}(x)}.$$
- *If $|m(0)| < M(0)$, then there exists $t^\star$ such that $0 < t^\star \leq -\frac{1}{m(0)} - \frac{1}{M(0)}$ and $m(t^\star) = -M(t^\star)$. Therefore,*
$$t^\star + \frac{1}{\sup_{x\in\mathbb{R}} u_{0,x}(x)} \leq T^\star \leq -\frac{2}{\inf_{x\in\mathbb{R}} u_{0,x}(x)}.$$

Finally, the global existence of conservative solutions,[f] not necessarily vanishing as $|x| \to +\infty$ was also established in Ref. 38, Section §4:

Theorem 4.3. *Let $u_0 \in \dot{H}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, where the homogeneous Sobolev space $\dot{H}^1(\mathbb{R})$ is defined as*

$$\dot{H}^1(\mathbb{R}) \stackrel{\text{def}}{:=} \{f \mid \|f'\|_2 < +\infty\}.$$

If there exists a Lipschitz function ϕ such that $\phi' \in L^1(\mathbb{R})$ with $u_0 - \phi \in H^1(\mathbb{R})$, then there exists a global conservative solution[f] u to Eq. (4.1) such that $u(t,\cdot) - \phi \in H^1(\mathbb{R})$ for $\forall t > 0$. In addition, for $\forall T^\star > 0$, if $\lim_{t\uparrow T^\star} \inf_{x\in\mathbb{R}} u_x(t,x) = -\infty$, then $\lim_{t\downarrow T^\star} \sup_{x\in\mathbb{R}} u_x(t,x) = +\infty$. If $u_0 \in H^1(\mathbb{R})$, then $\forall t > 0$ we have

$$\int_{\mathbb{R}} \left(u(t,x)^2 + u_x(t,x)^2\right) \mathrm{d}x = \int_{\mathbb{R}} \left(u_0(x)^2 + u_{0,x}(x)^2\right) \mathrm{d}x.$$

The formation of singularity (*i.e.* the derivative blow-up) is observed below by our numerical simulations.

4.2. Symmetries and conservation laws

The three point symmetries of the eBBM equation (4.1) coincide with symmetries of the iBBM equation given by (3.6). In particular, the Galilean invariance is preserved.

We now seek the CLs of the eBBM equation (4.1) with up to the third-order multipliers:

$$\Lambda = \Lambda(x, t, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, u_{xxt}, u_{xtt}, u_{ttt}).$$

It can be shown that the full set of such CLs is given by the four conservation laws with multipliers

$$\Lambda_1 = 1,$$
$$\Lambda_2 = u,$$
$$\Lambda_3 = u^2 - u_x^2 - 2(u_{xt} + uu_{xx}), \tag{4.3a}$$
$$\Lambda_4 = x - t(u+1). \tag{4.3b}$$

We observe that no third-order CLs arise; otherwise, the first two multipliers coincide with the multipliers (2.10) for the BBM equation, while the multiplier Λ_3 (4.3a) differs from the corresponding multiplier for the BBM equation by the expression in parentheses. The multiplier Λ_4 (4.3b) is new. The divergence forms of the local CLs, up to a CL equivalence, are given by

$$\mathcal{D}_t(u - u_{xx}) + \mathcal{D}_x\left(\frac{1}{2}(u^2 - u_x^2) + u(1 - u_{xx})\right) = 0, \tag{4.4}$$

$$\mathcal{D}_t\left(\frac{1}{2}(u^2 + u_x^2)\right) + \mathcal{D}_x\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 - u(u_{xt} + uu_{xx})\right) = 0, \tag{4.5}$$

$$\begin{aligned}
&\mathcal{D}_t\left(\frac{1}{3}u^3 + (u-1)u_x^2\right) \\
&+ \mathcal{D}_x\Big(\frac{1}{4}(u^4 + u_x^4) + \frac{1}{3}u^3 - (u^2 - u_x^2)u_{xt} + \\
&u^2u_{xx}^2 - u_t^2 + u_{xt}^2 - uu_x^2 - \frac{1}{2}u^2u_x^2 - 2uu_tu_x + \\
&(u_x^2 - u^2 + 2u_{xt} + uu_{xx})uu_{xx}\Big) = 0,
\end{aligned} \tag{4.6}$$

[e] More precisely, Equation (3.17) in Ref. 39 is given in physical variables while Equation (3.20) is its scaled version with a few extra typos.

[f] For the definition of a *conservative solution* we refer to Ref. 38, Definition 4.1.

$$\mathcal{D}_t\left(\left(t - x + \frac{1}{2}tu\right)(u - u_{xx})\right) + \mathcal{D}_x\left(\frac{1}{3}tu^3 - \left(\frac{1}{2}x + t(u_{xx} - 1)\right)u^2 + \left((t - x)(u_{xx} - 1) + \frac{1}{2}(u_x + tu_{tx})\right)u + \frac{1}{2}\left(2 + tu_t + (x - t)u_x\right)u_x\right) = 0. \tag{4.7}$$

The first CL (4.4) of the eBBM equation (4.1), with $\Lambda_1 = 1$, corresponds to the conservation of momentum, and has the same conserved quantity $\mathcal{M}(t)$ (2.11) as the BBM equation (1.1). Similarly, the second CL (4.5) with $\Lambda_2 = u$ has the conserved quantity $\mathcal{E}(t)$ (2.12) which is the total kinetic energy, equal to that for the BBM equation (1.1). For the third CL (4.6), the conserved quantity is

$$\mathcal{N}(t) = \int_a^b \left(\frac{1}{3}u^3 + (u - 1)u_x^2\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t} = 0, \tag{4.8}$$

which is conserved provided the suitable boundary conditions at $\infty \leq a < b \leq +\infty$. The conserved quantity (4.8) is related to the Hamiltonian structure of the eBBM equation (4.1), as shown below. The final CL (4.7) has the associated conserved quantity

$$C(t) = \int_a^b \left(t\left(1 + \frac{1}{2}u\right) - x\right)(u - u_{xx})\mathrm{d}x, \tag{4.9}$$

with an explicitly x- and t-dependent conserved density. Under suitable boundary conditions, (4.9) is a constant, and can be written as $C(t) =$ const, or

$$C(t) = C_0(t) + t \cdot C_1(t), \tag{4.10}$$

with

$$C_0 \stackrel{\text{def}}{:=} -\int_a^b x(u - u_{xx})\mathrm{d}x,$$

$$C_1 \stackrel{\text{def}}{:=} \int_a^b \left(1 + \frac{1}{2}u\right)(u - u_{xx})\mathrm{d}x.$$

making (4.10) similar to the "*centre of mass theorem*" arising in continuum mechanics and describing the motion of the centre of mass (see, *e.g.*, Ref. 42, Equation (3.24)).

It is possible to show that the eBBM equation (4.1) can be written in a Hamiltonian form.[25, Section §2.3] The Hamiltonian is a linear combination of the conserved energy $\mathcal{E}(t)$ (2.12) and the quantity $\mathcal{N}(t)$ (4.8). Also, similarly to the BBM equation (1.1), the eBBM equation (4.1) has a Lagrangian variational formulation, specifically, a potential form that is an Euler–Lagrange equation for some Lagrangian density. Details are presented in Section 5 below, where we consider a more general one-parameter family of PDEs that includes both the BBM equation (1.1) and eBBM equation (4.1), sharing common Lagrangian and Hamiltonian structures.

5. The α-family of BBM-type equations

Let us consider a one-parameter family of equations

$$u_t + u_x + uu_x - u_{xxt} - \alpha(uu_{xxx} + 2u_xu_{xx}) = 0, \tag{5.1}$$

with $\alpha \in \mathbb{R}$. The PDE (5.1) reduces to the BBM equation (1.1) when $\alpha = 0$, and to the eBBM equation (4.1) when $\alpha = 1$. It is straightforward to show that the parameter α in Eq. (5.1) cannot be removed by a rescaling or other simple equivalence transformation (see *e.g.*,[43] and references therein). In dimensionless variables, the PDE family (5.1) has the form

$$u_t + u_x + \frac{3}{2}\varepsilon uu_x - \frac{\delta^2}{6}u_{xxt} - \alpha\frac{\varepsilon\delta^2}{4}(uu_{xxx} - 2u_xu_{xx}) = 0,$$

the α-term having the highest asymptotic order $\mathcal{O}(\varepsilon^2) \sim \mathcal{O}(\delta^4)$ in the so-called Boussinesq regime $\varepsilon \sim \delta^2$, *c.f.* Ref. 17.

In this section, we study the properties of the PDE PDE family (5.1) common for all equations, including the BBM equation (1.1) and eBBM equation (4.1), such as common symmetries, conservation laws, Lagrangian and Hamiltonian structures, and classify with respect to α to reveal cases that have special properties.

5.1. Common symmetries, conservation laws, and hamiltonian structure of the α-family

The point symmetries of the family of PDEs (5.1) holding for all α are given by the generators

$$\mathrm{X}_1 = \partial_t, \tag{5.2a}$$

$$\mathrm{X}_2 = \partial_x, \tag{5.2b}$$

$$\mathrm{X}_3 = (\alpha - 1)t\partial_t + \alpha t\partial_x + (1 + (1 - \alpha)u)\partial_u, \tag{5.2c}$$

matching (2.4) for $\alpha = 0$, and (3.6) for $\alpha = 1$. In particular, when $\alpha = 0$, the generator X_3 yields the scaling symmetry group (2.5) holding for the BBM equation (1.1), and when $\alpha = 1$, X_3 yields the Galilei group (3.7) holding for the eBBM equation (4.1). The latter is the only Galilei-invariant representative of the PDE family (5.1).

A direct computation shows that there are no higher-order symmetries for any member of the family (5.1) depending on derivatives of u up to the second-order, *i.e.*, with generators of the form

$$\hat{\mathrm{X}} = \zeta(x, t, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx})\partial_u, \tag{5.3}$$

except the point symmetries (5.2). Moreover, one can show there are no symmetries holding for an arbitrary α with components depending on x, t, u, and x-derivatives of u up to order five.

In the case of an arbitrary α, the α-family (5.1) admits three local conservation laws with multipliers

$$\Lambda_1 = 1, \tag{5.4a}$$

$$\Lambda_2 = u, \tag{5.4b}$$

$$\Lambda_3 = u^2 - 2u_{xt} - \alpha(u_x^2 + 2uu_{xx}). \tag{5.4c}$$

It follows that all PDEs from the family (5.1) conserve the momentum (multiplier (5.4a), conserved quantity $\mathcal{M}(t)$ (2.11)) and energy (multiplier (5.4b), conserved quantity $\mathcal{E}(t)$ (2.12)). These conservation laws are shared with the BBM equation (1.1) and the eBBM equation (4.1). The multiplier Λ_3, also common for the whole α-family (5.1), yields a modified version of the conserved quantity (4.8):

$$\mathcal{N}_\alpha(t) = \int_a^b \left(\frac{1}{3}u^3 + (\alpha u - 1)u_x^2\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{N}_\alpha}{\mathrm{d}t} = 0.$$

The latter, taken in a linear combination with $\mathcal{E}(t)$ (2.12), leads to a conserved quantity

$$\mathcal{H}_\alpha(t) = \frac{1}{2}\int_a^b \left(u^2 + \frac{1}{3}u^3 + \alpha uu_x^2\right)\mathrm{d}x, \qquad \frac{\mathrm{d}\mathcal{H}_\alpha}{\mathrm{d}t} = 0.$$

which defines a Hamiltonian for the whole α-family (5.1). Indeed, the PDEs (5.1) can be written in the Hamiltonian form (2.14) with the same symplectic operator $\mathbb{J}$ (2.15) as the one for the original the BBM equation (1.1).

5.2. The potential α-family and its Lagrangian structure

It is straightforward to show that similarly to the KdV and the BBM equations, the PDEs of the α-family (5.1) as it stands does not have a self-adjoint Fréchet derivative, and hence does not directly arise from a variational principle for any Lagrangian functional. This can also be seen from the obvious differences between conservation law multipliers (5.4) and the evolutionary forms of symmetries (5.2), which contradicts the first Noether theorem (*c.f.*[31]).

However, the α-family (5.1) has a *potential* Lagrangian formulation. In fact, upon the introduction of the potential variable $w = w(x, t)$ defined by

$$u \stackrel{\text{def}}{:=} w_x, \tag{5.5}$$

the α-family (5.1) is written in the potential form

$$w_{tx} + w_{xx} + w_x w_{xx} - w_{xxxt} - \alpha(w_x w_{xxxx} + 2w_{xx}w_{xxx}) = 0, \tag{5.6}$$

which, for every α, is nonlocally related to the corresponding PDE (5.1) (that is, the correspondence between solutions $u(x,t)$ and $w(x,t)$ is one-to-many; see *e.g.*, Ref. 29). One can show that the linearization of (5.6) *is* self-adjoint. The homotopy formula[31, Theorem 5.92] can be used to construct the Lagrangian density, which up to equivalence is readily found to be

$$\mathcal{L}[w] \overset{\text{def}}{:=} -\frac{1}{2}\int_{\mathbb{R}}\left(\frac{1}{3}w_x^3 + w_x^2 + w_x w_t + w_{xx}w_{xt} + \alpha w_x w_{xx}^2\right)\mathrm{d}x, \tag{5.7}$$

The Lagrangian (5.7) is different from the Lagrangian (2.18) of the potential BBM equation (2.17) only by the last term. It is straightforward to check that the Euler–Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta w} = 0$$

yields the potential equation (5.6) for each value of the parameter α.

It is of interest to briefly consider symmetries and conservation laws of the potential α-family (5.6). The first reason for that is that the potential α-family equations (5.6) are nonlocally related to the PDEs (5.1), and hence the local symmetries and conservation laws of (5.1) *vs.* (5.6) might differ (*c.f.*[29]). Secondly, since the PDEs (5.6) arise from a variational principle, the first Noether theorem will provide an evident relation between their local symmetries and conservation laws. Seeking local symmetries of the potential α-family equations (5.6) in evolutionary form

$$\hat{\mathrm{X}} = \zeta(x,t,w,w_x,w_{xx},w_{xxt},w_{xxx})\partial_w, \tag{5.8}$$

holding for all $\alpha \in \mathbb{R}$, we find the admitted symmetry generator components

$$\zeta_1 := w_t, \tag{5.9a}$$
$$\zeta_2 := w_x, \tag{5.9b}$$
$$\zeta_3 := x - (\alpha-1)(w + tw_t) - \alpha t w_x, \tag{5.9c}$$
$$\zeta_4 := 2(x - tw_x) + (\alpha-1)^2 t(w_{xx}^2 + 2w_x w_{xxx}) + (\alpha-1)\big(2w + t(2w_{xxt} + w_{xx}^2 + 2w_x w_{xxx} - w_x^2)\big), \tag{5.9d}$$
$$\zeta_5 := w_x^2 - 2w_{xxt} - \alpha(w_{xx}^2 + 2w_x w_{xxx}), \tag{5.9e}$$
$$\zeta_F := F(t). \tag{5.9f}$$

The generators (5.8) with ζ_1 and ζ_2 correspond to the time and space translation point symmetries $\mathrm{X}_1 = \partial_t$, $\mathrm{X}_2 = \partial_x$ of the α-family (5.1) (*c.f.* (5.2)). The local symmetry with ζ_3 (5.9c) corresponds to the point symmetry $\mathrm{X}_3 = \alpha t\partial_x + (\alpha-1)t\partial_t + (x - (\alpha-1)w)\partial_w$ of (5.1), which includes the Galilei group only when $\alpha = 1$, that is, only for the potential form of the eBBM equation (4.1) itself. The generator with ζ_4 (5.9d) is generally a higher-order symmetry generator, which also degenerates into the Galilei group $\mathrm{X}_4 = t\partial_x + \partial_w$ when $\alpha = 1$. For $\alpha \neq 1$, the symmetry component ζ_4 yields a nonlocal higher-order symmetry of the corresponding PDE from the α-family (5.1) (*c.f.* Ref. 29). The local generator $\hat{\mathrm{X}}_5$ with symmetry component ζ_5 (5.9e) is a higher-order symmetry of the potential equations (5.6) for each α. It corresponds to the local conservation law multiplier Λ_3 (5.4) of the PDE family (5.1), and also to a point symmetry $\mathrm{X} = \partial_t + \partial_x$ of the eBBM family (5.1), which can be verified by a prolongation and a substitution of u_{xxt} on solutions (5.1). The local symmetry generator with ζ_F (5.9f) admitted by the potential family (5.6) is its point symmetry $w := w + F(t)$, with no projection on the space of variables (x,t,u) of the eBBM equation (4.1), related to the non-uniqueness of the definition (5.5) of the potential up to an arbitrary function of time. Higher than second order local symmetries and conservation laws of the potential α-family (5.6) can be sought and classified with respect to the parameter α (see Section 5.3 below).

Since the potential α-family (5.6) arises from a Lagrangian, by Noether theorem, its variational symmetries yield conservation law multipliers. A brief computation shows that indeed,

$$\Lambda_1 = \zeta_1, \quad \Lambda_2 = \zeta_2, \quad \Lambda_5 = \zeta_5, \quad \Lambda_F = \zeta_F$$

are local conservation law multipliers of the potential α-family (5.6), whereas the local symmetries with ζ_3 and ζ_4 are not variational, and have no corresponding conservation laws. The multiplier Λ_F corresponds to a conservation law

$$\mathcal{D}_t\big(F(t)(w_x - w_{xxx})\big) + \mathcal{D}_x\Big(-F'(t)(w - w_{xx}) + F(t)\Big(w_x(1-\alpha w_{xxx}) + \frac{1}{2}(w_x^2 - \alpha w_{xx}^2)\Big)\Big) = 0, \tag{5.10}$$

holding for an arbitrary function $F(t)$, which is a *nonlocal conservation law* of the α-family (5.1) (see *e.g.*[29]). Indeed, the spatial flux in (5.10) is explicitly dependent on the potential $w = \int u\,\mathrm{d}x$ itself, and hence is not equivalent to any local expression in terms of u. The conserved density in (5.10),

$$\rho_F \overset{\text{def}}{:=} F(t)(w_x - w_{xxx}) \equiv F(t)(u - u_{xx})$$

is, however, a local quantity in terms of the dependent variable u of (5.6).

An additional important conservation law arises for the potential α-family (5.6). In fact, the left-hand side of each Eq. (5.6) is a *total x-derivative*:

$$\mathcal{D}_x\left(w_t + w_x + \frac{1}{2}w_x^2 - w_{xxt} - \alpha\left(\frac{1}{2}w_{xx}^2 + w_x w_{xxx}\right)\right) = 0,$$

which can be written as

$$w_t + w_x + \frac{1}{2}w_x^2 - w_{xxt} - \alpha\left(\frac{1}{2}w_{xx}^2 + w_x w_{xxx}\right) = C, \tag{5.11}$$

for an arbitrary constant C. For every fixed value of the parameter α, the solution set integrated potential α-family (5.11) is in a not-one-to-one relationship with solutions of PDEs from the α-family (5.1). Indeed, every solution $w(x,t)$ of (5.11) for a fixed C yields a solution $u = w_x$ of (5.1). Conversely, any solution $u(x,t)$ of (5.1) corresponds to a family of solutions of (5.11) for some specific C, given by $w(x,t) = \int u\,\mathrm{d}x$ and defined up to a free additive constant.

We note that in addition to (5.6), other potential systems nonlocally related to the α-family (5.1) can be constructed, using each of the three linearly independent conservation law multipliers (5.4) to obtain three independent pairs of potential equations. Such potential equations can be used by themselves (*singlet potential systems*), in pairs (*couplet potential systems*), and all together (the *triplet potential system*), and may lead to new nonlocal symmetries and/or nonlocal conservation laws of the α-family of PDEs (5.1), including new results for the original BBM equation (1.1) and the eBBM (4.1).[29,44]

5.3. Local conservation law and symmetry classification of the α-family

Since the PDE family (5.1), as well as its potential form (5.6), involve a arbitrary element α, its local symmetries and conservation laws, as well as other mathematical properties can be classified according to α (see *e.g.* Ref. 29 and references therein). In Section 5.1, symmetries and conservation laws that arise for an arbitrary α were listed. We now use the `GeM` package for `Maple`,[36,37,43,45] along with `Maple rifsimp` routine, to generate, classify, and solve local symmetry and conservation law determining equations for the PDE families (5.1) and (5.6). We refer also to Ref. 46 for an alternative approach to the computation of conservation laws based on symmetries and adjoint symmetries. Namely, this work brought a new one-to-one correspondence between conservation laws and pairs of symmetries and adjoint symmetries for non-variational equations.

First, classifying local conservation laws of the α-family (5.1) with third-order multipliers, the following cases are distinguished:

1. In the general case with arbitrary α, as reported above, only three common local conservation laws arise, with multipliers (5.4).

Remark 5.1. We note that the BBM equation (1.1) with $\alpha = 0$ and its three conserved quantities (2.11)–(2.13) belong to this general case.

2. An additional local conservation law (4.7) with multiplier (4.3b) arises when $\alpha = 1$; this is the case of the eBBM equation (4.1).

3. When $\alpha = \frac{1}{3}$, the PDE

$$u_t + u_x + uu_x - u_{xxt} - \frac{1}{3}uu_{xxx} - \frac{2}{3}u_xu_{xx} = 0. \tag{5.12}$$

turns out to be the only other special case when additional conservation laws arise. We shall refer to the PDE (5.12) as the eBBM$_{1/3}$ equation.

The PDE equation (5.12) admits three common-family multipliers (5.4) and additional conservation laws, including ones with local multipliers

$$\begin{aligned}\Lambda_4^{(1/3)} &= 5u^3 + (9 - 4u_{xx})u^2 \\ &\quad - (u_x^2 + 6u_{xt})u + 6(u_xu_t + 3u_{tt}),\end{aligned} \tag{5.13a}$$

$$\Lambda_5^{(1/3)} = \frac{1}{(2(u - u_{xx}) + 3)^{1/2}}, \tag{5.13b}$$

$$\Lambda_6^{(1/3)} = \frac{u_{xxxx} - \frac{1}{2}(2u+3)}{\big(2(u - u_{xx}) + 3\big)^{5/2}} + \frac{5}{2}\frac{u_{xxx}^2 + u_x(u_x - 2u_{xxx})}{\big(2(u - u_{xx}) + 3\big)^{7/2}}. \tag{5.13c}$$

Writing these conservation laws as

$$\mathcal{D}_t\mathcal{T}_j[u] + \mathcal{D}_x\mathcal{F}_j[u] = 0, \quad j = 4,5,6,$$

the conserved densities and fluxes are found to be, up to equivalence,

$$\begin{aligned}\mathcal{T}_4[u] &\overset{\text{def}}{:=} 5u^4 + 12u^3 + (26u_x^2 + 4u_{xx}^2)u^2 + \\ &(u_x^2u_{xx} + 24u_x^2 + 36u_{tt})u - 36u_{tt}u_{xx},\end{aligned} \tag{5.14a}$$

$$\begin{aligned}\mathcal{F}_4[u] &\overset{\text{def}}{:=} 4u^5 - \frac{2}{3}(4u_{xx} + 2u_{xxxx} - 21)u^4 \\ &- \frac{1}{3}(20u_xu_{xxx} + 24u_{xx} + 84u_{xt} - 32u_x^2 - 36)u^3 + \\ &2(6u_tu_{xxx} + 8u_{xt}u_{xx} - 2u_x^2u_{xx} - 24u_{xt} \\ &+ 12u_{tt} + 3u_x^2 - 22u_xu_t)u^2 \\ &+ \big(36(u_{tt} - u_{xttt}) - 24(u_{xx}u_{tt} + u_xu_t) \\ &+ 8u_xu_{xx}u_t + 3u_x^2u_{xt}\big)u \\ &+ 36u_xu_{ttt} - 3u_x^3u_t,\end{aligned} \tag{5.14b}$$

$$\mathcal{T}_5[u] \overset{\text{def}}{:=} \sqrt{2(u - u_{xx}) + 3}, \tag{5.14c}$$

$$\mathcal{F}_5[u] \overset{\text{def}}{:=} \frac{1}{3}\left(u\sqrt{2(u - u_{xx}) + 3}\right), \tag{5.14d}$$

$$\begin{aligned}\big(2(u - u_{xx}) + 3\big)^{5/2}\cdot\mathcal{T}_6[u] &\overset{\text{def}}{:=} 2\big(2(u - u_{xx}) + 3\big)^2 + \\ &(2u+3)\big(2(u - u_{xx}) + 3\big) + 3(u_{xxx}^2 - u_x^2)\end{aligned} \tag{5.14e}$$

$$\begin{aligned}&\big(2(u - u_{xx}) + 3\big)^{5/2}\cdot\mathcal{F}_6[u] \\ &\overset{\text{def}}{:=} \big(2(u - u_{xx}) + 3\big)^3 + (u+3)\big(2(u - u_{xx}) + 3\big)^2 + \\ &2(u_x^2 - u_xu_{xxx} + u_{xt})\big(2(u - u_{xx}) + 3\big) \\ &- 2\big(uu_x + 3(u_t - u_{xxt})\big)u_{xxx} + uu_x^2.\end{aligned} \tag{5.14f}$$

Further higher-order conservation laws exist for the eBBM$_{1/3}$ equation (5.12) but are technically challenging to obtain by the multiplier method. Local symmetry classification of the α-family (5.1) in the ansatz (5.3) reveals no additional cases compared to the symmetries (5.2). A higher-order symmetry classification

$$\hat{\mathrm{X}} = \zeta(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx})\partial_u \tag{5.15}$$

yields two additional higher-order symmetries arising in the case $\alpha = 1/3$, that is, for the eBBM$_{1/3}$ equation (5.12). The first additional symmetry is a third-order symmetry given by

$$\hat{\mathrm{X}}_4 = \frac{u_x - u_{xxx}}{\big(2(u - u_{xx}) + 3\big)^{3/2}}\partial_u, \tag{5.16a}$$

and the second is fifth-order symmetry of the form

$$\hat{\mathrm{X}}_5 = \frac{A[u]}{2\big(2(u - u_{xx}) + 3\big)^{9/2}}\partial_u, \tag{5.16b}$$

where

$$\begin{aligned}A[u] = {} &(8u_{xx}^2 - 8(2u+3)u_{xx} + 8u^2 + 24u + 18)u_{xxxxx} \\ &+ 20(2(u - u_x) + 3)(u_{xxx} - u_x)u_{xxxx} + 35u_{xxx}^2(u_{xxx} - 3u_x) \\ &+ (20u_{xx}^2 + 105u_x^2 - 20u^2 - 60u - 45)u_{xxx} - 28u_xu_{xx}^2 \\ &+ 8(2u+3)u_xu_{xx} - (35u_x^2 - 12u^2 - 36u - 27)u_x.\end{aligned}$$

5.4. *Equivalence between eBBM $_{1/3}$ and the Camassa–Holm equation*

The additional conservation law multipliers (5.13b), (5.13c) and symmetry generators (5.16) arising for the eBBM$_{1/3}$ equation (5.12) involving fractional-order powers are similar to higher-order symmetries and conservation laws of the integrable Short Pulse equation (SP equation)

$$v_{xt} - v - \frac{1}{6}(v^3)_{xx} = 0 \tag{5.17}$$

and the Camassa–Holm (CH) equation

$$u_t + 3uu_x - 2u_xu_{xx} - u_{xxt} - uu_{xxx} = 0. \tag{5.18}$$

The SP equation (5.17) arises in nonlinear optics.[26] It is an $\mathcal{S}$-integrable equation possessing a Lax pair and a bi-Hamiltonian structure.[47–49] While small H^2 norm solutions of the SP equation exist globally for infinite time, large norm solutions blow up in finite time, with the amplitude remaining bounded and solution slope becoming steeper.[50] It is related to the quasilinear Klein–Gordon equation (see *e.g.* Refs. 51, 52 and references therein). In particular, the PDE (5.17) is known to possess local conservation laws with multipliers[53]

$$\Lambda_1 = \frac{v_x}{(v_x^2 + 1)^{1/2}}, \tag{5.19a}$$

$$\Lambda_2 = \frac{v_{xxx}}{(v_x^2 + 1)^{5/2}} - \frac{5}{2}\frac{v_xv_{xx}^2}{(v_x^2 + 1)^{7/2}}. \tag{5.19b}$$

The Camassa–Holm equation (5.18) is a part of the family (3.5); it models unidirectional wave propagation in shallow water over a flat bottom. It is also a bi-Hamiltonian $\mathcal{S}$-integrable equation, admitting infinite sequences of local and nonlocal conservation laws, with local densities involving fractional-order denominators of increasing orders,[27] highly similar to (5.14c) and (5.14e). In particular, the first two conservation law multipliers of the local sequence for the CH equation (5.18) are given by

$$\Lambda_1 = \frac{2}{(u - u_{xx})^{1/2}}, \tag{5.20a}$$

$$\Lambda_2 = \frac{u - u_{xxxx}}{(u - u_{xx})^{5/2}} - \frac{5}{4}\frac{(u_x - u_{xxx})^2}{(u - u_{xx})^{7/2}}. \tag{5.20b}$$

The local conservation law multiplier pairs (5.19), (5.20), and (5.13b), (5.13c) are very similar. While the short pulse equations and the CH are not equivalent, one can observe that indeed, the eBBM$_{1/3}$

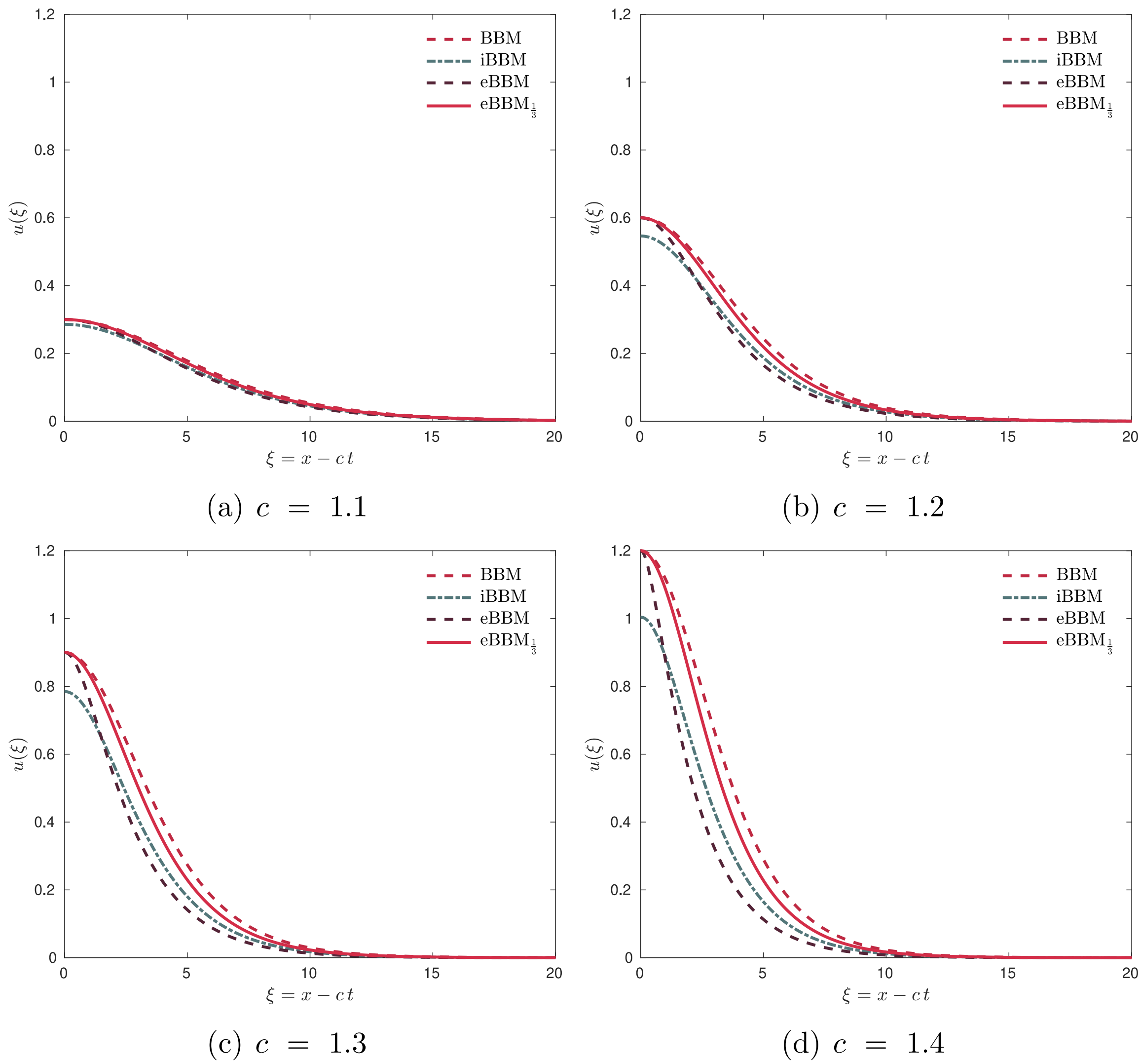


Fig. 1. Comparison of solitary wave solutions to four BBM-type equations (1.1), (3.1), (4.1), and (5.12) for various values of the propagation celerity c. Only the right half of solutions is shown. These solutions have to be extended by symmetry to negative values of ξ.

equation is related to the CH by a local scaling transformation. For example, a time scaling $t = 3\tau$ and the change of notation $\tau \to t$ maps the PDE (5.12) into the Camassa–Holm equation with $\kappa = 3/2$:

$$u_t + 3u_x + 3uu_x - 2u_x u_{xx} - u_{xxt} - uu_{xxx} = 0,$$

which can be further mapped into the form (5.18) without the u_x term by a Galilean transformation

$$x = x^* - \kappa t^*, \quad t = t^*, \quad u = u^* - \kappa, \quad \kappa = \frac{3}{2},$$

and a subsequent omission of the asterisks. This demonstrates that the PDE eBBM$_{1/3}$ the only integrable member of the BBM α-family (5.1). We would like to mention also that the Hamiltonian and quasi-Hamiltonian structures associated with integrable equations can be constructed based on the trace identity[54] and the variational identity.[55] Ref. 55 presents a variational identity which generates Hamiltonian structures. This variational identity is a generalization of the one presented earlier in Ref. 54.

6. Numerical investigations

In this section we perform a comparative numerical studies of the models discussed above. Namely, we shall compare the BBM equation (1.1), the iBBM equation (3.1), the eBBM equation (4.1), and the eBBM$_{1/3}$ equation (5.12).

6.1. Solitary wave solutions

After substituting the travelling wave ansatz $u(x, t) := u(\xi)$, with $\xi := x - ct$, into the governing equations (1.1), (3.1), (4.1), (5.12) we obtain four following ODE reductions correspondingly:

$$(1 - c)u' + cu' + \left(\frac{1}{2}u^2\right)' = 0,$$

$$(1 - c)u' + cu' + \left(\frac{1}{2}u^2\right)' - uu' = 0,$$

$$(1 - c)u' + cu' + \left(\frac{1}{2}u^2\right)' - uu' - 2u'u'' = 0,$$

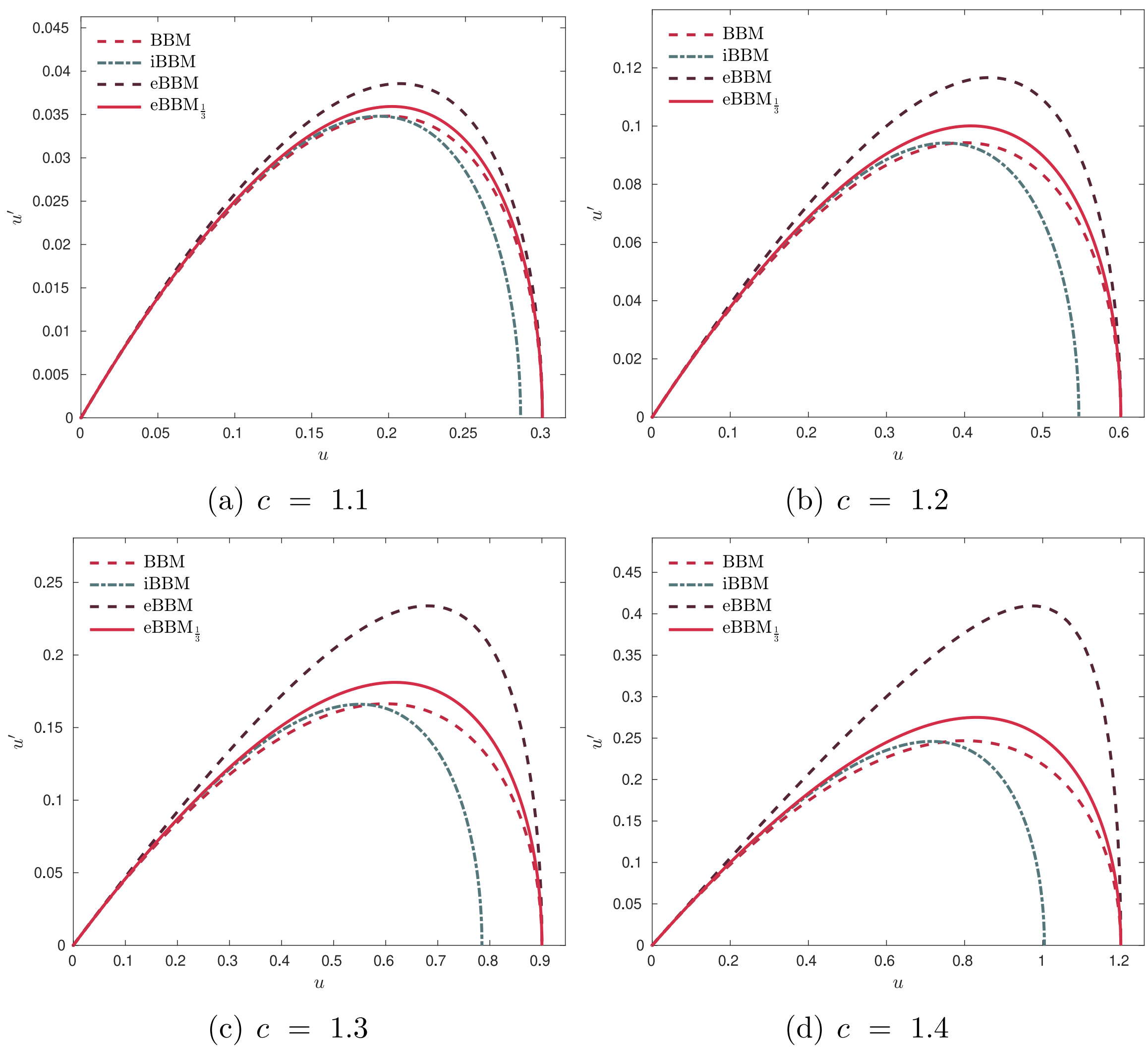
$$(1 - c)u' + cu' + \left(\frac{1}{2}u^2\right)' - \frac{1}{3}uu' - \frac{2}{3}u'u'' = 0,$$

where the primes denote the derivatives with respect to the variable ξ. In this Section we are interested in solitary wave solutions, which have the characteristic property that the function $u(\xi)$ together with all its derivatives decays exponentially at infinity. It allows us to integrate once each equation above with respect to ξ to lower the order of ODEs:

$$(c - 1)u - cu'' = \frac{1}{2}u^2 \overset{\text{def}}{=:} \mathcal{N}_1(u),$$

$$(c - 1)u - cu'' = \frac{1}{2}u^2 - uu'' + \frac{1}{2}(u')^2 \overset{\text{def}}{=:} \mathcal{N}_2(u),$$

$$(c - 1)u - cu'' = \frac{1}{2}u^2 - uu'' - \frac{1}{2}(u')^2 \overset{\text{def}}{=:} \mathcal{N}_3(u),$$



(a) $c = 1.1$ (b) $c = 1.2$ (c) $c = 1.3$ (d) $c = 1.4$

Fig. 2. Comparison of solitary wave solutions in the phase plane to four BBM-type equations (1.1), (3.1), (4.1), and (5.12) for various values of the propagation celerity c. Only the upper half is shown. These portraits have to be extended by symmetry to negative values of u'.

$$(c-1)u - cu'' = \frac{1}{2}u^2 - \frac{1}{3}uu'' - \frac{1}{6}(u')^2 \overset{\text{def}}{=:} \mathcal{N}_4(u).$$

It can be seen that each equation above has the following form:

$$\mathcal{L} \cdot u = \mathcal{N}_i(u),$$

where $\mathcal{L} \overset{\text{def}}{:=} (c-1)\mathbb{1} - c\,\partial^2_{\xi\xi}$ is the common linear operator ($\mathbb{1}$ is the identity operator) and $\mathcal{N}_i(u)$ is a quadratically nonlinear operator, which depends on the model equation. In order to find numerically the solitary wave profiles, it is natural to employ the classical Petviashvili iteration.[56–59] We note that for the original BBM equation the solitary waves are available analytically. We have not been able to find *analytical* solutions to other equations considered in this study.

The numerically computed profiles to all four models are presented in Fig. 1. The numerical computations show that the BBM equation, eBBM and $\mathrm{eBBM}_{1/3}$ share exactly the same speed–amplitude relation, while the iBBM equation stands apart from this point of view. At the level of the SW shape, the nonlinearly enhanced models seem to have faster decay properties for the same celerity parameter than the classical BBM equation. We would like to underline the fact that the classical BBM equation and the integrable $\mathrm{eBBM}_{1/3}$ model have SWs with a very similar shape. In order to illustrate better the shape of obtained SW solutions, we depict them on the phase plane in Fig. 2.

6.2. Dynamics: the transient solutions

For the sake of completeness of our numerical study, we would like to shed some light on the behaviour of unsteady solutions under the dynamics of considered systems. In order to solve numerically the family of BBM-type equations (1.1), (3.1), (4.1), and (5.12) we employ the classical Fourier-type pseudo-spectral method with periodic boundary conditions (similar to one we employed in Ref. 60). Moreover, with these boundary conditions all the integrals of locally conserved quantities do not evolve in time. In the simulations presented below we use $N = 8\,192$ Fourier modes. For the time stepping we use a Variable Step Variable Order (VSVO) Adams–Bashforth–Moulton Predict Evaluate Correct Evaluate (PECE) solver of orders from 1 to 13. The highest order used in practice is 12, however, a formula of order 13 is used to form the error estimate and the routine `ode113` does local extrapolation to advance the integration up to the order 13. This solver was found to be the most efficient for this problem among `Matlab` ODE suite.[61]

6.2.1. Bump evolution

As the first test case, we study the evolution under governing equations dynamics of a non-monotonic initial condition. More precisely, we choose as the initial condition the same infinitely smooth bump for

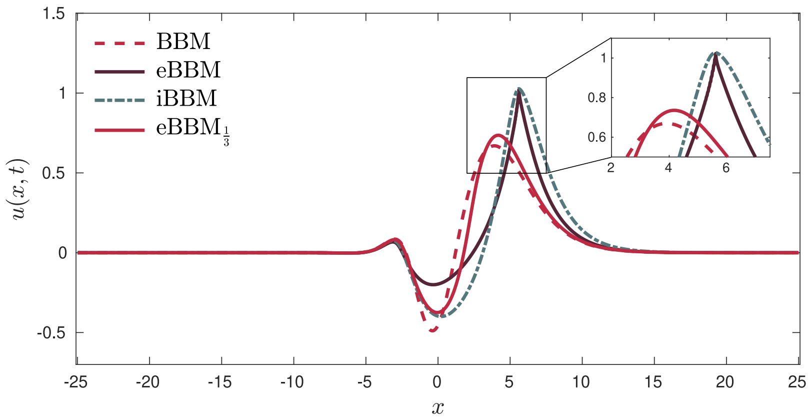


Fig. 3. Unsteady solution to four BBM-type models (1.1), (3.1), (4.1), and (5.12) taken at $t = 5$. The slight 'wiggles' visible on the zoom around the singularity in the eBBM solution come from the loss of the regularity.

all four models (1.1), (3.1), (4.1), and (5.12):

$$u(x, 0) = \mathrm{e}^{-\frac{1}{2}x^2}.$$

The computational domain is chosen so that the solution decays below the machine precision before reaching the boundary for the time horizon of our unsteady simulations. A snapshot of four numerical solutions taken at the same final moment of time $t = 5$ is shown in Fig. 3. It can be seen that all numerical solutions, except one, are smooth. The inspection of the Fourier power spectrum (more precisely, its asymptotic decay properties) confirms this conclusion. The solution to the iBBM equation developed to this time much sharper gradients than the original BBM or $\mathrm{eBBM}_{1/3}$ solutions. However, the most interesting transformations happened in the eBBM model. In perfect agreement with the well-posedness theory, this solution developed a singularity in the derivative. We may conjecture a peakon (or cuspon) emergence from a smooth non-monotonic initial condition. The inspection of the Fourier power spectrum also indicates the loss of regularity since its decay becomes algebraic. Unfortunately, the pseudo-spectral method becomes much less efficient for such (weakly-)singular solutions. A similar phenomenon of the peakon emergence from smooth initial conditions has been observed earlier in the capillary–gravity Serre–Green–Naghdi system (SGN system) for the critical Bond number.[62] The existence of cusped solitary waves was studied in Hamiltonian regularizations of NSWE.[63] Indeed, in the critical regime, this system becomes dispersionless. However, in the present case the dispersion is present, which makes it pretty interesting. We may reasonably conjecture also that the iBBM equation will also develop a finite time singularity on larger times because we were able to find analytical peakon-type solutions (3.4) in it.

The evolution of the energy $\mathcal{E}(t)$ (2.12) during the transient simulation is shown in Fig. 4. It is interesting to discuss the behaviour of this quantity in different models. First of all, without any surprise, this quantity was conserved to the machine precision in the BBM and $\mathrm{eBBM}_{1/3}$ equations. The quantity $\mathcal{E}(t)$ is not conserved by the iBBM equation dynamics, which explains a very fast grow of this quantity along the numerical solution to (3.1). Finally, this quantity is perfectly conserved by the eBBM dynamics until $t \approx 3$, where the substantial loss of solution regularity happens, which explains the slight grow (about 0.015%) of this quantity in the numerical solution. The non-conservation of the energy after singularity formation has been observed numerically and confirmed theoretically before in Ref. 63 for a different model, which gives more confidence in our numerical results. Moreover, this observation confirms one more time the finite time singularity formation in the eBBM equation.

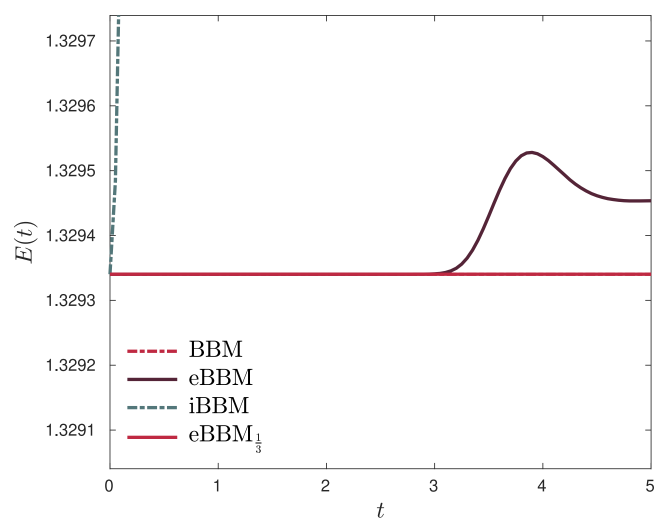


Fig. 4. Evolution of the total energy in four models (1.1), (3.1), (4.1), and (5.12) considered in our numerical study during the unsteady bump evolution simulation.

6.2.2. Solitary waves collisions

In order to test the eventual integrability of considered models, we set up the test-case with solitary wave interactions. It is already known that the BBM equation is not integrable. However, the question remains open for three other models under consideration.

More precisely, since we are dealing with unidirectional models, we consider the overtaking collision of two right-going solitary waves in a periodic domain $[-140, 140]$. Initially at $t = 0$, the solitary waves are located at $x = \pm 50$. The left solitary wave has the velocity $c_1 = 1.3$ and the right one $c_r = 1.05$. The initial configuration is depicted in Fig. 5. The solitary waves are computed in all models with the Petviashvili iteration employed above (see Section 6.1). Then, this initial condition was simulated in all four models (1.1), (3.1), (4.1), and (5.12) until the final time $t = T := 750$, when the interaction happened and two solitary waves had time to separate. A zoom on the computational domains at the final simulation time is shown in Fig. 6. It can be seen on panel (*a*) that the interaction of solitary waves in the BBM equation is inelastic as it was correctly discovered numerically in Ref. 12 and described theoretically in Ref. 64. We see also that the interaction is inelastic and iBBM and eBBM equations (see panels (*b, c*) in Fig. 6). On the other hand, the collision appears to be perfectly elastic in the $\mathrm{eBBM}_{1/3}$ equation as the radiative component is totally absent on the corresponding panel (*d*) of Fig. 6. This fact is not surprising since thanks to the existing relation between the $\mathrm{eBBM}_{1/3}$ equation (5.12) and the integrable CH equation.

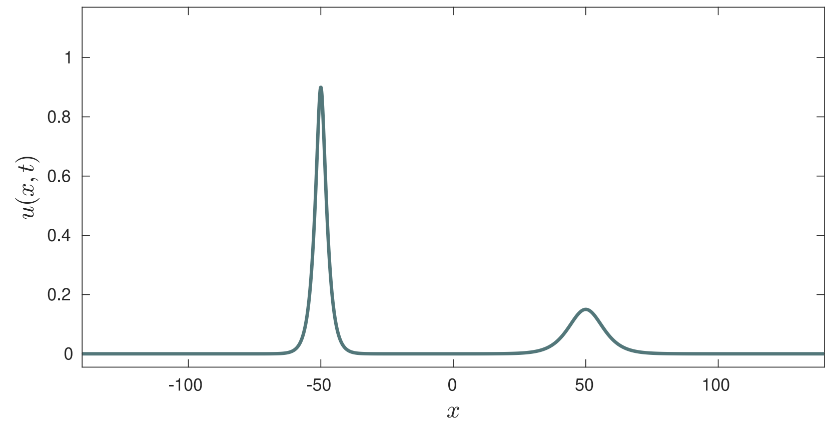


Fig. 5. The initial condition ($t = 0$) in all overtaking collision simulations involving the BBM-type models (1.1), (3.1), (4.1), and (5.12). Higher solitary wave moves faster, which explains the name of this type of collision.

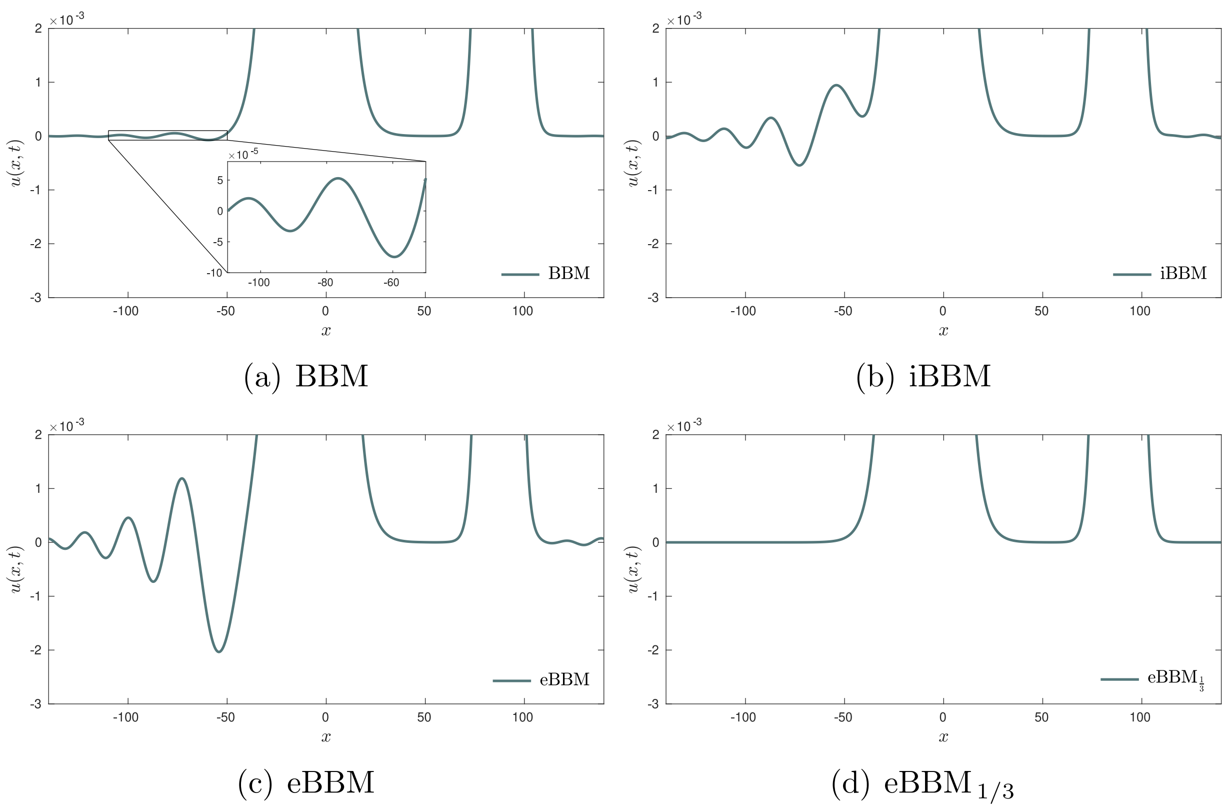


Fig. 6. Zoom on numerical solutions after the overtaking collision at $t = 750$ of two solitary waves with initial speeds of $c_1 = 1.3$ and $c_1 = 1.05$.

7. Conclusions and perspectives

The well known Benjamin–Bona–Mahony equation (BBM equation) (1.1), (2.1) is a physically interesting nonlinear long wave model possessing a number of remarkable properties, including an improved dispersion relation (compared to the KdV), energy conservation and Hamiltonian and Lagrangian formulations, but lacking the physically important Galilei symmetry that corresponds to the invariance with respect to the choice of a physical frame of reference. We show how the Galilean invariance property can be recovered in the framework of BBM equation while preserving simultaneously the energy conservation, as well as other important analytical properties (Section 2). In Section 3, the Galilei-invariant extension of the BBM equation, the iBBM equation model (3.1), is introduced. However, it turns out that iBBM equation does not conserve energy, preserving rather a related exponential quantity (3.8b). Using a multiplier argument, a nonlinear term of the order $\mathcal{O}(\varepsilon\,\delta^2)$ is added to the iBBM equation. The resulting energy-preserving Benjamin–Bona–Mahony equation (eBBM) Eq. (4.1) is similar to the famous p-family of PDEs (3.5), has four local conservation laws including energy and a new time-dependent conserved quantity, has a Hamiltonian structure, and a potential formulation arising from a variational principle. For all equations we systematically discuss the infinitesimal point symmetries and conservation laws.

In Section 5, it was shown that both BBM and eBBM equations belong to a one-parameter PDE family (5.1) whose member PDEs share common local symmetry, conservation law, Hamiltonian and Lagrangian structures. Local symmetry and conservation law classifications of the α-family (5.1) show two particular cases with additional structure: $\alpha = 1$ corresponding to the eBBM equation, and $\alpha = 1/3$ leading to the new PDE (5.12) called the $\mathrm{eBBM}_{1/3}$ equation. The latter model admits additional higher-order symmetries and conservation laws, including ones with fractional-order components. These structures are similar to the properties of the integrable Short Pulse equation (SP equation) (5.17) and the well known Camassa–Holm (CH) (CH) equation (5.18). In Section 5.4, it was shown that indeed, a point transformation relating eBBM$_{1/3}$ and CH exists.

Finally, in Section 6, comparative numerical studies of the BBM equation, iBBM equation, eBBM, and eBBM$_{1/3}$ models were performed. While the solitary wave interaction was elastic in the eBBM$_{1/3}$ (CH) case, it was inelastic in all other models under consideration. Numerical tracking of the evolution of energy (2.12) during the transient simulation displayed machine-precision conservation for the BBM and eBBM$_{1/3}$ equations, lack of conservation for the iBBM equation dynamics, and perfect conservation by the eBBM dynamics until the loss of solution regularity, confirming the finite time singularity formation in the eBBM equation.

7.1. Perspectives

Open problems for future work include the study of possible physical applications and further analytical properties of the α-family of PDEs (5.1). In particular, one can use local conservation laws with multipliers (5.4) holding for all members of the α-family (5.1) to construct potential systems, and classify nonlocal symmetries and CLs of this PDE family within the framework of non-locally related systems.[44]

Regarding the perspectives of this study, the same type of techniques can be applied to various Boussinesq-type equations, since many of the systems of this kind also lack the Galilean invariance and/or the energy conservation properties. Some of such systems have been addressed in Refs. 25, 65. Nevertheless, there are still many models that can be improved using the methods presented in this study. Moreover, the extension to two-dimensional[g] systems will be another step towards more physically sound modelling of long dispersive water wave propagation. The symmetry analysis of these models can be performed similarly to these recent studies.[66–68]

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CRediT authorship contribution statement

A. Cheviakov: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing – review & editing. **D. Dutykh:** Conceptualization, Software, Formal analysis, Investigation, Data curation, Writing – original draft, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

[g] Here we mean the two horizontal dimensions in space or $(2+1)$, 2DH in other notations.

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