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Conservation Laws and Nonlocally Related Systems of Two-Dimensional Boundary Layer Models

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Abstract: Local conservation laws, potential systems, and nonlocal conservation laws are systematically computed for three-equilibrium two-component boundary layer models that describe different physical situations: a plate flow, a flow parallel to the axis of a circular cylinder, and a radial jet striking a planar wall. First, local conservation laws of each model are computed using the direct method. For each of the three boundary layer models, two local conservation laws are found. The corresponding potential variables are introduced, and nonlocally related potential systems and subsystems are formed. Then nonlocal conservation laws are sought, arising as local conservation laws of nonlocally related systems. For each of the three physical models, similar nonlocal conservation laws arise. Further nonlocal variables that lead to further potential systems are considered. Trees of nonlocally related systems are constructed; their structure coincides for all three models. The three boundary layer models considered in this work provide rich and interesting examples of the construction of trees of nonlocally related systems. In particular, the trees involve spectral potential systems depending on a parameter; these spectral potential systems lead to nonlocal conservation laws. Moreover, potential variables that are not locally related on solution sets of some potential systems become local functions of each other on solution sets of other systems. The point symmetry analysis shows that the plate and radial jet flow models possess infinite-dimensional Lie algebras of point symmetries, whereas the Lie algebra of point symmetries for the cylinder flow model is three-dimensional. The computation of nonlocal symmetries reveals none that arise for the original model equations, which is common for partial differential equations (PDE) systems without constitutive parameters or functions, but does reveal nonlocal symmetries for some nonlocally related PDE systems.

Keywords: Boundary Layer; Direct Method; Nonlocal Conservation Laws; Potential Systems.

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1 Introduction

For a mathematical model given in terms of partial differential equations (PDE), its local conservation laws contain essential coordinate-independent information about the structure of the model. Local divergence-type conservation laws are given by divergence expressions that vanish on solutions of a model; globally, for time-dependent problems, they yield a rate of change of the total amount of the conserver density in every domain in terms of boundary fluxes (e.g. [1]). In many cases, conservation principles, such as the conservation of mass, energy, and charge, serve as a cornerstone for the formulation of the mathematical models themselves. On the other hand, when a model is already prescribed, one can systematically seek its local conservation laws, and obtain additional conserved physical quantities.

Local conservation laws are employed in the analysis of PDE solution behavior, such as stability, existence, and uniqueness of solutions (e.g. [2–4]). Infinite discrete sequences of local and nonlocal conservation laws involving derivatives of increasing orders may be related to the existence of a Lax pair, and consequently, to the integrability of a model [5]. Invertible mappings of nonlinear PDEs to linear PDEs through conservation laws may exist for PDE systems that admit families of conservation laws parameterized by arbitrary functions [6]. Nonlocally related potential systems arise directly from local conservation laws, and may lead to the discovery of new (local and nonlocal) symmetries, conservation laws, invariant reductions, and exact solutions of a model at hand (see, e.g. [1] and references therein). For variational models, the first Noether's theorem provides a connection between local variational symmetries and local conservation laws; for nonvariational systems, relationships between symmetries and local conservation law structure can also be established [1, 7]. Conservation laws play an important role in numerical computations, where they are key elements in finite-volume and finite-element numerical methods [8, 9]; they are also used to construct conservative discretizations in finite-difference numerical methods [10].

In addition to local conservation laws, for certain models, nonlocal conservation laws have been obtained (see, e.g. [1, 11–15] and references therein). Nonlocal conservation laws can arise as formal expressions involving integration/inverse differentiation, or can be systematically sought as local conservation laws of potential systems of a given model. Nonlocal conservation laws can be used for the same purposes as the local ones, and bear particular importance when a model lacks a sufficient number of local conservation laws. As local conservation laws of potential systems, nonlocal conservation laws arise relatively rarely, usually in special cases of classifications, within families of models that involve arbitrary constitutive functions and/or parameters. It is less common to find a nonlocal conservation law for a given nonlinear differential equation model with fixed parameters, outside a classification problem.

In the literature, local and nonlocal conservation laws have been studied in general as well as for specific models and classes of equations; relevant works are included but are not limited to Refs. [1, 7, 12, 14, 16–25].

PDE systems equivalent but nonlocally related to a given one arise in different contexts, and have been shown to be useful for a variety of purposes. In [26], an iterative procedure is suggested, where at each step, a known conservation law is used to introduce a new potential variable; then local conservation laws of such potential system are studied. This procedure has been recently extended and simplified (see, e.g. [1, 14, 21, 27] and related works). Potential variables introduced in this way often have a direct physical meaning; simplest examples include the stream function in fluid dynamics, and electric/magnetic potentials. Potential systems and nonlocally related subsystems can yield nonlocal symmetries of a given model, and may lead to invariant solutions that do not arise as symmetry-invariant solutions with respect to any point symmetries of the model (e.g. [1, 14, 28–30]). A nonlocally related system may yield to a noninvertible linearization of a nonlinear PDE system [31]. The introduction of nonlocal variables can lead to a variational formulation of a PDE model, as is the case with the Korteweg-de Vries equation (e.g. [1, 32]). Nonlocally related systems can be constructed by other means, for example, using pseudopotentials (e.g. [33–35]) or local symmetries [36], and from other considerations, such as the construction of nonlocal mappings between PDEs [37], or the identification of symmetry-integrable equations [38]).

In the current paper, we consider Prandtl-type boundary layer equations in Cartesian and cylindrical coordinates. The models of interest represent asymptotic reductions describing three constant-density viscous flow types: a boundary layer flow in the vicinity of a plate, an axisymmetric flow near a circular cylinder parallel to its axis, and a radial wall jet flow. Various properties of these and related models, including self-similar, asymptotic, and group-invariant solutions and basic conservation laws, have been studied in the literature [39–50].

The main goal of the current paper is the systematic construction of extended trees of nonlocally related PDE systems [1, 14, 27] for the three boundary layer models. In particular, for each model, we use its local conservation laws to introduce potential variables, and consider nonlocally related singlet, multiplet, and spectral potential systems, as well as subsystems, that arise. A "tree" is consequently formed; it is further extended through the computation of nonlocal conservation laws, which arise as local conservation laws of potential systems, and the introduction of further potentials. Remarkably, for the considered dimensionless models involving no parameters, several nonlocal conservation laws arise. For each model, all nonlocally related PDE systems obtained using the above procedure are equivalent to the given one in the sense that the solution set of each PDE system in the extended tree yields the solution set of any other PDE system within the extended tree.

The paper is organized as follows. Basic facts concerning the conservation laws, the direct conservation law construction method, and the notions of potential and other nonlocally related systems are reviewed in Section 2 (for a more general description of various types of local conservation laws in multi-dimensions, see, e.g. [1, 51, 52]). Section 3 introduces the three boundary layer models of interest. In Section 4, local and nonlocal conservation laws are computed, and an extended tree of nonlocally related PDE systems is presented, for the classical Prandtl boundary layer model describing a two-dimensional steady flow near a plate. A similar analysis for the axially symmetric cylinder and the wall jet boundary layer models is performed in Sections 5 and 6. Remarkably, we find that the sets of local and nonlocal conservation laws, and the corresponding trees of nonlocally related systems, are very similar for the three models. Point symmetries are computed, and nonlocal symmetries are sought (none is found), in each corresponding section.

The paper is concluded with a discussion in Section 7, where it is shown that the three boundary layer models are not related by a local transformation, and some open problems and research directions are outlined.

The conservation law and symmetry computations within the current work were performed using the `Maple`-based symbolic software package `GeM` [53–56].

2 Local and Nonlocal Conservation Laws, Nonlocally Related PDE Systems

Let U$\{x; u\}$ be a system of PDE given by

$$R^{\sigma}[u] \equiv R^{\sigma}(z, u, \partial u, \dots, \partial^{\mu_\sigma} u) = 0, \qquad \sigma = 1, \dots, N, \tag{1}$$

with $n \geq 2$ independent variables $z = (z^1, \dots, z^n)$ and $m \geq 1$ dependent variables $u(z) = (u^1(z), \dots, u^m(z))$. The symbol $\partial^q u$ is used to denote all partial derivatives of order q of all components of u.

The *solution set* $\mathcal{G}_U$ of (1) consists of vector functions $u(z)$ that satisfy all PDEs (1). Suppose that (1) has differential orders $0 < \mu_\sigma < \mu$ for some $\mu > 0$. Then for any $\nu \geq \mu$, the set of PDEs (1) and all their independent differential consequences up to order ν corresponds to a manifold $\mathcal{G}_U^J$ in the jet space $J^\nu(z; u)$ of order ν (the coordinate space of z, u, and derivatives of u up to order ν). A *differential function* $f[u]$ is a smooth function defined on a domain in $J^V(z; u)$.

The total derivative of a differential function $f[u]$ with respect to z^i is given by $\mathrm{D}_i f[u]$, where

$$\mathrm{D}_i \equiv \mathrm{D}_{z^i} = \frac{\partial}{\partial z^i} + u_i^j \frac{\partial}{\partial u^j} + u_{ii_1}^j \frac{\partial}{\partial u_{i_1}^j} + u_{ii_1i_2}^j \frac{\partial}{\partial u_{i_1i_2}^j} + \cdots$$

denotes the total derivative operator. The summation in repeated indices is assumed where appropriate, and

$$u_i^j \equiv u_{z^i}^j \equiv \frac{\partial u^j}{\partial z^i}, \qquad u_{i\ell}^j \equiv \frac{\partial^2 u^j}{\partial z^i \partial z^\ell}, \qquad \dots$$

is a short-hand notation for partial derivatives.

2.1 Local Conservation Laws

A local (divergence-type) conservation law of the model (1) is a divergence expression

$$\mathrm{D}_i Z^i[u] = 0 \tag{2}$$

holding for every solution $u(z)$ of the given system (1). One may denote $\mathbf{Z}[u] = (Z^1[u], \dots, Z^n[u])$ to be the density-flux vector.

For ordinary differential equations (ODE), local conservation laws (2) take the form $\mathrm{D}_Z Z[u] = 0$ and yield first integrals $Z[u] = \text{const.}$

When one of the independent variables, e.g. $z^1 = t$, is time, the conservation law (2) takes the form

$$\mathrm{D}_t T[u] + \sum_{j=2}^{n} \mathrm{D}_j Z^j[u] = 0, \tag{3}$$

and corresponds to the global integral expression

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} T[u]\, \mathrm{d}V = -\oint_{\partial \mathcal{V}} \mathbf{X}[u] \cdot \hat{\mathbf{n}}\, \mathrm{d}S, \tag{4}$$

where $\mathbf{X}[u] = (Z^2[u], \dots, Z^n[u]$ is the spatial flux vector, $\mathcal{V}$ is any closed volume within the model domain, having a piecewise-smooth boundary surface $\partial\mathcal{V}$ with a outward unit normal normal vector $\hat{\mathbf{n}}$. The global relationship (4) holds on the solution set of the given system (1); its physical meaning is the fact that the rate of change of the volume quantity

$$C = \int_{\mathcal{V}} T[u]\, \mathrm{d}V$$

in every subdomain $\mathcal{V}$ is balanced by the net flux through the boundary surface $\partial\mathcal{V}$.

2.1.1 Topological Conservation Laws

In models involving only spatial variables z, the vanishing divergence expression div $\mathbf{Z}[u] = 0$ (2) corresponds to a *topological conservation law* (see [57]). In particular, for the spatial divergence conservation law (2), the global form is obtained by integration of (2) over any connected closed volume $\mathcal{V}$ within the physical domain of the given model. The Gauss' theorem then yields

$$\oint_{\partial \mathcal{V}} \mathbf{Z}[u] \cdot \mathrm{d}\mathbf{S} = 0, \tag{5}$$

holding on the solution set of the given PDE system. Here $\mathrm{d}\mathbf{S} = \boldsymbol{\nu}\mathrm{d}A$ is the outward-directed area element. When the domain $\mathcal{V}$ is not simply connected, and its boundary $\partial\mathcal{V}$ of a consists of two disjoint surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, then from (5), on solutions of the given model (1), one has the relationship

$$\oint_{\mathcal{S}_1} \mathbf{Z}[u] \cdot \mathrm{d}\mathbf{S} = \oint_{\mathcal{S}_2} \mathbf{Z}[u] \cdot \mathrm{d}\mathbf{S}. \tag{6}$$

Here the unit normal vectors $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen so that one is inward-directed and the other is outward-directed with respect to the volume $\mathcal{V}$. The equality (6) does not change if $\mathcal{S}_1$, $\mathcal{S}_2$ are continuously deformed in a topology-preserving manner, for example, in the sense of homotopy equivalence.

2.1.2 Trivial and Equivalent Conservation Laws

Both from the practical and theoretical point of view, it is highly important to distinguish between trivial and nontrivial conservation laws (2). In particular, the former hold as identities and provide no information about solutions of the model. A conservation law (2) is *trivial* if its density-flux vector has the form

$$\mathbf{Z}[u]=\mathbf{Z}^{\mathrm{I}}_{\mathrm{triv}}[u]+\mathbf{Z}^{\mathrm{II}}_{\mathrm{triv}}[u], \tag{7}$$

where $\mathbf{Z}^{\mathrm{I}}_{\mathrm{triv}}[u]$ vanishes on solutions of the given system (1), and $\mathbf{Z}^{\mathrm{II}}_{\mathrm{triv}}[u]$ satisfies (2) as a differential identity, for all functions $u(z)$, not only the solutions of (1) (similarly to div curl $\equiv 0$).

Two local conservation laws (2) of the PDE system (1) are *equivalent* if they differ by a trivial conservation law. This notion defines an *equivalence class* of conservation laws; the conservation laws within one equivalence class have the same physical meaning, as they correspond to the same global conservation principle (4). One is consequently interested in finding, for a given model, a maximal set of nonequivalent linearly independent local conservation laws.

2.1.3 The Direct Construction Method

The direct conservation law construction method [1, 7, 16–18] is the most general, coordinate-independent systematic way to seek conservation laws of *any* PDE system. In particular, it generalizes the Noether's theorem onto nonvariational system. Similarly, Ibragimov's "new conservation theorem" and related constructs are restricted versions of the direct method [58]. In addition to the direct method, other computational techniques for finding conservation laws of PDEs and first integrals of ODEs exist (see, e.g. [59–61]), these may be practically useful in specific situations.

The direct method is based on seeking local conservation laws (2) for a given system in a *characteristic form*

$$\mathrm{D}_i Z^i[U]=\Lambda_\sigma[U]R^\sigma[U] \tag{8}$$

for some set of conservation law multipliers (characteristics) $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$, holding for an arbitrary vector function $U(z)$. Then on solution of the given model (1), one has a local conservation law

$$\mathrm{D}_i Z^i[u]=\Lambda_\sigma[u]R^\sigma[u]=0. \tag{9}$$

For ordinary differential equations (ODE), local conservation law multipliers are the integrating factors.

The computation of local conservation laws in the characteristic form proceeds by setting a dependence ansatz for each of the multipliers (same or different), and formulating the multiplier determining equations. The latter arise from the well-known fact that the Euler differential operators

$$\mathcal{E}_{U^j}=\frac{\partial}{\partial U^j}-\mathrm{D}_i\frac{\partial}{\partial U^j_i}+\cdots+(-1)^s\mathrm{D}_{i_1}\ldots\mathrm{D}_{i_s}\frac{\partial}{\partial U^j_{i_1\ldots i_s}}+\cdots, \tag{10}$$

with respect to the functions U^j, $j=1, \ldots, m$, annihilate a differential function $F[U]$ if and only if it is a divergence expression (see, e.g. [1, 62]). Therefore a set of functions $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ defines a set of local conservation law multipliers if and only if

$$\mathcal{E}_{U^j}(\Lambda_\sigma[U]R^\sigma[U])\equiv 0, \quad j=1, \ldots, m, \tag{11}$$

holding for all $U(z)$. Setting to zero coefficients at highest-order derivatives not present in multiplier dependence, one consequently splits the determining equations (11), and obtains a linear overdetermined system of determining equations for the unknown multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$.

Importantly, for totally nondegenerate PDE systems, *any* nontrivial local conservation law (2) has an equivalent local conservation law in the characteristic form [1, 7]. Moreover, when the numbers of equations and dependent variables coincide, $N=m$, and the given PDE system (1) is in an extended Kovalevskaya form (i.e. is solved for highest derivatives with respect to some independent variable), then the multipliers $\Lambda_\sigma[U]$ involve neither these leading derivatives nor their differential consequences (See Lemma 3 of [16], and also [7]). Consequently, for such models, by specifying a sufficiently general ansatz for the multipliers $\Lambda_\sigma[U]$, one can in principle find *all* local conservation laws of the model. Moreover, the multipliers cannot vanish on the solution set of the model, as they do not involve leading derivatives and their differential consequences. As a result, conservation laws computed in this way for extended Kovalevskaya PDE systems always are nontrivial.

The *order* of a local conservation law commonly refers to the highest order of a derivative of a dependent variable present in the density-flux vector (the minimum such number in the equivalence class). For the computations using the direct method, the *order* may refer to the highest order of a derivative present in multipliers for a given local conservation law. In the current work, we use the latter definition.

2.1.4 Symbolic Computation

Within the direct conservation law construction method, the split overdetermined linear systems of determining equations for the unknown multiplies often consist of hundreds or thousands of interdependent linear equations, which makes computations by hand feasible only in elementary cases. The algorithm therefore has been implemented in symbolic software packages, including the `GeM` package for `Maple` [53–56]. For any given PDE system, linear determining equations for multipliers Λ^σ are efficiently generated and subsequently solved with `Maple` routines. Conservation law density/fluxes are subsequently computed via direct inegration, homotopy, or scaling formulas implemented in `GeM` routines [54]. A number of other symbolic software packages for conservation law computation exist (see, e.g. [59, 63–65]).

2.2 Nonlocally Related PDE Systems

The vast majority of examples in the theory of nonlocally related PDE systems based on conservation laws and potential equations has been obtained for the case of two independent variables, to which we now mainly restrict our attention. It is convenient to denote PDE systems using the names of dependent variables they involve. Let **U**$\{x, y; u\}$ be a system of PDE given by

$$\mathbf{U}\{x, y; u\}:\ R^\sigma[u]=0, \qquad \sigma=1, \ldots, N, \tag{12}$$

with $m\geq 1$ dependent variables $u(x, y)=(u^1(x, y), \ldots, u^m(x, y))$. Suppose one knows $K\geq 1$ nontrivial linearly independent local conservation laws of (12), having a form

$$\mathrm{D}_x X^k[u]+\mathrm{D}_y Y^k[u]=0, \qquad k=1, \ldots, K. \tag{13}$$

holding for every solution $u(x, y)$ of the given system (12). Every conservation law (13) yields a pair of *potential equations*

$$(w^k)_y = X^k[u], \quad (w^k)_x=-Y^k[u], \tag{14}$$

where $w^k(x, y)$ is a nonlocal (potential) variable. A corresponding *singlet potential system* $\mathbf{UW^k}\{x, y; u, w^k\}$ [1, 14] involving a single potential variable is formed as a union of the given PDEs (12) and the potential equations (14). For example, for a given conservation law $\mathrm{D}_t T[u]+\mathrm{D}_x X[u]=0$, a singlet potential system with a potential variable w has the form

$$\mathbf{UW}\{x, y; u, w\}:\begin{cases} R^\sigma[u]=0, \quad \sigma=1, \ldots, N,\\ w_x = X[u],\\ w_t=-Y[u]. \end{cases} \tag{15}$$

Singlet potential systems arising from linearly independent sets of conservation laws are nonlocally related to each other [66].

We note that without loss of generality, the potential equations can be used to replace the original conservation law equation in the given system, or one of the PDEs in the given system which yielded the corresponding conservation law with a nonzero multiplier.

Similarly, using more than one potential, *multiplet potential systems* are formed. For example, using two local conservation laws (13) ($k=1$, 2), one obtains a couplet potential system

$$\mathbf{UW^1W^2}\{x, y; u, w^1, w^2\}:\begin{cases} R^\sigma[u]=0, \quad \sigma=1, \ldots, N,\\ (w^1)_y = X_1[u],\\ (w^1)_x=-Y_1[u],\\ (w^2)_y = X_2[u],\\ (w^2)_x=-Y_2[u]. \end{cases} \tag{16}$$

When a given system has two or more linearly independent conservation laws, in addition to a couplet potential system, one can use linear combinations of these conservation laws, with nonzero coefficients, to form a *spectral potential system*. For example, for two conservation laws, it can be written as

$$\mathbf{UW_\alpha}\{x, y; u, w_\alpha\}:\begin{cases} R^\sigma[u]=0, \quad \sigma=1, \ldots, N,\\ (w_\alpha)_y = X^1[u]+\alpha X^2[u],\\ (w_\alpha)_x=-(Y^1[u]+\alpha Y^2[u]), \end{cases} \tag{17}$$

where $\alpha\in\mathbb{R}\backslash\{0\}$ is a continuous parameter. The PDE system $\mathbf{UW_\alpha}\{x, y; u, w_\alpha\}$ is nonlocally related to both singlet potential systems with potentials w^1, w^2; two spectral potential systems $\mathbf{UW_\alpha}\{x, y; u, w_\alpha\}$ and $\mathbf{UW_\beta}\{x, y; u, w_\beta\}$ with $\alpha\neq\beta$ are also nonlocally related. When more than two local conservation laws are available, similar more general spectral potential systems can be constructed.

Another way of obtaining a PDE system nonlocally related to a given system **U**$\{x, y; u\}$ is forming a nonlocally related *subsystem*, by excluding one or more of the dependent variables $u^i(x, y)$ through differential substitutions [1].

An initial *tree* $\mathcal{T}$ of nonlocally related systems for a given system **U**$\{x, y; u\}$ (12) is obtained by considering the given system and available systems nonlocally related to it. The tree can be extended by seeking *nonlocal conservation laws* of the given system, and using them to introduce additional potential variables [1].

A somewhat formal definition of locally and nonlocally related PDE systems can be given in the following way.

Definition 1. *Let* **UV**$\{x, y; u, v\}$ *and* **UW**$\{x, y; u, w\}$ *be two PDE systems with independent variables* x, y *and the respective sets of dependent variables, where*

$$u(x, y)=(u^1(x, y), \ldots, u^m(x, y)), \quad m\geq 1;$$
$$v(x, y)=(v^1(x, y), \ldots, v^p(x, y)), \quad p\geq 0;$$
$$w(x, y)=w^1(x, y), \ldots, w^r(x, y)), \quad r\geq 0, \quad r+p>0. \tag{18}$$

$u(x, y)$ denote common, and $v(x, y)$, $w(x, y)$ additional dependent variables. The two systems have *equivalent solution sets* if every solution $(u(x, y), v(x, y))$ of the system **UV**$\{x, y; u, v\}$ yields a solution $(u(x, y), w(x, y))$ of the system **UW**$\{x, y; u, w\}$, and vice versa.

We note that within trees of nonlocally related PDE systems, the sets of dependent variables often significantly overlap. Moreover, it is commonly the case that in Definition (1), the relationship between the solution sets is not one-to-one; this is so, for example, when some of the variables within $v(x, y)$ and/or $w(x, y)$ are nonlocal (potential) variables.

Definition 2. *Let* **UV**$\{x, y; u, v\}$ *and* **UW**$\{x, y; u, w\}$ *be two PDE systems with dependent variables* (18). *Suppose* **UV**$\{x, y; u, v\}$ *and* **UW**$\{x, y; u, w\}$ *have equivalent solution sets. Then these systems are locally related if* $v^i(x, y)=f^i[u, w]$ *for all* i, $0\leq i\leq p$, *and* $w^j(x, y)=g^j[u, v]$ *for all* j, $0\leq j\leq r$; *in other words, all additional dependent variables of one system are local expressions on the jet space of the other system, and vice versa. Otherwise, the PDE systems* **UV**$\{x, y; u, v\}$ *and* **UW**$\{x, y; u, w\}$ *are nonlocally related.*

A nonlocal relationship between any two PDE systems in the tree of nonlocally related systems holds both ways, as it implies a non-one-to-one relationship between solution sets of the two systems. The conservation law, symmetry, or other local analysis for any two locally related systems would yield the same results. For nonlocally related systems, new results may be obtained.

We note that in addition to introducing nonlocal (potential) variables, or removing dependent variables by differential elimination, other ways of obtaining PDE systems nonlocally related to a given one exist; they include, for example, *inverse potential systems* obtained via a symmetry-based method [36], potential systems and subsystems obtained after point transformations [1, 20], and systems with pseudopotentials [33–35]. Moreover, nonlocal potential variables are sometimes defined not through the use of conservation laws, but from other considerations, such as the construction of nonlocal mappings between PDEs, or the identification of symmetry-integrable equations (e.g. [37, 38]).

2.3 Nonlocal Conservation Laws

A nontrivial zero divergence expression

$$\mathrm{D}_i\tilde{Z}^i=0 \tag{19}$$

holding on solutions of a given PDE system **U**$\{x, y; u\}$ (1) defines a *nonlocal conservation law* of (1) if it is not equivalent to any local conservation law of the system (1).

In other words, a conservation law (19) is nonlocal if and only if there does not exist a conservation law $\mathrm{D}_i\hat{Z}^i=0$ equivalent to (19) such that for all j, $\hat{Z}^i=\hat{Z}^i[u]$, i.e. the density-flux vector of a nonlocal conservation law is not equivalent to a vector differential function on a jet space associated with the solution set of the given PDE system **U**$\{x, y; u\}$ (1), but involves *nonlocal variables* given by integrals of such differential functions. Nonlocal conservation laws arise as local conservation laws of potential systems of a given system, or nonlocally related subsystems that involve potentials. For example, for the potential system **UW¹W²**$\{x, y; u, w^1, w^2\}$ (16), one can have a local conservation law (19) with the density-flux vector components $\tilde{Z}^i=\tilde{Z}^i[u, w^1, w^2]$ being local differential functions on an extended jet space corresponding to the solution set of the potential system **UW¹W²**$\{x, y; u, w^1, w^2\}$; such a conservation law would correspond to a nonlocal conservation law of the given model **U**$\{x, y; u\}$ (12) if there does not exist an equivalent conservation law with components $Z^i=Z^i[u]$.

Nonlocal conservation laws are systematically constructed through the direct method applied to a potential system, or a nonlocally related subsystem involving potentials. The following important result gives a necessary condition for such conservation laws to be nonlocal [14, 21].

Theorem 1. *A conservation law of a potential system* **UW**$\{x, y; u, w\}$ (15), *arising from multipliers independent of the potential variable w, is equivalent to a local conservation law of the given system* **U**$\{x, y; u\}$ (12).

[A more general statement of this theorem, containing four equivalent statements, is found in [21] (Theorem 7).

We note that Theorem 1 only holds for potential systems. One can, for example, have nonlocal conservation laws for a given model arising from local multipliers, if the direct method is applied to a *subsystem* of a potential system. (An example is given in Section 4.2 below.)

2.4 Local and Nonlocal Symmetries

In this subsection, we again consider a general n-dimensional PDE system $\mathbf{U}\{z;\ u\}$ (1). A Lie group of point symmetries of the system (1) is a Lie group of point transformations that maps the solution set $\mathcal{G}_U$ into itself. A Lie group of point symmetries corresponds to a Lie algebra of point symmetry generators

$$\mathrm{X} = \xi^i(z,\ u)\frac{\partial}{\partial z^i} + \eta^\mu(z,\ u)\frac{\partial}{\partial u^\mu}, \tag{20}$$

which may be equivalently written in the evolutionary form (e.g. [1])

$$\hat{\mathrm{X}} = \zeta^\mu(z,\ u,\ \partial u)\frac{\partial}{\partial u^\mu}. \tag{21}$$

Higher-order local symmetries, which arise for some PDE models, have the form (21) with $\zeta^\mu = \zeta^\mu[u]$ being differential functions that depend on higher derivatives of u. Point and higher-order symmetries of a given model are computed using the standard Lie's algorithm.

Lie groups of nonlocal symmetries do not arise from an application of the local Lie's algorithm to a given PDE system. For nonlocal symmetries, the components of symmetry generators generators are not differential functions on a jet space associated with $\mathcal{G}_U$. Nonlocal symmetries were first explicitly derived as local symmetries of a PDE system nonlocally related to the given one; there, the infinitesimal generator components corresponding to the variables of the original model have an essential dependence on nonlocal variables (see [1, 27, 28]; also [11–13]). For example, let $\mathbf{U}\{x,\ y;\ u\}$ (12) be a given PDE system, and $\mathbf{UW}\{x,\ y;\ u,\ w\}$ (15) its potential system. A point symmetry of the latter, given by

$$\mathrm{Y} = \xi^1(x,\ y,\ u,\ w)\frac{\partial}{\partial x} + \xi^2(x,\ y,\ u,\ w)\frac{\partial}{\partial y} + \eta^\mu(x,\ y,\ u,\ w)\frac{\partial}{\partial u^\mu} + \kappa(x,\ y,\ u,\ w)\frac{\partial}{\partial w}, \tag{22}$$

corresponds to a nonlocal symmetry of $\mathbf{U}\{x,\ y;\ u\}$ (12) if at least one of the components $\xi^i(x,\ y,\ u,\ w)$, $\eta^\mu(x,\ y,\ u,\ w)$ depends on the nonlocal variable w.

The consideration of nonlocal symmetries significantly enhances the applicability of symmetry methods (see, e.g. [1] and references therein for theoretical results and multiple examples pertaining to computation and applications of nonlocal symmetries). We mention the following facts concerning nonlocal symmetries.

- Nonlocal symmetries do not arise for underdetermined potential systems (see [1, 67]).
- A local symmetry of a given PDE system may correspond to a nonlocal symmetry of a nonlocally related system.
- For a given PDE system (12) in two dimensions which has precisely n linearly independent local conservation laws, all its local symmetries are preserved in the n-plet potential system [66, 68].

3 Boundary Layer Models

In the current section and all subsequent sections, upper indices will not be used; upper space in the notation will be reserved for powers.

The dimensionless Navier–Stokes equations of incompressible constant-density viscous fluid flow without external forces in three dimensions are given by

$$\nabla \cdot \mathbf{u} = 0, \tag{23a}$$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \hat{\nu}\nabla^2\mathbf{u} = 0, \tag{23b}$$

where the fluid velocity vector $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$ and the hydrostatic pressure p are functions of x, y, z, t. The dimensionless viscosity is given by $\hat{\nu} = \text{const} = 1/\text{Re}$, where Re is the Reynolds number. The inviscid case $\hat{\nu} = 0$ corresponds to the Euler model.

3.1 Prandtl Equations for Steady Plate Flows

The classical Prandtl boundary layer equations for the steady plate flow are obtained under the assumption of the two-dimensional time-independent flow [40]

$$u = u(x,\ y),\quad v = v(x,\ y),\quad w = 0,$$

using a scaling change of variables

$$x^* = x,\quad y^* = y/\delta,\quad u(x,\ y) = u^*(x^*,\ y^*),$$
$$v(x,\ y) = v^*(x^*,\ y^*)/\delta,\quad p(x,\ y) = p^*(x^*,\ y^*),$$

where $\delta = (\text{Re})^{-1/2} \ll 1$ is a small parameter, and the starred variables are of the order of magnitude $\mathcal{O}(1)$. As a result, keeping the highest-order terms in the PDEs (23a, 23b), using the equilibrium solution asymptotics

$$u^*(x^*,\ y^*) \to u_0,\quad v^*(x^*,\ y^*) \to 0,\quad p^*(x^*,\ y^*) \to p_0$$

as $y^* \to \infty$ (outside of the boundary layer), and omitting the asterisks, one obtains the dimensionless Prandtl equations describing a two-dimensional steady plate boundary layer flow:

$$u_x + v_y = 0, \tag{24a}$$

$$uu_x + vu_y = u_{yy}. \tag{24b}$$

(In the dimensional form, the right-hand side of the PDE (24b) is νu_{yy}, where ν is the viscosity coefficient.)

3.2 Boundary Layer Equations for Axially Symmetric Flows

In order to write (23a, 23b) in the axially symmetric setting, the velocity is represented, in cylindrical coordinates, as

$$\mathbf{u} = u_1\mathbf{e}_r + u_2\mathbf{e}_\varphi + u_3\mathbf{e}_z. \tag{25}$$

Upon a transformation to cylindrical coordinates and imposing the rotational invariance $\partial/\partial\varphi \equiv 0$, the continuity equation (23a) and the three components of the momentum equation (23b) yield the following system of four scalar PDEs, commonly referred to as the *dimensionless axially symmetric Navier–Stokes system in primitive variables* (cf. [25]):

$$(ru_1)_r + (ru_3)_z = 0, \tag{26a}$$

$$(u_1)_t + u_1(u_1)_r + u_3(u_1)_z - \frac{1}{r}(u_2)^2 = -p_r + \hat{\nu}\left[\frac{1}{r}(r(u_1)_r)_r + (u_1)_{zz} - \frac{1}{r^2}u_1\right], \tag{26b}$$

$$(u_2)_t + u_1(u_2)_r + u_3(u_2)_z + \frac{1}{r}u_1u_2 = \hat{\nu}\left[\frac{1}{r}(r(u_2)_r)_r + (u_2)_{zz} - \frac{1}{r^2}u_2\right], \tag{26c}$$

$$(u_3)_t + u_1(u_3)_r + u_3(u_3)_z = -p_z + \hat{\nu}\left[\frac{1}{r}(r(u_3)_r)_r + (u_3)_{zz}\right]. \tag{26d}$$

In the PDEs (26a, 26b, 26c, 26d), the four dependent variables u_1, u_2, u_3, p are functions of the three independent variables t, r, z. We note that the PDEs (26a, 26b, 26c, 26d) were obtained in [25] in a more general setting of helical invariance, of which the axial symmetry is a special case.

3.2.1 A Two-Component Boundary Layer Flow About a Circular Cylinder

When the time-independent boundary layer flow around an infinite vertical circular cylinder $r=\text{const}$ around the z-axis is considered, one can, similarly to the plate flow, denote

$$u_1 = v(x, y), \quad u_2 = 0, \quad u_3 = u(x, y),$$

and employ a rescaling

$$z^* = z, \quad r^* = r/\delta, \quad u(r, z) = u^*(r^*, z^*), \quad v(r, z) = v^*(r^*, z^*)/\delta, \quad p(r, z) = p^*(r^*, z^*).$$

Retaining the highest-order terms in the PDEs (26a–d), and using the asymptotics to a steady equilibrium flow solution outside of the boundary layer

$$u^*(r^*, z^*) \to u_0, \quad v^*(r^*, z^*) \to 0, \quad p^*(r^*, z^*) \to p_0$$

as $r^* \to \infty$, one obtains the dimensionless cylindrical Prandtl equations given by

$$(rv)_r + (ru)_z = 0, \tag{27a}$$

$$uu_z + vu_r = u_{rr} + \frac{1}{r}u_r, \tag{27b}$$

where the asterisks have been omitted.

(In the dimensional form, the right-hand side of the PDE (27b) acquires a factor ν.) The above model has been derived by Schlichting [39, 40], and is sometimes referred to as an "axisymmetric jet without a swirl".

3.2.2 A Radial Wall Jet Boundary Layer Model

The radial wall flow that forms when a circular fluid jet strikes a planar wall normally and spreads out over it was considered in [41] (see also [42]). In particular, a boundary layer approximation was derived. Let the wall be described by $z=\text{const}$. Assuming no polar flow, $u_2=0$, a constant viscosity ν, and denoting the mean velocities $u_1=u(r, z)$, $u_3=v\,(r, z)$, one arrives at the dimensionless radial wall jet boundary layer equations

$$(ru)_r + (rv)_z = 0, \tag{28a}$$

$$uu_r + vu_z = u_{zz}. \tag{28b}$$

(In the dimensional form, again, the right-hand side of the PDE (28b) has an extra factor ν.)

4 Conservation Laws and Nonlocally Related Systems of the Prandtl Plate Flow Model

We now study the conservation laws and nonlocally related PDE systems for the Prandtl equations (24a, 24b) describing for steady plate boundary layer flows (Section 3.1). The corresponding PDE system can be written as

$$\mathbf{UV}\{x, y; u, v\}: \begin{cases} u_{yy} = uu_x + vu_y, \\ v_y = -u_x \end{cases} \tag{29}$$

in a solved (extended Kovalevskaya) form with respect to the highest derivatives in y. One observes that the first PDE of (29) can be solved for v explicitly,

$$v = (u_{yy} - uu_x)/u_y, \tag{30}$$

and v can be excluded by the substitution into the second PDE of (29). As a result, one obtains a subsystem given by a single PDE

$$\mathbf{U}\{x, y; u\}: u_x + ((u_{yy} - uu_x)/u_y)_y = 0, \tag{31}$$

which is locally related to the given PDE system $\mathbf{UV}\{x, y; u, v\}$ (29), as $v = v[u]$ as per (30).

4.1 Local Conservation Laws and Potential Systems

First we compute all local conservation laws of the Prandtl plate flow model $\mathbf{UV}\{x, y; u, v\}$ (29) with fourth-order multipliers. According to Lemma 3 of [16], the leading derivatives u_{yy}, v_y of (29) and their differential consequences can be excluded from the multiplier dependence without loss of generality. The fourth-order multiplier form becomes

$$\Lambda_\sigma[u, v] = \Lambda_\sigma(x, y, u, v, u_x, v_x, u_y, u_{xx}, u_{xy}, v_{xx}, u_{xxx}, u_{xxy}, v_{xxx}, u_{xxxx}, u_{xxxy}, v_{xxxx}), \quad \sigma = 1, 2. \tag{32}$$

The following result is proven by a direct computation (cf. [46]).

Proposition 4.1. *The linear space of inequivalent nontrivial local conservation laws of the PDE system* (29) *arising from the fourth-order multipliers* (32) *is spanned by the two conservation laws*

$$\mathrm{D}_x(u) + \mathrm{D}_y(v) = 0, \tag{33}$$

$$\mathrm{D}_x(u^2) + \mathrm{D}_y(uv - u_y) = 0, \tag{34}$$

corresponding to the zeroth-order multiplier pairs $(\Lambda_1, \Lambda_2) = (0, 1)$ and $(\Lambda_1, \Lambda_2) = (-1, u)$.

One can introduce the corresponding potentials, and establish potential systems, as follows. Using the first conservation law (33), i.e. the first PDE of the model (29) itself, we introduce the first potential ψ, the stream function $\psi(x, y)$, and obtain a singlet potential system

$$\mathbf{UV\Psi}\{x, y; u, v, \psi\}: \begin{cases} \psi_y = u, \\ \psi_x = -v, \\ u_{yy} = uu_x + vu_y. \end{cases} \tag{35}$$

This potential system is not in an extended Kovalevskaya form as it stands. However, one may exclude, for example, the variable v by a local substitution, to obtain a locally related subsystem

$$\mathbf{U\Psi}\{x, y; u, \psi\}: \begin{cases} \psi_y = u, \\ u_{yy} = uu_x - \psi_x u_y, \end{cases} \tag{36}$$

which is in the extended Kovalevskaya form with respect to y, and is more suitable for direct conservation law computations. Similarly, one can exclude the variable u, which leads to a locally related subsystem $\mathbf{V\Psi}\{x, y; v, \psi\}$. Excluding both u and v, or equivalently, excluding u from (36) by a local substitution, we arrive at a further locally related subsystem, given by a scalar stream function equation

$$\mathbf{\Psi}\{x, y; \psi\}: \psi_{yyy} = \psi_y \psi_{xy} - \psi_x \psi_{yy}. \tag{37}$$

The potential system (35) and its subsystems are nonlocally related to the given plate flow model (29), but are locally related to each other.

The second singlet potential system for the Prandtl equations (29) is obtained using the second conservation law (34). Defining the potential variable ϕ according to the potential equations $\phi_y = u^2$, $\phi_x = u_y - uv$, one may write the resulting potential system in the extended Kovalevskaya form with respect to y:

$$\mathbf{UV\Phi}\{x, y; u, v, \phi\}: \begin{cases} \phi_y = u^2, \\ u_y = \phi_x + uv, \\ v_y = -u_x. \end{cases} \tag{38}$$

Again, from the potential system $\mathbf{UV\Phi}\{x, y; u, v, \phi\}$, the dependent variables u, v can be excluded by local substitutions, yielding locally related subsystems $\mathbf{V\Phi}\{x, y; v, \phi\}$, $\mathbf{U\Phi}\{x, y; u, \phi\}$, and $\mathbf{\Phi}\{x, y; u, \phi\}$.

Using both potentials ψ, ϕ, one obtains a couplet potential system

$$\mathbf{UV\Psi\Phi}\{x, y; u, v, \psi, \phi\}: \begin{cases} \psi_y = u, \\ \psi_x = -v, \\ \phi_y = u^2, \\ \phi_x = u_y - uv \end{cases} \tag{39}$$

nonlocally related to all previously considered systems. Its locally related subsystem

$$\mathbf{U\Psi\Phi}\{x,\ y;\ u,\ \psi,\ \phi\}:\begin{cases}\psi_y = u,\\ \phi_y = u^2,\\ u_y = \phi_x - u\psi_x\end{cases} \tag{40}$$

obtained by a substitution exclusion of v has an extended Kovalevskaya form with respect to y and is suitable for further conservation law analysis using the direct method. One can further exclude from (30) the variable u by a local substitution, to obtain a subsystem

$$\mathbf{\Psi\Phi}\{x,\ y;\ \psi,\ \phi\}:\begin{cases}\phi_y = \psi_y^2,\\ \psi_{yy} = \phi_x - \psi_x\psi_y,\end{cases} \tag{41}$$

which is also nonlocally related to the given model, and has an extended Kovalevskaya form with respect to y.

Finally, a linear combination of the conservation laws (33), (34), without loss of generality with factors 1 and $\alpha \in \mathbb{R}\backslash\{0\}$,

$$\mathrm{D}_x(u+\alpha u^2) + \mathrm{D}_y(v+\alpha(uv-u_y)) = 0 \tag{42}$$

is used to formulate a spectral potential system with potential equations

$$(q_\alpha)_y = u + \alpha u^2, \quad (q_\alpha)_x = -(v+\alpha(uv-u_y)).$$

The variable q_α is a local function of the other two potential variables ψ, ϕ:

$$q_\alpha = \psi + \alpha\phi + c, \tag{43}$$

where c is an arbitrary constant, and hence q_α is not a nonlocal variable on solutions of the couplet potential system $\mathbf{UV\Psi\Phi}\{x,\ y;\ u,\ v,\ \psi,\ \phi\}$ (30), or any of its subsystems that involve both ψ and ϕ.

As the factors 1, $\alpha \neq 0$, either PDE of the given system (29) can be replaced by the potential equations; we leave the first PDE, and obtain the spectral potential system in an extended Kovalevskaya form:

$$\mathbf{UVQ}_\alpha\{x,\ y;\ u,\ v,\ q_\alpha\}:\begin{cases}(q_\alpha)_y = u + \alpha u^2,\\ u_y = ((q_\alpha)_x + v)/\alpha + uv,\\ v_y = -u_x.\end{cases} \tag{44}$$

The dependent variables u and/or v can be eliminated from (44) by local substitutions, leading to locally related subsystems $\mathbf{UQ}_\alpha\{x,\ y;\ u,\ q_\alpha\}$, $\mathbf{VQ}_\alpha\{x,\ y;\ v,\ q_\alpha\}$, and $\mathbf{Q}_\alpha\{x,\ y;\ q_\alpha\}$; those subsystems are nonlocally related to the original model $\mathbf{UV}\{x,\ y;\ u,\ v\}$ (29). We exclude these systems from the consideration below since they do not have a simple form, and will not lead to new results compared to the analysis of the potential system $\mathbf{UVQ}_\alpha\{x, y; u, v, q_\alpha\}$ due to their local relationship with it.

The preliminary tree $\mathcal{T}_1$ of PDE systems for the Prandtl plate boundary layer model (24a, 24b) is summarized in Figure 1. In particular, groups of PDE systems of different colors are nonlocally related to each other, whereas systems shown by boxes of the same color are locally related to each other.

Remark 4.1. The tree $\mathcal{T}_1$ may be immediately extended by considering couplet potential systems $\mathbf{UV\Psi Q}_\alpha\{x,\ y;\ u,\ v,\ \psi,\ q_\alpha\}$, $\mathbf{UV\Phi Q}_\alpha\{x,\ y;\ u,\ v,\ \phi,\ q_\alpha\}$, and their subsystems; Figure 1 does not show these extensions for the sake of compactness. Note that the triplet potential system $\mathbf{UV\Psi\Phi Q}_\alpha\{x,\ y;\ u,\ v,\ \psi,\ \phi,\ q_\alpha\}$ is locally related to $\mathbf{UV\Psi\Phi}\{x, y; u, v, \psi, \phi\}$ (30) due to (43) and therefore is not considered.

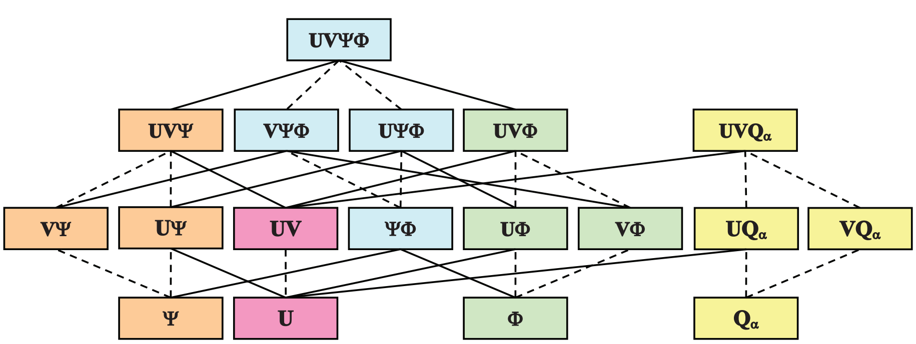

Figure 1: A preliminary tree $\mathcal{T}_1$ of locally and nonlocally related systems for the Prandtl plate flow boundary layer model (24a, 24b). Groups of the same color (color online) correspond to PDE systems locally related to each other; groups of PDE systems of different colors are nonlocally related. Nonlocal relations between PDE systems are illustrated with solid lines, and local relations with dashed lines. [The same tree structure, in terms of different notation, also describes locally and nonlocally related systems of the cylinder and radial jet boundary layer models (Sections 5 and 6).]

4.2 Nonlocal Conservation Laws and Further Potential Systems

For the Prandtl plate boundary layer model given by the PDE system **UV**$\{x, y; u, v\}$ (29), we now seek nonlocal conservation laws that arise as local conservation laws of its potential systems within the preliminary tree discussed in Section 4.1.

4.2.1 A Nonlocal Conservation Law Arising from the Potential System UVΦ$\{x, y; u, v, \phi\}$

First, we compute local conservation laws of the potential system **UVΦ**$\{x, y; u, v, \phi\}$ (38) using second-order multipliers, i.e.

$$\Lambda_\sigma[u, v] = \Lambda_\sigma(x, y, u, v, \phi, u_x, v_x, \phi_x, u_{xx}, v_{xx}, \phi_{xx}),\quad \sigma = 1, 2, 3.$$

The general solution involves two linearly independent multiplier sets. The first set $(0, 0, 1)$ yields the local conservation law (33), and the second set $(v, -u, \phi)$ a conservation law

$$\mathrm{D}_x(u\phi) + \mathrm{D}_y\left(v\phi - \frac{1}{2}u^2\right) = 0, \tag{45}$$

It is clear that (45) is a nonlocal conservation law of the Prandtl boundary layer system **UV**$\{x, y; u, v\}$ (29), as its fluxes explicitly involve the potential variable.

Using (45), one can introduce a further potential variable $\gamma(x, y)$ satisfying

$$\gamma_y = u\phi, \quad \gamma_x = \frac{1}{2}u^2 - v\phi,$$

and thereby obtain a potential system

$$\mathbf{UV\Phi\Gamma}\{x, y; u, v, \phi, \gamma\}: \begin{cases} \phi_y = u^2, \\ \phi_x = u_y - uv, \\ v_y = -u_x, \\ \gamma_y = u\phi, \\ \gamma_x = \dfrac{1}{2}u^2 - v\phi. \end{cases} \tag{46}$$

[Any of the first three PDEs of (46) can be dropped from this system, as the corresponding multipliers that yield the conservation law (45) are nonzero.]

Through obvious local substitutions, the variables ϕ and/or u and/or v can be excluded, to obtain PDE systems **UVΓ**$\{x, y; u, v, \gamma\}$, **VΦΓ**$\{x, y; v, \phi, \gamma\}$, **UΦΓ**$\{x, y; u, \phi, \gamma\}$, **VΓ**$\{x, y; v, \gamma\}$, **ΦΓ**$\{x, y; \phi, \gamma\}$, and **UΓ**$\{x, y; u, \gamma\}$, locally related to **UVΦΓ**$\{x, y; u, v, \phi, \gamma\}$ (46). In Figure 2, the corresponding tree extension is outlined.

4.2.2 A Nonlocal Conservation Law Arising from the Potential System UVΨ$\{x, y; u, v, \psi\}$

We now seek local conservation laws of PDE systems involving the nonlocal variable ψ. The computations for

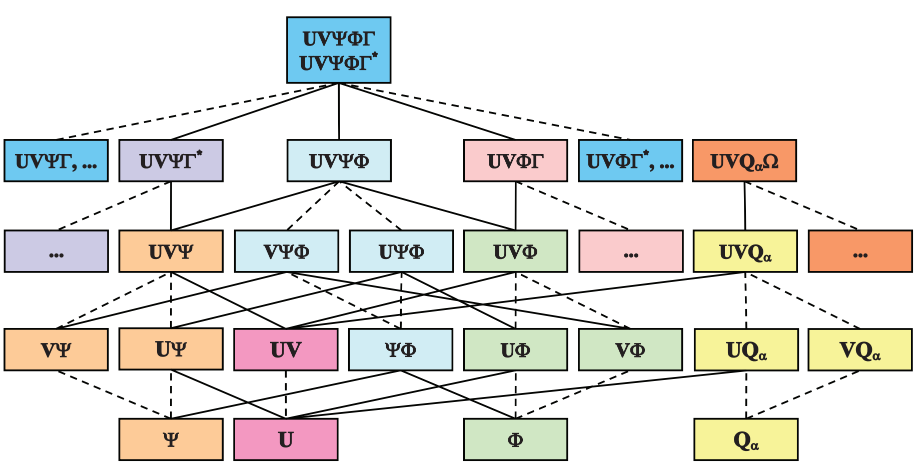

Figure 2: An extended tree $\mathcal{T}_3$ of nonlocally related systems for the Prandtl plate flow boundary layer model (24a, 24b) obtained using the nonlocal conservation laws (45) and (47). Groups of the same color (color online) correspond to PDE systems locally related to each other; groups of PDE systems of different colors are nonlocally related. Nonlocal relations between PDE systems are illustrated with solid lines, and local relations with dashed lines. [The same extended tree structure, in terms of different notation, also describes locally and nonlocally related systems of the cylinder and radial jet boundary layer models (Sections 5 and 6).]

the system **UΨ**$\{x, y; u, \psi\}$ (36) involving third-order multipliers leads to two first-order multiplier pairs $(-u_x, 1)$ and $(-u_x\psi, \psi)$. The first one is equivalent to the local conservation law (34) of the given model **UV**$\{x, y; u, v\}$ (29), and the second one can be equivalently written as

$$\mathrm{D}_x(u^2\psi)+\mathrm{D}_y\left((uv-u_y)\psi+\frac{1}{2}u^2\right)=0. \tag{47}$$

The divergence expression (47) yields a nonlocal conservation law of the plate flow Prandtl model (29). (The same result is obviously obtained if one analyzes the potential system **UVΨ**$\{x, y; u, v, \psi\}$ (35), or any other locally related system, instead.)

From the conservation law (47), one introduces a further potential $\gamma^*(x, y)$ satisfying

$$\gamma^*_y=u^2\psi, \quad \gamma^*_x=-\left((uv-u_y)\psi+\frac{1}{2}u^2\right).$$

As a result, one has a potential system

$$\mathbf{UV\Psi\Gamma^*}\{x, y; u, v, \psi, \gamma^*\}:\begin{cases}\psi_y=u,\\ \psi_x=-v,\\ u_{yy}=uu_x+vu_y,\\ \gamma^*_y=u^2\psi,\\ \gamma^*_x=-\left((uv-u_y)\psi+\frac{1}{2}u^2\right).\end{cases} \tag{48}$$

[Any of the first three PDEs of (48) can be dropped from this system, as the multipliers that lead to the conservation law (47) for **UVΨ**$\{x, y; u, v, \psi\}$ are given by $(-u_x\psi+u_y, \psi u_y, \psi)$ and are nonzero.]

Through local substitutions, the variables ψ and/or u and/or v can be excluded, to obtain PDE systems **UVΓ***$\{x, y; u, v, \gamma^*\}$, **VψΓ***$\{x, y; v, \psi, \gamma^*\}$, **UΨΓ***$\{x, y; u, \psi, \gamma^*\}$, **UΓ***$\{x, y; u, \gamma^*\}$, and **ΨΓ***$\{x, y; \psi, \gamma^*\}$ locally related to **UVΨΓ***$\{x, y; u, v, \psi, \gamma^*\}$ (48). Figure 2 shows the corresponding tree extension.

4.2.3 A Local Relationship between the Potentials in a Combined Potential System

Instead of considering the PDE systems **UΨ**$\{x, y; u, \psi\}$ (36) and **UVΨ**$\{x, y; u, v, \phi\}$ (38) independently, as we did above, one could start from the couplet potential system **UVΨΦ**$\{x, y; u, v, \psi, \phi\}$, and use its two local conservation laws (45) and (47) to introduce the potential variables γ and γ^*, obtaining the potential systems

$$\mathbf{UV\Psi\Phi\Gamma}\{x, y; u, v, \psi, \phi, \gamma\}:\begin{cases}\psi_y=u,\\ \psi_x=-v,\\ \phi_y=u^2,\\ \phi_x=u_y-uv,\\ \gamma_y=u\phi,\\ \gamma_x=\frac{1}{2}u^2-v\phi\end{cases} \tag{49}$$

and

$$\mathbf{UV\Psi\Phi\Gamma^*}\{x, y; u, v, \psi, \phi, \gamma^*\}:\begin{cases}\psi_y=u,\\ \psi_x=-v,\\ \phi_y=u^2,\\ \phi_x=u_y-uv,\\ \gamma^*_y=u^2\psi,\\ \gamma^*_x=-\left((uv-u_y)\psi+\frac{1}{2}u^2\right).\end{cases} \tag{50}$$

as well as the joint potential system **UVΨΦΓΓ***$\{x, y; u, v, \psi, \phi, \gamma, \gamma^*\}$. The latter, however, appears redundant, as the following statement holds.

Lemma 4.1. *The potentials γ and γ^* are locally related on the solution set of the PDE systems* **UVΨΦΓ**$\{x, y; u, v, \psi, \phi, \gamma\}$ (49) *and* **UVΨΦΓ***$\{x, y; u, v, \psi, \phi, \gamma^*\}$ (49):

$$\gamma(x, y)=-\gamma^*(x, y)+\psi\phi+\text{const.} \tag{51}$$

The PDE systems (49) and (49) are consequently locally related.

The relationship (70) is verified by a direct computation. For example, the x-flux of (45) can be written as

$$u\phi=\psi_y\phi=(\psi\phi)_y-\psi\phi_y=(\psi\phi)_y-u^2\psi,$$

which is an x-flux equivalent to that of (47). The same is true for the y-flux. It follows that the conservation laws (45) and (47) are equivalent on solutions of the PDE system **UVΨΦ**$\{x, y; u, v, \psi, \phi\}$ (30), or any subsystem that includes the dependent variables ψ and ϕ. In particular, a local conservation law equivalent to (45), (47) can be found for the PDE system **ΨΦ**$\{x, y; \psi, \phi\}$ (41), using a pair of multipliers $(\psi_x, \psi_y)\equiv(-v, u)$; the resulting divergence expression

$$\mathrm{D}_x(\psi\phi_y-\psi_y\phi)+\mathrm{D}_y(\psi_x\phi-\psi\phi_x+\psi_y^2)=0, \tag{52}$$

is equivalent to (45). The latter also illustrates that Theorem 1 does not hold for subsystems: *local* multipliers [on solutions of the Prandtl equations **UV**$\{x, y; u, v\}$ (29)]

for a nonlocally related subsystem yield a *nonlocal* conservation law of the Prandtl model (29).

The potentials γ, γ^* are not locally related on the solution set of $\mathbf{UV\Phi}\{x, y; u, v, \phi\}$ (38) and $\mathbf{UV\Psi}\{x, y; u, v, \psi\}$ (35), so the PDE systems $\mathbf{UV\Phi}\{x, y; u, v, \phi\}$ and $\mathbf{UV\Psi}\{x, y; u, v, \psi\}$, with their corresponding subsystems, are nonlocally related. Yet $\mathbf{UV\Psi\Phi\Gamma}\{x, y; u, v, \psi, \phi, \gamma\}$ (49) and $\mathbf{UV\Psi\Phi\Gamma^*}\{x, y; u, v, \psi, \phi, \gamma^*\}$ (49), and their subsystems obtained by local substitutions of u and/or v and/or ψ and/or ϕ as appropriate, are locally related to each other (Fig. 2).

In order to seek further nonlocal conservation laws, nonlocal symmetries, etc., arising from PDE systems involving the new nonlocal variable γ, one may use the PDE system $\mathbf{UV\Psi\Phi\Gamma}\{x, y; u, v, \psi, \phi, \gamma\}$ (49) or any system locally related to it. A direct symbolic computation of conservation laws for the PDE system (49) with second-order multipliers reveals no additional conservation laws of the plate flow Prandtl equations (23a, 23b).

4.2.4 A Nonlocal Conservation Law Arising from the Potential System UVQ$_\alpha\{x, y; u, v, q_\alpha\}$

We now compute local conservation laws of the spectral potential system $\mathbf{UVQ}_\alpha\{x, y; u, v, q_\alpha\}$ (44). Seeking third-order local multipliers, we find the multiplier triple $(v, -\alpha u, q_\alpha)$, corresponding to a local conservation law

$$\mathrm{D}_x(uq_\alpha)+\mathrm{D}_y\left(vq_\alpha-\frac{1}{2}\alpha u^2\right)=0. \tag{53}$$

The conservation law (53) generally ($\alpha\neq 0$) yields a nonlocal conservation law of the plate flow Prandtl model (29), linearly independent of (47).

Using the new conservation law (53), one can introduce a further potential variable ω, and obtain a potential system

$$\mathbf{UVQ}_\alpha\mathbf{\Omega}\{x, y; u, v, q_\alpha, \omega\}:\begin{cases}\omega_y=uq_\alpha,\\ \omega_x=\dfrac{1}{2}\alpha u^2-vq_\alpha,\\ u_y=((q_\alpha)_x+v)/\alpha+uv,\\ v_y=-u_x.\end{cases} \tag{54}$$

also involving the spectral parameter $\alpha\in\mathbb{R}\backslash\{0\}$.

The extended tree $\mathcal{T}_3$ for the plate flow Prandtl model (24a, 24b) including the potential system (55) and its locally related subsystems obtained by local substitutions of the dependent variables u and/or v and/or q_α is shown in Figure 2.

In order to study nonlocal conservation laws of the plate flow model (24a, 24b) arising from the above-described new nonlocally related systems, one may consider, for example, a PDE system

$$\mathbf{UQ}_\alpha\mathbf{\Omega}\{x, y; u, q_\alpha, \omega\}:\begin{cases}\omega_y=uq_\alpha,\\ (q_\alpha)_y=u+\alpha u^2,\\ u_y=\alpha^{-1}\left((q_\alpha)_x+q_\alpha^{-1}\left[\dfrac{1}{2}\alpha u^2-\omega_x\right](1+\alpha u)\right),\end{cases} \tag{55}$$

in an extended Kovalevskaya form with respect to y, where v has been eliminated by a local substitution

$$v=q_\alpha^{-1}\left(\frac{1}{2}\alpha u^2-\omega_x\right).$$

By a direct computation, one can show that for the third-order multipliers, only one conservation law of (55) arises. It corresponds to the multiplier triple $((q_\alpha)_x/(q_\alpha)^2, v/q_\alpha, -u/q_\alpha)$, and is equivalent to the local conservation law (the continuity equation) (33) of the plate flow model (24a, 24b). Thus no nonlocal conservation laws arise for the plate flow model (24a, 24b) in the chosen multiplier ansatz.

We have established the following result.

Proposition 4.2. *The plate flow boundary layer Prandtl model* (29) *admits nonlocal conservation laws* (45), (47), *and* (53). *Moreover, the conservation laws* (45) *and* (47) *are locally equivalent on solution sets of the PDE systems nonlocally related to the Prandtl model, which include the dependent variables* ψ *and* ϕ.

The full set of local and nonlocal conservation laws arising for the Prandtl boundary layer model (24a, 24b), the corresponding multipliers, and nonlocal variables are summarized in Table 1.

4.3 Symmetry Analysis of the Plate Prandtl Model

For basic facts about point symmetries of PDEs, an unfamiliar reader is referred to any standard book on the subject, for example, [1, 7, 69].

First, the point symmetries of the PDE system (24a, 24b) are found from a direct symbolic computation; they are given by

Table 1: Conservation laws and nonlocal variables of the plate flow boundary layer model (24a, 24b).

PDE system	Multipliers	Conservation law	Nonlocal variable
UV (29)	$(1, 0)$	(33)	ψ
	$(u, 1)$	(34)	ϕ
	$(1+\alpha u, \alpha)$	(42)	q_α
UVΦ (38)	$(v, -u, \phi)$	(45)	γ
UVΨ (35)	$(-u_x\psi+u_y, \psi u_y, \psi)$	(47)	γ^*
UVQ$_\alpha$ (44)	$(v, -\alpha u, q_\alpha)$	(53)	ω

$$X_1 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v}, \quad X_2 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$
$$X_3 = \frac{\partial}{\partial x}, \quad X_4(f) = f(x)\frac{\partial}{\partial y} + uf'(x)\frac{\partial}{\partial v}, \tag{56}$$

corresponding to two scalings, a translation, and a family of generalized translations

$$x^* = x, \quad y^* = y + f(x), \quad u^* = u, \quad v^* = v + f'(x)u,$$

involving an arbitrary function $f(x)$. We note that when $f(x)=\text{const}$, the generator $X_4 = \partial/\partial y$ correspond to a pure translation in y.

The symmetry commutator table for the generators (55) is shown in Table 2.

In order to seek nonlocal symmetries of the Prandtl plate flow model (24a, 24b), one can compute point symmetries of each of the nonlocally related PDE system groups within the tree $\mathcal{T}_3$, (shown using different colors in Fig. 2) and compare them with point symmetries of the PDE system (24a, 24b).

Direct symbolic symmetry computations using the `Maple/GeM` software demonstrate that no nonlocal symmetries arise within the extended tree $\mathcal{T}_3$. In particular:

– All symmetries of the potential systems **UVΨ**$\{x, y; u, v, \psi\}$ (35), **UVΦ**$\{x, y; u, v, \phi\}$ (38), **UVΨΦ**$\{x, y; u, v, \psi, \phi\}$ (30), **UVΨΦΓ**$\{x, y; u, v, \psi, \phi, \gamma\}$ (49), **UVQ$_\alpha$**$\{x, y; u, v, q_\alpha\}$ (44), and **UVQ$_\alpha$Ω**$\{x, y; u, v, q_\alpha, \omega\}$ (55) project on point symmetries of the given model **UV**$\{x, y; u, v\}$ (29).
– The local symmetry X_2 of the Prandtl model yields a nonlocal symmetry of the potential systems **UVQ$_\alpha$**$\{x, y; u, v, q_\alpha\}$ (44) and **UVQ$_\alpha$Ω**$\{x, y; u, v, q_\alpha, \omega\}$ (55).
– The PDE system **UVΨΦΓ**$\{x, y; u, v, \psi, \phi, \gamma\}$ (49) has a local symmetry

$$X = \psi\frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \phi}, \tag{57}$$

which corresponds to a nonlocal symmetry of all PDE systems in the tree $\mathcal{T}_3$ that include the potential γ but not the potential ψ (cf. Fig. 2).

Table 2: Symmetry commutators for two-dimensional plate Prandtl boundary layer model (24a, 24b).

	X_1	X_2	X_3	$X_4(f(x))$
X_1	0	0	$-2X_3$	$X_4(g)$
X_2	0	0	0	$-X_4(f)$
X_3	$2X_3$	0	0	$X_4(f')$
$X_4(f)$	$-X_4(g)$	$X_4(f)$	$-X_4(f')$	0

Here $f=f(x)$ is an arbitrary function; $g(x)=2xf'(x)-f(x)$. Note that $[X_4(p), X_4(q)]=0$ for any $p(x)$, $q(x)$.

5 Conservation Laws and Nonlocally Related Systems of the Cylinder Boundary Layer Model

We now consider (27a, 27b) for a two-component boundary layer flow about a circular cylinder. The notation for all PDE systems and variables in the current section is not in any way related to that for the plate flow model discussed in Section 4.

Writing the corresponding PDE system in an extended Kovalevskaya form with respect to r, we obtain the system

$$\mathbf{UV}\{r, z; u, v\}: \begin{cases} u_{rr} = uu_z + vu_r - \dfrac{1}{r}u_r, \\ v_r = -u_z - \dfrac{1}{r}v. \end{cases} \tag{58}$$

Similar to what happens in Section 4, the first PDE of (58) can be solved for v; the latter is thus given by a local differential function in terms of u:$v=v[u]$. The substitution of v into the second PDE of (58) consequently leads to a single equation **U**$\{r, z; u\}$ locally related to the PDE system **UV**$\{r, z; u, v\}$ (58).

5.1 Local Conservation Laws and Potential Systems

In order to compute local conservation laws of (58), we use the direct method, and establish the following result which is proven by a direct computation.

Proposition 5.1. *The linear space of inequivalent nontrivial local conservation laws of the PDE system* (58) *arising from* *the fourth-order multipliers is spanned by the two vanishing divergence expressions*

$$D_r(rv)+D_z(ru)=0, \tag{59}$$

$$D_r(r(uv-u_r))+D_z(ru^2)=0, \tag{60}$$

corresponding to the zeroth-order multiplier pairs $(\Lambda_1, \Lambda_2)=(0, r)$ *and* $(\Lambda_1, \Lambda_2)=(-r, ru)$.

In a manner similar to that of Section 4, the conservation laws (59) and (60) can be used to introduce potential variables. In particular, the first conservation law (59) yields the potential equations

$$\psi_r=ru, \quad \psi_z=-rv \tag{61}$$

for the stream function $\psi(r, z)$. The second conservation law (60) leads to the potential equations

$$\phi_r=ru^2, \quad \phi_z=r(u_r-uv) \tag{62}$$

for the nonlocal variable $\phi(r, z)$. A linear combination of the conservation laws (59) and (60) with factors 1, α, given by

$$D_r(r(v-\alpha(u_r-uv)))+D_z(ru(1+\alpha u))=0, \tag{63}$$

$\alpha\in\mathbb{R}\backslash\{0\}$ is a continuous parameter, yields a spectral potential system with potential equations

$$(q_\alpha)_r=ru(1+\alpha u), \quad (q_\alpha)_z=r(\alpha(u_r-uv)-v) \tag{64}$$

for the nonlocal variable $q_\alpha(r, z)$.

For the cylinder flow boundary layer equations (58), one consequently has the singlet potential systems **UVΨ**$\{r, z; u, v, \psi\}$, **UVΦ**$\{r, z; u, v, \phi\}$, the spectral potential system $\mathbf{UVQ}_{\boldsymbol{\alpha}}\{r, z; u, v, q_\alpha\}$, and a couplet potential system **UVΨΦ**$\{r, z; u, v, \psi, \phi\}$, nonlocally related to each other and to the given model (58). Exclusions of dependent variables by an explicit substitution yield the subsystems in a manner exactly parallel to that of the Section 4. The initial tree $\mathcal{T}_1$ of nonlocally related PDE systems coincides with that for the two-dimensional plate flow Prandtl equations (see Fig. 1 and also Remark 4.1 which is relevant here as well).

5.2 Nonlocal Conservation Laws and Further Potential Systems

We now seek further conservation laws of the cylinder flow boundary layer equations (58), arising as local conservation laws of its potential systems.

5.2.1 Preliminary Analysis

The following conservation laws are found within the second-order multiplier ansatz.

1. For the potential system **UVΦ**$\{r, z; u, v, \phi\}$ and its locally related systems, one finds a local conservation law equivalent to (59), and an additional local conservation law

$$D_r\left((1+rv)\phi-\frac{1}{2}r^2u^2\right)+D_z(ru\phi)=0; \tag{65}$$

the latter is a second nonlocal conservation law of the cylinder flow boundary layer equations (58). The conservation law (65) yields a new nonlocal variable γ satisfying the potential equations

$$\gamma_r=ru\phi, \qquad \gamma_z=-(1+rv)\phi+\frac{1}{2}r^2u^2, \tag{66}$$

and leads to the potential system **UVΦΓ**$\{r, z; u, v, \phi, \gamma\}$ and, as usual, its locally related subsystems.

2. For the potential system **UVΨ**$\{r, z; u, v, \psi\}$ and the locally related systems, one finds a local conservation law equivalent to (60), and an additional local conservation law

$$D_r\left(r(uv-u_r)(\psi-z)+\frac{1}{2}r^2u^2\right)+D_z(ru^2(\psi-z))=0; \tag{67}$$

the latter is a nonlocal conservation law of the cylinder flow boundary layer equations (58). The conservation law (67) yields a nonlocal variable γ^* satisfying the potential equations

$$\gamma^*_r=ru^2(\psi-z), \quad \gamma^*_z=-\left(r(uv-u_r)(\psi-z)+\frac{1}{2}r^2u^2\right) \tag{68}$$

and leads to the potential system $\mathbf{UV\Psi\Gamma^*}\{r, z; u, v, \psi, \gamma^*\}$.

3. For the potential system $\mathbf{UVQ}_{\boldsymbol{\alpha}}\{r, z; u, v, q_\alpha\}$, using the second-order multiplier ansatz in the direct method, one obtains two conservation laws; one of them is equivalent to (59), and another one is given by

$$D_r\left((1+rv)q_\alpha-rvz-\frac{\alpha}{2}r^2u^2\right)+D_z(ru(q_\alpha-z))=0; \tag{69}$$

The divergence expression (69) yields a nonlocal conservation law of the cylindrical boundary layer model **UV**$\{r, z; u, v\}$ (58). Using this conservation law, one can introduce a further potential variable ω satisfying

$$\omega_r=ru(q_\alpha-z), \quad \omega_z=\frac{\alpha}{2}r^2u^2-(1+rv)q_\alpha+rvz.$$

One consequently obtains a higher-level potential system $\mathbf{UVQ_\alpha\Omega}\{r, z; u, v, q_\alpha, \omega\}$ and further locally and nonlocally related subsystems, also involving the spectral parameter $\alpha \in \mathbb{R}\backslash\{0\}$.

5.2.2 The Relationship between the Potential Variables. An Extended Tree for the Cylindrical Model

For the couplet potential system $\mathbf{UV\Psi\Phi}\{r, z; u, v, \psi, \phi\}$ of the the cylindrical model, one has two local conservation laws (65) and (67). Similarly to Lemma 4.1, the following statement is proven by a direct computation.

Lemma 5.1. *The potentials γ and γ^* given by* (66) *and* (68) *are locally related on the solution set of the PDE systems* $\mathbf{UV\Psi\Phi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$ *and* $\mathbf{UV\Psi\Phi\Gamma^*}\{r, z; u, v, \psi, \phi, \gamma^*\}$*:*

$$\gamma(r, z) = -\gamma^*(r, z) + \phi(\psi - z) + \text{const.} \tag{70}$$

The PDE systems $\mathbf{UV\Psi\Phi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$ *and* $\mathbf{UV\Psi\Phi\Gamma^*}\{r, z; u, v, \psi, \phi, \gamma^*\}$ *are consequently locally related.*

We observe that the structure of conservation laws and potential systems of the two-component cylindrical boundary layer flow model (58) is parallel to that of the basic plate flow equations (Section 4). The notation in the current section has been chosen to underline this fact. The tree of nonlocally related PDE systems for the boundary layer about the cylinder discovered so far is equivalent to the one shown in Figure 2.

In summary, the following result has been obtained.

Proposition 5.2. *The boundary layer model* (58) *for the flow around the circular cylinder admits nonlocal conservation laws* (65), (67), *and* (69). *Moreover, the conservation laws* (65) *and* (67) *are locally equivalent on solution sets of the PDE systems which include the dependent variables ψ and ϕ.*

5.2.3 Further Conservation Laws

In order to complete the conservation law analysis of the cylinder boundary layer model (58), we seek local conservation laws of the potential systems $\mathbf{UVQ_\alpha\Omega}\{r, z; u, v, q_\alpha, \omega\}$ and $\mathbf{UV\Psi\Phi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$.

The PDE system $\mathbf{UQ_\alpha\Omega}\{r, z; u, q_\alpha, \omega\}$ obtained by a local substitution of v from $\mathbf{UVQ_\alpha\Omega}\{r, z; u, v, q_\alpha, \omega\}$ can be written in the Kovalevskaya form with respect to r. A complete direct conservation law computation with second-order multipliers yields only a previously known local conservation law (59).

Table 3: Conservation laws and nonlocal variables of the cylinder flow boundary layer model (58).

PDE system	Conservation law	Nonlocal variable
UV	(59)	ψ
	(60)	ϕ
	(63)	q_α
$\mathbf{UV\Phi}$	(65)	γ
$\mathbf{UV\Psi}$	(67)	γ^*
$\mathbf{UVQ_\alpha}$	(69)	ω

A computation for the PDE system $\mathbf{UV\Psi\Phi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$ with first-order multipliers reveals no additional conservation laws.

The local and nonlocal conservation laws arising for the cylinder flow boundary layer model (58), with the corresponding nonlocal variables, are summarized in Table 3.

5.3 Symmetry Analysis of the Cylinder Boundary Layer Model

The point symmetries of the cylinder boundary layer model (58) are found from a direct symbolic computation, and are given by the three generators

$$\mathrm{Y}_1 = \frac{\partial}{\partial z}, \quad \mathrm{Y}_2 = z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}, \quad \mathrm{Y}_3 = 2z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} - v\frac{\partial}{\partial v}. \tag{71}$$

corresponding to a translation and two scalings. Table 4 shows the commutators of the generators (71).

In order to seek nonlocal symmetries of the cylinder boundary layer model (58), we can compute point symmetries of all independent nonlocally related PDE systems found so far. Direct symbolic computations using the `Maple/GeM` software demonstrate that no nonlocal symmetries arise. In particular:

– All symmetries of the potential systems $\mathbf{UV\Psi}\{r, z; u, v, \psi\}$, $\mathbf{UV\Phi}\{r, z; u, v, \phi\}$, $\mathbf{UV\Psi\Phi}\{r, z; u, v, \psi, \phi\}$, $\mathbf{UV\Psi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$, $\mathbf{UV\Psi\Gamma^*}\{r, z; u, v, \psi, \phi, \gamma^*\}$, $\mathbf{UV\Psi\Phi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$, $\mathbf{UVQ_\alpha}\{r, z; u, v, q_\alpha\}$ and $\mathbf{UVQ_\alpha}\{r, z; u, v, q_\alpha, \omega\}$ project on point symmetries of the given model (58).

Table 4: Symmetry commutators for the axisymmetric cylinder boundary layer model (58).

	Y_1	Y_2	Y_3
Y_1	0	Y_1	$2\mathrm{Y}_1$
Y_2	$-\mathrm{Y}_1$	0	0
Y_3	$-2\mathrm{Y}_1$	0	0

– The local symmetry Y_2 in (71) yields a nonlocal symmetry of the potential systems $\mathbf{UVQ}_{\boldsymbol{\alpha}}\{r, z; u, v, q_{\boldsymbol{\alpha}}\}$ and $\mathbf{UVQ}_{\boldsymbol{\alpha}}\mathbf{\Omega}\{r, z; u, v, q_\alpha, \omega\}$.
– The PDE system $\mathbf{UV\Psi\Phi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$ has a local symmetry

$$\mathrm{Y} = (\psi - z)\frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \phi}, \tag{72}$$

which corresponds to a nonlocal symmetry of all PDE systems within the tree that include the potential γ but not the potential ψ (cf. Fig. 2).

6 Conservation Laws and Nonlocally Related Systems of the Radial Jet Boundary Layer Model

We now consider the third model: the radial jet boundary layer equations (28a, 28b). The notation in the current section is again independent of the notation in the previous sections.

We start from writing the dimensionless PDEs (28a, 28b) in an extended Kovalevskaya form with respect to z:

$$\mathbf{UV}\{r, z; u, v\}: \begin{cases} u_{zz} = uu_r + vu_z, \\ v_z = -u_r - \dfrac{1}{r}u. \end{cases} \tag{73}$$

The substitution of v from the first PDE of (58) yields a locally related PDE $\mathbf{U}\{r, z; u\}$.

6.1 Local Conservation Laws and Potential Systems

The direct method is now used to compute local conservation laws of (73). The following result holds.

Proposition 6.1. *The linear space of inequivalent nontrivial local conservation laws of the PDE system* (73) *arising from the fourth-order multipliers is spanned by the two conservation laws*

$$\mathrm{D}_r(ru) + \mathrm{D}_z(rv) = 0, \tag{74}$$

$$\mathrm{D}_r(ru^2) + \mathrm{D}_z(r(uv - u_z)) = 0, \tag{75}$$

corresponding to the zeroth-order multiplier pairs $(\Lambda_1, \Lambda_2) = (0, r)$ *and* $(\Lambda_1, \Lambda_2) = (-r, ru)$.

Similar to the above models, the conservation laws (74) and (75) are used to introduce potential variables $\psi(r, z)$ and $\phi(r, z)$, satisfying as it was done in the previous sections, and define the singlet potential systems

$$\mathbf{UV\Psi}\{r, z; u, v, \psi\}: \begin{cases} \psi_r = -rv, \\ \psi_z = ru, \\ u_{zz} = uu_r + vu_z, \end{cases} \tag{76}$$

$$\mathbf{UV\Phi}\{r, z; u, v, \phi\}: \begin{cases} \phi_r = r(u_z - uv), \\ \phi_z = ru^2, \\ v_z = -u_r - \dfrac{1}{r}u, \end{cases} \tag{77}$$

the couplet potential system $\mathbf{UV\Psi\Phi}\{r, z; u, v, \psi, \phi\}$, and locally related subsystems $\mathbf{U\Psi\Phi}\{r, z; u, \psi, \phi\}$, $\mathbf{V\Psi\Phi}\{r, z; v, \psi, \phi\}$, $\mathbf{U\Psi}\{r, z; u, \psi\}$, $\mathbf{V\Psi}\{r, z; v, \psi\}$, $\mathbf{\Psi}\{r, z; \psi\}$, $\mathbf{U\Phi}\{r, z; u, \phi\}$, $\mathbf{V\Phi}\{r, z; v, \phi\}$, and $\mathbf{\Phi}\{r, z; \phi\}$. The use of a linear combination of the conservation laws (74), (75)

$$\mathrm{D}_r(ru(1 + \alpha u)) + \mathrm{D}_z(r(v + \alpha(uv - u_z))) = 0, \quad \alpha \in \mathbb{R} \setminus \{0\} \tag{78}$$

leads to the spectral potential system $\mathbf{UVQ}_{\boldsymbol{\alpha}}\{r, z; u, v, q_\alpha\}$

$$\mathbf{UVQ}_{\boldsymbol{\alpha}}\{r, z; u, v, q_\alpha\}: \begin{cases} (q_\alpha)_r = r(-v + \alpha(u_z - uv)), \\ (q_\alpha)_z = r(u + \alpha u^2), \\ v_z = -u_r - \dfrac{1}{r}u, \end{cases} \tag{79}$$

and the locally related subsystems $\mathbf{UQ}_{\boldsymbol{\alpha}}\{r, z; u, q_\alpha\}$, $\mathbf{VQ}_{\boldsymbol{\alpha}}\{r, z; v, q_\alpha\}$, and $\mathbf{Q}_{\boldsymbol{\alpha}}\{r, z; q_\alpha\}$. The resulting preliminary tree $\mathcal{T}_1$ of nonlocally related systems for the radial jet model (28a, 28b) and (73) coincides with the one for the Prandtl plate flow boundary layer model shown in Figure 1 (see also Remark 4.1 which holds here as well).

6.2 Nonlocal Conservation Laws and Further Potential Systems

We now seek further conservation laws of the radial jet boundary layer model (73) arising as local conservation laws of its potential systems.

The following conservation laws are found within the second-order multiplier ansatz.

4. For the potential system $\mathbf{UV\Phi}\{r, z; u, v, \phi\}$ (79) and its locally related systems, one finds a local conservation law equivalent to (74), and an additional local conservation law

$$\mathrm{D}_r(ru\phi) + \mathrm{D}_z\left(rv\phi - \frac{1}{2}r^2u^2\right) = 0; \tag{80}$$

the latter is a nonlocal conservation law of the radial jet boundary layer equations (28a, 28b) and (73). The conservation law (80) yields a new nonlocal variable γ satisfying the potential equations

$$\gamma_z = ru\phi, \quad \gamma_r = \frac{1}{2}r^2u^2 - rv\phi, \tag{81}$$

and yields the potential system $\mathbf{UV\Phi\Gamma}\{r, z; u, v, \phi, \gamma\}$ with its locally related subsystems.

5. Using the potential system $\mathbf{UV\Psi}\{r, z; u, v, \psi\}$ (76), one finds a local conservation law equivalent to (75), and an additional local conservation law

$$\mathrm{D}_r(ru^2\psi) + \mathrm{D}_z\left(\frac{1}{2}r^2u^2 + r(uv - u_z)\psi\right) = 0; \tag{82}$$

this is a nonlocal conservation law of the radial jet model (73). The conservation law (82) yields a nonlocal variable γ^* satisfying the potential equations

$$\gamma^*_z = ru^2\psi, \qquad \gamma^*_z = -\left(\frac{1}{2}r^2u^2 + r(uv - u_z)\psi\right), \tag{83}$$

and leads to the corresponding potential system $\mathbf{UV\Psi\Gamma^*}\{r, z; u, v, \psi, \gamma^*\}$.

6. For the potential system $\mathbf{UVQ_\alpha}\{r, z; u, v, q_\alpha\}$ (79), using the second-order multiplier ansatz in the direct method, one obtains two conservation laws, the first being equivalent to (74), and the second given by

$$\mathrm{D}_r(ruq_\alpha) + \mathrm{D}_z\left(rvq_\alpha - \frac{\alpha}{2}r^2u^2\right) = 0. \tag{84}$$

The divergence expression (84) yields a nonlocal conservation law of the cylindrical boundary layer model $\mathbf{UV}\{r, z; u, v\}$ (73). Using this conservation law, one can introduce a further potential variable ω satisfying

$$\omega_z = ruq_\alpha, \quad \omega_r = \frac{\alpha}{2}r^2u^2 - rvq_\alpha.$$

One obtains a further potential system $\mathbf{UVQ_\alpha\Omega}\{r, z; u, v, q_\alpha, \omega\}$ and further locally and nonlocally related subsystems, also involving the spectral parameter $\alpha \in \mathbb{R}\backslash\{0\}$.

Similar to the plate and cylindrical boundary layer models, for the couplet potential system $\mathbf{UV\Psi\Phi}\{r, z; u, v, \psi, \phi\}$ of the radial model, the potentials γ, γ^* arising from its two local conservation laws (80), (82), again turn out to be locally related. The following statements hold.

Lemma 6.1. *The potentials γ and γ^* given by* (81) *and* (83) *are locally related on the solution set of all PDE systems that include the dependent variables ψ, ϕ. The relationship is given by*

$$\gamma(r, z) = -\gamma^*(r, z) + \phi\psi + \text{const.} \tag{85}$$

Proposition 6.2. *The boundary layer model* (73) *for the radial jet boundary layer flow admits nonlocal conservation laws* (80), (82), *and* (84). *Moreover, the conservation laws* (80) *and* (82) *are locally equivalent on solution sets of the PDE systems which include the dependent variables ψ and ϕ.*

The extended tree of nonlocally related PDE systems for the radial jet boundary layer model described so far is equivalent to the one for the other two models, and is shown in Figure 2.

The local and nonlocal conservation laws computed for the radial jet equations (73) and the respective nonlocal variables are summarized in Table 5. The direct computation with first-order multipliers reveals no additional conservation laws analysis of the radial jet equations (73) arising from the potential systems $\mathbf{UV\Psi\Phi\Gamma}\{r, z; u, v, \psi, \phi, \gamma\}$ or $\mathbf{UQ_\alpha\Omega}\{r, z; u, q_\alpha, \omega\}$.

6.3 Symmetry Analysis of the Radial Jet Boundary Layer Model

The point symmetries of the radial jet equations (73) are found by a direct symbolic calculation; the infinite-dimensional Lie symmetry algebra is spanned by the generators

$$\begin{aligned} &\mathrm{Z} = 2r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} - v\frac{\partial}{\partial v}, \qquad \mathrm{Z}_2 = z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \\ &\mathrm{Z}_3 = -\frac{1}{r^2}\frac{\partial}{\partial r} + \frac{z}{r^3}\frac{\partial}{\partial z} - \frac{3zu + rv}{r^4}\frac{\partial}{\partial v}, \qquad \mathrm{Z}_4 = f(r)\frac{\partial}{\partial z} + uf'(r)\frac{\partial}{\partial v}, \end{aligned} \tag{86}$$

corresponding to a translation and two scalings. Table 6 contains the commutator relations generators (86).

Table 5: Conservation laws and nonlocal variables of the radial jet boundary layer model (73).

PDE system	Conservation law	Nonlocal variable
UV	(74)	ψ
	(75)	ϕ
	(78)	q_α
$\mathbf{UV\Phi}$	(80)	γ
$\mathbf{UV\Psi}$	(82)	γ^*
$\mathbf{UVQ_\alpha}$	(84)	ω

Table 6: Symmetry commutators for the radial boundary layer model (73).

	Z_1	Z_2	Z_3	$Z_4[f(r)]$
Z_1	0	0	$-6Z_3$	$Z_4[g(r)]$
Z_2	0	0	0	$-Z_4[f(r)]$
Z_3	$6Z_3$	0	0	$Z_4[h(r)]$
$Z_4[f(r)]$	$-Z_4[g(r)]$	$Z_4[f(r)]$	$-Z_4[h(r)]$	0

Here $f=f(r)$ is an arbitrary function; $g(r)=2rf'(r)-f(r)$; $h(r)=-[f'(r)/r^2+f(r)/r^3]$. Note that $[Z_4(p), Z_4(q)]=0$ for any $p(r)$, $q(r)$.

In order to seek nonlocal symmetries of the radial jet boundary layer model (73), we now seek point symmetries of all independent nonlocally related PDE systems for this model as described above. Symbolic computations using the `Maple/GeM` software show that no nonlocal symmetries arise. In particular, the following relationships hold.

- All symmetries of the potential systems **UVΨ**$\{r, z; u, v, \psi\}$, **UVΦ**$\{r, z; u, v, \phi\}$, **UVΨΦ**$\{r, z; u, v, \psi, \phi\}$, **UVΦΓ**$\{r, z; u, v, \psi, \phi, \gamma\}$, **UVΨΓ***$\{r, z; u, v, \psi, \phi, \gamma^*\}$, **UVΨΦΓ**$\{r, z; u, v, \psi, \phi, \gamma\}$, $\mathbf{UVQ}_{\boldsymbol{\alpha}}\{r, z; u, v, q_\alpha\}$ and $\mathbf{UVQ}_{\boldsymbol{\alpha}}\mathbf{\Omega}\{r, z; u, v, q_\alpha, \omega\}$ project on point symmetries of the given model (73).
- The local symmetry Z_2 in (86) yields a nonlocal symmetry of the potential systems $\mathbf{UVQ}_{\boldsymbol{\alpha}}\{r, z; u, v, q_\alpha\}$ and $\mathbf{UVQ}_{\boldsymbol{\alpha}}\mathbf{\Omega}\{r, z; u, v, q_\alpha, \omega\}$.
- The PDE system **UVΨΦΓ**$\{r, z; u, v, \psi, \phi, \gamma\}$ has a local symmetry

$$Z=\psi\frac{\partial}{\partial\gamma}+\frac{\partial}{\partial\phi}, \tag{87}$$

which yields a nonlocal symmetry of all PDE systems within the tree that include the potential γ but not the potential ψ (cf. Fig. 2).

7 Discussion

In the current study, local and nonlocal conservation laws were systematically obtained, and trees of nonlocally related systems were constructed, for three models of two-dimensional, two-component boundary layer fluid flow: the Prandtl equations for steady plate flows (24a, 24b) (Section 4), a layer flow about a circular cylinder (27a, 27b) (Section 5), and a radial wall jet boundary layer model (28a, 28b) (Section 6).

It is remarkable that for these three physically different models, the analysis reveals essentially the same structure of local conservation laws, potential systems and subsystems, and nonlocal conservation laws. In particular, for each model, two linearly independent local conservation laws arise, leading to the introduction of the stream function ψ, and a second potential ϕ. Further, using the corresponding potential systems **UVΨ** and **UVΦ**, for each of the three models, two local conservation laws were found, corresponding to nonlocal conservation laws of each boundary layer model.

Another interesting relationship is observed: for each of the three models, the independent nonlocal variables γ and γ^* introduced using the two nonlocal conservation laws appear to be *locally related* on the solution set of the most general potential systems (Lemmas 4.1, 5.1, and 6.1): one has, for example, $\gamma=\gamma[\gamma^*, \phi, \psi]$, a local differential function.

For each of the three boundary layer models, a spectral potential system $\mathbf{UVQ}_{\boldsymbol{\alpha}}$ for the potential q_α, arising from a linear combination of the two local conservation laws and depending on a continuous parameter $\alpha\in\mathbb{R}\backslash\{0\}$ (e.g. (44)), has been also shown to possess a local conservation law that corresponds to a nonlocal conservation law of a given PDE system; this yields a family of nonlocal conservation laws parameterized by α. The conservation laws and potential variables for the three PDE systems (24a, 24b), (27a, 27b), and (28a, 28b) have been summarized in Tables 1, 3, and 5. The three physical models consequently possess the same trees of locally nonlocally related systems; the preliminary version of such a tree was shown in Figure 1, and a more extended version involving potentials that arise from nonlocal conservation laws was presented in Figure 2. These trees can be possibly further extended using other techniques, such as exclusion of variables following a local coordinate change to obtain nonlocally related subsystems (e.g. [1, 20]), and/or inverse potential systems obtained using symmetries [36].

As a further observation, the computations carried out in the current work provide an illustration that complements the important result of [14, 21] (Theorem 1). The theorem establishes that if a direct conservation law construction method is applied to a potential system of some given PDE system, then local multipliers lead only to local conservation laws of the given system. In Section 4.2, for example, for the plate flow model **UV** (29), we derive its potential system **UVΦ** (38). The latter has a local conservation law (45) arising from multipliers $(v, -u, \phi)$, which are not local functions for the given model **UV** and indeed, the conservation law (45) is nonlocal for the PDE system **UV**. Yet an equivalent conservation law (52) arises from a locally related subsystem **ΨΦ** (41) of the potential system, and here, the multipliers are given by $(\psi_x, \psi_y)\equiv(-v, u)$, which are *local* on solutions of the given model **UV**. This illustrates that the statement of Theorem 1 does not hold for subsystems.

It is natural to ask the question about the origins of the complete correspondence observed in the conservation law structure and the trees of nonlocally related systems for the three boundary layer models. One may expect that they belong to a general family of PDE systems possessing the same local and nonlocal conservation law structure; it is an open problem to specify such a family. From the physical point of view, this is challenging because the systems describe substantially different physical situations. Yet the similarity of conservation laws and nonlocally related systems is striking; one might conjecture that the three models are related by a coordinate change. However, the following result holds.

Proposition 7.1. *Neither pair of the three boundary layer models* (24a, 24b), (27a, 27b), *and* (28a, 28b) *is related by a local transformation.*

For two PDE systems having more than one independent variable to be related by an invertible transformation, the transformation must be point [70] (see also Section 2.2.1 in [1], and [71]). One can show that there is no point transformation mapping any of the three PDE systems to another one. First, it is well known that if two PDE systems are related by a point transformation, the Lie algebras of their point symmetries are isomorphic. Comparing the respective symmetry generators (56), (71), and (86) for the three boundary layer models, we conclude that the model (27a, 27b) of equilibrium two-component boundary layer flow about a circular cylinder is not connected to any of the other two models by an invertible transformation. On the other hand, the symmetry structures (56) and (86) of the plate and radial jet flows do not preclude the possible existence of a point coordinate transformation between the systems. In order to prove that it does not exist, we take the following steps.

1. Pose a general point transformation

$$\begin{aligned} r &= F(x, y, A(x, y), B(x, y)), \\ z &= G(x, y, A(x, y), B(x, y)), \\ u(r, z) &= H(x, y, A(x, y), B(x, y)), \\ v(r, z) &= K(x, y, A(x, y), B(x, y)), \end{aligned} \tag{88}$$

   satisfying the nondegeneracy condition $|\partial(F, G, H, K)/\partial(x, y, u, v)| \neq 0$.
2. Change the variables in the radial jet model (28a, 28b) according to (88).
3. Assume $u = A(x, y)$, $v = B(x, y)$ is a solution of the plate flow model (24a, 24b). Substitute the PDEs (24a, 24b) (solved, for example, with respect to the leading derivatives A_{yy}, B_y) and their differential consequences into the transformed equations.
4. Set to zero coefficients of all nonleading derivatives. Solve the resulting PDEs (with the nonzero Jacobian constraint) for the unknown F, G, H, K.

A symbolic computation that employs `GeM` and `Maple rifsimp` routines shows that such transformations (86) do not exist. This completes the proof. An alternative proof can likely be obtained using the general Cartan's method of equivalence [72–74].

For each of the three models, we have computed their point symmetries, as well as point symmetries of all nonlocally related PDE systems within each extended tree (Sections 4.3, 5.3, and 6.3). No nonlocal symmetries were found for the given physical equations, but nonlocal symmetries for some nonlocally related PDE systems within the tree were identified.

Open problems related to the subject of this contribution include the physical interpretation of the nonlocal topological conservation laws obtained for the three models, as well as the derivation and analysis of exact solutions, including symmetry-invariant solutions, in particular, those arising from the symmetries that involve an arbitrary function.

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