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A recursion formula for the construction of local conservation laws of differential equations

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ABSTRACT

A simple formula is presented that, for any given local divergence-type conservation law of a system of partial or ordinary differential equations (PDE, ODE), generates a divergence expression involving an arbitrary function of all independent variables. In the cases when the new flux vector is a local expression inequivalent to the initial local conservation law flux vector, a new local conservation law is obtained. For ODEs, this can yield additional integrated factors. Examples of systems of differential equations are presented for which the proposed new relationship yields important local conservation laws starting from basic ones. Examples include a nonlinear ODE and several fundamental physical PDE models, in particular, general classes of nonlinear wave and diffusion equations, vorticity-type equations, and a shear wave propagation model in hyper-viscoelastic fiber-reinforced solids.

1. Introduction

Consider a system of differential equations (DE) $\mathcal{R}$ given by

$$R^\sigma[u] = 0, \qquad \sigma = 1, \dots, N, \tag{1.1}$$

with independent variable(s) $z = \{z^i\}_{i=1}^n$ and dependent variable(s) $u = \{u^k(z)\}_{k=1}^m$.

Definition 1. A *local divergence-type conservation law* of $\mathcal{R}$ is given by a divergence expression

$$\mathrm{D}_i \Psi^i[u] = 0 \tag{1.2}$$

that vanishes for all solutions $u(z)$ of the system (1.1).

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In (1.2), $\Psi^i[u]$ are the conservation law *fluxes*, and

$$\mathrm{D}_i \equiv \mathrm{D}_{z^i} = \frac{\partial}{\partial z^i} + u^j_i \frac{\partial}{\partial u^j} + u^j_{ii_1} \frac{\partial}{\partial u^j_{i_1}} + u^j_{ii_1i_2} \frac{\partial}{\partial u^j_{i_1i_2}} + \cdots \tag{1.3}$$

denote the total derivative operators. In (1.1), (1.2) and throughout the paper, symbols $F[u]$ denote differential functions, i.e., functions that may depend on independent variables, dependent variables, and derivatives of dependent variables of a given DE system, to some finite order. In (1.2), (1.3), and below where appropriate, summation in repeated indices is used, as well as the notation

$$u^j_i \equiv u^j_{z^i} \equiv \frac{\partial u^j}{\partial z^i}, \qquad u^j_{i\ell} \equiv \frac{\partial^2 u^j}{\partial z^i \partial z^\ell}, \qquad \ldots$$

for partial derivatives. When the independent variables are time and space variables, $z = \{t, \{x^i\}_{i=1}^n\}$, local conservation laws (1.2) take the form

$$\mathrm{D}_t \Theta[u] + \mathrm{D}_{x^i} \Phi^i[u] = 0, \tag{1.4}$$

where $\Theta[u]$ is the local *conserved density*, and $\Phi^i[u]$ are the *spatial fluxes*.

Nontrivial local conservation laws of PDE systems are coordinate-invariant expressions incorporating fundamental information about the system at hand. They are widely used in analysis and numerical simulation (see, e.g., [9,36] and references therein). For a local conservation law (1.4) of a time-dependent PDE system holding in the spatial domain $\mathcal{D}$, under appropriate boundary conditions where fluxes vanish on the boundary of the domain or at infinity, the divergence theorem yields a global conserved quantity

$$\mathcal{J} = \int_{\mathcal{D}} \Theta[u]\, dV, \qquad \frac{d}{dt}\mathcal{J} = 0. \tag{1.5}$$

For ODEs, local conservation laws $\mathrm{D}_t\Theta[u] = 0$ yield first integrals $\Theta[u] = \text{const}$.

Basic local conservation laws of classical physical PDE systems, such as conservation of mass, energy, momentum, vorticity, etc., have often been either the building blocks of the models, or have been derived by inspection. For a variational PDE system, the first Noether's theorem provides a correspondence between its local variational symmetries and its local conservation laws; Lie symmetries thus can be employed to compute conservation laws (see, e.g., [8,9,35,36]). For general PDE systems, a number of systematic, or somewhat systematic, methods of conservation law computation exist (see, e.g., [31–33,41]).

In [29,36,40], a systematic method of construction of local conservation laws applicable for general classes of DE systems, sometimes referred to as the *direct method*, has been developed (see also [2,3,9]). It is based on the fact that for totally nondegenerate PDE systems, any local conservation law (1.2) can be written, up to equivalence, in a *characteristic form*

$$\mathrm{D}_i \Psi^i[u] = \Lambda_\sigma[u] R^\sigma[u] \tag{1.6}$$

for some set of conservation law *multipliers* $\{\Lambda_\sigma[u]\}_{\sigma=1}^N$, which are also sometimes referred to as *characteristics*, or *generating functions*, or *integrating factors*. (The notion of equivalence of conservation laws is discussed in Section 2.2 below.) In particular, the relations (1.6) hold for an arbitrary vector function $u = u(z)$. Then on solutions of the given DE system, a local conservation law (1.2) holds. Within the direct method, the multipliers $\{\Lambda_\sigma[u]\}_{\sigma=1}^N$ are found from the determining equations obtained through the action of Euler differential operators with respect to all dependent variables of the system. The multiplier determining equations form an overdetermined system of linear PDEs, similar to determining equations arising in local symmetry analysis. The generation and solution of determining equations, as well as subsequent conservation law density and flux computations, have been implemented in the symbolic package `GeM` for `Maple` (see [15–17]), as well as a number of other symbolic software packages (e.g., [41]).

Within the direct method, in order to avoid trivial conservation law multipliers, one needs to exclude from their dependencies some leading derivatives of the given DE system, as well as their differential consequences (see, e.g., [9]). For PDE systems that admit a *Kovalevskaya form* [36], or more generally, an *extended Kovalevskaya form* [29], multipliers may indeed be chosen independent of appropriately defined leading derivatives and their differential consequences, and hence for such systems, the direct method is *complete*, yielding all linearly independent, nontrivial local conservation laws that arise for the multipliers within any *a priori* specified ansatz.

The direct method has been used to compute local conservation laws of multiple PDE systems; see, e.g., [6,7,9–11,21,25,26] and references therein. For a comparative discussion of the direct method vs. the first Noether's theorem, see, for example, [8,9,20].

Local divergence-type conservation laws (1.2) can be used to systematically introduce *nonlocal variables*, commonly referred to as potentials, vector potentials, stream functions, etc. A *framework of nonlocally related PDE systems* has been developed in [6,11] (see also [9]). PDE systems that are nonlocally related to the given one but have an equivalent solution set are systematically derived within this framework [9]. Subsequently, standard local techniques such as symmetry and conservation law analysis may be applied to nonlocally related PDE systems, yielding new results, including non-invertible linearizations, and nonlocal symmetries and nonlocal conservation laws which essentially involve nonlocal (potential) variables. For the direct conservation law construction method, conditions under which local conservation law multipliers of a potential system lead to nonlocal conservation laws of a given PDE system have been established in [11,28].

In the current contribution, a recursion formula is presented that formally maps a local conservation law of a given DE system into a family of divergence expressions, whose densities/fluxes include an arbitrary function of all variables, and may involve local or nonlocal (integral) expressions (Section 2). It is shown that under certain conditions, in particular, for certain forms of the initial conservation law and the arbitrary function, the new conservation law(s) will be nontrivial local conservation laws of a given system, linearly independent of the given one. The benefit of the suggested formula is the possibility of immediate construction of one or more additional local conservation laws from a basic known local conservation law of a given DE system, without the need to solve any determining equations that arise in general procedures, such as the direct method or Noether-like methods. This possibility of avoiding the generation and solution of determining equations is particularly important for complicated systems of nonlinear PDEs.

Several physical PDE examples, for two- and multi-dimensional situations, are presented in Section 3. The recursion formula is applied there to derive, in a straightforward manner, basic local conservation laws for the nonlinear wave and diffusion equations, vorticity-type equations (arising in fundamental fluid, plasma and electrodynamics models), and the nonlinear equation describing wave propagation in a hyper-viscoelastic fiber-reinforced solid. An ODE example appears in Section 3.5.

In Section 4, a relationship between the new conservation law recursion formula and the inverse point symmetry action on conservation laws is discussed.

Conclusions are provided in Section 5.

2. The conservation law recursion formula

2.1. The basic recursion

We start with the following elementary argument. Consider a PDE system with two independent variables x, y and dependent variable(s) u. Suppose it has a nontrivial local conservation law

$$\mathrm{D}_x A[u] + \mathrm{D}_y B[u] = 0, \tag{2.1}$$

for some differential functions A, B. Then the multiplication by y and the application of the product rule yield a formal divergence expression

$$\mathrm{D}_x(yA[u]) + \mathrm{D}_y\left(yB[u] - \int B[u]\,dy\right) = 0. \tag{2.2}$$

(An expression of the type (2.2) appeared in [34] in the context of Prandtl boundary layer PDE systems.) Importantly, when $B[u]$ has a form of a total derivative in y, the divergence expression (2.2) yields a nontrivial local conservation law linearly independent of the original one given by (2.1). When the y-flux in (2.2) is an essentially nonlocal expression, there is no obvious interpretation of this divergence expression; we discuss this question later in a more general setting.

Example. Let $u = u(t, x)$. The linear wave equation that models, for example, small transverse oscillations of an elastic string, is given by

$$u_{tt} - u_{xx} = 0. \tag{2.3}$$

It is well known that any linear PDE or PDE system admits an infinite set of conservation laws [4,9]. Here we, however, consider only conservation laws arising from the recursion formula (2.2). The PDE (2.3) is a conservation law as it stands,

$$\mathrm{D}_t u_t - \mathrm{D}_x u_x = 0, \tag{2.4}$$

expressing the local conservation of linear momentum. The x-flux is evidently a total derivative by x. The application of the formula (2.2) with respect to x and t yields two conservation laws

$$\mathrm{D}_t(xu_t) - \mathrm{D}_x(xu_x - u) = 0, \tag{2.5}$$

$$\mathrm{D}_t(tu_t - u) - \mathrm{D}_x(tu_x) = 0. \tag{2.6}$$

Both (2.5) and (2.6) are physical local conservation laws, expressing, respectively, the conservation of angular momentum of the string with respect to the origin, and the motion of the center of mass (see, e.g., [36], p. 279; [19]). A further application of the recursion formula (2.2) to the local conservation law (2.5) with respect to t, or to the conservation law (2.6) with respect to x, yields a "symmetric" local conservation law

$$\mathrm{D}_t(x(tu_t - u)) - \mathrm{D}_x(t(xu_x - u)) = 0. \tag{2.7}$$

It is easy to see that the three local conservation laws (2.5), (2.6), (2.7) arise from the direct conservation law construction method, through the multiplication of the given PDE (2.3) by the multipliers $\Lambda = x$, $\Lambda = t$, and $\Lambda = xt$.

When the local flux or density in a corresponding variable is not a total derivative, the formula (2.2) yields formal divergence local containing nonlocal terms. For example, the application of the recursion formula to the local conservation law (2.5) with respect to x leads to a zero divergence expression

$$\mathrm{D}_t(x^2 u_t) - \mathrm{D}_x\left(x(xu_x - u) - \int (xu_x - u)\,dx\right) = 0. \tag{2.8}$$

Here the spatial flux has the form

$$\Phi = \Phi[u, w] = x(xu_x - u) - w,$$

and contains a nonlocal variable w defined by $w_x = xu_x - u$. We note that the conservation law (2.5) can be used to introduce a potential variable q through

$$q_x = xu_t, \qquad q_t = xu_x - u,$$

and $w_x = q_t$. The nonlocal variable w itself is not a potential variable arising from any local conservation law of the given PDE (2.3).

A related example for a nonlinear wave equation is considered in Section 3.

2.2. A multi-dimensional generalization

For the two-dimensional case, a sequential application of (2.2) to a given local conservation law (2.1) yields divergence expressions in the form

$$\mathrm{D}_x\Theta_n[u] + \mathrm{D}_y\Psi_n[u] = 0, \qquad n = 1, 2, \dots,$$

with $\Theta_n[u] = y^n A[u]$. For an arbitrary analytic function $f(y)$, a linear combination of such expressions leads to a divergence expressions with the “density” $\Theta_n[u] = f(y)A[u]$. This observation leads to a formulation of a more general result holding for the multi-dimensional case, as follows.

Lemma 1. *Let $\mathcal{R}$ (1.1) be a DE system with independent variables $z = \{z^i\}_{i=1}^n$ and dependent variables $u = \{u^k(z)\}_{k=1}^m$. Suppose that it admits a nontrivial local conservation law (1.2). Then for an arbitrary differentiable function $f = f(z)$, the following formal divergence expression vanishes on any given solution $u(z)$ of the system $\mathcal{R}$:*

$$\mathrm{D}_i\Xi^i \equiv \mathrm{D}_i\left(f\Psi^i[u] - \int \frac{\partial f}{\partial z^i}\,\Psi^i[u]\,dz^i\right) = 0. \tag{2.9}$$

We now discuss the properties of the recursion formulas (2.9).

A. Locality conditions; Relationship to the direct conservation law construction method

The divergence expressions (2.9) are generally nonlocal, however in multiple instances, including the above ones and several important examples of the following Section 3, the formula (2.9) yields useful local conservation laws of the given DE system.

For the formal divergence expression (2.9), one has, for an arbitrary vector function u,

$$\mathrm{D}_i\Xi^i = f(z)\,\mathrm{D}_i\Psi^i[u], \tag{2.10}$$

the function $f(z)$ in (2.9) playing the role of a conservation law multiplier. Therefore in a manner parallel to the ideas used in the direct conservation law construction method, one may employ Euler differential operators to provide a condition that is satisfied when the left-hand side of (2.10) is a local differential function. In that case, for a given local conservation law (1.2) and a given $f(z)$, the recursion formula (2.9) provides a *new local conservation law.*

It is well known that a necessary and sufficient condition that a differential function $F[u]$ is a divergence expression, i.e., $F[u] = \mathrm{D}_iQ^i[u]$ for some differential vector function $Q[u]$ (off of the solution space of the given DE system $\mathcal{R}$) if and only if it vanishes identically under the action of Euler operators with respect to all the dependent variables [9,22]:

$$\mathrm{E}_{u^j} F[u] \equiv \left(\frac{\partial}{\partial u^j} - \mathrm{D}_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s \mathrm{D}_{i_1} \dots \mathrm{D}_{i_s} \frac{\partial}{\partial u^j_{i_1 \dots i_s}} + \cdots \right) F[u] \equiv 0, \qquad j = 1, \dots, m.$$

The following statement holds.

Lemma 2. *The divergence expression (2.9) yields a local conservation law of the DE system $\mathcal{R}$ if and only if*

$$\mathrm{E}_{u^j} (f(z)\, \mathrm{D}_i \Psi^i[u]) = 0 \tag{2.11}$$

for an arbitrary vector function u, for all $j = 1, \dots, m$.

Note that the condition (2.11) generally restricts not only the form of the arbitrary function $f(z)$ but also the form of the initial conservation law (1.2). We also remark that if (2.11) holds, the fluxes of the divergence expression (2.9) as written are not necessarily local expressions, yet there exists an equivalent flux vector that is local. An example of such possibility is provided in Section 3.4 below.

As a simple example, let $\mathcal{R}$ be a linear wave equation (2.3), $z = (t, x)$, and let the initial conservation law be given by (2.4), the equation itself. In order to answer the question when the divergence expression arising from the formula (2.9),

$$\mathrm{D}_t \Xi^1 + \mathrm{D}_x \Xi^2 = f(t, x)(\mathrm{D}_t \Psi^1[u] + \mathrm{D}_x \Psi^2[u]) = 0,$$

yields a local conservation law of the PDE (2.3), we apply the condition (2.11) with respect to $u = u(t, x)$:

$$\mathrm{E}_u \Big(f(t, x)(\mathrm{D}_t \Psi^1[u] + \mathrm{D}_x \Psi^2[u]) \Big) = (\mathrm{D}_t^2 - \mathrm{D}_x^2) f(t, x) = f_{tt} - f_{xx} = 0.$$

In this case, due to the linearity of (2.3), the condition does not involve u. We observe that any $f = f(t, x)$ satisfying the linear wave equation $f_{tt} - f_{xx} = 0$ is indeed a conservation law multiplier, and for all such f, the recursion formula (2.9) yields local conservation laws of the wave equation (2.3).

It is challenging to provide a general interpretation of the formula (2.9) when the fluxes Ξ^i involve essentially nonlocal terms. Such nonlocal divergence expressions are generally not equivalent to local conservation laws of potential systems in the usual sense, where potentials are introduced based on local conservation laws of a given PDE system (cf. [9,11]). Similar constructs arise, for example, for evolution-type PDEs

$$u_t = F(x, u, \partial u, \dots, \partial^n u) \tag{2.12}$$

involving two independent variables (t, x), with $\partial u = u_x$, etc. Then one can formally write a conservation law-type expression

$$\mathrm{D}_t\, u - \mathrm{D}_x \left(\int F\, dx \right) = 0, \tag{2.13}$$

which, one may say, has just as much meaning as the expressions (2.9) in the general nonlocal case.

In the case when the formula (2.9) yields a *local* conservation law, one is naturally interested in its mathematical origins, as well as its possible equivalence to, or dependence on, the original local conservation law (1.2) of the DE system (1.1). These questions are discussed below.

B. Non-triviality and linear independence of conservation laws (2.9)

Suppose that a DE system $\mathcal{R}$ (1.1) has a local conservation law (1.2). The following definitions hold (e.g., [9,36,37]).

Definition 2. The conservation law (1.2) is *trivial* if its fluxes have the form $\Psi^i[u] = M^i[u] + H^i[u]$, where $M^i[u]$ and $H^i[u]$ are differential functions such that $M^i[u]$ vanishes on the solutions of $\mathcal{R}$, and $\mathrm{D}_i H^i[u] \equiv 0$ is a null divergence (i.e., it vanishes identically).

Definition 3. Two conservation laws $\mathrm{D}_i\Psi^i[u] = 0$ and $\mathrm{D}_i\Gamma^i[u] = 0$ are *equivalent* if $\mathrm{D}_i(\Psi^i[u] - \Gamma^i[u]) = 0$ is a trivial conservation law. An *equivalence class* of conservation laws consists of all conservation laws equivalent to some given nontrivial conservation law.

If two conservation laws are written in a characteristic form (1.6), their multiplier sets are called *equivalent* when they differ by a trivial multiplier set, i.e., one vanishing on the solutions of the given DE system. The following theorem holds [36].

Theorem 1. *Let $\mathcal{R}$ (1.1) be a normal, totally nondegenerate DE system. Let $\mathrm{D}_i\Psi^i[u] = 0$ and $\mathrm{D}_i\Gamma^i[u] = 0$ be its two local conservation laws, with multiplier sets $\{\Lambda_\sigma[u]\}_{\sigma=1}^N$ and $\{\tilde{\Lambda}_\sigma[u]\}_{\sigma=1}^N$. Then the two local conservation laws are equivalent if and only if $\{\Lambda_\sigma[u]\}_{\sigma=1}^N$ and $\{\tilde{\Lambda}_\sigma[u]\}_{\sigma=1}^N$ are equivalent multiplier sets.*

Definition 4. A set of p conservation laws $\{\mathrm{D}_i\Psi^i_{(j)}[u] = 0\}_{j=1}^p$ is *linearly dependent* if there exists a set of constants $\{a^{(j)}\}_{j=1}^p$, not all zero, such that the linear combination

$$\mathrm{D}_i(a^{(j)}\Psi^i_{(j)}[u]) = 0 \tag{2.14}$$

is a trivial conservation law.

In practice, one is interested in computing the largest possible set of nontrivial, linearly independent conservation laws of a given DE system. A conservation law that, up to equivalence, can be expressed as a linear combination of the other known ones, is naturally disregarded.

A natural question to ask is whether or not *local* conservation laws constructed through the recursion formula (2.9) are trivial or linearly dependent on the given conservation law (1.2) of the DE system at hand. The following statement holds.

Lemma 3. *Let (1.1) be a normal, totally nondegenerate DE system, with a nontrivial local conservation law $\mathrm{D}_i\Psi^i[u] = 0$ (1.2) written in a characteristic form (1.6). Suppose also that for some $f(z) \neq 0$, the formula (2.9) applied to that conservation law yields a local conservation law $\mathrm{D}_i\Xi^i[u] = 0$. Then the latter is a nontrivial conservation law, linearly independent of the given conservation law (1.2).*

The proof follows from Theorem 1. Indeed, if the initial conservation law (1.2) has the multiplier set $\{\Lambda_\sigma[u]\}$, then the new conservation law (2.9) corresponds to the multipliers $\{\tilde{\Lambda}_\sigma[u]\} = \{f(z)\Lambda_\sigma[u]\}$. The multiplier set

$$a\Lambda_\sigma[u] + b\tilde{\Lambda}_\sigma[u] = (a + bf(z))\Lambda_\sigma[u]$$

is nontrivial, except when $a = b = 0$. It follows that $\mathrm{D}_i(a\Psi^i[u] + b\,\Xi^i[u]) = 0$ is a nontrivial conservation law unless $a = b = 0$, which completes the proof.

Similarly, if the formula (2.9) yields several new local conservation laws $\mathrm{D}_i\,\Xi^i_{(j)}[u] = 0$, $j = 1, \ldots, s$, corresponding to linearly independent nonconstant functions $f^j(z)$, then a set composed of these conservation laws and the original conservation law (1.2) is linearly independent.

3. Examples

3.1. Nonlinear wave equations

Local conservation laws, and a related tree of nonlocally related PDE systems, were considered in [7] for the class of nonlinear wave equations

$$u_{tt} = (c^2(u)u_x)_x \tag{3.1}$$

with the dependent variable $u = u(t, x)$. It was shown that for an arbitrary constitutive function $c(u)$, there are exactly four zeroth-order multipliers $\Lambda = 1, t, xt, x$, with corresponding linearly independent local conservation laws given by

$$\mathrm{D}_t(u_t) - \mathrm{D}_x(c^2(u)u_x) = 0, \tag{3.2}$$

$$\mathrm{D}_t(tu_t - u) - \mathrm{D}_x(tc^2(u)u_x) = 0, \tag{3.3}$$

$$\mathrm{D}_t(xu_t) - \mathrm{D}_x\Big(xc^2(u)u_x - \int c^2(u)du\Big) = 0, \tag{3.4}$$

$$\mathrm{D}_t\Big(x[tu_t - u]\Big) - \mathrm{D}_x\Big(t\big[xc^2(u)u_x - \int c^2(u)du\big]\Big) = 0. \tag{3.5}$$

The first conservation law (3.2) is the PDE (3.1) itself. We observe that the three remaining conservation laws can be obtained in a straightforward way by the application of the recursion formula (2.9) to the wave equation (3.1) (conservation law (3.2)), without invoking the direct construction or any other generic method. Applying (2.9) to (3.2) with respect to t and x, one obtains the conservation laws (3.3) and (3.4). Applying the recursion formula to (3.3) with respect to x, one gets (3.5).

Other choices of $f(t, x)$ in the recursion formula of Lemma 1 lead to nonlocal divergence expressions (2.9); indeed, the locality condition (2.11) is given by

$$f_{tt} - c^2(u)f_{xx} = 0,$$

which, for an arbitrary u, has the indicated four linearly independent solutions $f(t, x) = 1, t, x, xt$.

3.2. Nonlinear diffusion equations

Consider a class of nonlinear diffusion equations for $u = u(t, x)$, given by

$$u_t = (L(u))_{xx}. \tag{3.6}$$

Every local conservation law of the PDEs (3.6) is equivalent to a local conservation law where u_t and its differential consequences have been excluded through the substitution of (3.6). Since (3.6) is an evolution PDE of order two (even order) in space, and also is quasilinear, it follows that the conserved densities of its local conservation laws are of order zero, i.e., involve no derivatives of u [1,24,39]. There are consequently exactly two local conservation laws of (3.6) holding for an arbitrary $L(u)$ [12]; they are given by

$$\mathrm{D}_t(u) - \mathrm{D}_x\Big((L(u))_x\Big) = 0, \tag{3.7}$$

$$\mathrm{D}_t(xu) - \mathrm{D}_x\Big(x(L(u))_x - L(u)\Big) = 0. \tag{3.8}$$

The first conservation law (3.7) is the PDE (3.6) itself. One can apply the recursion formula (2.9) to (3.7) for an arbitrary $f(t,x)$ to obtain divergence expressions

$$\mathrm{D}_t\left(fu - \int f_t u\,dt\right) - \mathrm{D}_x\left(f(L(u))_x - \int f_x (L(u))_x\,dx\right) = 0, \tag{3.9}$$

which yield local conservation laws if and only if $f(t,x) = C_1 x + C_2$, $C_1, C_2 = \text{const}$, correspond to the first and the second conservation laws (3.7), (3.8).

3.3. The vorticity-type equations

Consider a system of vorticity-type equations in 3+1 dimensions [18], given by four scalar PDEs

$$\operatorname{div}\mathbf{N} = 0, \qquad \mathbf{N}_t + \operatorname{curl}\mathbf{M} = 0, \tag{3.10}$$

in Cartesian coordinates. The system (3.10) involves two vector functions $\mathbf{N} = (N_1, N_2, N_3)$, $\mathbf{M} = (M_1, M_2, M_3)$ which depend on the time t and the spatial variables x, y, z.

The system (3.10) is not closed; it can, however, be considered as it stands for the sake of conservation law computations. PDEs (3.10) form a part of several fundamental linear and nonlinear physical PDE models in various fields, including Maxwell's equations, vorticity dynamics of Euler and Navier–Stokes equations of fluid motion, and magnetohydrodynamics (MHD) equations describing ideal plasmas as well as plasmas with nonzero resistivity [18,21]. In Maxwell's and MHD equations, the vector field $\mathbf{N} = \mathbf{B}$ represents the magnetic induction, and in vorticity equations, $\mathbf{N} = \boldsymbol{\omega}$ is the fluid vorticity. In many cases, the vector function $\mathbf{M}$ is a nonlinear function of physical parameters.

The three components of the vector equation in (3.10) are given by

$$\begin{aligned} &\mathrm{D}_t N_1 + \mathrm{D}_y M_3 - \mathrm{D}_z M_2 = 0,\\ &\mathrm{D}_t N_2 + \mathrm{D}_z M_1 - \mathrm{D}_x M_3 = 0,\\ &\mathrm{D}_t N_3 + \mathrm{D}_x M_2 - \mathrm{D}_y M_1 = 0. \end{aligned} \tag{3.11}$$

The general application of the recursion formula of Lemma 1 to each of the three PDEs (3.11) using arbitrary functions $f_i = f_i(x,y,z,t)$, $i = 1,2,3$, respectively, leads to a formal divergence expression

$$\mathrm{D}_t\left(f_1 N_1 - \int f_{1\,t} N_1\,dt\right) + \mathrm{D}_y\left(f_1 M_3 - \int f_{1\,y} M_3\,dy\right) - \mathrm{D}_z\left(f_1 M_2 - \int f_{1\,z} M_2\,dz\right) = 0, \tag{3.12}$$

and the two other respective ones obtained from (3.12) by cyclic permutations $1 \to 2 \to 3 \to 1$ and $x \to y \to z \to x$. These three divergence expressions generally involve nonlocal terms.

The first basic example when the conservation law (3.12) (with the two permuted ones) is local is when one takes $f_1 = f_1(x)$ in (3.12) (and $f_2 = f_2(y)$, $f_3 = f_3(z)$ in the two related formulas). In this case, (3.12) is given by

$$\mathrm{D}_t\left(f_1(x) N_1\right) + \mathrm{D}_y\left(f_1(x) M_3\right) - \mathrm{D}_z\left(f_1(x) M_2\right) = 0. \tag{3.13}$$

Denoting $\mathbf{f} = (f_1, f_2, f_3)$, one may write the three conservation laws together as a vector formula

$$\mathrm{D}_t\left(\mathbf{N}\cdot\mathbf{f}\right) + \operatorname{div}\left(\mathbf{M}\times\mathbf{f}\right) = 0. \tag{3.14}$$

The general case when the conservation laws given by (3.12) and the related expressions can yield a local conservation law of (3.10) is when the arbitrary functions f_i are the three components of a gradient of an arbitrary function $F = F(t,x,y,z)$:

$$(f_1, f_2, f_3) = \operatorname{grad} F. \tag{3.15}$$

In this case, using the basic differential identities, one can show that the integral terms vanish on the solution set of the system (3.10), and the resulting linear combination of the three scalar conservation laws has an equivalent form

$$(\mathbf{N} \cdot \operatorname{grad} F)_t + \operatorname{div} (\mathbf{M} \times \operatorname{grad} F - F_t \, \mathbf{N}) = 0. \tag{3.16}$$

In the context of vorticity equations in gas and fluid dynamics, the conserved density $\Theta = \mathbf{N} \cdot \operatorname{grad} F$ is known in literature as *potential vorticity* [13,18,21,23,30].

The local conservation laws (3.16) are known to hold in a more general context, when the function F is an arbitrary differential function, i.e., it may depend not only on independent variables, but also on physical parameters of the model and their derivatives [18]. The existence of such a wide class of conservation laws has to do with the fact that any PDE system involving equations (3.10) is abnormal. To variational abnormal PDE systems, Noether's second theorem applies [21,35,36].

3.4. Wave propagation in a hyper-viscoelastic fiber-reinforced material

Wave models for elastic solids within the nonlinear incompressible hyperelasticity and viscoelasticity frameworks were considered in [19]. A third-order PDE governing anti-plane shear fiber-aligned displacements $G(t, x)$ of a fiber-reinforced viscoelastic solid was derived; it has a dimensionless form

$$\begin{aligned} G_{tt} = \;& \left(G_x^2 + 1\right) G_{xx} \\ & + \eta\, G_x \Big[2\,(4\alpha G_x^2 + 3) G_{xx} G_{tx} + (2\alpha G_x^2 + 3) G_x G_{txx}\Big] \\ & + \zeta G_x^3 \Big[12(2\alpha G_x^2 + 1) G_{xx} G_{tx} + (4\alpha G_x^2 + 3) G_x G_{txx}\Big], \end{aligned} \tag{3.17}$$

involving three constant constitutive parameters α, η, ζ. It was shown in [19] that the PDE (3.17) can be written in the form of a local conservation law

$$\mathrm{D}_t\Big(G_t - \left[\eta(3 + 2\alpha G_x^2) + \zeta(3 + 4\alpha G_x^2) G_x^2\right] G_x^2 G_{xx}\Big) - \mathrm{D}_x \left(\left[1 + \tfrac{1}{3}\, G_x^2\right] G_x\right) = 0. \tag{3.18}$$

To this conservation law, we now apply the recursion formula of Lemma 1 with respect to t. It is convenient to rewrite (3.18) as

$$\mathrm{D}_t\Big(G_t - A(G_x) G_{xx}\Big) - \mathrm{D}_x \Big(B(G_x)\Big) = 0, \tag{3.19}$$

with

$$A(G_x) = \left[\eta(3 + 2\alpha G_x^2) + \zeta(3 + 4\alpha G_x^2) G_x^2\right] G_x^2, \qquad B(G_x) = \left[1 + \tfrac{1}{3}\, G_x^2\right] G_x.$$

Using the formula (2.9), one can write a divergence expression

$$\mathrm{D}_t\Big[t(G_t - A(G_x) G_{xx}) - \int (G_t - A(G_x) G_{xx})\, dt\Big] - \mathrm{D}_x \Big(t B(G_x)\Big) = 0. \tag{3.20}$$

It can be shown that (3.20) is equivalent to a local conservation law. Indeed, it can be rewritten as

$$\begin{aligned} 0 &= \mathrm{D}_t\Big(t(G_t - A(G_x)G_{xx}) - G\Big) + A(G_x)G_{xx} - \mathrm{D}_x\left(tB(G_x)\right) \\ &= \mathrm{D}_t\Big(t(G_t - A(G_x)G_{xx}) - G\Big) - \mathrm{D}_x\left(tB(G_x) - \textstyle\int A(G_x)\,dG_x\right). \end{aligned} \tag{3.21}$$

Explicitly, we get the second local conservation law of the PDE (3.17) given by

$$\begin{aligned} &\mathrm{D}_t\Big(tG_t - G - t\left[\eta(3+2\alpha G_x^2) + \zeta(3+4\alpha G_x^2)G_x^2\right] G_x^2 G_{xx}\Big) \\ &\quad + \mathrm{D}_x\left(\eta\left[1+\tfrac{2}{5}\alpha G_x^2\right]G_x^3 + \zeta\left[\tfrac{3}{5}+\tfrac{4}{7}\alpha G_x^2\right]G_x^5 - t\left[1+\tfrac{1}{3}G_x^2\right]G_x\right) = 0, \end{aligned} \tag{3.22}$$

linearly independent of the first conservation law (3.18).

3.5. Computation of a new first integral from a known one for a nonlinear third-order ODE

Formula (2.9) can be used for ODEs, and thus lead to further reduction of order, when the new conservation law density (first integral) is also a local quantity. For example, consider an ODE for $K(x)$,

$$K''' = \frac{2\,(K'')^2\,K - (K')^2\,K''}{KK'}, \tag{3.23}$$

arising in a symmetry classification problem. (In (3.23), primes denote derivatives.) It can be shown that there is a first integral of (3.23) given by

$$\mathrm{D}_x\left(\frac{KK''}{(K')^2}\right) = 0. \tag{3.24}$$

We apply the formula (2.9) with $f = x$ to (3.24) to get an independent first integral, which indeed appears to be a local quantity:

$$0 = \mathrm{D}_x\left(x\frac{KK''}{(K')^2} - \int \frac{KK''}{(K')^2}\,dx\right) = \mathrm{D}_x\left(\frac{xKK''}{(K')^2} + \frac{K}{K'} - x\right). \tag{3.25}$$

Generally, using the direct method, one can construct three independent integrating factors and therefore three first integrals of the ODE (3.23), thus completely integrating it; the three first integrals are given by

$$\frac{KK''}{(K')^2} = C_1, \quad \frac{xKK''}{(K')^2} + \frac{K}{K'} - x = C_2, \quad \frac{KK''\ln K}{(K')^2} - \ln K' = C_3, \tag{3.26}$$

however, the transition from (3.24) to (3.25) via the formula (2.9) is much simpler since it only requires an elementary calculation.

4. Relationship between the recursion formula (2.9) and symmetry action on conservation laws

It has been long established that local symmetries of DEs can be used to map their local conservation laws into local conservation laws (e.g., [5,14,27,36–38,40]). In particular, suppose that a DE system $\mathcal{R}$ (1.1) has a point symmetry with an infinitesimal operator

$$\mathrm{X} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu}\ ,$$

and $\mathrm{X}^{(k)}$ denotes its k-th prolongation. Let (1.2) be a nontrivial local conservation law of $\mathcal{R}$, with fluxes $\Psi^i[u]$ involving derivatives of u up to order r. Then the differential functions

$$\Omega^i[u] = \mathcal{M}^i_{\mathrm{X}}[\Psi] \equiv -\mathrm{X}^{(r)}\Psi^i + (\mathrm{D}_j\xi^i)\Psi^j[u] - (\mathrm{D}_j\xi^j)\Psi^i \tag{4.1}$$

are fluxes of a local conservation law $\mathrm{D}_i\Omega^i[u] = 0$ of the system (1.1). Formulas similar to (4.1) for infinitesimal symmetry generators in evolutionary form appear in [24,36]. It is worth noting that conservation laws arising from a symmetry action (4.1) on nontrivial conservation law(s) may be trivial or linearly dependent on the original conservation law(s). Nevertheless, this approach may be practically beneficial for complicated DE systems where a complete conservation law analysis is not feasible, yet some basic symmetries and conservation laws are known.

We observe that the recursion formula (2.9), in the case when it yields a *local* conservation law (2.9), is related to finding a pre-image of the local symmetry mapping (4.1). As an example, consider the nonlinear wave PDE family (3.1). For an arbitrary $c(u)$, the PDE (3.1) has conservation laws (3.3)–(3.5), which are obtained through the formula (2.9) applied to the obvious conservation law (3.2),

$$\mathrm{D}_t\Theta + \mathrm{D}_x\Phi \equiv \mathrm{D}_t(u_t) - \mathrm{D}_x(c^2(u)u_x) = 0.$$

Every wave equation (3.1) is invariant with respect to time and space translations, with the generators $\mathrm{X}_1 = \partial/\partial t$ and $\mathrm{X}_2 = \partial/\partial x$. Denote the respective densities and fluxes of the conservation laws (3.3)–(3.5) by $(\Theta_{(i)}, \Phi_{(i)})$, $i = 1, 2, 3$. Then, for example, the symmetry action of these generators on the respective density and flux of the conservation laws (3.3), (3.4) yields

$$\mathcal{M}_{\mathrm{X}_1}\big(\Theta_{(1)}, \Phi_{(1)}\big) = \mathcal{M}_{\mathrm{X}_2}\big(\Theta_{(2)}, \Phi_{(2)}\big) = \big(u_t,\ -c^2(u)u_x\big) = \big(\Theta, \Phi\big).$$

Similarly, for the conservation law (3.5),

$$\mathcal{M}_{\mathrm{X}_1}\mathcal{M}_{\mathrm{X}_2}\big(\Theta_{(3)}, \Phi_{(3)}\big) = \big(\Theta, \Phi\big).$$

Thus the conservation laws (3.3)–(3.5) essentially arise from an inversion of the symmetry action (4.1). The computation of the pre-image under the mapping (4.1) naturally involves integration, in agreement with the form of the recursion formula (2.9).

In the case of a general $f(z)$, it is not evident how one can identify a local symmetry that maps a (local) conservation law (2.9) into the simpler conservation law $\mathrm{D}_i\Psi^i[u] = 0$ according to the symmetry mapping like (4.1); additional coordinate transformations may be involved in such a mapping.

To our knowledge, it remains an open problem to find out in which cases an inverse symmetry action, similar to the one discussed above, can be applied to a local conservation law of a given DE system, and yield independent local conservation law(s) of that system.

5. Discussion and conclusions

The current contribution is concerned with the basic properties and applications of the simple conservation law recursion formula (2.9) (Lemma 1), which, for a known local divergence-type conservation law of a given system of differential equations, yields a set of formal divergence expressions that vanish on solutions of that system.

The practical value of the suggested formula (2.9) is the possibility of immediate construction of additional local conservation law(s) of DE systems from its known local conservation law(s), without the need to solve determining equations that arise in general conservation law construction procedures, such as the direct method, or methods related to Noether-type results. The result therefore can be useful to efficiently derive basic conservation laws of even rather complicated models.

Lemma 2 provides an explicit condition (2.11) for the recursion formula (2.9) to yield a local conservation law. In fact, it is shown that new local conservation laws obtained through the recursion formula (2.9) are equivalent to those computed by the application of the direct conservation law construction method to one object, the initial conservation law of the given DE system, rather than to all its equations. Though the use of linear combinations of subsets of equations of a given system to obtain conservation laws has been used in literature for various applications (see, e.g., [18,26]), the result of Lemma 2 contains a general, explicit, and rather practically useful instance of this approach.

When the given system is a normal, totally nondegenerate system of differential equations, and when a divergence expression arising from the recursion formula (2.9) ($f \neq$ const) is local, this divergence expression provides a new nontrivial conservation law of the given DE system, linearly independent of the initial conservation law (Lemma 3).

Local conservation laws following from the recursion relation (2.9) for basic models have a transparent physical meaning. Examples of fundamental physical PDEs, including 1+1-dimensional nonlinear wave and diffusion equations, were considered. In both cases, the application of the formula to the given equation as it stands has led directly to all its known local conservation laws holding for an arbitrary nonlinearity (Sections 3.1, 3.2). In Section 3.3, for vorticity-type equations in three space dimensions, the recursion formula was shown to give rise to the important family of potential vorticity local conservation laws, involving an arbitrary function of all variables. In Section 3.4, for a nonlinear third-order PDE describing wave propagation in a hyper-viscoelastic fiber-reinforced material, an additional local conservation law was derived from the basic one using the recursion formula (2.2). An example where the formula (2.9) was used to derive an independent first integral of a nonlinear ODE from a known one was presented in Section 3.5.

It is worth mentioning that within the framework of nonlocally related PDE systems [6,7,9,11], for many models, the majority of useful results, such as nonlocal symmetries and conservation laws, new exact solutions, etc., are known to arise from simplest potential systems. The latter, in turn, follow from basic local conservation laws of a given PDE system. This is the case, for example, for the nonlinear telegraph (NLT) equations [11] given by

$$u_{tt} - (F(u)u_x)_x - (G(u))_x = 0, \tag{5.1}$$

$u = u(t, x)$. The first conservation law holding for arbitrary constitutive functions $F(u)$, $G(u)$ is the PDE (5.1) itself, with the conserved density given by u_t. The second conservation law follows from Lemma 1 and has the conserved density given by $tu_t - u$. It is these two conservation laws that yield the couplet potential system having the largest set of Lie symmetries for the case of power nonlinearities, including a nonlocal symmetry of the NLT equations that does not arise as a local symmetry of any other potential system [11].

The presented general recursion formula (2.9) and even its elementary version (2.2) are practically useful to immediately derive, without tedious computations, additional local conservation laws from basic ones for equations where the integrals in (2.2), (2.9) yield local quantities. These formulas provide an additional insight into the structure of many known conservation law classifications, in particular, those for the models in fluid and solid mechanics (e.g., [19,20,26]).

A relationship between the conservation law formula (2.9) and the action of point symmetries on local conservation laws of DEs is discussed in Section 4.

In the recursion formula (2.9), instead of $f(z)$, one may consider a general differential function $f[u]$ (then in (2.9), one replaces $\partial f/\partial z^i$ by D_i). In this case, the result is generalized, however, from the practical point of view, the locality condition (2.11) becomes more complicated to verify. The development of useful examples that employ such a generalization is in the realm of future research.

Possible applications of nonlocal divergence expressions obtained from the recursion formulas (2.9) when the locality condition (2.11) is not satisfied also remain to be studied. The nonlocal terms that arise do not obviously correspond to classical nonlocal potential variables constructed from a potential PDE system. It is of interest to study cases when the nonlocal quantities appearing in (2.9) can be endowed with a geometrical meaning; this would lead to an extension of the framework of nonlocally related PDE systems, and may lead to new analytical results for a given PDE system, such as new nonlocal symmetries, reductions, and exact solutions.

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