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On new stability modes of plane canonical shear flows using symmetry classification

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In the work of Nold and Oberlack [Phys. Fluids **25**, 104101 (2013)], it was shown that three different instability modes of the linear stability analysis perturbing a linear shear flow can be derived in the common framework of Lie symmetry methods. These modes include the normal-mode, the Kelvin mode, and a new mode not reported before. As this was limited to linear shear, we now present a full symmetry classification for the linearised Navier-Stokes equations which are employed to study the stability of an arbitrary plane shear flow. If viscous effects for the perturbations are neglected, then we obtain additional symmetries and new *Ansatz* functions for a linear, an algebraic, an exponential, and a logarithmic base shear flow. If viscous effects are included in the formulation, then the linear and a quotient-type base flow allow for additional symmetries. The symmetry invariant solutions derived from the new and classical generic symmetries for all different flow types naturally lead to algebraic growth and decay for all cases except for two linear base flow cases. In turn this leads to the formulation of a novel eigenvalue problem in the analysis of the transition to turbulence for the respective flows, all of which are very distinct from the classical Orr-Sommerfeld eigenvalue problems. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4934726]

I. INTRODUCTION

The crucial parameter for describing fluid flows is the Reynolds number, which represents the ratio between inertial and viscous forces in a flow. For low Reynolds numbers, i.e., relatively slow flows, the fluid flows are linearly stable. However, increasing the Reynolds number leads to instabilities which may lead to a new laminar state or turbulence. The nature of this transition is of great interest for industrial applications, as an effective influence or control of the onset of turbulence could help bring up efficiency and lower costs. The study of the onset of turbulence is also of paramount importance to better understand flows in nature.

A classical approach to analyse the onset of turbulence is to assume a laminar base flow, and to linearly superpose a small perturbation. Stability analysis of this perturbation then leads to a range of stable/unstable modes and corresponding Reynolds numbers. The most famous approach for the perturbations is the normal mode approach, leading to the famous Orr-Sommerfeld equation.[1] However, for many flows, the results obtained employing the normal mode approach do not coincide with experimental observations.[2,3] Alternatively, for linear base flows, Kelvin modes can be employed as a perturbation.[4] In this case, analytical results[5,6] show that the flows are always stable, clearly contradicting experimental observations.

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In the 1990s, a novel approach revealed an explanation for this apparent paradox. It was shown that the eigenfunctions for the perturbations of non-uniform flows can be non-orthogonal, meaning that the eigenfunctions interact: they are spectrally stable, but perturbations are able to gain the basic (shear) flow energy transiently and, consequently, exhibit strong growth during a limited time interval.[7–10] In the case of large enough initial perturbations, the strong short-term non-normal growth allows for non-linear effects to take place, which regenerate the transiently growing perturbations. This positive feedback-loop allows for the onset of turbulence and is usually denoted as bypass-transition.[11–13]

The limitations of the modal approach in linear stability analysis[14] lead us to revisit the linear stability analysis by further generalizing the normal mode and the Kelvin mode approach. Recently, it was shown that for a linear shear flow, a systematic application of symmetry analysis allows to obtain a new invariant solution for the perturbations, which differs significantly from the latter two classical approaches.[15] Here, we perform a complete point symmetry classification and present a whole set of base shear flows which allow for novel invariant solutions for the perturbations describing new modes not captured by the classical approaches.

A symmetry of a system of differential equations (DEs) is a transformation which maps the solution set into itself. For example, any linear homogeneous DE has a scaling symmetry. Similarly, any DE that is autonomous in some independent variable has a translational symmetry in that variable. In particular, physical systems which are autonomous in time and space allow for a translational symmetry in time and space. An *invariant solution* is a solution curve that is mapped into itself upon the application of a given symmetry transformation. For systems of DEs involving arbitrary functions and/or parameters, symmetry classification is applied to isolate special cases for which additional symmetries arise.

Local symmetries and invariant solutions reflect the mathematical structure of a given DE system. Invariant solutions often have a clear physical meaning. In fluid mechanics, symmetry methods have been applied successfully in various specific areas; see, e.g., the work of Boisvert, Ames, and Srivastava;[16] Simonsen and Meyer-ter Vehn;[17] Oberlack, Wenzel, and Peters;[18] Oberlack;[19] Avramenko *et al.*;[20] Barenblatt, Galerkina, and Luneva;[21] and Grebenev.[22] One of the most well-known examples is the invariant solution of the Prandtl boundary layer equations derived by Blasius, which is often referred to as similarity solution because of the exclusive use of scaling symmetries.

In the current paper, we present several base shear flows for which the linearized Navier-Stokes equations (LNSEs) admit additional symmetries: four base flows for the inviscid setting, and the linear shear flow plus one additional base flow for the viscous setting. For each base flow, we present the corresponding point symmetries, introduce the respective invariant solution forms, and obtain the corresponding reduced equations by substituting the invariant solution form into the linearized Navier-Stokes equations. We briefly discuss physical properties of such invariant solutions.

In Sec. II, we give a brief introduction to the symmetry analysis procedure for the linearized Navier-Stokes equations for the perturbations, in the streamfunction formulation. We present the results of the symmetry classification in Sec. III, before presenting the invariant solutions for each base flow: In Sec. IV, we consider inviscid base flows, whilst we consider viscous base flows in Sec. V. In both of the latter sections, we present the respective novel eigenvalue equations. We then summarize the main results and review possible research directions in Sec. VI.

II. SYMMETRY ANALYSIS

Consider an unbounded incompressible parallel two-dimensional shear flow $(U(y),0)^T$ with a perturbation of the form $(u(x,y,t),v(x,y,t))^T$. Assuming that the Navier-Stokes equations hold for the base flow, one obtains the following set of equations for the perturbations:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{dU}{\mathrm{d}y} + \left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\Delta u, \tag{1}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + \left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right\} = -\frac{1}{\varrho}\frac{\partial p}{\partial y} + \nu\Delta v, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

where ν is the kinematic viscosity and Δ is the Laplace operator. Since only incompressible flows are considered, the density ϱ will formally be included in the pressure perturbation p. Introducing a stream function ψ for the velocity perturbations

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x} \tag{4}$$

eliminates the continuity equation. Applying the curl on two resulting Equations (1) and (2) eliminates the pressure and we obtain a nonlinear equation for the streamfunction of the perturbations, which in its linearized form gives the LNSE

$$\frac{\partial}{\partial t}\Delta\psi - \frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\frac{\partial \psi}{\partial x} + U\frac{\partial}{\partial x}\Delta\psi = \nu\Delta\Delta\psi. \tag{5}$$

For the simplicity of presentation, we only give results for streamfunction formulation (5). All results obtained in this section also apply if the linearized form of Equations (1)-(3) is used.

The main step in the following analysis is to seek Lie point symmetries of partial differential equation (PDE) (5). Symmetries are given by point transformations $\mathbf{T} = \left(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\psi}\right)$ of the form

$$\tilde{x} = \tilde{x}\left(x, y, t, \psi; \varepsilon\right), \qquad \tilde{y} = \tilde{y}\left(x, y, t, \psi; \varepsilon\right), \tag{6}$$

$$\tilde{t} = \tilde{t}\left(x, y, t, \psi; \varepsilon\right), \qquad \tilde{\psi} = \tilde{\psi}\left(x, y, t, \psi; \varepsilon\right), \tag{7}$$

for which the transformed quantities satisfy transformed Equation (5),

$$\frac{\partial}{\partial \tilde{t}}\tilde{\Delta}\tilde{\psi} - \frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\frac{\partial \tilde{\psi}}{\partial \tilde{x}} + U\frac{\partial}{\partial \tilde{x}}\tilde{\Delta}\tilde{\psi} = \nu\tilde{\Delta}\tilde{\Delta}\tilde{\psi}. \tag{8}$$

In other words, we search for transformations (6) and (7) which leave PDE (5) invariant. In (6) and (7), $\varepsilon \in \mathbb{R}$ is the group parameter, and the transformations are assumed to be smooth functions of ε.

Analytically, it is particularly useful to relate the global transformation group $\mathbf{T}$ with the tangent vector field $(\xi^x, \xi^y, \xi^t, \eta)$ at $\varepsilon = 0$, defined as

$$\xi^x = \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \xi^y = \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \xi^t = \left.\frac{\partial \tilde{t}}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \eta = \left.\frac{\partial \tilde{\psi}}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{9}$$

Tangent vector field components (9) determine $\mathbf{T}$ uniquely[23] through the relation

$$\mathbf{T} = e^{\varepsilon X}\mathbf{x}, \tag{10}$$

where $\mathbf{x} = (x, y, t, \psi)$ and X is the tangent vector field of $\mathbf{T}$ written as an infinitesimal generator

$$X := \xi^x\frac{\partial}{\partial x} + \xi^y\frac{\partial}{\partial y} + \xi^t\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \psi} \equiv \xi^x\partial_x + \xi^y\partial_y + \xi^t\partial_t + \eta\partial_\psi. \tag{11}$$

(The short-hand notation $\partial/\partial x \equiv \partial_x$, etc., is used here and below for symmetry generators.)

The infinitesimal generator X related to Equation (5) can be found by requiring that its corresponding symmetry transformation does not change the form of the PDE. In particular, denote PDE (5) by $F\left(\mathbf{x}, \psi, \psi_{[1]}, \psi_{[2]}, \ldots\right) = 0$, where $\psi_{[n]}$ stands for a vector of partial n derivatives of order n, $n = 1, 2, 3, 4$. In order to determine the point symmetries of (5), one solves a system of determining equations

$$X^{(4)}F|_{F=0} = 0, \tag{12}$$

where $X^{(4)}$ is the 4th prolongation of the symmetry generator X defined in (11).[23]

One of the most powerful tools of symmetry analysis is the ability to construct invariant solutions, based on the invariants of the symmetry transformation $\mathbf{T}$. These invariant solutions have been shown to provide special physical solutions of the equations at play.[16–22] The mathematical condition for invariance of a solution is that it should not change its functional form after application of the infinitesimal generator,

$$X\left(\psi - \psi_1(x,y,t)\right)|\psi = \psi_1(x,y,t) = \eta - \xi^x \frac{\partial \psi_1}{\partial x} - \xi^y \frac{\partial \psi_1}{\partial y} - \xi^t \frac{\partial \psi_1}{\partial t} = 0. \tag{13}$$

From the practical point of view, it is often beneficial to consider combinations of point symmetries. Since symmetry generators form a linear vector space, for any n symmetries $X_1, \ldots, X_n$ admitted by a given DE system, a linear combination $X := \sum_i a_i X_i$ is also a generator of a point symmetry. For example, invariant solutions with respect to a combination of a space translation $X_1 = \partial_x$ and a time translation $X_2 = \partial_t$ are determined by the condition

$$X(\psi - \psi_1(x,y,t)) = (a_1 X_1 + a_2 X_2)(\psi - \psi_1(x,y,t)) = -a_1 \frac{\partial \psi}{\partial x} - a_2 \frac{\partial \psi}{\partial t} = 0. \tag{14}$$

The corresponding invariant solution represents a traveling wave $\psi = f(a_2 x - a_1 t)$, for which the advection speed depends on the ratio of the coefficients a_1 and a_2.

In this work, we show that normal modes, Kelvin modes, as well as a new types of base solutions can be systematically derived by seeking invariant solutions of linearized perturbation equation (5) with respect to combinations of its point symmetries for different base flows. For more details about point and local symmetries and related extensions, see, e.g., the work of Bluman, Cheviakov, and Anco;[23] Bluman and Kumei;[24] Bluman and Anco;[25] Cantwell;[26] Steeb;[27] and references therein. The point symmetries presented in the current work were derived using the GeM software package of Cheviakov[28] and the DESOLVE package of Carminati and Vu.[29]

III. SYMMETRY CLASSIFICATION

For general shear base flows $\left(U(y), 0)^T\right)$, LNSE (5) admits four symmetries: superposition, translation in x and t and scaling of ψ, all obtained by solving condition (13) for LNSE (5). The respective infinitesimal generators and the global transformation groups are given below. For simplicity of notation, we denote $\partial/\partial x \equiv \partial_x$, etc.,

$$X_0 = f(x,y,t)\partial_\psi \quad \Leftrightarrow \quad [\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{t} = t, \quad \tilde{\psi} = \psi + f(x,y,t)], \tag{15}$$

$$X_1 = \partial_x \quad \Leftrightarrow \quad [\tilde{x} = x + x_0, \quad \tilde{y} = y, \quad \tilde{t} = t, \quad \tilde{\psi} = \psi], \tag{16}$$

$$X_2 = \partial_t \quad \Leftrightarrow \quad [\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{t} = t + t_0, \quad \tilde{\psi} = \psi], \tag{17}$$

$$X_3 = \psi\partial_\psi \quad \Leftrightarrow \quad [\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{t} = t, \quad \tilde{\psi} = C\psi]. \tag{18}$$

Here, x_0 and t_0 are the space and time shift, respectively; $f(x,y,t)$ is an arbitrary solution of PDE (5); $C \neq 0$ is an arbitrary scaling constant. We note that the superposition symmetry X_0 and the scaling symmetry X_3 were induced by the linearization, whereas the symmetries X_1 and X_2 are already admitted before the linearization leading to (5).

The general set of symmetries (15)-(18) and, in turn, the set of corresponding invariant solutions, can be considerably extended for some special cases of the base flow $U(y)$, as well as for the case of zero viscosity. In the Appendix, details of the symmetry classification are presented.

In Figure 1, we present the special cases of the symmetry classification modulo to equivalence transformations, which do not change the differential structure of the equations (see, e.g., Refs. 23

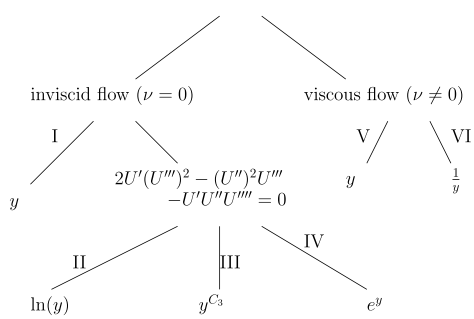

FIG. 1. Base flows $U(y)$ allowing for additional symmetries in the viscous and inviscid case, modulo to equivalence transformations, where C_3 is an arbitrary constant. The nodes of the tree represent the pivot elements of the classification.

and 30). In the following, for practical purposes of reconstructing solutions for specific cases, we analyze each case of the symmetry classification in its general form including parameters of the equivalence transformation, leading to base flows such as listed in Table I.

For viscous flows, we find two special base flows allowing for additional symmetries. Inviscid flows allow for a richer variety of symmetries (cases V and VI). Besides the linear inviscid shear flow, which admits six symmetries, algebraic, exponential, and logarithmic base flows also allow for one additional symmetry each (cases I–IV). Out of these base flows, only the linear shear flow and a parabolic channel flow profile satisfy the Navier-Stokes equation.

The infinitesimal generators of the additional symmetries for the respective base flows are given by

$$\mathbf{I}\quad U(y) = A_L y + V_L: \qquad X_{4,\mathrm{I}} = A_L t\partial_x + \partial_y, \tag{19}$$

$$X_{5,\mathrm{I}} = (x - tV_L)\partial_x + y\partial_y, \tag{20}$$

$$\mathbf{II}\quad U(y) = A_2 \ln(y+L_2) + V_2: \qquad X_{4,\mathrm{II}} = (x + A_2 t)\partial_x + (y+L_2)\partial_y + t\partial_t, \tag{21}$$

$$\mathbf{III}\quad U(y) = A_3 (y+L_3)^{C_3} + V_3: \qquad X_{4,\mathrm{III}} = (C_3 V_3 t - x)\partial_x - (y+L_3)\partial_y + (C_3-1)t\partial_t, \tag{22}$$

$$\mathbf{IV}\quad U(y) = A_4 e^{B_4 y} + V_4: \qquad X_{4,\mathrm{IV}} = -B_4 V_4 t\partial_x + \partial_y - B_4 t\partial_t, \tag{23}$$

$$\mathbf{V}\quad U(y) = A_L y + V_L: \qquad X_{4,\mathrm{V}} = A_L t\partial_x + \partial_y, \tag{24}$$

$$\mathbf{VI}\quad U(y) = \frac{A_6}{y+L_6} + V_6: \qquad X_{4,\mathrm{VI}} = (V_6 t + x)\partial_x + (y+L_6)\partial_y + 2t\partial_t, \tag{25}$$

and the corresponding global transformations are given by

$$T_{4,\mathrm{I}}: \quad [\tilde{x} = x - y_0 A_L t, \quad \tilde{y} = y - y_0, \quad \tilde{t} = t, \quad \tilde{\psi} = \psi], \tag{26}$$

$$T_{5,\mathrm{I}}: \quad [\tilde{x} = k(x - V_L t) + V_L t, \quad \tilde{y} = ky, \quad \tilde{t} = t, \quad \tilde{\psi} = \psi], \tag{27}$$

$$T_{4,\mathrm{II}}: \quad [\tilde{x} = kx + A_2 t\frac{\ln k}{\ln(e/k)}, \quad \tilde{y} = k(y+L_2) - L_2, \quad \tilde{t} = kt, \quad \tilde{\psi} = \psi], \tag{28}$$

$$T_{4,\mathrm{III}}: \quad [\tilde{x} = \frac{x - V_3 t}{k} + V_3 t k^{C_3-1}, \quad \tilde{y} = \frac{y+L_3}{k} - L_3, \quad \tilde{t} = k^{C_3-1}t, \quad \tilde{\psi} = \psi], \tag{29}$$

$$T_{4,\mathrm{IV}}: \quad [\tilde{x} = x + (k^{-B_4} - 1)V_4 t, \quad \tilde{y} = ky, \quad \tilde{t} = k^{-B_4}t, \quad \tilde{\psi} = \psi], \tag{30}$$

$$T_{4,\mathrm{V}}: \quad [\tilde{x} = x - y_0 A_L t, \quad \tilde{y} = y - y_0, \quad \tilde{t} = t, \quad \tilde{\psi} = \psi], \tag{31}$$

$$T_{4,\mathrm{VI}}: \quad [\tilde{x} = kx + tV_6(k^2 - k), \quad \tilde{y} = k(y+L_6) - L_6, \quad \tilde{t} = k^2 t, \quad \tilde{\psi} = \psi], \tag{32}$$

where y_0, $k \neq 0$, A_i, B_i, C_i, V_i, and L_i are arbitrary constants.

A general symmetry generator for every special case (base shear flow and viscosity) is given by a linear combination of four general symmetry generators (15)-(18) and the corresponding additional symmetry generators for that case. In Secs. III A–V, we derive invariant solutions for different symmetry combinations. An overview, for both viscous and inviscid setups, is given in Table I.

A. Derivation of the Orr-Sommerfeld equation using symmetry methods

The classical normal mode approach turns out to be an invariant solution with respect to the combination of the three symmetries $X_1, \ldots, X_3$ in (16)-(18),

$$X^{(0)} := a_1\partial_x + a_2\partial_t + a_3\psi\partial_\psi, \quad a_{1,2,3} \in \mathbb{C}. \tag{33}$$

Invariant solution condition (13) becomes

$$a_3\psi - a_1\frac{\partial\psi}{\partial x} - a_2\frac{\partial\psi}{\partial t} = 0. \tag{34}$$

If $a_2 \neq 0$, the method of characteristics yields the solution

$$\psi^{(0)}(x,y,t) = f^{(0)}(\xi, y)\, e^{\frac{a_3}{a_2}t}, \tag{35}$$

TABLE I. An overview of invariant solutions for viscous and inviscid cases (cf. (19)-(32)). Cases I-IV are for inviscid perturbations, while symmetries V-VI are for viscous flows. In this table, we present simplified versions of the invariant solutions by setting some parameters to zero. For the complete solutions, see the respective sections. In the present table, f is a function of the indicated arguments, and $\alpha, \beta, \gamma, \delta, c, k$ are constant parameters.

Case	$U(y)$	a_1	a_2	a_3	a_4	a_5	*Ansatz* for $\psi(x,y,t)$
0	General	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	...	...	$f(y)e^{i\alpha(x-ct)}$
I.a	A_Ly+V_L	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	$e^{ct}f(x/y, t-\ln y)$ (for $V_L=0$)
I.b	A_Ly+V_L	$\mathbb{C}$	0	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	$y^k f(x/y,t)$ (for $A_L=1, V_L=0$)
II	$A_2\ln(y+L_2)+V_2$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	...	$t^\beta f(x/t-A_2\ln t, y/t)$ (for $A_2=1$, $L_2=V_2=0$)
III	$A_3(y+L_3)^{C_3}+V_3$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	...	$t^\beta f(xt, yt)$ (for $L_3=V_3=0, C_3=2$)
IV	$A_4e^{B_4y}+V_4$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	...	$t^\beta f(x-\gamma\ln t, y-\ln t)$ (for $V_4=0, B_4=-1$)
V.a	A_Ly+V_L	$\mathbb{C}$	0	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	...	$f(t)e^{\delta(x-yt)+\beta y}$ (for $A_L=1$)
V.b	A_Ly+V_L	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	...	$f(y-t)e^{\delta(x-\frac{t^2}{2})+\beta t}$ (for $A_L=1, V_L=0, a_2=a_4$)
VI	$\frac{A_6}{y+L_6}+c$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\setminus\{0\}$	...	$t^\beta f\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)$ (for $V_6=0$)

depending on the traveling wave coordinate in the x-direction,

$$\xi = x - \frac{a_1}{a_2}t. \tag{36}$$

The function $f^{(0)}$ satisfies the linear fourth order differential equation

$$\left(U - \frac{a_1}{a_2}\right)\frac{\partial}{\partial\xi}\Delta f^{(0)} + \frac{a_3}{a_2}\Delta f^{(0)} - U''\frac{\partial}{\partial\xi}f^{(0)} = \nu\Delta\Delta f^{(0)} \tag{37}$$

obtained by inserting form (35) into the LNSE (5). While the number of variables has been reduced by one, the PDE is still of the fourth order, and the number of parameters has increased by two: in addition to the viscosity ν, we now also have a_1/a_2 and a_3/a_2. Since linear homogeneous PDE (37) does not involve ξ explicitly, it clearly admits two basic symmetries: the scaling $f^{(0)}\partial_{f^{(0)}}$ and the translation ∂_ξ. In order to further reduce the number of independent variables in (37), we require that the solution $f^{(0)}(\xi, y)$ is invariant with respect to a linear combination

$$\tilde{X}^{(0)} = b_1\frac{\partial}{\partial\xi} + b_2 f\frac{\partial}{\partial f}, \quad b_{1,2} \in \mathbb{C}. \tag{38}$$

This leads to an invariant *Ansatz*

$$f^{(0)}(\xi, y) = g^{(0)}(y)e^{\frac{b_2}{b_1}\xi}, \quad b_1 \neq 0, \tag{39}$$

while $b_1 = 0$ leads to the trivial solution $\psi = 0$. General solution *Ansatz* (35) becomes

$$\psi^{(0)}(x,y,t) = g^{(0)}(y)\exp\left(\frac{b_2}{b_1}x + \frac{a_3b_1 - a_1b_2}{a_2b_1}t\right). \tag{40}$$

A natural assumption that the solution is bounded for $x \to \pm\infty$, i.e., $\mathrm{Re}\,(b_2/b_1) = 0$, leads to the classical normal mode approach

$$\psi^{(0)}(x,y,t) = g^{(0)}(y)e^{i\alpha(x-ct)}, \tag{41}$$

with the wavelength $\alpha = \mathrm{Im}\,(b_2/b_1)$ and the wave speed $c = \mathrm{Im}\,((a_3b_1)/(a_2b_2) - a_1/a_2)$. The use of *Ansatz* (41) in the LNSE (5) leads to the well-known *Orr-Sommerfeld equation*

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)g^{(0)} - U''g^{(0)} = \frac{\nu}{i\alpha}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)^2 g^{(0)}. \tag{42}$$

(Note that a similar analysis can be done for $a_2 = 0$ and $a_1 \neq 0$, yielding the same result.)

Summarizing this section, the Orr-Sommerfeld equation is derived through a successive symmetry reduction of linearized Navier-Stokes equation (5) with for an arbitrary $U(y)$. This holds true for both the viscous and the inviscid case and is usually referred to as normal mode or modal approach. In this work, we will repeatedly apply the method of successive symmetry reductions in special flow cases, in order to reproduce further known *Ansätze*, and introduce new ones.

IV. INVISCID PERTURBATIONS

In this section, we consider the inviscid equations for the perturbations by setting $\nu = 0$ in Equation (5). Let us note that while the base flows may satisfy the viscous Navier-Stokes equations, the additional symmetries presented in this sections only apply for the inviscid equations for the perturbations. This simplification has a very long tradition and traces back to the famous work of Rayleigh, where details may be taken from Refs. 31 and 32. Still, it limits the analysis to inviscid modes, but could be of interest for scenarios where the stress tensor of the perturbations is small. In fact, this condition has to be verified *a posteriori* after obtaining the invariant solution.

A. Solution I.a for a linear shear flow

For case I.a for a linear base flow

$$U(y) = A_L y + V_L, \tag{43}$$

in Table I, we choose $a_5 \neq 0$, thus employing the general symmetry

$$X^{(\mathrm{I.a})} = a_1\partial_x + a_2\partial_t + a_3\psi\partial_\psi + a_4\left(A_L t\partial_x + \partial_y\right) + a_5\left((x - tV_L)\,\partial_x + y\partial_y\right). \tag{44}$$

For the case $a_2 \neq 0$, Equation (13) yields the corresponding invariant solution form

$$\psi^{(\mathrm{I.a})}(x,y,t;c,\lambda_2,A_L) := e^{ct} f^{(\mathrm{I.a})}(\xi,\eta;c,\lambda_2,A_L), \tag{45}$$

$$\text{with} \qquad \xi := \frac{x - x_0 - (y_0A_L + V_L)\,t}{y - y_0} \tag{46}$$

$$\text{and} \qquad \eta := \lambda_2 t - \ln(y - y_0), \tag{47}$$

where

$$c = \frac{a_3}{a_2}, \qquad \lambda_2 = \frac{a_5}{a_2} \neq 0, \qquad x_0 = \frac{a_2}{a_5}(y_0A_L + V_L) - \frac{a_1}{a_5}, \qquad y_0 = -\frac{a_4}{a_5}. \tag{48}$$

Substituting this *Ansatz* into LNSE (5) yields

$$\left\{\frac{c}{A_L} + \frac{\lambda_2}{A_L}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}\right\}\left\{(\xi^2+1)\frac{\partial^2}{\partial\xi^2} + \left(1 + \frac{\partial}{\partial\eta}\right)\left(2\xi\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\right\} f^{(\mathrm{I.a})}(\xi,\eta;c,\lambda_2,A_L) = 0. \tag{49}$$

This allows for one additional symmetry reduction,

$$f^{(\mathrm{I.a})}(\xi,\eta;c,\lambda_2,A_L,k) = g^{(\mathrm{I.a})}(\xi;c,\lambda_2,A_L,k)\,e^{k\eta}. \tag{50}$$

Insertion into Equation (49) yields

$$\left(\frac{c}{A_L} + \frac{\lambda_2}{A_L}k + \frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left\{(\xi^2+1)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + (1+k)\left(2\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + k\right)\right\} g^{(\mathrm{I.a})}(\xi;c,\lambda_2,k) = 0. \tag{51}$$

The latter equation can be integrated once to give an ordinary differential equation (ODE)

$$\left\{\left(\xi^2+1\right)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+(1+k)\left(2\xi\frac{\mathrm{d}}{\mathrm{d}\xi}+k\right)\right\}g^{(\mathrm{I.a})}(\xi;c,\lambda_2,k)=e^{-\frac{(c+\lambda_2 k)}{A_L}\xi}, \tag{52}$$

which is solved exactly by

$$g^{(\mathrm{I.a})}(\xi;c,\lambda_2,k)=\begin{cases}C_1+C_2\arctan(\xi)+ & \\ \quad +\dfrac{iA_L}{2c}\left\{e^{\frac{ic}{A_L}}\mathrm{Ei}_1\left(\dfrac{c}{A_L}(\xi+i)\right)-e^{-\frac{ic}{A_L}}\mathrm{Ei}_1\left(\dfrac{c}{A_L}(\xi-i)\right)\right\} & \text{for } k=0,\\ C_1(\xi+i)^{-k}+C_2(\xi-i)^{-k} & \\ \quad -\dfrac{i}{2k}\displaystyle\int_0^{\xi}\left\{\left(\dfrac{\hat{\xi}+i}{\xi+i}\right)^k-\left(\dfrac{\hat{\xi}-i}{\xi-i}\right)^k\right\}e^{-\frac{(c+\lambda_2 k)}{A_L}\hat{\xi}}d\hat{\xi} & \text{for } k\neq 0,\end{cases} \tag{53}$$

where Ei is the exponential integral.

B. Solution I.b for a linear shear flow

This case can be obtained by a simple reformulation of *Ansatz* (45)-(47). Redefining the second variable of the problem gives the *Ansatz*

$$\psi^{(\mathrm{I.b})}(x,y,t;x_0,y_0,k)=(y-y_0)^k f^{(\mathrm{I.b})}(\xi,\eta;k), \tag{54}$$

$$\text{with}\qquad \xi=\frac{x-x_0-(y_0A_L+V_L)t}{y-y_0} \tag{55}$$

$$\text{and}\qquad \eta=A_L\left(t-\frac{1}{\lambda_2}\ln(y-y_0)\right)\overset{a_2=0}{=}A_Lt, \tag{56}$$

with $\frac{1}{\lambda_2}=\frac{a_2}{a_5}=0$,

$$k=\frac{a_3}{a_5}, \tag{57}$$

and x_0, y_0 such as defined in (48). In particular, for $a_2=0$, the second term of (56) vanishes and substitution of *Ansatz* (54) into LNSE (5) yields the following third order partial differential equation for $f(\xi,\eta;k)$:

$$\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\left(k(k-1)+2\xi(1-k)\frac{\partial}{\partial\xi}+(1+\xi^2)\frac{\partial^2}{\partial\xi^2}\right)f^{(\mathrm{I.b})}(\xi,\eta;k)=0. \tag{58}$$

Integrating to eliminate the first differential operator in (58) yields

$$\left(k(k-1)+2\xi(1-k)\frac{\partial}{\partial\xi}+(1+\xi^2)\frac{\partial^2}{\partial\xi^2}\right)f^{(\mathrm{I.b})}(\xi,\eta;k)=h(\xi-\eta), \tag{59}$$

which is essentially an ODE with respect to ξ, with h an arbitrary function. The full solution is of the form

$$f^{(\mathrm{I.b})}(\xi,\eta;k)=\begin{cases}g_{0,1}^{(\mathrm{I.b})}(\eta)+g_{0,2}^{(\mathrm{I.b})}(\eta)\arctan(\xi) & \\ \quad +\displaystyle\int^{\xi}\frac{\int^{\xi'}h(\xi''-\eta)\mathrm{d}\xi''}{1+\xi'^2}\mathrm{d}\xi' & \text{for } k=0,\\ g_{k,1}^{(\mathrm{I.b})}(\eta)(\xi+i)^k+g_{k,2}^{(\mathrm{I.b})}(\eta)(\xi-i)^k & \\ \quad +\dfrac{i}{2k}\displaystyle\int_0^{\xi}\left\{\left(\dfrac{\xi+i}{\hat{\xi}+i}\right)^k-\left(\dfrac{\xi-i}{\hat{\xi}-i}\right)^k\right\}h(\hat{\xi}-\eta)d\hat{\xi} & \text{for } k\neq 0,\end{cases} \tag{60}$$

for arbitrary functions $g_{k,1}^{(\mathrm{I.b})}(\eta)$ and $g_{k,2}^{(\mathrm{I.b})}(\eta)$. Given that η is proportional to the time for $a_2=0$ (see Eq. (56)), this means that the homogeneous part of solution (60), given by its first two terms for both cases $k=0$ and $k\neq 0$, yields modes whose absolute values can either increase or decrease, independent of the initial condition.

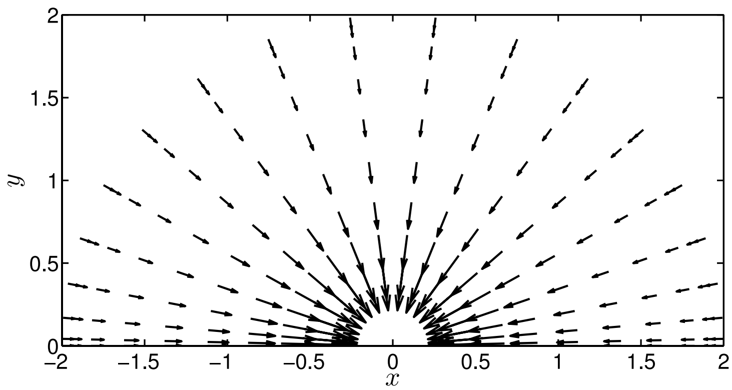

FIG. 2. Velocity field induced by the homogeneous part of solution of (59) for $k=0$ such as given by $\psi_0^{(\mathrm{I.b})}$ in (61). The velocity field is that of a point sink — or, for a prefactor of inverse sign, a point source — in the half plane.

Insertion into (54) and setting for simplicity $x_0 = y_0 = V_L = 0$ and $A_L = 1$ yields for the homogeneous part of the solution

$$\psi^{(\mathrm{I.b})}(x,y,t;k) = \begin{cases} g_{0,1}^{(\mathrm{I.b})}(t) + g_{0,2}^{(\mathrm{I.b})}(t)\arctan\left(\dfrac{x}{y}\right) & \text{for } k = 0, \\ g_{k,1}^{(\mathrm{I.b})}(t)\,(x+iy)^k + g_{k,2}^{(\mathrm{I.b})}(t)\,(x-iy)^k & \text{for } k \neq 0. \end{cases} \tag{61}$$

For $k < 1$, the solution is not continuous at $y = 0$, in particular, the velocities, defined through relation (4), diverge for $x = 0$, $y \to 0$. For $k > 1$, the velocities resulting from the stream function $\psi^{(\mathrm{I.b})}(x,y,t;k)$ are unbounded as $y \to \infty$. A special case is given by $k = 0$, where the second term in (61) exhibits a discontinuity on the line $y = 0$ as $x \to \pm 0$. By considering a linear shear flow in the half plane $y \geq 0$, the discontinuity can be avoided. Note that the first term $g_{0,1}^{(\mathrm{I.b})}(t)$ corresponds to zero velocities, as it carries no x and y dependency. The velocity field for the case $k = 0$ is depicted in Figure 2.

Another physical solution in the homogeneous case is obtained for $k = 1$. It represents a superposition of the base flow with a velocity which is constant in space and varies in time,

$$u_1^{(\mathrm{I.b})} = i\left(g_{k,1}^{(\mathrm{I.b})}(t) - g_{k,2}^{(\mathrm{I.b})}(t)\right), \tag{62}$$

$$v_1^{(\mathrm{I.b})} = -\left(g_{k,1}^{(\mathrm{I.b})}(t) + g_{k,2}^{(\mathrm{I.b})}(t)\right). \tag{63}$$

For the inhomogeneous part of the solution for $k = 1$, Equation (59) simplifies to

$$\left(1+\xi^2\right)\frac{\partial^2}{\partial \xi^2} f^{(\mathrm{I.b})}(\xi,\eta;1) = h(\xi - \eta) \tag{64}$$

leading to the inhomogeneous contribution of Equation (60) for $k = 1$,

$$f^{(\mathrm{I.b})}(\xi,\eta;1) = \int_0^{\xi} \frac{\xi - \tilde{\xi}}{\tilde{\xi}^2 + 1} h\left(\tilde{\xi} - \eta\right) \mathrm{d}\tilde{\xi}. \tag{65}$$

Insertion of this solution into *Ansatz* (54) then defines a velocity field for the perturbations. In Figure 3, a solution is given for the right-hand side $h(\zeta) = \zeta e^{-\zeta^2}$.

C. Solution *Ansatz* II and a reduced PDE for a logarithmic shear flow

For a logarithmic base flow, the inviscid equations for the perturbations, i.e., Equation (5) with $\nu = 0$, admit additional symmetries (21) and (28). Obviously, a logarithmic base flow of the form

$$U(y) = A_2 \ln(y + L_2) + V_2 \tag{66}$$

for $y > -L_2$ does not satisfy the momentum balance in horizontal direction of the full (viscous) Navier-Stokes equation $-\frac{\partial p}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} = 0$ for the base flow, and hence is, strictly speaking, not

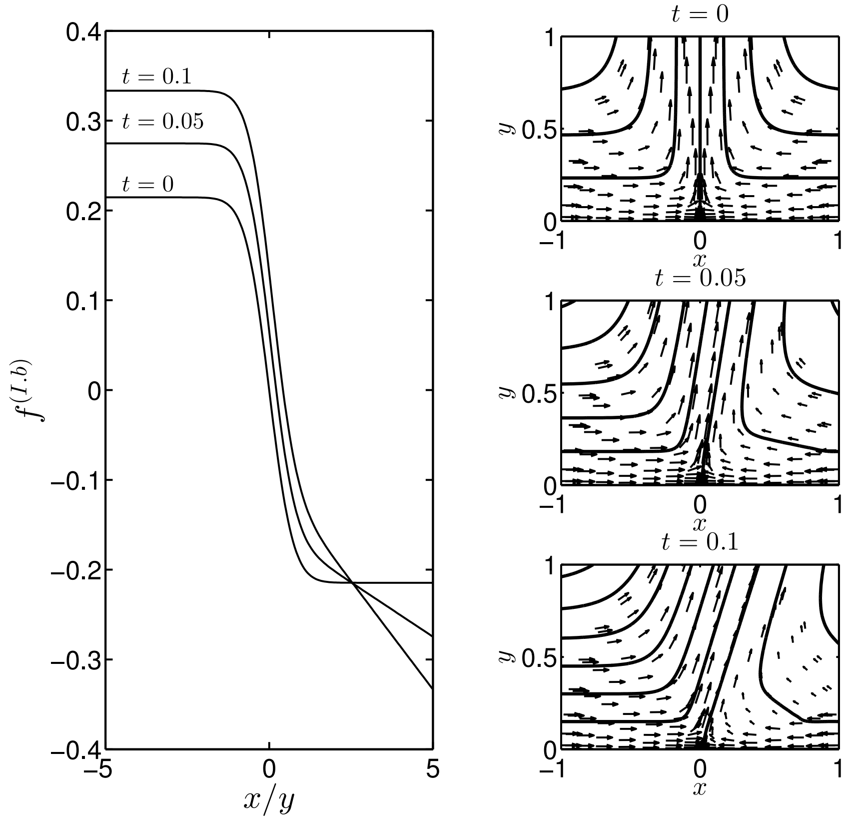

FIG. 3. The left figure depicts a numerical solution of (59) for $k = 1$, a base flow $U(y) = y$ for $y \geq 0$, and $h(\zeta) = \zeta e^{-\zeta^2}$. The right figures depict the streamlines and the velocity field of the solution for different points in time (here, $\eta = t$). Parameters for the invariant solution are $x_0 = y_0 = 0$. The perturbed velocity field satisfies the condition $v|_{y=0} = 0$ at the wall.

suitable for a further analysis. Nevertheless, it is reasonable to investigate (66), since at the large distance from the wall, $y \gg 1$, a log-law region is known to exist in a turbulent wall parallel shear flow,[33]

$$U(y) = \frac{1}{\kappa} \ln y + B, \tag{67}$$

where B = const, and κ is the Kármán constant. In this case, the order of magnitude of the perturbations in (1) is larger than the departure from the Navier-Stokes-Equation for the base flow, i.e.,

$$\left|\frac{\partial u}{\partial t}\right|, \left|U \frac{\partial u}{\partial x}\right|, \left|v \frac{\mathrm{d}U}{\mathrm{d}y}\right| \gg \nu \left|\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right| = \nu \left|\frac{1}{\kappa y^2}\right|. \tag{68}$$

Keeping these simplifications and assumptions in mind, we compute an invariant solution with respect to a general combination of all point symmetries available for the logarithmic base flow, i.e., (16)-(18) and (21), (28),

$$X^{(\mathrm{II})} = a_1 \partial_x + a_2 \partial_t + a_3 \partial_\psi + a_4 \left((x + A_2 t)\, \partial_x + (y + L_2)\, \partial_y + t \partial_t\right). \tag{69}$$

This yields an invariant solution *Ansatz*

$$\psi^{(\mathrm{II})}(x, y, t; x_0, t_0, \beta) = (t - t_0)^\beta f^{(\mathrm{II})}(\xi, \eta; \beta), \tag{70}$$

$$\text{with} \qquad \xi = \frac{x - x_0}{t - t_0} - A_2 \ln(t - t_0) - V_2 \tag{71}$$

$$\text{and} \qquad \eta = \frac{y + L_2}{t - t_0}, \tag{72}$$

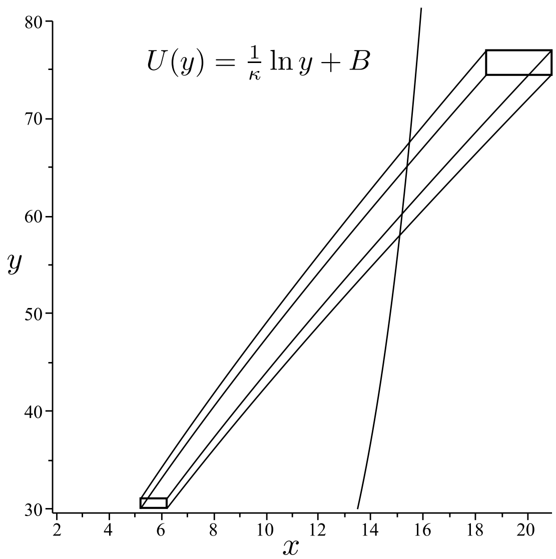

FIG. 4. Movement of a self-similar perturbation for $t = 1$–2.5 and for a logarithmic base flow $U(y) = \frac{1}{\kappa}\ln(y) + B$ with the typical constants for a log-law region $\kappa = 0.41$ and $B = 5.2$ (Pope[33]) and with *Ansätze* (70)-(72) for $x_0 = t_0 = 0$. Note that $\frac{1}{\kappa}$ and B correspond with A_2 and V_2 in Equation (66), respectively. The box is translated such that the similarity variables ξ and η are constant in $[0,1]\times[30,31]$. Hence, the form of the perturbations is conserved inside the box, which is stretched with time. The magnitude of the velocities increases/decreases algebraically with $t^{\beta-1}$, depending on the sign of $\beta - 1$.

where

$$x_0 = A_2\frac{a_2}{a_4} - \frac{a_1}{a_4}, \qquad t_0 = -\frac{a_2}{a_4}, \qquad \text{and} \qquad \beta = \frac{a_3}{a_4}, \tag{73}$$

and where $f^{(\mathrm{II})}(\xi,\eta;\beta)$ satisfies the linear PDE

$$\eta^2\left(\beta - 2 + (A_2\ln(\eta) - (\xi + A_2))\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)\Delta f^{(\mathrm{II})} + A_2\frac{\partial}{\partial\xi}f^{(\mathrm{II})} = 0. \tag{74}$$

Figure 4 depicts the movement of a box in which the values of the self-similar variables ξ and η are kept constant. The shape of the streamlines is stretched with the box, while the magnitude of the velocity perturbations scales algebraically with $t^{\beta-1}$, as the stream function ψ scales as t^{β} according to (70). Following *Ansatz* (70), it can be observed at which speed invariant perturbations are transported away from the wall and in direction of the base flow,

$$x(t) - x_0 = (t - t_0)\left(U(y) + \xi - A_2\ln\eta\right) \qquad \text{and} \qquad y(t) + L_2 = \eta\,(t - t_0)\,. \tag{75}$$

Hence, the perturbations travel with the base flow superposed with a constant speed in longitudinal and vertical directions. Two key results may be taken from (74) and (75). First, any invariant perturbation, no matter if amplified or suppressed due to the value of β in (70), travels away from the wall, since η is always positive. Second, (74) represents an eigenvalue problem similar to the Orr-Sommerfeld equation. The eigenvalue β represents the rate of algebraic growth or decay of the perturbations. We note, however, that any boundary condition applied on (74) acts on moving boundaries shown in Figure 4.

D. Solution *Ansatz* III and a reduced PDE for an algebraic shear flow

For an algebraic base shear flow (see case III in Table I for the inviscid case)

$$U(y) = A_3\,(y + L_3)^{C_3} + V_3, \tag{76}$$

one has an additional symmetry (22) and (29), and the general symmetry generator

$$X^{\mathrm{III}} = a_1\partial_x + a_2\partial_t + a_3\partial_\psi + a_4\left((C_3V_3t - x)\,\partial_x - (y + L_3)\,\partial_y + (C_3 - 1)t\partial_t\right). \tag{77}$$

In the following, we restrict ourselves to the exponent $C_3 = 2$, and obtain the invariant solution

$$\psi^{(\mathrm{III})}(x, y, t; x_0, t_0, \beta) = (t - t_0)^{\beta} f^{(\mathrm{III})}(\xi, \eta), \tag{78}$$

$$\text{with} \qquad \xi = (t - t_0)(x - x_0 - V_3(t - t_0)) \tag{79}$$

$$\text{and} \qquad \eta = (y + L_3)(t - t_0), \tag{80}$$

where

$$x_0 = \frac{a_1}{a_4} - \frac{2a_2V_3}{a_4}, \qquad t_0 = -\frac{a_2}{a_4}, \qquad \text{and} \qquad \beta = \frac{a_3}{a_4}. \tag{81}$$

The substitution of (78)-(80) into the LNSE (5) with $\nu = 0$ leads to the equation

$$\left(\left(\xi + A_3\eta^2\right)\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta} + (2 + \beta)\right)\Delta f^{(\mathrm{III})} - 2A_3\frac{\partial f^{(\mathrm{III})}}{\partial\xi} = 0. \tag{82}$$

For simplicity, we set $x_0 = t_0 = 0$ and consider the classical Poiseuille flow as the base shear flow $U(y) = 1 - y^2$, where the length scales have been normalized by the channel half width h, and the velocities have been non-dimensionalized by the maximum velocity at the channel center $-\frac{h^2}{2\mu}\frac{\partial p}{\partial x}$. For this case, we have $A_3 = -1$, $L_3 = 0$, $C_3 = 2$, and $V_3 = -1$. The values $\xi(x(t), t) = \xi$ and $\eta(y(t), t) = \eta$ will stay constant on the characteristics defined by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\xi}{t} + t \\ \dfrac{\eta}{t} \end{pmatrix}. \tag{83}$$

Various conclusions may be drawn from Equations (79) and (80). For this, we trace a box representing a disturbance with time where the similarity variables ξ and η are constant (see Figure 5). Inside this box, the shape of the streamlines is preserved, while the magnitude of the velocities scales with $t^{\beta+1}$. Furthermore, the linear component in $x(t)$ in (83) derived from (79) suggests that the disturbance velocity tends to a constant velocity in x-direction, where the velocity corresponds to the maximal velocity of the channel flow, here chosen to equal 1. Hence, the wave speed of the perturbation is 50% faster then the bulk velocity, which is 2/3 for the normalized velocity profile $1 - y^2$. This appears to be an interesting coincidence to the observation that puffs and slugs in pipe flows exhibit an increased wave speed approximately of the same order and first observed in Ref. 34.

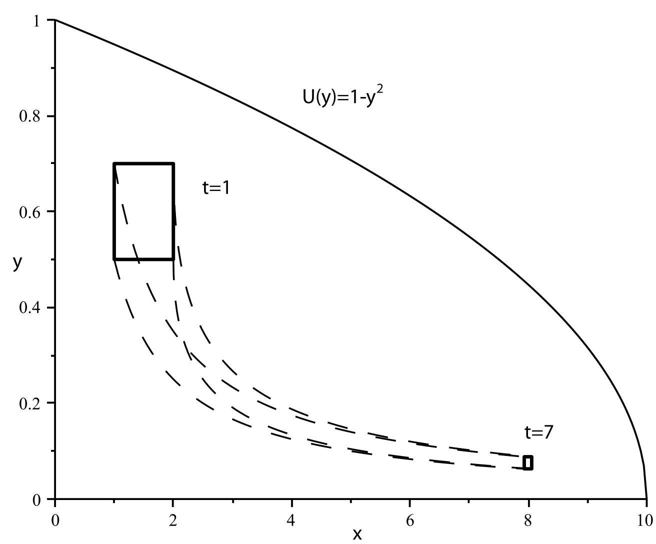

FIG. 5. Translation of the box $(\xi, \eta) \in [0, 1] \times [0.5, 0.7]$ in the physical space with time for $x_0 = t_0 = 0$ for a parabolic channel flow with *Ansätze* (78)-(80). The magnitude of the velocities within the box scales with $t^{\beta+1}$.

Consider also that conceptually, the perturbations are transported from the margins towards the center of the flow, as we may conclude from (83). At the same time, the box itself shrinks in time which relates to the fact that the wavelength of the perturbations decrease in time, i.e., we observe a front steepening (see Figure 5). Equation (82) represents an eigenvalue problem in which the eigenvalue β represents the rate of algebraic growth or decay of the perturbations. We also note that given a specific Poiseuille flow with fixed values for A_3, L_3, C_3, and V_3, there remain no further undefined parameters in Equation (82). Similar to eigenvalue problem (74) for a logarithmic base flow, the boundary conditions act on a moving boundary, as discussed above and shown in Figure 5. We note that no successive symmetry reduction of Equation (82) can be found, such as done, e.g., in the derivation Orr-Sommerfeld equation in Section III A. This means that we cannot further reduce the complexity of this equation with the help of symmetry methods.

E. Solution *Ansatz* IV and a reduced PDE for an exponential shear flow

Let us now consider an exponential base flow of the form

$$U(y) = A_4 e^{B_4 y} + V_4. \tag{84}$$

An exponential mean velocity profile for the wake region of a turbulent flat-plate boundary-layer flow was first found by Oberlack.[19] The exponential base flow may also be taken as a model for a laminar boundary layer profile in order to discuss how disturbances travel. Hence, similar to the case of a logarithmic base flow, here the exponential base flow, written in dimensionless form as $U(y) = 1 - e^{-y}$ is analysed though it does not satisfy the viscous version of perturbation equation (5). However, it is reasonable to apply perturbation methods if the departure from the momentum balance in longitudinal direction is assumed to be small compared to the contributions from the perturbations, in particular, if

$$\left|\frac{\partial u}{\partial t}\right|, \left|U\frac{\partial u}{\partial x}\right|, \left|v\frac{\mathrm{d}U}{\mathrm{d}y}\right| \gg \nu\left|\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right| = \frac{1}{\mathrm{Re}}\left|e^{-y}\right| \tag{85}$$

for $\mathrm{Re} \to \infty$ and $y \to \infty$. Under the restriction of the above mentioned assumptions, we obtain the general symmetry from (16)-(18) and (23),

$$X = a_1\partial_x + a_2\partial_t + a_3\partial_\psi + a_4\left(-B_4V_4t\partial_x + \partial_y - tB_4\partial_t\right). \tag{86}$$

The general invariant solution is derived by employing (86) in (13) to obtain

$$\psi^{(\mathrm{IV})}(x,y,t;x_0,t_0,\beta) = (t-t_0)^\beta f^{(\mathrm{IV})}(\xi,\eta;\beta), \tag{87}$$

$$\text{with} \qquad \xi = x - V_4t - \gamma\ln(t-t_0) \tag{88}$$

$$\text{and} \qquad \eta = y + \frac{1}{B_4}\ln(t-t_0), \tag{89}$$

where

$$t_0 = \frac{a_2}{B_4a_4}, \qquad \gamma = \frac{V_4a_2 - a_1}{B_4a_4}, \qquad \beta = -\frac{a_3}{B_4a_4}. \tag{90}$$

Substitution into the LNSE (5) leads to

$$\left(\beta B_4 + \frac{\partial}{\partial\eta} + B_4\left(A_4e^{B_4\eta} - \gamma\right)\frac{\partial}{\partial\xi}\right)\Delta f^{(\mathrm{IV})} - A_4(B_4)^3e^{B_4\eta}\frac{\partial f^{(\mathrm{IV})}}{\partial\xi} = 0. \tag{91}$$

In Figure 6, we trace a box which has constant self-similar variables ξ and η over time. *Ansatz* (87) suggests that perturbations travel with decreasing velocity away from the wall, while in downstream direction it approaches a constant velocity,

$$x(t) = \xi + V_4t + \gamma\ln(t-t_0) \qquad \text{and} \qquad y(t) = \eta - \frac{1}{B_4}\ln(t-t_0), \tag{92}$$

and for the perturbations a bending effect is observed towards the x-axis, when traveling downstream. Again, the velocities exhibit algebraic growth or decay with $(t-t_0)^\beta$, depending on the sign of β.

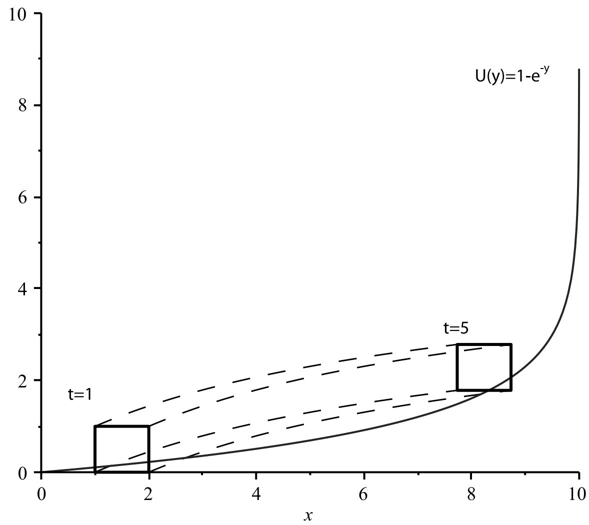

FIG. 6. Movement of a self-similar perturbation for an exponential base flow $U(y) = 1 - e^{-y}$. The box is translated such that the similarity variables ξ and η are constant. Hence, the form of the perturbations is conserved inside the box. The magnitude of the velocities increases/decreases algebraically with $(t - t_0)^{\beta}$, depending on the sign of β.

Note that (91) represents an eigenvalue problem where — similar to Sections IV C and IV D — the eigenvalue β stands for the rate of algebraic growth or decay of the perturbations. The boundary conditions needed in order to solve this eigenvalue problem are set in terms of the variables ξ and η, which in the Cartesian x-y-coordinate system will move such as depicted in Figure 6. We also note that in (91), coordinates cannot be rescaled such that any of the appearing parameters may be eliminated. Hence, the eigenvalue β and the eigenfunctions depend on the parameters A_4 and B_4. Analogous to Sec. IV D, we note that no successive symmetry reduction of Equation (91) can be found, such that we cannot reduce the complexity of this equation further or indeed find solutions to this equation by using symmetry methods.

V. VISCOUS FLOWS

We now consider the viscous equation for perturbations satisfying the LNSE (5) with $\nu \neq 0$. In this case, only the linear shear flow and a quotient type flow allow for additional symmetries and lead to new modes for the perturbations.

A. Solution V.a for a linear shear flow

For a linear base flow

$$U(y) = A_L y + V_L \tag{93}$$

we have four symmetries (16)-(18) and (24). Excluding time-translation symmetry (17) leads to

$$X^{(\mathrm{V.a})} = a_1\partial_x + a_3\partial_\psi + a_4\left(A_L t\partial_x + \partial_y\right). \tag{94}$$

With this the respective invariant solution from (13) is given by

$$\psi^{(\mathrm{V.a})}(x, y, t; t_0, \beta) = e^{\beta y} f^{(\mathrm{V.a})}(\xi, \eta; \beta), \tag{95}$$

$$\text{with} \qquad \xi = x - A_L y\,(t - t_0) \tag{96}$$

$$\text{and} \qquad \eta = t - t_0, \tag{97}$$

where

$$t_0 = -\frac{a_1}{A_L a_4} \qquad \text{and} \qquad \beta = \frac{a_3}{a_4}. \tag{98}$$

Using (95)-(97) in the LNSE (5), we obtain a fourth-order linear PDE for $f^{(\mathrm{V.a})}$, given by

$$\left(V_L\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\left(\beta^2-2A_L\eta\beta\frac{\partial}{\partial\xi}+\left(1+{A_L}^2\eta^2\right)\frac{\partial^2}{\partial\xi^2}\right)f^{(\mathrm{V.a})}=$$
$$=\nu\left(\beta^2-2A_L\eta\beta\frac{\partial}{\partial\xi}+\left(1+{A_L}^2\eta^2\right)\frac{\partial^2}{\partial\xi^2}\right)^2 f^{(\mathrm{V.a})}. \tag{99}$$

This equation has a point symmetry

$$X=b_1\partial_\xi+b_2 f\partial_f,\qquad b_1,b_2=\mathrm{const}, \tag{100}$$

leading to the invariant solution

$$f^{(\mathrm{V.a})}(\xi,\eta)=g^{(\mathrm{V.a})}(\eta)\,e^{\delta\xi}, \tag{101}$$

where $\delta=\frac{b_2}{b_1}$. The function $g^{(\mathrm{V.a})}(\eta)$ must solve the ODE

$$\left(V_L\delta+\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\left((\beta-\delta A_L\eta)^2+\delta^2\right)g^{(\mathrm{V.a})}=\nu\left((\beta-\delta A_L\eta)^2+\delta^2\right)^2 g^{(\mathrm{V.a})}. \tag{102}$$

Up to an arbitrary multiplicative constant, the solution of the latter is given by

$$g^{(\mathrm{V.a})}(\eta)=\frac{\exp\left(\frac{\eta}{3}\left(\nu\delta^2{A_L}^2\eta^2-3\nu\beta\delta A_L\eta+3\nu\beta^2+3\nu\delta^2-3V_L\delta\right)\right)}{(\beta-\delta A_L\eta)^2+\delta^2}. \tag{103}$$

Note that *Ansatz* (95) together with (103) corresponds exactly to the classical Kelvin modes.[4]

B. Solution V.b for a linear shear flow

We note that formula (94) is not the most general form of a point symmetry generator for linear base flow (93), since time-translation symmetry (17) was excluded. Including the time-translation symmetry leads to the following form of the generator:

$$X^{(\mathrm{V.b})}=a_1\partial_x+a_2\partial_t+a_3\psi\partial_\psi+a_4\left(A_L t\partial_x+\partial_y\right). \tag{104}$$

The corresponding invariant function is given by

$$\psi^{(\mathrm{V.b})}(x,y,t;t_0,c,\beta)=e^{\beta(t-t_0)}f^{(\mathrm{V.b})}(\xi,\eta;\beta), \tag{105}$$

$$\text{with}\qquad \xi=x-\frac{A_L c}{2}(t-t_0)^2 \tag{106}$$

$$\text{and}\qquad \eta=y+\frac{V_L}{A_L}-c\,(t-t_0), \tag{107}$$

with

$$\beta=\frac{a_3}{a_2},\qquad c=\frac{a_4}{a_2},\qquad\text{and}\qquad t_0=-\frac{a_1}{A_L a_4}. \tag{108}$$

Insertion of this approach into (5) leads to the partial differential equation

$$\left(\beta+A_L\eta\frac{\partial}{\partial\xi}-c\frac{\partial}{\partial\eta}\right)\Delta f^{(\mathrm{V.b})}=\nu\Delta^2 f^{(\mathrm{V.b})}. \tag{109}$$

Similar to Sec. V A, PDE (109) can be further reduced by looking for its invariant solutions with respect to its point symmetry

$$X=b_1\partial_\xi+b_2 f\partial_f,\quad b_1,b_2=\mathrm{const}. \tag{110}$$

Such invariant solutions have the form

$$f^{(\mathrm{V.b})}(\xi,\eta)=g^{(\mathrm{V.b})}(\eta)\,e^{\delta\xi}, \tag{111}$$

where $g^{(\mathrm{V.b})}(\eta)$ satisfies the ODE

$$\left(\delta A_L\eta+\beta-c\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\left(\delta^2+\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\right)g^{(\mathrm{V.b})}=\nu\left(\delta^2+\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\right)^2 g^{(\mathrm{V.b})}. \tag{112}$$

The solution of this equation in the inviscid and the viscous setting is extensively discussed in Ref. 15. In summary, the inclusion of viscous effects in the equations for the perturbations leads to unphysical solutions, which diverge for $y \to \infty$, while the exclusion of viscous effects leads to novel invariant modes which conserve energy and travel in parabola-shaped trajectories.

C. Solution VI for a quotient flow

The final form of the base shear flow that leads to additional symmetries admitted by perturbation equation (5) is the quotient flow, given by

$$U(y) = \frac{A_6}{y + L_6} + V_6. \tag{113}$$

This power law for the viscous case is a part of general power law base flow family (76) that arises for the inviscid case.

The general symmetry generator for the quotient flow is a linear combination of symmetries (16)-(18) and (25), given by

$$X = a_1\partial_x + a_2\partial_t + a_3\partial_\psi + a_4\big((V_6 t + x)\partial_x + (y + L_6)\partial_y + 2t\partial_t\big). \tag{114}$$

The corresponding invariant solution is given by

$$\psi^{(\mathrm{VI})}(x, y, t; x_0, t_0, \beta) = (t - t_0)^\beta f^{(\mathrm{VI})}(\xi, \eta; \beta), \tag{115}$$

$$\text{with} \qquad \xi = \frac{x - x_0 - V_6(t - t_0)}{\sqrt{t - t_0}} \tag{116}$$

$$\text{and} \qquad \eta = \frac{y + L_6}{\sqrt{t - t_0}}, \tag{117}$$

where

$$t_0 = -\frac{a_2}{2a_4}, \qquad x_0 = \frac{V_6 a_2 - 2a_1}{2a_4}, \qquad \beta = \frac{a_3}{2a_4}. \tag{118}$$

The substitution of (115)-(117) into the LNSE (5), $f^{(\mathrm{VI})}$ yields

$$\eta^2\left((2A_6 - \eta\xi)\frac{\partial}{\partial\xi} - \eta^2\frac{\partial}{\partial\eta} + 2\eta(\beta - 1)\right)\Delta f^{(\mathrm{VI})} - 4A_6\frac{\partial f^{(\mathrm{VI})}}{\partial\xi} = 2\eta^3\nu\Delta^2 f^{(\mathrm{VI})}. \tag{119}$$

The characteristic curves $\xi, \eta = \text{const}$ are given by parametric equations

$$x(t) = x_0 + V_6(t - t_0) + \xi\sqrt{t - t_0}, \qquad y(t) = -L_6 + \eta\sqrt{t - t_0}, \tag{120}$$

which correspond to parabolas

$$x = x_0 + \frac{V_6}{\eta^2}(y + L_6)^2 + \frac{\xi}{\eta}(y + L_6).$$

We note that no further symmetry reduction of Equation (119) can be found, such that symmetry methods cannot be used to further simplify this equation.

VI. CONCLUSION

In conclusion, we have presented a systematic symmetry classification for the linearised Euler and Navier-Stokes equations for perturbations of a laminar base flow. We have found that a general base shear flow without any restrictions allows for four symmetries: time and space translations, a scaling symmetry, and a superposition symmetry. A systematic symmetry reduction based on these symmetries leads to the classical normal mode approach, which yields the famous Orr-Sommerfeld equation.[1] A symmetry classification with respect to the viscosity ν being zero or non-zero, as well as the type of the base flow $U(y)$, leads to special cases of base shear flows which admit additional symmetries not reported before.

Assuming that for the perturbations viscous terms can be neglected leads to four different base shear flows which admit additional symmetries. In particular, the linear base flow case admits two extra symmetries, while the logarithmic, the algebraic, and the exponential base flows lead to one extra symmetry. Respective invariant solutions for the perturbations were presented and discussed. Each invariant solution leads to an eigenvalue-type problem of a nature similar to that of the Orr-Sommerfeld equation.

In particular, in the case of the linear base flow, the symmetry reduction with respect to new symmetry (44) with $a_2 \neq 0$ leads to an ODE (52), which is further integrated completely as per (53). The reduction with respect to same symmetry (44) with $a_2 = 0$ yields another ODE (59), with exact solution (60). For logarithmic flow (66), under the reduction with respect to a new general symmetry (69), a reduced linear PDE (74) is obtained. Similarly, for algebraic and exponential base flows (76) and (84), the use of special symmetries (77) and (86) leads to reduced PDEs (82) and (91). For the algebraic, the logarithmic, and the exponential base flows, the eigenvalue represents the rate of algebraic growth or decay of the perturbations. The eigenvalue problem is formulated in a coordinate system which is translated over time in a way characteristic for each base flow.

If viscous terms are included in the equations for the perturbations, then only two base flows allow for additional symmetries: a linear base flow and a quotient flow of the form $U(y) = a/(y+b)+c$. The invariant solution for the linear base flow turns out to lead directly to classical Kelvin mode *Ansatz* (95).[4] The other invariant reduction (105) for the linear base flow, with a further reduction (111), leads to an ODE (112). The latter, as discussed in Ref. 15, can be solved analytically, leading to stable modes which travel in parabola-shaped trajectories. Finally, we presented an invariant solution for the quotient flow, yielding a reduced linear PDE (119).

In summary, the systematic application of symmetry analysis performed in the current contribution has been shown to lead directly not only to the two classical approaches of perturbation stability analysis, namely, the normal mode and the Kelvin mode approach, but also to different, new *Ansätze* of the same nature, arising from special point symmetries admitted for particular classes of base flows. Open questions remain, in particular, the formulation of an appropriate set of boundary conditions for the newly derived eigenvalue problems. This would naturally lead to a numerical study of the stability of the novel modes in terms of the algebraic growth rate of the perturbations.

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APPENDIX: CLASSIFICATION OF SYMMETRIES OF LNSE (5)

The set of symmetries of the LNSE (5) is computed by solving invariance condition (12). Comparison of coefficients yields independent equations for the infinitesimals ξ^x, ξ^y, ξ^t, and η. Out of these, the following symmetry determining equations are equally valid in the inviscid and the viscous case:

$$\partial_{\psi x}\eta = \partial_{\psi y}\eta = \partial_{\psi t}\eta = \partial_{\psi\psi}\eta = 0, \tag{A1}$$

$$\partial_\psi \xi^x = \partial_\psi \xi^y = \partial_\psi \xi^t = 0, \tag{A2}$$

$$\partial_x \xi^t = \partial_y \xi^t = 0, \tag{A3}$$

$$\partial_x \xi^x - \partial_y \xi^y = 0, \tag{A4}$$

$$\partial_y \xi^x = \partial_x \xi^y = 0, \tag{A5}$$

$$\partial_{xx}\xi^x = \partial_{xy}\xi^x = \partial_{yy}\xi^x = 0, \tag{A6}$$

$$\partial_{xx}\xi^y = \partial_{xy}\xi^y = \partial_{yy}\xi^y = 0, \tag{A7}$$

$$\partial_t\Delta\eta - U''\partial_x\eta + U\partial_x\Delta\eta = \nu\Delta\Delta\eta, \tag{A8}$$

$$\partial_t\xi^y = 0. \tag{A9}$$

Assuming $U(y)$ to be completely generic, Equations (A1)-(A8) have the general solution

$$\xi^x(x,t) = \alpha(t)x + f_1(t) + a_1, \tag{A10}$$

$$\xi^y(y) = \alpha(t)y + \gamma, \tag{A11}$$

$$\xi^t(t) = f_3(t) + a_2, \tag{A12}$$

$$\eta = a_3\psi + f(x,y,t), \tag{A13}$$

where $f(x,y,t)$ has to satisfy (A8), which is equivalent to the LNSE (5).

1. The inviscid case

In the case of an inviscid flow, the set of determining Equations (A1)-(A9) is extended by two additional equations

$$\partial_{xt}\xi^x = 0, \tag{A14}$$

$$U\partial_t\xi^t - \partial_t\xi^x - U\partial_x\xi^x + U'\xi^y = 0. \tag{A15}$$

This leads to

$$\alpha = \text{const.} \tag{A16}$$

Note that implementing the latter into (A15) and differentiating (A15) with respect to y and t give that $f_3(t)$ has to be a linear function in t, i.e., $f_3(t) = \delta t$, provided that U is non-constant. Inserting this result back into (A15) and differentiating once with respect to t yields that $f_1(t)$ also exhibits linear behavior in t, i.e., $f_1(t) = \epsilon t$. For simplicity, we incorporate the constant contributions to $f_1(t)$ and $f_3(t)$ into a_1 and a_2, respectively. This gives

$$\xi^x(x,t) = \alpha x + \epsilon t + a_1, \tag{A17}$$

$$\xi^y(y) = \alpha y + \gamma, \tag{A18}$$

$$\xi^t(t) = \delta t + a_2, \tag{A19}$$

$$\eta = a_3\psi + f(x,y,t). \tag{A20}$$

The insertion of the above equations into (A15) yields

$$\epsilon = U(\delta - \alpha) + U'(\alpha y + \gamma), \tag{A21}$$

which gives, after taking the derivative with respect to y,

$$0 = U'\delta + U''(\alpha y + \gamma). \tag{A22}$$

The latter equation leads to the following two special cases.

1. In the case of a linear base flow $U(y) = A_L y + V_L$, i.e., if $U'' = 0$, δ has to vanish. We can choose α and γ independently of each other. (A15) then gives the respective value for $\epsilon = A_L\gamma - V_L\alpha$. This yields five symmetries, given here as a combination of their infinitesimals with prefactors $\{a_1, a_2, a_3, \alpha, \gamma\}$,

$$\xi^x(x,t) = \alpha(x - V_L t) + \gamma A_L t + a_1, \tag{A23}$$

$$\xi^y(y) = \alpha y + \gamma, \tag{A24}$$

$$\xi^t(t) = a_2, \tag{A25}$$

$$\eta = a_3\psi + f(x,y,t). \tag{A26}$$

2. If δ and U'' do not vanish, we may divide (A22) by U''. Then, the fraction U'/U'' has to exhibit linear behavior in y, i.e., its second derivative has to vanish. This yields the defining equation

$$2U'U'''^2 - U''^2U''' - U'U''U^{IV} = 0, \tag{A27}$$

which has the three classes of nontrivial solutions

$$\{A_2 \ln(y + L_2) + V_2, A_3(y + L_3)^{C_3} + V_3, A_4 e^{B_4 y} + V_4\}. \tag{A28}$$

Let us insert these solutions into (A21) and (A22). A comparison of coefficients defines three of the parameters α, γ, δ, and ϵ in terms of the coefficients of the base flows in (A28). In particular, (A17)-(A20) transform through (A21) to the following set of equations:

- $U(y) = A_2 \ln(y + L_2) + V_2$. In this case, we obtain

$$\xi^x(x,t) = \alpha\,(x + A_2 t) + a_1, \tag{A29}$$
$$\xi^y(y) = \alpha\,(y + L_2)\,, \tag{A30}$$
$$\xi^t(t) = \alpha t + a_2, \tag{A31}$$
$$\eta = a_3\psi + f(x,y,t), \tag{A32}$$

where α is kept as an independent parameter, defining one additional symmetry.

- $U(y) = A_3(y + L_3)^{C_3} + V_3$. In this case, we find

$$\xi^x(x,t) = \alpha\,(x - C_3V_3t) + a_1, \tag{A33}$$
$$\xi^y(y) = \alpha\,(y + L_3)\,, \tag{A34}$$
$$\xi^t(t) = -\alpha\,(C_3 - 1)\,t + a_2, \tag{A35}$$
$$\eta = a_3\psi + f(x,y,t), \tag{A36}$$

where α is kept as an independent parameter, defining one additional symmetry.

- $U(y) = A_4 e^{B_4 y} + V_4$. In this case, the infinitesimals write

$$\xi^x(x,t) = -B_4V_4\gamma t + a_1, \tag{A37}$$
$$\xi^y(y) = \gamma, \tag{A38}$$
$$\xi^t(t) = -B_4\gamma t + a_2, \tag{A39}$$
$$\eta = a_3\psi + f(x,y,t), \tag{A40}$$

where γ is kept as an independent parameter, defining one additional symmetry.

2. The viscous case

In the case of a viscous flow, the set of symmetry determining Equations (A1)-(A9) is extended by three additional equations

$$2\partial_x\xi^x - \partial_t\xi^t = 0, \tag{A41}$$
$$\partial_{ty}\xi^y = 0, \tag{A42}$$
$$\xi^y U' + U\partial_x\xi^x - \partial_t\xi^x = 0. \tag{A43}$$

(A41) and (A42) lead to

$$\alpha = \text{const.}, \qquad f_3'(t) = 2\alpha. \tag{A44}$$

We distinguish between two qualitatively distinct cases:

1. If the base flow is linear, i.e., if $U(y)$ is of the form $A_L y + V_L$, then insertion of (A10)-(A13) with (A44) into (A43) and comparing coefficients yields

$$\alpha = f_3' = 0, \qquad f_1' = A_L\gamma.$$

Insertion into (A10)-(A13) and keeping γ as the parameter defining the additional symmetry yields

$$\xi^x(x,t) = \gamma A_L t + a_1, \tag{A45}$$

$$\xi^y(y) = \gamma, \tag{A46}$$

$$\xi^t(t) = a_2, \tag{A47}$$

$$\eta = a_3\psi + f(x,y,t), \tag{A48}$$

where f_3, being constant, has been adsorbed in a_2.

2. In the case of a nonlinear base flow, inserting (A10)-(A13) with (A44) into (A43) yields

$$(\alpha y + \gamma)U' + U\alpha - f_1' = 0. \tag{A49}$$

By comparison of coefficients for t, we see that $f_1' = \text{const}$. We conclude by solving (A49) with

$$U(y) = \frac{A_6}{y + L_6} + V_6, \tag{A50}$$

$$\text{where} \qquad L_6 = \frac{\gamma}{\alpha} \quad \text{and} \quad V_6 = \frac{f_1'}{\alpha}. \tag{A51}$$

Insertion into (A10)-(A13) and keeping α as the parameter defining the additional symmetry yields

$$\xi^x(x,t) = \alpha\,(x + V_6 t) + a_1, \tag{A52}$$

$$\xi^y(y) = \alpha\,(y + L_6)\,, \tag{A53}$$

$$\xi^t(t) = 2\alpha t + a_2, \tag{A54}$$

$$\eta = a_3\psi + f(x,y,t). \tag{A55}$$

With this the symmetry classification of Equation (5) with respect to $U(y)$ is complete.

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