J. Fluid Mech. (2014), vol. 760, pp. 368–386.
doi:10.1017/jfm.2014.611

Generalized Ertel's theorem and infinite hierarchies of conserved quantities for three-dimensional time-dependent Euler and Navier–Stokes equations

Alexei F. Cheviakov[1,†] **and Martin Oberlack**[2]

[1]Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, SK, S7N 5E6, Canada

[2]Chair of Fluid Dynamics, Technische Universität Darmstadt, Otto-Berndt-Strasse 2, 64287 Darmstadt, Germany

(Received 29 November 2013; revised 9 October 2014; accepted 16 October 2014; first published online 7 November 2014)

Local conservation laws are systematically constructed for three-dimensional time-dependent viscous and inviscid incompressible fluid flows, in primitive variables and vorticity formulation, using the direct construction method. Complete sets of local conservation laws in primitive variables are derived for the case of conservation law multipliers depending on derivatives up to the second order. In the vorticity formulation, there exists an infinite family of vorticity-dependent conservation laws involving an arbitrary differentiable function of space and time, holding for both viscous and inviscid cases. The infinite conservation law family is used to generate further independent hierarchies of conservation laws that essentially involve vorticity and arbitrary flow parameters, which are determined by known evolution equations such as those for momentum, energy or helicity, though not necessarily in the form of a conservation law. The new conservation laws are not restricted to any reduced flow geometry such as planar or axisymmetric limits. Examples are considered.

Key words: general fluid mechanics, mathematical foundations, Navier–Stokes equations

1. Introduction

Conservation laws holding for a system of partial differential equations (PDEs) that describe a mathematical model provide critically important information about the model and the underlying physical system. Local divergence-type conservation laws have the form

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \boldsymbol{\Phi} = 0, \tag{1.1}$$

where Θ is the conservation law density, the components of $\boldsymbol{\Phi}$ are spatial fluxes, and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the divergence operator. If the fluxes $\boldsymbol{\Phi}$ vanish on the

† Email address for correspondence: cheviakov@math.usask.ca

boundary of the fluid domain $\mathscr{D}$ or at infinity, as well as in the periodic case, Gauss's theorem provides a global conserved quantity given by

$$\mathscr{J} = \iiint\limits_{\mathscr{D}} \Theta \mathrm{d}V, \quad \frac{\mathrm{d}\mathscr{J}}{\mathrm{d}t} = 0. \tag{1.2}$$

(For solutions periodic in one or more dimensions, the domain $\mathscr{D}$ can be restricted to one such period.) Moreover, knowledge of local conservation laws (1.1) admitted by a given model is important from the point of view of numerical simulation, since modern finite-element methods, such as discontinuous Galerkin methods, require the PDEs in divergence form. Conservation laws are also useful in PDE analysis, including existence, uniqueness and global solution behaviour of nonlinear PDEs (e.g. Lax 1968; Benjamin 1972; Holm *et al.* 1985; Anco, Bluman & Wolf 2008). An important feature of local conservation laws is their coordinate invariance; in particular, a zero-divergence expression (1.1) is mapped into a zero-divergence expression under non-degenerate coordinate transformations.

A related important concept in fluid dynamics is the notion of material conservation laws, sometimes called Lagrange invariants, given by vanishing material derivatives,

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} \equiv \frac{\partial \Theta}{\partial t} + \boldsymbol{u} \boldsymbol{\cdot} \nabla \Theta = 0, \tag{1.3}$$

where $\boldsymbol{u}$ is the flow velocity vector. The material conservation law (1.3) expresses the conservation of the total amount of the quantity Θ initially contained in a moving fluid parcel. For an incompressible flow with the mass conservation constraint $\nabla \boldsymbol{\cdot} \boldsymbol{u} = 0$, each material conservation law (1.3) is equivalent to a local conservation law (1.1) with $\boldsymbol{\Phi} = \Theta \boldsymbol{u}$. Material conservation laws are well known and widely used in the literature (e.g. Moiseev *et al.* 1982; Bowman 2009; Kelbin, Cheviakov & Oberlack 2013).

The current work is devoted to the study of conservation laws of the equations of incompressible constant-density fluid dynamics, in both the constant-viscosity and the inviscid settings. With this, the Navier–Stokes equations in three dimensions, in the absence of external forces, are given by

$$\nabla \boldsymbol{\cdot} \boldsymbol{u} = 0, \quad \boldsymbol{u}_t + (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{u} + \nabla p - \nu \nabla^2 \boldsymbol{u} = 0, \tag{1.4a,b}$$

where the fluid velocity vector $\boldsymbol{u} = u^1 \boldsymbol{e}^x + u^2 \boldsymbol{e}^y + u^3 \boldsymbol{e}^z$ and the fluid pressure p are functions of x, y, z, t. (Where appropriate, throughout the paper, we use subscripts to denote partial derivatives.) Further, the inviscid case $\nu = 0$ yields the Euler equations

$$\nabla \boldsymbol{\cdot} \boldsymbol{u} = 0, \quad \boldsymbol{u}_t + (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{u} + \nabla p = 0. \tag{1.5a,b}$$

We generally consider systems (1.4) and (1.5) in the fully three-dimensional time-dependent situation, with all three velocity components non-zero. A set of important conserved quantities involves the fluid vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. The vorticity dynamics equation obtained by taking the curl of the momentum equation in (1.4) is given by

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) - \nu \nabla^2 \boldsymbol{\omega} = 0, \tag{1.6}$$

with $\nu = 0$ corresponding to the inviscid case.

The well-known conservation laws of the Euler system (1.5) in three dimensions, with no forcing, include the local conservation of kinetic energy, linear and angular momenta, generalized momenta in primitive variables, and the local conservation of vorticity and helicity in the vorticity formulation (e.g. Moffatt 1969; Batchelor 2000; Kelbin *et al.* 2013). The conservation of mass is obviously satisfied, since the density is constant; the divergence-free condition $\nabla \cdot \boldsymbol{u} = 0$ is commonly referred to as the mass conservation equation for constant-density incompressible flows.

For the Navier–Stokes equations, the kinetic energy and helicity are not locally conserved; linear and angular momenta are still locally conserved in the sense of satisfying a local divergence-type conservation law (1.1).

In the case of physical settings involving flow symmetry that leads to reduction in the number of physical dimensions, such as translational, axial, or helical symmetry, respectively, resulting in plane, axisymmetric and helically symmetric flows, additional conservation laws are known to arise, including the famous vorticity-related conservation laws for plane flows (e.g. Batchelor 2000; Bowman 2009). A rather complete set of conservation laws for the above three settings including many new conservation laws for all the cases was published in Kelbin *et al.* (2013).

A classical result involving material conservation laws is Ertel's theorem for the Euler equations (1.5) (Truesdell 1954), establishing the existence of a set of vorticity-related material conservation laws based on one special material conservation law (see also Salmon 1982). The connections between Ertel's theorem with the relabelling symmetry for Euler flows and Noether's second theorem are discussed in Newcomb (1967) and Padhye & Morrison (1996*a*,*b*).

A natural question that can be posed is whether or not it is possible to describe the full set of conservation laws for a certain class of fluid models. No conclusive answer is available in the literature; many textbooks only contain subsets of the above-mentioned basic conservation laws. The main direction of the current contribution is the systematic construction of conservation laws for constant-density incompressible flows in primitive variables and the vorticity formulation, using the direct construction method (Anco & Bluman 1997).

Suppose that a mathematical model involves N PDEs denoted

$$R^\sigma = 0, \quad \sigma = 1, \ldots, N, \tag{1.7}$$

with m dependent variables $\boldsymbol{v}(\boldsymbol{x})$ and n independent variables $\boldsymbol{x}$, one of which can be time. The direct construction method seeks linear combinations of the given equations with a set of multipliers $\Lambda_\sigma[\boldsymbol{v}]$, $\sigma = 1, \ldots, N$, that yield divergence expressions

$$\Lambda_\sigma R^\sigma \equiv \frac{\partial \Phi^i}{\partial x^i}, \tag{1.8}$$

for some functions Φ^i, $i = 1, \ldots, n$ (summation over repeated indices is assumed throughout the paper). The multipliers $\Lambda_\sigma[\boldsymbol{v}] \equiv \Lambda_\sigma(\boldsymbol{x}, \boldsymbol{v}, \ldots)$ may be chosen to depend on independent variables $\boldsymbol{x}$, dependent variables $\boldsymbol{v}$ and partial derivatives of $\boldsymbol{v}$ up to a certain order. It follows that, on solutions of the given equation (1.7), one has a local divergence expression

$$\frac{\partial \Phi^i}{\partial x^i} = 0, \tag{1.9}$$

equivalent to the local conservation law (1.1) when one of the variables is time. Sets of multipliers $\{\Lambda_\sigma[\boldsymbol{v}]\}$ yielding divergence expressions arise as solutions of overdetermined systems of linear determining equations. The latter are obtained through the action of Euler differential operators $\mathscr{E}_{v^i}$ with respect to each scalar dependent variable v^i, $i = 1, \ldots, m$, applied to (1.8) (for details, see e.g. Bluman, Cheviakov & Anco 2010). The key idea behind this is that an Euler operator identically annihilates an expression if and only if it is a divergence expression. Hence applying $\mathscr{E}_{v^i}$ to (1.8) leads to an identically vanishing right-hand side, and the set of linear PDEs determining the multipliers $\{\Lambda_\sigma[\boldsymbol{v}]\}$ is given by

$$\mathscr{E}_{v^i}[\Lambda_\sigma R^\sigma] = 0, \quad i = 1, \ldots, m. \tag{1.10}$$

Importantly, the determining equations (1.10) have to be solved off of the solution space of (1.7), i.e. without assuming that the vector function $\boldsymbol{v}$ is the solution of the given system (1.7) (see Bluman *et al.* 2010).

Unlike Noether's theorem, the direct construction method is applicable to a wide class of models, including dissipative ones. For the majority of physical systems, in particular, for equations that can be solved for some set of derivatives, the direct construction method is complete: all local divergence-type conservation laws (1.1) arise through linear combinations of the given equations with corresponding sets of multipliers $\{\Lambda_\sigma[\boldsymbol{v}]\}$. Since all fluid dynamics models considered in the current paper are in the solved form, the direct construction method is a natural method of choice to seek conservation laws of such systems.

In practice, the direct construction method requires an *a priori* specification of the multiplier dependence. If multipliers depend on derivatives up to an order $\ell \geqslant 0$ (ℓth-order multipliers), taking higher ℓ can generally yield additional (higher-order) conservation laws. However, the size of the system of multiplier determining equations grows drastically with ℓ, and can easily reach the numbers of hundreds, thousands and more determining equations. Specialized symbolic software can be used in such situations (e.g. Cheviakov 2007). It is important to note that, for equations of fluid dynamics, no upper bound for the differential order of the conservation law has been established to date. Partial results establishing the highest possible conservation law order have been derived for certain types of scalar equations; for others, like the Korteweg–de Vries, there exist infinite hierarchies of conservation laws of increasing order (for details and examples, see e.g. Bluman *et al.* (2010), and references therein). For a class of vorticity-related conservation laws considered in this paper, the conserved densities can contain derivatives of flow variables up to an arbitrary order.

In seeking local conservation laws of normal PDE systems, trivial conservation laws are avoided since they carry no physical meaning. Two types of trivial conservation laws are distinguished (Bluman *et al.* 2010). A conservation law (1.1) is a trivial conservation law of the first type when its density and fluxes vanish on solutions. Trivial conservation laws of the second type are expressions (1.1) that vanish identically, e.g. $\operatorname{div}(\operatorname{curl}(\cdot)) \equiv 0$. The direct construction method with non-trivial multipliers, applied to a PDE system in a solved form, which is possible for (1.4) and (1.5), yields its non-trivial conservation laws up to a prescribed differential order (Anco & Bluman 1997; Bluman *et al.* 2010). For abnormal PDE systems, the situation with trivial conservation laws is less transparent, as discussed in appendix A.

It is straightforward to apply the direct construction method to compute all local conservation laws of the Navier–Stokes and Euler equations (1.4) and (1.5) arising from, for example, zeroth-order multipliers. In § 2 of the current contribution, we find a complete set of conservation laws admitted by these two models, in the full time-dependent three-dimensional setting, involving up to second-order multipliers, i.e. multipliers that contain only derivatives of the dependent variables up to the second order. This ansatz includes all first-order and some second-order conservation law densities and fluxes. The resulting systems of multiplier determining equations consist of 31 918 and 58 273 equations, respectively, for the Euler and the Navier–Stokes model, presenting a significant computational challenge. The determining equations are reduced and solved using a combination of iterative and Gröbner basis techniques. Full sets of conservation laws are derived and listed. It is shown that no second-order conservation laws arise for either of these models; the analysis, though negative, establishes an important completeness result.

Finally, in § 3, we analyse local conservation laws of the full three-dimensional vorticity system (1.6), in both viscous and inviscid settings. It is shown that, in both settings, the vorticity dynamics equation can be written in a generic form, which admits an infinite family of vorticity-dependent conservation laws involving arbitrary differentiable functions of space, time, flow parameters, and spatial and temporal derivatives of the latter. Any physical field pertaining to the fluid flow can be used in the place of the arbitrary function. Moreover, operations that preserve the generic vorticity system structure can be used to generate further hierarchies of vorticity-related conservation laws of increasing differential orders. Specific forms of the infinite conservation law family for axially and helically symmetric flows in cylindrical and helical coordinates are presented.

Importantly, the new infinite families of conservation laws presented in the current contribution hold for both Navier–Stokes and Euler equations. These conservation laws are physically non-trivial, and are not equivalent to any conservation law families that have previously occurred in the literature. In particular, they are not equivalent to the Cauchy invariants of the Euler equations (see e.g. Salmon 1988; Kuznetsov 2008), and do not present a re-definition of the Lagrangian coordinates (initial positions of particles). Unlike the family of Cauchy invariants, the new vorticity-dependent conserved quantities hold for both the viscous and the inviscid case, and involve an arbitrary differentiable function that does not have to be a conserved quantity itself. Unlike Lagrangian coordinates and other Lagrange invariants, the new conservation laws are not material conservation laws of the form (1.3).

The presented infinite families of vorticity conservation laws involve the time derivative of the arbitrary function appearing as a part of the flux vector. If the arbitrary function is chosen to represent a flow parameter satisfying an evolution equation (not necessarily a conservation law), the time derivative in the spatial flux can be replaced accordingly, leading to conservation law forms common in fluid dynamics, where a time derivative acts on the conserved density, and the spatial fluxes only contain spatial derivatives. Examples are considered, including kinetic-energy-dependent vorticity conservation laws.

The paper is concluded with a discussion of results and open problems in § 4.

2. Direct construction of conservation laws for inviscid and viscous incompressible flows using second-order multipliers

We now apply the direct construction method to equations of fluid dynamics in primitive variables.

2.1. *Conservation laws of the Euler equations*

In order to seek the conservation laws, the four scalar equations (1.5) are multiplied by multipliers Λ_σ, $\sigma = 1, 2, 3, 4$, and a linear combination is formed. The Euler equations (1.5) can be solved for the first derivatives,

$$u^1_x, \quad u^i_t, \quad i = 1, 2, 3, \tag{2.1}$$

while the second derivatives,

$$u^1_{xx}, u^1_{xy}, u^1_{xz}, \quad u^i_{tt}, u^i_{tx}, u^i_{ty}, u^i_{tz}, \quad i = 1, 2, 3, \tag{2.2}$$

are subsequently computed from the differential consequences of the Euler equations. To avoid trivial multipliers vanishing on solutions of equations, the dependence of the multipliers on the variables (2.1) and (2.2) must be avoided. Finally, each of the four unknown second-order multipliers in Λ_σ is a function of 45 variables, given by

$$\left.\begin{array}{c} t,x,y,z, \quad u^1,u^2,u^3,p, \quad u^1_y,u^1_z, \quad u^2_x,u^2_y,u^2_z, \quad u^3_x,u^3_y,u^3_z, \quad p_t,p_x,p_y,p_z, \\ u^1_{yy},u^1_{yz},u^1_{zz}, \quad u^2_{xx},u^2_{xy},u^2_{xz},u^2_{yy},u^2_{yz},u^2_{zz}, \quad u^3_{xx},u^3_{xy},u^3_{xz},u^3_{yy},u^3_{yz},u^3_{zz}, \\ p_{tt},p_{tx},p_{ty},p_{tz},p_{xx},p_{xy},p_{xz},p_{yy},p_{yz},p_{zz}. \end{array}\right\} \tag{2.3}$$

The determining equations are obtained by applying four Euler operators, corresponding to the dependent variables u^1, u^2, u^3, p of the system, to the linear combination of the four equations (1.5) with multipliers $\{\Lambda_\sigma\}$, and splitting of the resulting equations. (For the details of the application of the direct construction method, the interested reader is referred to Bluman *et al.* (2010).) As a result, one obtains a system of 31 918 linear homogeneous PDEs for the unknown multipliers.

The determining equation system is essentially overdetermined, i.e. it contains a large number of redundant equations. It was first simplified iteratively by removing multiples of existing equations and substituting simpler equations of the form 'monomial = 0' into more complicated equations. As a result, the number of equations was reduced by a factor of two. A powerful `Maple` function `rifsimp` based on a differential Gröbner basis reduction method was subsequently used to reduce the number of determining equations to 189, the majority being in the form 'partial derivative = 0'. The `rifsimp` operation took approximately 24 h on a workstation with a 3.6 GHz Xeon processor and 128 Gb RAM, running `Maple` 17.

The resulting determining equations were solved directly to yield the following multipliers and local conservation laws. The list below is an exhaustive list of conservation law multipliers admitted by the Euler equations, when the multiplier dependence can involve derivatives of at most second order. In particular, it follows that no conservation laws with second-order multipliers are admitted for the incompressible Euler system (1.5).

2.1.1. *Conservation of generalized momentum*

The local conservation of the generalized linear momentum in the x direction is given by a conservation law

$$\frac{\partial}{\partial t}(f(t)u^1)+\frac{\partial}{\partial x}((u^1f(t)-xf'(t))u^1+f(t)p) \\ +\frac{\partial}{\partial y}((u^1f(t)-xf'(t))u^2)+\frac{\partial}{\partial z}((u^1f(t)-xf'(t))u^3)=0, \tag{2.4}$$

with two analogous expressions holding for the projections on the y and z directions. In (2.4), $f(t)$ is an arbitrary differentiable function of time. Multipliers corresponding to (2.4) are given by $\Lambda_1=f(t)u^1-xf'(t)$, $\Lambda_2=f(t)$ and $\Lambda_3=\Lambda_4=0$.

2.1.2. *Conservation of angular momentum*

The angular momentum vector is locally conserved for Euler flows. The x projection of the conservation law is given by the divergence expression

$$\frac{\partial}{\partial t}(zu^2-yu^3)+\frac{\partial}{\partial x}((zu^2-yu^3)u^1) \\ +\frac{\partial}{\partial y}((zu^2-yu^3)u^2+zp)+\frac{\partial}{\partial z}((zu^2-yu^3)u^3-yp)=0. \tag{2.5}$$

The corresponding multipliers are given by $\Lambda_1 = u^2_z - u^3_y$, $\Lambda_2 = 0$, $\Lambda_3 = z$ and $\Lambda_4 = -y$. Two additional scalar conservation laws for the y and z projections of the angular momentum are given by similar expressions, obtained by simultaneous cyclic permutations of the variables (x, y, z) and indices $(1, 2, 3)$.

2.1.3. *Conservation of kinetic energy*

This scalar conservation law is given by

$$\frac{\partial}{\partial t}K + \nabla \cdot ((K + p)\,\boldsymbol{u}) = 0, \tag{2.6}$$

where $K = |\boldsymbol{u}|^2/2$ is the kinetic energy density; it arises from the multipliers $\Lambda_1 = K + p$ and $\Lambda_{i+1} = u^i$, $i = 1, 2, 3$.

2.1.4. *Conservation of helicity*

The helicity density is given by $h = \boldsymbol{u} \cdot \boldsymbol{\omega}$. The local conservation law has the form

$$\frac{\partial}{\partial t}h + \nabla \cdot (\boldsymbol{u} \times \nabla E + (\boldsymbol{\omega} \times \boldsymbol{u}) \times \boldsymbol{u}) = 0, \tag{2.7}$$

where $E = K + p$ is the total energy density. This conservation law is the only one that involves first-order multipliers, namely, $\Lambda_1 = 0$ and $\Lambda_{i+1} = \omega^i$, $i = 1, 2, 3$.

2.1.5. *Generalized continuity equation*

The time-independent vanishing divergence

$$\nabla \cdot (k(t)\,\boldsymbol{u}) = 0, \tag{2.8}$$

holding for an arbitrary differentiable function of time $k(t)$, is an obvious consequence of the continuity equation. It corresponds to conservation law multipliers $\Lambda_1 = k(t)$ and $\Lambda_2 = \Lambda_3 = \Lambda_4 = 0$.

In total, for the incompressible Euler equations, there exist nine sets of second-order multipliers yielding nine scalar conservation laws, four of them being families involving arbitrary functions of time.

2.2. *Conservation laws of the Navier–Stokes equations*

Now consider the incompressible Navier–Stokes system (1.4). Since it is dissipative, it is expected to have fewer conservation laws than the Euler equations, which indeed turns out to be the case.

The Navier–Stokes equations can be solved, for example, for the derivatives

$$u^2_y, \quad u^i_{xx}, \quad i = 1, 2, 3, \tag{2.9}$$

and the second-order differential consequences u^2_{ty}, u^2_{xy}, u^2_{yy}, u^1_{yz}. Excluding them, one finds that the four second-order multipliers Λ_σ for the Navier–Stokes equations are functions of 56 variables, given by

$$\left.\begin{array}{c} t, x, y, z, \quad u^1, u^2, u^3, p, \quad u^1_t, u^1_x, u^1_y, u^1_z, \quad u^2_t, u^2_x, u^2_z, \quad u^3_t, u^3_x, u^3_y, u^3_z, \quad p_t, p_x, p_y, p_z, \\ u^1_{tt}, u^1_{tx}, u^1_{ty}, u^1_{tz}, u^1_{xy}, u^1_{xz}, u^1_{yy}, u^1_{yz}, u^1_{zz}, \quad u^2_{tt}, u^2_{tx}, u^2_{tz}, u^2_{xz}, u^2_{zz}, \\ u^3_{tt}, u^3_{tx}, u^3_{ty}, u^3_{tz}, u^3_{xy}, u^3_{xz}, u^3_{yy}, u^3_{yz}, u^3_{zz}, \quad p_{tt}, p_{tx}, p_{ty}, p_{tz}, p_{xx}, p_{xy}, p_{xz}, p_{yy}, p_{yz}, p_{zz}. \end{array}\right\} \tag{2.10}$$

Application of the four Euler operators and splitting yields a linear homogeneous overdetermined system of 58 273 PDEs on the unknown Λ_σ. Using the above-described iterative reduction procedure, the system was reduced to 9644 linear PDEs, which were subsequently reduced to 227 simple equations through `rifsimp`. The latter operation took approximately 24 h of computation time.

Similarly to the Euler case, it follows that no conservation laws with second-order multipliers are admitted by the incompressible Navier–Stokes system (1.4). The admissible conservation laws are listed below.

2.2.1. *Conservation of generalized momentum*

The local conservation of the generalized linear momentum in the x direction is given by the conservation law

$$\begin{aligned}&\frac{\partial}{\partial t}(f(t)u^1)+\frac{\partial}{\partial x}((u^1 f(t)-xf'(t))u^1+f(t)(p-\nu u^1_x))\\&\quad+\frac{\partial}{\partial y}((u^1 f(t)-xf'(t))u^2-\nu f(t)u^1_y)+\frac{\partial}{\partial z}((u^1 f(t)-xf'(t))u^3-\nu f(t)u^1_z)=0,\end{aligned} \tag{2.11}$$

with two analogous expressions holding for the projections on the y and z directions, obtained by cyclically permuting (x, y, z).

2.2.2. *Conservation of angular momentum*

The local conservation of the angular momentum is expressed by three scalar conservation laws. The x projection is given by the divergence expression

$$\begin{aligned}&\frac{\partial}{\partial t}(zu^2-yu^3)+\frac{\partial}{\partial x}((zu^2-yu^3)u^1+\nu(yu^3_x-zu^2_x))\\&\quad+\frac{\partial}{\partial y}((zu^2-yu^3)u^2+zp+\nu(yu^3_y-zu^2_y-u^3))\\&\quad+\frac{\partial}{\partial z}((zu^2-yu^3)u^3-yp+\nu(yu^3_z-zu^2_z+u^2))=0.\end{aligned} \tag{2.12}$$

The y and z projections are given by expressions obtained by simultaneous cyclic permutations of the variables (x, y, z) and indices $(1, 2, 3)$ in (2.12).

The multipliers corresponding to the sets of conservation laws given by (2.11) and (2.12) are the same as for the Euler system. Neither the kinetic energy nor helicity conservation holds for viscous flow. The generalized continuity equation (2.8) carries over without change.

3. An infinite family of vorticity conservation laws for viscous and inviscid flows

3.1. *The vorticity system and trivial conservation laws*

We now consider the systematic construction of local conservation laws of fluid dynamics equations in the vorticity formulation. Since the Laplacian ∇^2 and the curl operator commute, the vorticity equations can be written as

$$\nabla\cdot\boldsymbol{\omega}=0,\quad \boldsymbol{\omega}_t+\nabla\times(\boldsymbol{\omega}\times\boldsymbol{u}-\nu\nabla^2\boldsymbol{u})=0, \tag{3.1a,b}$$

or, letting $\boldsymbol{\beta}=\boldsymbol{\omega}\times\boldsymbol{u}-\nu\nabla^2\boldsymbol{u}$, in the form

$$\nabla\cdot\boldsymbol{\omega}=0,\quad \boldsymbol{\omega}_t+\nabla\times\boldsymbol{\beta}=0. \tag{3.2a,b}$$

Further, (3.1) may be rewritten in the usual local conservation form

$$\boldsymbol{\omega}_t + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{\omega} - \boldsymbol{\omega} \otimes \boldsymbol{u} - \nu \nabla \boldsymbol{\omega}) = 0, \tag{3.3}$$

known as the Helmholtz vorticity transport equation, with $\otimes$ denoting the dyadic product.

From the point of view of local conservation law construction, the vorticity system (3.1) is unusual in two important ways. Firstly, being the differential consequences of the Navier–Stokes equations, the vorticity dynamics equations (3.1) are themselves, by definition, trivial conservation laws of the first type on solutions of Navier–Stokes equations (see appendix A). Hence neither the vorticity equation itself, nor its local conservation laws arise from applying the direct construction method to the Navier–Stokes equations in primitive variables. It follows that, in order to seek additional vorticity conservation laws, one must apply the direct construction method to the vorticity system (3.1) itself. Secondly, the fact that the vorticity system is abnormal must be taken into account (for details, see appendix A).

3.2. *An infinite family of vorticity conservation laws*

With zeroth-order multipliers $\Lambda_i = \Lambda_i(t, x, y, z)$, the application of the direct construction method to the vorticity system (3.2) yields an infinite family of admissible multipliers,

$$\Lambda_1 = -F_t, \quad \Lambda_2 = F_x, \quad \Lambda_2 = F_y, \quad \Lambda_2 = F_z, \tag{3.4a–d}$$

where $F = F(t, x, y, z)$ is an arbitrary sufficiently smooth function. Moreover, using higher-order (first-, second-, etc.) conservation law multipliers, one obtains higher-order admissible multipliers

$$\Lambda_1 = -D_t F, \quad \Lambda_2 = D_x F, \quad \Lambda_2 = D_y F, \quad \Lambda_2 = D_z F, \tag{3.5a–d}$$

where

$$D_i = \frac{\partial}{\partial x_i} + q_i^j \frac{\partial}{\partial q^j} + q_{ik}^j \frac{\partial}{\partial q_k^j} + \cdots \tag{3.6}$$

denote total derivatives (the chain rule), x_i ($i = 1, 2, 3, 4$) denote all independent variables (x, y, z, t), q^j ($j = 1, \ldots, 7$) denote all flow parameters ($u^1, u^2, u^3, p, \omega^1, \omega^2, \omega^3$), $q_i^j \equiv \partial q^j / \partial x_i$ denote first derivatives, etc. The arbitrary function

$$F = F(t, x, y, z, \boldsymbol{u}, p, \boldsymbol{\omega}, \ldots) \tag{3.7}$$

in (3.5) may depend on any combination of t, x, y, z, any flow parameters, and their derivatives. The multipliers (3.5) yield the local conservation laws of equations (3.2) given by

$$(\boldsymbol{\omega} \cdot \nabla F)_t + \nabla \cdot (\boldsymbol{\beta} \times \nabla F - F_t \boldsymbol{\omega}) = 0. \tag{3.8}$$

In the notation of the vorticity equations (3.1), one has an infinite family of conservation laws

$$(\boldsymbol{\omega} \cdot \nabla F)_t + \nabla \cdot ([\boldsymbol{\omega} \times \boldsymbol{u} - \nu \nabla^2 \boldsymbol{u}] \times \nabla F - F_t \boldsymbol{\omega}) = 0, \tag{3.9}$$

with the local conserved density

$$\mathcal{Q}_F = \boldsymbol{\omega} \cdot \nabla F \tag{3.10}$$

involving vorticity and an arbitrary combination of variables and flow parameters incorporated within the free function (3.7).

We note that the conservation law family (3.9) does not include helicity conservation, since, in general, the velocity vector $\boldsymbol{u}$ may not be written as a gradient of a scalar. Further, it is important to mention that, in principle, the family (3.9) is not limited to scalar conservation laws but may represent local conservation of vector- or tensor-valued flow parameters.

Conservation laws involving arbitrary functions of all variables are known to exist for abnormal PDE systems that follow from a variational principle (Noether's second theorem, e.g. Olver 2000). The above computations demonstrate that the same holds for the Euler and Navier–Stokes vorticity equations, even though the system (3.1) is not variational.

3.3. *Preliminary discussion*

The infinite set of conservation laws (3.9) does not overlap with any basic vorticity-related conservation laws holding for inviscid or viscous flows, such as helicity or enstrophy; in other words, independent functions F generate physically independent conservation laws. For an arbitrary F, the conservation laws (3.9) are essentially non-material. Geometrically, the conservation laws (3.9) describe the local rate of change (and the global conservation, under appropriate boundary conditions) of the amount of local alignment between the flow vorticity and the gradient of the flow parameter F. For instance, in the simplest case of a linear time-independent $F = \alpha x + \beta y + \gamma z$, with $\alpha, \beta, \gamma = \text{const.}$, the conservation law (3.9) reads

$$\frac{\partial}{\partial t}(\alpha\omega^1 + \beta\omega^2 + \gamma\omega^3) + \nabla\cdot([\boldsymbol{\omega}\times\boldsymbol{u} - \nu\,\nabla^2\boldsymbol{u}]\times[\alpha, \beta, \gamma]) = 0, \tag{3.11}$$

expressing the local conservation of an arbitrary linear combination of vorticity components.

Though unusual in mechanics, the time derivative F_t in the spatial fluxes of (3.9) does not present an irregularity. Let F be a mechanical or thermodynamic parameter of the model (either a scalar parameter, or a component of a vector- or tensor-valued function), satisfying an evolutionary differential equation

$$F_t = \mathscr{M}(t, x, y, z, \boldsymbol{u}, p, \boldsymbol{\omega}, F, \ldots) \tag{3.12}$$

for some right-hand side $\mathscr{M}$, which may involve spatial derivatives. Then (3.12) can be substituted into the conservation law (3.9) to yield a canonical conservation expression

$$\frac{\partial}{\partial t}\mathscr{Q}_F + \nabla\cdot([\boldsymbol{\omega}\times\boldsymbol{u} - \nu\,\nabla^2\boldsymbol{u}]\times\nabla F - \boldsymbol{\omega}\,\mathscr{M}) = 0, \tag{3.13}$$

providing the rate of change of the amount of local alignment between the flow vorticity and the gradient of the flow parameter F.

As initial candidates for the flow parameters F, one may consider, for instance, functions that obey transport equations common in fluid dynamics, heat transfer and thermodynamics, i.e.

$$F_t + \nabla\cdot(F\,\boldsymbol{u}) = \varGamma_F. \tag{3.14}$$

Here $\varGamma_F$ may comprise the sum of various physical processes modelled as diffusive or sink/source terms, i.e. $\varGamma_F = \nabla\cdot(\lambda\nabla F) + H(F)$. Employing this in (3.9) and rearranging terms, we obtain

$$(\boldsymbol{\omega}\cdot\nabla F)_t + \nabla\cdot((\boldsymbol{\omega}\cdot\nabla F)\boldsymbol{u}) + \nabla\cdot(\nu\,(\nabla\times\boldsymbol{\omega})\times\nabla F - \varGamma_F\,\boldsymbol{\omega}) = 0, \tag{3.15}$$

where the viscous term has been rewritten in vorticity notation. If we limit (3.14) to a material conservation law ($\Gamma_F = 0$) and neglect viscosity, (3.15) collapses to the generalized Euler–Ertel theorem (3.19) below, where $\boldsymbol{\omega} \cdot \nabla F$ is materially conserved.

This property may still be retained in the limit of vanishing viscosity and if Γ_F is in sink/source term form. In this case, $\Gamma_F = H(F)$, and the quantity

$$C = \int \frac{\mathrm{d}F}{\Gamma(F)} - t$$

satisfies a material conservation law $C_t + \boldsymbol{u} \cdot \nabla C = 0$. One consequently obtains a materially conserved quantity $\boldsymbol{\omega} \cdot \nabla C$.

Further, the flow parameter F may be identified by any of the densities listed in § 2. For example, one may choose the fluid kinetic energy density $F = K = |\boldsymbol{u}|^2/2$. The dynamic equation for K is obtained by multiplying the momentum equation by $\boldsymbol{u}$, and reads

$$K_t = -\nabla \cdot ((K + p)\,\boldsymbol{u}) + \nu\, \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u}, \tag{3.16}$$

with the dissipation term vanishing for inviscid flows, as per (2.6). Using (3.16) in (3.9) or alternatively in (3.15), one obtains a conservation law

$$\frac{\partial}{\partial t}\mathcal{Q}_K + \nabla \cdot \boldsymbol{\Phi}_K = 0, \tag{3.17a}$$

$$\boldsymbol{\Phi}_K = (\boldsymbol{\omega} \cdot \nabla K)\boldsymbol{u} + \nu\,(\nabla \times \boldsymbol{\omega}) \times \nabla K + \nu\, \boldsymbol{\omega}\,(\nabla \times \boldsymbol{\omega}) \cdot \boldsymbol{u} + \boldsymbol{\omega}\, \nabla \cdot (p\,\boldsymbol{u}), \tag{3.17b}$$

reflecting the local conservation of $\mathcal{Q}_K = \boldsymbol{\omega} \cdot \nabla K$, the alignment between the flow vorticity and the kinetic energy gradient. We further observe that $\mathcal{Q}_K$ is a pseudo-scalar, which changes sign under reflections. This very specific property is shared with helicity h in (2.7), though h is only conserved for ideal fluids. Further implications of this are discussed below. We may note that other dynamic flow quantities of arbitrary tensor order can be used in the same manner.

3.4. *The Euler–Ertel theorem, its generalization, the Cauchy invariants and the potential vorticity*

The generalized Euler–Ertel conservation theorem pertaining to vorticity-related conserved quantities of the inviscid Euler model (1.5) can be stated as follows (Truesdell 1954).

THEOREM 1. *Let Θ be a material conserved quantity satisfying (1.3) and an additional condition*

$$(\nabla \Theta) \cdot (\nabla \times \boldsymbol{a}) = 0, \quad \boldsymbol{a} \equiv \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}. \tag{3.18}$$

Then $\boldsymbol{\omega} \cdot \nabla \Theta$ is also a material conserved quantity:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \cdot \nabla \Theta) = 0. \tag{3.19}$$

The above statement is a consequence of the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \cdot \nabla \Theta) = \boldsymbol{\omega} \cdot \nabla \left(\frac{\mathrm{d}\Theta}{\mathrm{d}t}\right) + (\nabla \times \boldsymbol{a}) \cdot \nabla \Theta, \tag{3.20}$$

where the right-hand side vanishes when $\mathrm{d}\Theta/\mathrm{d}t = 0$ and (3.18) is satisfied.

For inviscid flows only, a similar result holds with fewer restrictions on Θ.

THEOREM 2. *Let Θ be a material conserved quantity satisfying (1.3) for an Euler flow (1.5). Then $\boldsymbol{\omega}\cdot\nabla\Theta$ is also a material conserved quantity satisfying (3.19).*

The result follows since the given material conservation law (1.3) and the vorticity equation (1.6) with $\nu=0$ yield the commutator relation

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}, \boldsymbol{\omega}\cdot\right]=0. \tag{3.21}$$

In this case, quantities $(\boldsymbol{\omega}\cdot\nabla)^n\Theta$, $n=1, 2, 3, \ldots$, are referred to as the Cauchy invariants. As remarked in Kuznetsov (2008), these invariants characterize the 'frozenness' of the vorticity into the fluid, which is only the case for inviscid flows.

A different result, which goes far beyond the Euler–Ertel formula, but which is still sometimes referred to as 'Ertel's theorem', states that, if Θ is any flow parameter, not necessarily a material conserved quantity, then $\boldsymbol{\omega}\cdot\nabla\Theta$ is still locally conserved as a density of a conservation law (1.1) (Haynes & McIntyre 1990; see also Müller 1995). In the atmospheric sciences community, the set of conserved quantities $\boldsymbol{\omega}\cdot\nabla\Theta$ is referred to as the 'potential vorticity'. The local conservation law form for the 'potential vorticity' is equivalent to the formula (3.9) derived using a direct method, with $\Theta\equiv F$. The potential vorticity was used to study quasi-conservation laws for compressible Navier–Stokes equations in Gibbon & Holm (2012*a*), with the mass density ρ used in the place of the arbitrary function F. From the physical point of view, it is interesting to note that, in the latter paper, it was observed that, in the resulting conservation law, the pressure and other thermodynamic terms cancel completely, without any further simplification, such as the barotropic approximation. The dynamics of the gradient of the potential vorticity was considered in Gibbon & Holm (2010, 2012*b*); possible applications to atmospheric analysed data at the tropopause were discussed.

It is important to note that, unlike 'Ertel's theorem' known to hold for the vorticity equations (3.1), the conservation law family (3.9) has been derived for a more general PDE system of the form (3.2). This fact is used in § 3.6 below to generate additional sets of vorticity-related conservation laws, different from the 'potential vorticity' (3.10).

The new locally conserved quantities (3.9) hold for both viscous and inviscid flows, and do not represent material conservation laws (Lagrange invariants) (1.3). They are therefore clearly different from the Cauchy invariants, or the Lagrangian coordinates of fluid, or any other geometric quantities known in the literature.

3.5. *An infinite family of vorticity conservation laws for axially and helically symmetric flows*

We now specify the forms of the conservation laws (3.9) for axially and helically symmetric flows, which are of special interest due to their wide appearance in applications (cf. Kelbin *et al.* 2013).

In cylindrical coordinates (r, φ, z), the velocity and vorticity vectors are given by

$$\boldsymbol{u}=u^r\boldsymbol{e}^r+u^\phi\boldsymbol{e}^\phi+u^3\boldsymbol{e}^z, \quad \boldsymbol{\omega}=\omega^r\boldsymbol{e}^r+\omega^\phi\boldsymbol{e}^\phi+\omega^3\boldsymbol{e}^z, \tag{3.22}$$

where the vector components of an axially symmetric flow are functions of (r, z, t). A local axially symmetric conservation law (1.1) can be written as

$$\frac{\partial\Theta}{\partial t}+\nabla\cdot\boldsymbol{\Phi}\equiv\frac{\partial\Theta}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r)+\frac{\partial\Phi^3}{\partial z}=0. \tag{3.23}$$

The density and the fluxes of the infinite family of conservation laws (3.9) involving an arbitrary differentiable function $F=F(t, r, z)$ are given by

$$\Theta = F_r\,\omega^r + F_z\omega^3, \tag{3.24a}$$

$$\Phi^r = F_z(u^r\omega^3 - u^3\omega^r) - F_t\,\omega^r - \nu F_z\left[\frac{1}{r}(ru^\phi)_r + (u^\phi)_{zz}\right], \tag{3.24b}$$

$$\Phi^\xi = -F_r(u^r\omega^3 - u^3\omega^r) - F_t\omega^3 + \nu F_r\left[\frac{1}{r}(ru^\phi)_r + (u^\phi)_{zz}\right]. \tag{3.24c}$$

The helical coordinates (r, η, ξ) are given by $\xi = az + b\varphi$, $\eta = a\varphi - bz/r^2$, in terms of cylindrical coordinates; the velocity and vorticity vectors can be written in components as

$$\boldsymbol{u} = u^r\boldsymbol{e}^r + u^\eta\boldsymbol{e}^{\perp\eta} + u^\xi\boldsymbol{e}^\xi, \quad \boldsymbol{\omega} = \omega^r\boldsymbol{e}^r + \omega^\eta\boldsymbol{e}^{\perp\eta} + \omega^\xi\boldsymbol{e}^\xi, \tag{3.25}$$

with the orthogonal basis vectors

$$\boldsymbol{e}^r = \frac{\boldsymbol{\nabla} r}{|\boldsymbol{\nabla} r|}, \quad \boldsymbol{e}^\xi = \frac{\boldsymbol{\nabla}\xi}{|\boldsymbol{\nabla}\xi|}, \quad \boldsymbol{e}^{\perp\eta} = \frac{\boldsymbol{\nabla}_\perp\eta}{|\boldsymbol{\nabla}_\perp\eta|} = \boldsymbol{e}^\xi\times\boldsymbol{e}^r. \tag{3.26}$$

The helically invariant setting corresponds to the η independence of all flow parameters (for details, see Kelbin *et al.* 2013). Local conservation laws (1.1) of helically invariant flows have the form

$$\frac{\partial\Theta}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} \equiv \frac{\partial\Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{1}{B}\frac{\partial\Phi^\xi}{\partial\xi} = 0, \tag{3.27}$$

where $B = r/\sqrt{a^2r^2+b^2}$. Let $F = F(t, r, \xi, \ldots)$ represent an arbitrary differentiable function of time, coordinates and flow parameters. Then the vorticity conservation laws (3.9) have the form (3.27), with density and fluxes given by

$$\Theta = F_r\,\omega^r + \frac{F_\xi}{B}\omega^\xi, \tag{3.28a}$$

$$\begin{aligned}\Phi^r = {} & \frac{F_\xi}{B}(u^r\omega^\xi - u^\xi\omega^r) - F_t\,\omega^r + \nu\left[-\frac{F_{\xi\xi}}{B^2}\omega^r - \frac{F_r}{r}(r\omega^r)_r - \frac{2abBF_\xi}{r^2}\omega^\eta\right.\\ & \left. + \left(\frac{a^2BF_\xi}{r} + \frac{F_{r\xi}}{B}\right)\omega^\xi - \frac{F_\xi}{rB}(r\omega^\xi)_r\right],\end{aligned} \tag{3.28b}$$

$$\begin{aligned}\Phi^\xi = {} & -F_r(u^r\omega^\xi - u^\xi\omega^r) - F_t\,\omega^\xi + \nu\left[\left(\frac{2a^2B}{r} - \frac{2}{rB}\right)F_\xi\omega^r + \frac{F_{r\xi}}{B}\omega^r - \frac{F_r}{B}(\omega^r)_\xi\right.\\ & \left. + \frac{2abB^2F_r}{r^2}\omega^\eta - \left(\frac{2a^2B^2F_r}{r} + r\left(\frac{F_r}{r}\right)_r\right)\omega^\xi - \frac{F_\xi}{B^2}(\omega^\xi)_\xi\right].\end{aligned} \tag{3.28c}$$

The conservation laws (3.28) have not been listed in Kelbin *et al.* (2013) since they have been formally identified with trivial conservation laws of the first type. However, physically, the conservation laws (3.28) are non-trivial. This discrepancy between mathematical and physical definitions of triviality is due to the fact that the vorticity system is an abnormal system (see appendix A).

3.6. *Generalizations of Ertel-type conservation laws. Infinite hierarchies of higher-order conserved quantities*

3.6.1. *Higher-order hierarchies of vorticity conservation laws. Type I: iterative*

For an arbitrary flow parameter F, the corresponding 'potential vorticity' density is given by $\mathcal{Q}_F = \boldsymbol{\omega} \cdot \nabla F$ (3.10). Assume that the function F and the flow are sufficiently smooth, and that F involves spatial derivatives of flow parameters up to the order $k \geqslant 0$. Then $\mathcal{Q}_F$ has the differential order $k+1$. To obtain a higher-order conservation law, one can substitute $\mathcal{Q}_F$ instead of F in (3.9). The corresponding conserved density given by

$$\mathcal{Q}_F^{(1)} = \mathcal{I}[\mathcal{Q}_F] = \boldsymbol{\omega} \cdot \nabla\, (\boldsymbol{\omega} \cdot \nabla F) \tag{3.29}$$

has a spatial differential order $k+2$. Sequential application of the recursion operator $\mathcal{I}[\cdot] = \boldsymbol{\omega} \cdot \nabla(\cdot)$ yields a countable family of local scalar conserved quantities,

$$\mathcal{Q}_F \equiv \mathcal{Q}_F^{(0)} = \mathcal{I}[F], \quad \mathcal{Q}_F^{(1)} = \mathcal{I}^2[F], \quad \ldots, \quad \mathcal{Q}_F^{(n+1)} = \mathcal{I}^n[F], \quad \ldots, \tag{3.30}$$

where the power of the operator denotes the number of its repeated applications. By construction, for a general flow, the conserved quantities (3.30) are linearly independent.

As an example, consider the kinetic energy density $F = K = |\boldsymbol{u}|^2/2$, and the corresponding conservation law (3.17). The sequence of iterated conserved densities of the corresponding higher-order conservation laws is given by

$$\mathcal{Q}_K^{(0)} = \boldsymbol{\omega} \cdot \nabla K, \quad \mathcal{Q}_K^{(1)} = \boldsymbol{\omega} \cdot \nabla\, (\boldsymbol{\omega} \cdot \nabla K), \quad \mathcal{Q}_K^{(2)} = \boldsymbol{\omega} \cdot \nabla\, (\boldsymbol{\omega} \cdot \nabla\, (\boldsymbol{\omega} \cdot \nabla K)), \ldots. \tag{3.31}$$

The respective local conservation laws and spatial fluxes have the form

$$\frac{\partial}{\partial t} \mathcal{Q}_K^{(j)} + \nabla \cdot \boldsymbol{\Phi}_K^{(j)} = 0, \tag{3.32}$$

$$\boldsymbol{\Phi}_K^{(j)} = [\boldsymbol{\omega} \times \boldsymbol{u} - \nu\, \nabla^2 \boldsymbol{u}] \times \nabla\, (\mathcal{I}^j[F]) - (\mathcal{I}^j[F])_t\, \boldsymbol{\omega}, \quad j = 0, 1, 2, \ldots. \tag{3.33}$$

3.6.2. *Higher-order hierarchies of vorticity conservation laws. Type II: repeated differentiations of vorticity equations*

It is essential that the conservation laws (3.8) hold for PDEs of the form (3.2), where $\boldsymbol{\omega}$ and $\boldsymbol{\beta}$ are arbitrary vector fields; in particular, $\boldsymbol{\omega}$ is not constrained to have the meaning of the fluid vorticity. It follows that any transformation that preserves the form of the equations (3.2), leading to another $(\boldsymbol{\omega}, \boldsymbol{\beta})$ pair, will yield a different conservation law family (3.8). In particular, any operation applied to (3.2) that commutes with spatial and temporal differentiation will preserve the form of these PDEs.

As specific examples, we consider the curl operation and temporal differentiations.

Again, we start with a smooth flow and a flow parameter F of choice. Using the form (3.2) of the vorticity equations, by taking a curl, one obtains the differential consequences,

$$\widetilde{\boldsymbol{\omega}}^{(1)} \equiv \nabla \times \boldsymbol{\omega}, \quad \widetilde{\boldsymbol{\beta}}^{(1)} \equiv \nabla \times \boldsymbol{\beta}, \tag{3.34}$$

$$\nabla \cdot \widetilde{\boldsymbol{\omega}}^{(1)} = 0, \quad \widetilde{\boldsymbol{\omega}}_t^{(1)} + \nabla \times \widetilde{\boldsymbol{\beta}}^{(1)} = 0. \tag{3.35}$$

Equation (3.35) is of the form (3.2). Hence for the flow parameter F, one has a new conserved density

$$\widetilde{\mathcal{Q}}_F^{(1)} = \widetilde{\boldsymbol{\omega}}^{(1)} \cdot \nabla F \equiv (\nabla \times \boldsymbol{\omega}) \cdot \nabla F, \tag{3.36}$$

satisfying an appropriate conservation law (3.8)

$$\frac{\partial}{\partial t}(\widetilde{\mathscr{Q}}_F^{(1)}) + \nabla\cdot(\widetilde{\boldsymbol{\beta}}^{(1)}\times\nabla F - F_t\,\widetilde{\boldsymbol{\omega}}^{(1)}) = 0. \tag{3.37}$$

Iterating the curl operation on (3.34) and (3.35), one obtains a sequence of conserved quantities

$$\widetilde{\mathscr{Q}}_F^{(j)} = \widetilde{\boldsymbol{\omega}}^{(j)}\cdot\nabla F, \quad j = 1, 2, \ldots, \tag{3.38}$$

involving vorticity derivatives of increasing spatial orders,

$$\widetilde{\boldsymbol{\omega}}^{(1)} = \nabla\times\boldsymbol{\omega}, \quad \widetilde{\boldsymbol{\omega}}^{(2)} = \nabla\times\nabla\times\boldsymbol{\omega}, \ldots. \tag{3.39}$$

It is evident that the form (3.2) of the vorticity equations is also invariant with respect to the temporal differentiations. Hence one may similarly introduce a sequence of temporal vorticity derivatives

$$\widehat{\boldsymbol{\omega}}^{(k)} = \frac{\partial^k}{\partial t^k}\boldsymbol{\omega}, \quad \widehat{\boldsymbol{\beta}}^{(k)} = \frac{\partial^k}{\partial t^k}\boldsymbol{\beta}, \quad k = 1, 2, \ldots, \tag{3.40}$$

satisfying the same PDEs (3.2),

$$\nabla\cdot\widehat{\boldsymbol{\omega}}^{(k)} = 0, \quad \widehat{\boldsymbol{\omega}}_t^{(k)} + \nabla\times\widehat{\boldsymbol{\beta}}^{(k)} = 0. \tag{3.41}$$

For an arbitrary differentiable function F, equations (3.41) yield additional conserved quantities,

$$\widehat{\mathscr{Q}}_F^{(k)} = \widehat{\boldsymbol{\omega}}^{(k)}\cdot\nabla F, \quad k = 1, 2, \ldots. \tag{3.42}$$

The families of 'spatial' conserved quantities $\widetilde{\mathscr{Q}}_F^{(j)}$ (3.38) and the 'temporal' ones $\widehat{\mathscr{Q}}_F^{(k)}$ (3.42) are independent when k is odd. Using the PDE (3.1), it is straightforward to show that, when $j = 2k$, $\widetilde{\mathscr{Q}}_F^{(j)}$ and $\widehat{\mathscr{Q}}_F^{(k)}$ are linearly dependent. The conserved quantities $\widetilde{\mathscr{Q}}_F^{(j)}$ (3.38) and $\widehat{\mathscr{Q}}_F^{(k)}$ (3.42) are also independent of the iterated conserved quantities $\mathscr{Q}_F^{(j)}$ (3.30). Moreover, it is clear that the three iterative techniques suggested above can be used in various combinations (that generally do not commute), to yield further conserved quantities.

4. Summary and conclusions

Local conservation laws provide essential information necessary for analysis and numerical simulation of nonlinear mathematical models. Full three-dimensional time-dependent incompressible viscous and inviscid fluid flow models were considered in this work.

In § 2, the direct method was applied to derive all local conservation laws of these models involving up to second-order conservation law multipliers. The full lists of conserved quantities were given. For the Euler flows, linear and generalized momenta, angular momentum, kinetic energy and helicity are conserved. For the Navier–Stokes flows, only the generalized momenta and the angular momentum are locally conserved. Importantly, it was demonstrated that no second-order conservation law multipliers arise.

It remains an open problem to prove or disprove the existence of further higher-order conservation laws for Euler and Navier–Stokes equations. We note that it is currently beyond reach to perform computations involving third- and higher-order multipliers, corresponding to second-order densities and fluxes. Indeed, a function of n variables has $\binom{n+k-1}{k}$ derivatives of order k. Each of the four flow parameters u^i, p depends on $n=4$ variables (t, x, y, z), and thus has 10 second and 20 third derivatives. In total, third-order conservation law multipliers would have to depend on approximately $4 \times (1+1+10+20) = 128$ quantities, making the symbolic computations inaccessible for current platforms. Instead, analytic methods may be necessary, similar to the one used in Khor'kova & Verbovetsky (1996), where an upper bound was established for the conservation law orders in the k–ε turbulence model.

Unlike the fluid dynamics equations in primitive variables, the vorticity formulation yields an abnormal PDE system of the form (3.2) for both Euler and Navier–Stokes models. This gives rise to an infinite family of generalized Euler–Ertel local conservation laws (3.9), which have been derived using the direct multiplier approach in § 3. The Euler–Ertel family was generalized using iteration ideas and the invariance of the form of PDEs (3.2) with respect to differentiations. In particular, for any given flow parameter F, it was shown that, in addition to the conserved 'potential vorticity' $\mathscr{Q}_F = \boldsymbol{\omega} \cdot \nabla F$, one can construct families of independent conservation laws involving various differential orders of F and/or the vorticity field $\boldsymbol{\omega}$ (§ 3.6). The iterative procedures generally do not commute, and thus can be used in combination to yield additional locally conserved quantities. In general, any transformation that preserves the form of the PDEs (3.2) can be incorporated in the presented framework.

It is worth emphasizing that the new infinite hierarchies of vorticity-related conserved quantities constructed in §§ 3.2 and 3.6, holding for the general Navier–Stokes flows, are different from previously known conservation law families for Euler and Navier–Stokes models. In particular, the difference lies in the following aspects.

(*a*) The new conserved quantities are not of the material (Lagrange-invariant) type (1.3).
(*b*) They do not depend on the condition of the vorticity being 'frozen-in', and hold for both viscous and inviscid flows.
(*c*) The arbitrary function F involved does not have to be a conserved quantity itself, as it does in the original Ertel's theorem and in the Cauchy invariant construction for Euler equations (see § 3.4).

Importantly, the new local vorticity-related conserved quantities hold for a general PDE system of the form (3.2).

The presented vorticity-related conservation laws that hold for non-ideal fluids, and can incorporate both viscous/diffusive effects and sink/source terms, significantly extend Ertel's classical theorem. They may be used to expand classical results based on potential vorticity, which were limited to ideal fluids, for example, in the fields of meteorology and acoustics, where potential vorticity has been extensively used (see e.g. Haynes & McIntyre 1990; Chagelishvili 1997) and where non-ideal fluid properties have so far had to be excluded. In particular, in the field of meteorology, the potential vorticity has been considered in conjunction with thermodynamic quantities, such as the temperature. With the present extension, the analysis may be generalized to include not only diffusive effects but also sink and source terms, to account, for example, for the atmospheric chemistry.

The vorticity-related conservation laws may also give rise to a renewed discussion on pseudo-scalars, which so far has occurred primarily in the context of helicity

$h = \boldsymbol{\omega} \cdot \boldsymbol{u}$, which is a conserved quantity only for ideal fluids. Pseudo-scalars are quantities that change sign under a reflection symmetry, i.e. $\tilde{\boldsymbol{u}} = -\boldsymbol{u}$ when $\tilde{\boldsymbol{x}} = -\boldsymbol{x}$. Real scalars, such as pressure, do not change their sign under reflections. An immediate consequence is that, for systems admitting a mirror symmetry, any pseudo-scalar vanishes. A well-known example is isotropic turbulence, which is by definition a mirror-symmetric system, and hence any pseudo-scalar vanishes in a statistical sense. Interestingly, both real scalars and pseudo-scalars may be constructed from the infinite set of conservation laws (3.13). Associating F with the kinetic energy K, for example, leads to the pseudo-scalar $\boldsymbol{\omega} \cdot \nabla K$, while using the helicity leads to a proper scalar $\boldsymbol{\omega} \cdot \nabla(\boldsymbol{\omega} \cdot \boldsymbol{u})$. The latter, for example, does not statistically vanish in isotropic turbulence.

Future work will continue in the following important directions: (1) work towards a complete description of local conservation laws of fluid dynamics equations in primitive variables; (2) the physical interpretation of the 'potential vorticity' and higher-order hierarchies of the vorticity-dependent local conserved quantities; and (3) the application of local fluid dynamics conservation laws, in particular, infinite families of vorticity conservation laws studied in the current paper, to the development and testing of novel numerical methods.

Acknowledgements

The authors are grateful to V. Grebenev for discussion, and to NSERC of Canada, the Alexander von Humboldt Foundation, and DFG of Germany (grant number OB 96/35) for their financial support.

Appendix A. Abnormality of the vorticity system and trivial conservation laws

We recall the notion of a trivial conservation law of a normal PDE system (Bluman *et al.* 2010). A conservation law (1.1) is called trivial in two cases:

(*a*) each of its fluxes vanishes identically on the solutions of the given PDE system;

(*b*) the conservation law vanishes identically off solutions of the given PDE system, as a differential identity, such as $\operatorname{div}(\operatorname{curl}(\cdot)) \equiv 0$.

Two conservation laws (1.1) are equivalent if their difference is a trivial conservation law. An equivalence class of conservation laws consists of all conservation laws equivalent to some given non-trivial conservation law. A set of conservation laws is linearly dependent if their non-trivial linear combination is a trivial conservation law. In practice, one is interested in seeking linearly independent non-trivial conservation laws of a given PDE system.

The direct conservation law construction method with non-trivial multipliers, applied to a PDE normal system in the solved form, yields its non-trivial conservation laws up to a prescribed differential order (Anco & Bluman 1997; Bluman *et al.* 2010).

The situation is different if the given PDE system is abnormal (e.g. Olver 2000). In particular, a PDE system is abnormal if there exists a non-trivial linear combination of its differential consequences that vanishes identically. This is the case with the vorticity system (3.1) (for brevity, we will use its form (3.2)). Indeed, denoting the four PDEs (3.2) by $E^1, \ldots, E^4$, it can be readily verified that the linear combination of differential consequences given by

$$\frac{\partial}{\partial t} E^1 - \nabla \times (E^2, E^3, E^4) \equiv 0 \tag{A 1}$$

identically, which proves the abnormality of the vorticity equations.

For abnormal PDE systems, the notion of a trivial conservation law of the first type may not be well defined. Consider, for example, the first vorticity equation, i.e. the vorticity definition $\nabla \cdot \boldsymbol{\omega} = 0$. It can be rewritten as an equivalent conservation law

$$\nabla \cdot \boldsymbol{\omega} \equiv \frac{\partial}{\partial t}(t \nabla \cdot \boldsymbol{\omega}) - \nabla \cdot [t(\boldsymbol{\omega}_t + \nabla \times \boldsymbol{\beta})] = 0. \tag{A 2}$$

Clearly, the density and the fluxes of the equivalent conservation law in (A 2) are proportional to equations of the system (3.2) itself, and hence vanish on solutions. It follows that the vorticity definition is a trivial conservation law of the first type. However physically, this equation is an essential part of the vorticity PDE system, and is not trivial. Similarly, other conservation laws of the abnormal vorticity system (3.1) that are equivalent to trivial conservation laws of the first type should not be automatically dismissed.

REFERENCES

ANCO, S. & BLUMAN, G. 1997 Direct construction of conservation laws from field equations. *Phys. Rev. Lett.* **78** (15), 2869–2873.

ANCO, S., BLUMAN, G. & WOLF, T. 2008 Invertible mappings of nonlinear PDEs to linear PDEs through admitted conservation laws. *Acta Appl. Math.* **101** (1), 21–38.

BATCHELOR, G. K. 2000 *An Introduction to Fluid Dynamics*. Cambridge University Press.

BENJAMIN, T. B. 1972 The stability of solitary waves. *Proc. R. Soc. Lond.* A **328** (1573), 153–183.

BLUMAN, G. W., CHEVIAKOV, A. F. & ANCO, S. C. 2010 *Applications of Symmetry Methods to Partial Differential Equations*. *(Applied Mathematical Sciences*, vol. 168), Springer.

BOWMAN, J. C. 2009 Casimir cascades in two-dimensional turbulence. In *Advances in Turbulence XII*, Springer Proceedings in Physics, vol. 132, pp. 685–688. Springer.

CHAGELISHVILI, G. D., KHUJADZE, G. R., LOMINADZE, J. G. & ROGAVA, A. D. 1997 Acoustic waves in unbounded shear flows. *Phys. Fluids* **9**, 1955–1962.

CHEVIAKOV, A. F. 2007 GeM software package for computation of symmetries and conservation laws of differential equations. *Comput. Phys. Commun.* **176** (1), 48–61.

GIBBON, J. D. & HOLM, D. D. 2010 The dynamics of the gradient of potential vorticity. *J. Phys. A: Math. Theor.* **43** (17), 172001.

GIBBON, J. D. & HOLM, D. D. 2012*a* Quasiconservation laws for compressible three-dimensional Navier–Stokes flow. *Phys. Rev.* E **86** (4), 047301.

GIBBON, J. D. & HOLM, D. D. 2012*b* Stretching and folding diagnostics in solutions of the three-dimensional Euler and Navier–Stokes equations. In *Mathematical Aspects of Fluid Mechanics* (ed. J. C. Robinson, J. L. Rodrigo & W. Sadowski), London Mathematical Society Lecture Note Series, vol. 402, vol. 402, pp. 201–220. Cambridge University Press.

HAYNES, P. H. & MCINTYRE, M. E. 1990 On the conservation and impermeability theorems for potential vorticity. *J. Atmos. Sci.* **47** (16), 2021–2031.

HOLM, D. D., MARSDEN, J. E., RATIU, T. & WEINSTEIN, A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123** (1), 1–116.

KELBIN, O., CHEVIAKOV, A. F. & OBERLACK, M. 2013 New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. *J. Fluid Mech.* **721**, 340–366.

KHOR'KOVA, N. G. & VERBOVETSKY, A. M. 1996 On symmetry subalgebras and conservation laws for the k–ϵ turbulence model and the Navier–Stokes equations. In *Selected Papers on Number Theory and Algebraic Geometry*, pp. 61–90. American Mathematical Society.

KUZNETSOV, E. A. 2008 Mixed Lagrangian–Eulerian description of vortical flows for ideal and viscous fluids. *J. Fluid Mech.* **600**, 167–180.

LAX, P. D. 1968 Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Maths* **21** (5), 467–490.

MOFFATT, H. K. 1969 The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35** (1), 117–129.

MOISEEV, S. S., SAGDEEV, R. Z., TUR, A. V. & YANOVSKII, V. V. 1982 On the freezing-in integrals and Lagrange invariants in hydrodynamic models. *Sov. Phys. JETP* **56** (1), 117–123.

MÜLLER, P. 1995 Ertel's potential vorticity theorem in physical oceanography. *Rev. Geophys.* **33** (1), 67–97.

NEWCOMB, W. A. 1967 Exchange invariance in fluid systems. In *Magneto-Fluid and Plasma Dynamics*, Proceedings of Symposia in Applied Mathematics, vol. XVIII, pp. 152–161. American Mathematical Society.

OLVER, P. J. 2000 *Applications of Lie Groups to Differential Equations*. *(Graduate Texts in Mathematics*, vol. 107), Springer.

PADHYE, N. & MORRISON, P. J. 1996*a* Relabeling symmetries in hydrodynamics and magnetohydrodynamics. *Plasma Phys. Rep.* **22** (10), 869–877.

PADHYE, N. & MORRISON, P. J. 1996*b* Fluid element relabeling symmetry. *Phys. Lett.* A **219** (5), 287–292.

SALMON, R. 1982 Hamilton's principle and Ertel's theorem. *AIP Conf. Proc.* **88**, 127–136.

SALMON, R. 1988 Hamiltonian fluid mechanics. *Annu. Rev. Fluid Mech.* **20** (1), 225–256.

TRUESDELL, C. 1954 *The Kinematics of Vorticity*. Indiana University Press.