JOURNAL OF MATHEMATICAL PHYSICS **56**, 121505 (2015)

A comparison of conservation law construction approaches for the two-dimensional incompressible Mooney-Rivlin hyperelasticity model

A. F. Cheviakov and S. St. Jean
Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatoon S7N 5E6, Canada

(Received 23 February 2015; accepted 30 November 2015; published online 28 December 2015)

An extended Kovalevskaya form is derived for the two-dimensional incompressible Mooney-Rivlin nonlinear hyperelasticity equations and is used to compute a complete set of local conservation laws of the model through the direct method. Conserved densities and fluxes of the conservation laws are derived, and their physical interpretation is discussed. Since the model admits a variational formulation, the equations are rewritten in the self-adjoint form. Computation of local conservation laws through the direct method applied to the self-adjoint form, as well as a conservation law computation through the local symmetry analysis and the Noether's first theorem, is performed. A correspondence between local variational symmetries and conservation law multipliers is illustrated. It is argued that even though it leads to more complicated forms of multipliers, the direct conservation law construction method applied to the Kovalevskaya form of the equations is a preferred systematic way of conservation law computations for complicated physical models of the type considered in this work, since it yields complete results, and naturally avoids singular multipliers. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4937756]

I. INTRODUCTION

Conservation laws of differential equations (DE) contain fundamental, coordinate-invariant information about the properties of the system and are employed in analysis, computations, as well as exact and numerical solution of DE problems (e.g, Ref. 6 and references therein). For a system of partial differential equations (PDEs) $\mathcal{R}$: $\{R^\sigma[u] = 0\}_{\sigma=1}^N$ with independent variables $z = \{z^i\}_{i=1}^n$ and dependent variables $u = \{u^k(z)\}_{k=1}^m$, local conservation laws are given by divergence expressions

$$\mathrm{D}_i \Phi^i[u] = 0, \tag{1.1}$$

where differential functions $\Phi^i[u]$ are the conservation law fluxes, and summation in the repeated index is assumed. If one of the variables is time, the conservation law takes the form

$$\mathrm{D}_t \Theta[u] + \mathrm{D}_i \Phi^i[u] = 0,$$

with the conserved density $\Theta[u]$. The operators D_i and D_t stand for total derivatives by z^i and t, respectively. If the fluxes Φ^i vanish on the boundary of the spatial domain $\mathcal{D}$ or at infinity, as well as in the periodic case, a global conserved quantity follows:

$$\mathcal{J} = \iiint_{\mathcal{D}} \Theta \, dV, \qquad \frac{d\mathcal{J}}{dt} = 0. \tag{1.2}$$

Characteristic forms of local conservation laws (1.1) and the related direct (multiplier-based) conservation law construction method are discussed in Section II.

In the current paper, we consider (2+1)-dimensional incompressible dynamic nonlinear hyperelasticity equations. After a brief introduction, the PDE system of interest is presented in Section III.

It consists of two coupled second-order nonlinear wave equations and a first-order nonlinear differential constraint, the incompressibility condition. Even with the presence of the latter, hyperelasticity equations without viscous/dissipative effects are known to arise from a classical variational principle.[14]

In Section IV, we disregard the variational property of the hyperelasticity system and seek its local conservation laws using the direct method. We show that the equations admit extended Kovalevskaya forms with respect to both spatial coordinates, rewrite them in such a form, and derive all local conservation laws involving second-order multipliers. Conserved densities and fluxes of the conservation laws are presented, and their physical meaning is identified.

In Section V, we turn to the variational structure of the hyperelasticity equations. We first compute first-order local symmetries of the system and identify the variational ones. The conservation law multipliers computed in Section IV through the extended Kovalevskaya form are matched with multipliers of the system in the self-adjoint writing. Both the PDEs and the multipliers in the latter case turn out to have much simpler expressions than those for the extended Kovalevskaya formulation. A correspondence between the variational symmetry components and local conservation laws following from the Noether's first theorem is outlined. It is shown for the current example that for the sake of completeness, it is essential to work with a Kovalevskaya form of a PDE system when its conservation laws are computed through the direct construction method. Computationally, the direct method is also superior to Noether's theorem since it results in simpler determining equations.

The paper is concluded with a discussion in Section VI.

The symmetry and conservation law computations were performed using `GeM` software package for `Maple`.[9]

II. SYSTEMATIC COMPUTATION OF LOCAL CONSERVATION LAWS

Consider a PDE system $\mathcal{R}$ with independent variables $z = \{z^i\}_{i=1}^n$ and dependent variables $u = \{u^k(z)\}_{k=1}^m$, given by

$$R^\sigma[u] \equiv R^\sigma(z, u, \partial u, \dots, \partial^q u) = 0, \quad \sigma = 1, \dots, N, \tag{2.1}$$

where q is the highest derivative order in the system, and ∂u, …, $\partial^q u$ are sets of partial derivatives of u of corresponding orders. Let $\mathcal{R}_{(k)}$ denote a maximal set of algebraically independent differential consequences of $\mathcal{R}$ of order less than or equal to k. Then equations $\mathcal{R}_{(k)}$ correspond to a system of algebraic equations in the jet space $J^k(x|u)$. PDE system (2.1) is *totally nondegenerate*, if it is *locally solvable* and is *of maximal rank* in every point of $\mathcal{R}_{(k)}$ (for details, see, e.g., Refs. 20 and 22). Most physical systems are totally nondegenerate. For such PDE systems, it follows from the Hadamard lemma that every local conservation law (1.1) can be written in an equivalent *characteristic form*

$$\mathrm{D}_i \tilde{\Phi}^i[u] = \Lambda_\sigma[u] R^\sigma[u] = 0, \tag{2.2}$$

where $\{\Lambda_\sigma[u]\}_{\sigma=1}^N$ are the conservation law multipliers (characteristics), and $\mathrm{D}_i(\Phi^i[u] - \tilde{\Phi}^i[u]) = 0$ is a *trivial* conservation law (see, e.g., Refs. 6, 20, and 22).

In practice, one is interested in computing the complete set of independent, nontrivial local conservation laws of a given PDE system. An addition of a trivial conservation law to a given one yields a conservation law equivalent to the given one. Conservation law computations are therefore done modulo trivial conservation laws, i.e., one is interested in obtaining one representative of each conservation law equivalence class.

For variational PDE systems, that is, when all equations $R^\sigma[u] = 0$ arise as Euler-Lagrange equations of some action functional, the Noether's first theorem establishes a one-to-one correspondence between conservation law multipliers and variational symmetries of the given system. The necessary and sufficient conditions (the Helmholtz conditions) for a DE system to be variational are that its linearization operator (Fréchet derivative) is self-adjoint. (In particular, it follows that for variational PDE systems, the number of equations equals the number of unknowns, $N = m$.)

Relatively few models arising in modern applications turn out to be variational. Moreover, the property of a PDE system being variational is heavily dependent on the choice of variables and on the representation of a PDE system (e.g., Ref. 5). Generally, the "inverse problem of the calculus of variations," that is, the problem of determining whether or not a given DE system is equivalent to a system of Euler-Lagrange equations, remains open. For a detailed discussion including a literature review, see Ref. 20, p. 378.

For many classical, variational and non-variational PDE systems, conservation laws have been found by *ad hoc*, or somewhat systematic, methods, including the Noether's first theorem. An interesting review devoted to comparison of several methods of computation of conservation laws is found in Ref. 25. For non-variational systems, it is well-known that sets of local conservation laws and local symmetries can be quite different. A rich example is provided in Ref. 7, where local symmetries and conservation laws of a non-variational system of nonlinear telegraph equations were classified and compared.

A rather general systematic and practically efficient *direct construction* (multiplier) method of local conservation law construction, applicable to both variational and non-variational PDE systems, appears in Ref. 20, p. 330, and in a detailed fashion, in Refs. 2–4. The method consists in the application of Euler differential operators E_{U^α} to seek sets of local multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ yielding divergence expressions

$$\mathrm{D}_i\tilde{\Phi}[U] = \Lambda_\sigma[U]R^\sigma[U], \tag{2.3}$$

where $U = \{U^k(z)\}_{k=1}^m$ is an arbitrary vector function. Then on solutions $U^k(z) = u^k(z)$ of the PDE system, local conservation law (2.2) holds. Though the properties of Euler operators and conservation law characteristic form (2.2) have been known for a long time, the important aspect of the method is the consideration of the conservation law multiplier determining equations,

$$E_{U^\alpha}(\Lambda_\sigma[U]R^\sigma[U]) \equiv 0, \qquad \alpha = 1,\dots,N, \tag{2.4}$$

off of the solution space.

The direct construction method has been successfully applied to a large number of linear and nonlinear PDE systems arising in applications (see, e.g., Ref. 6 and references therein). The direct method and related methods of flux computation have been implemented in the symbolic package `GeM` for `Maple` (see Refs. 9 and 10). Other symbolic software packages for conservation law computations exist; see, e.g., Refs. 10 and 25.

Within the direct method, one obtains a linear overdetermined system of determining equations for the unknown multipliers, in many ways similar to symmetry determining equations arising in local symmetry analysis. Solution of such systems proceeds by usual techniques and software for differential elimination (see, e.g., Ref. 9 and references therein). In computations using symbolic software, the multiplier determining equations arising from (2.4) are solved more efficiently than, for example, determining equations for the unknown fluxes that one would get trying to directly solve (1.1). Indeed, the solution of the latter would involve arbitrary functions corresponding to trivial conservation law fluxes.

In computations that use the direct method, *singular multipliers* that vanish on all solutions of the given system may arise. Such multipliers correspond to no conservation laws of the original system, and should be excluded. For example, taking $\Lambda_1[U] = (\mathrm{D}_i F^i[U])/R^1[U]$ and $\Lambda_{\sigma\neq 1}[U] = 0$ yields a divergence expression $\Lambda_\sigma[U]R^\sigma[U] = \mathrm{D}_i F^i[U]$ for a *arbitrary* differential functions $\{F^i[U]\}$; yet $\mathrm{D}_i F^i[u] = 0$ is generally not a conservation law of the given PDE system.

In practical computations with the direct method, the main two related questions are as follows. (a) How does one compute a full set (or a largest possible set) of conservation laws for a given PDE system? (b) How does one avoid singular conservation law multipliers?

In order to obtain a larger set of conservation laws, one naturally allows the multipliers $\{\Lambda_\sigma[U]\}$ to depend on independent and dependent variables of the system, as well as on derivatives up to some highest order. (For some classes of PDE systems, the highest possible conservation law order is known theoretically.) However, in order to exclude singular multipliers that are singular functions on the solution manifold of $\mathcal{R}$, some set of *leading derivatives* and their differential consequences have to be excluded from the multiplier dependence.

A simultaneous answer to the above two questions is available for PDE systems written in a *general Kovalevskaya form*. The latter is a solved form with respect to leading derivatives of all dependent variables by a given independent variable, say z^n, such that the right-hand sides of the PDEs do not involve leading derivatives, as well as derivatives of orders higher than those of the leading derivatives. For PDE systems written in a general Kovalevskaya form, every local conservation law has equivalent characteristic form (2.2) with multipliers independent of the leading derivatives and their differential consequences.[20] (Another important result for systems written in the general Kovalevskaya form is the local existence and uniqueness of an analytic solution of a Cauchy problem with analytic initial data, e.g., Refs. 20 and 21).

The existence of a general Kovalevskaya form for a given PDE system is equivalent to the normality condition, in particular, the existence of a non-characteristic direction.[19,20] However, many examples of non-normal physical systems exist, for example, mechanical systems involving an incompressibility constraint. An extension of the above result, namely, the existence, for every local conservation law, of an equivalent conservation law with a characteristic form involving multipliers independent of the leading derivatives, has been proven for a wider class of PDE systems in Ref. 1 (see also Ref. 22). It is formulated as follows.

Definition 1. PDE system $\mathcal{R}$ (2.1) consisting of $N = m$ equations and involving m dependent variables $u = \{u^k(z)\}_{k=1}^m$ is in the *extended Kovalevskaya form* $\{\widehat{R}^\sigma[u] = 0\}_{\sigma=1}^m$ with respect to an independent variable, say, z^n, if each equation of $\mathcal{R}$ has the form

$$\widehat{R}^\sigma[u] = \frac{\partial^{\,r_\sigma} u^\sigma}{\partial\,(z^n)^{r_\sigma}} - H^\sigma[u] = 0, \quad \sigma = 1,\dots,m, \tag{2.5}$$

where the functions $H^\sigma[u]$ may involve z, u, and derivatives of u with respect to z up to some maximal order ρ, and moreover, each u^b is differentiated with respect to z^n at most $r_b - 1$ times, $b = 1,\dots,m$. In (2.5), $r_\sigma \leqslant \rho$.

In other words, a PDE system in extended Kovalevskaya form is solved for the leading derivatives

$$\frac{\partial^{\,r_\sigma} u^\sigma}{\partial\,(z^n)^{r_\sigma}}, \qquad \sigma = 1,\dots,m, \tag{2.6}$$

and the right-hand sides $H^\sigma[u]$ of the equations involve neither the leading derivatives nor their differential consequences. The form of PDE system (2.5) is broader than the general Kovalevskaya form, since the derivative orders in right-hand sides $H^\sigma[u]$ of the equations in (2.5) are not restricted by the orders r_σ of leading derivatives (2.6). The following theorem holds.[1]

Theorem 1. *Suppose $\mathcal{R}$ (2.1) is a locally solvable system of differential equations written in extended Kovalevskaya form (2.5). Then the divergence expression of every local conservation law (1.1) of $\mathcal{R}$ has an equivalent form*

$$\mathrm{D}_i\widehat{\Phi}^i[U] = \sum_{\sigma=1}^m \widehat{\Lambda}_\sigma[U]\left(\frac{\partial^{\,r_\sigma} U^\sigma}{\partial\,(z^n)^{r_\sigma}} - H^\sigma[U]\right),$$

where the multipliers $\{\widehat{\Lambda}_\sigma[U]\}_{\sigma=1}^m$ are independent of the leading derivatives (2.6) and their differential consequences.

Said differently, this means that every conservation law of a PDE system satisfying the requirements of Theorem 1 can be obtained (up to conservation law equivalence) through the direct construction method, and singular multipliers will not arise.

In a practical conservation law computation, if the PDE system of interest admits an extended Kovalevskaya form (2.5), one would normally use that form for the sake of completeness. On the other hand, for an initially complicated PDE system, its extended Kovalevskaya form can turn out significantly more complicated. This can result in more involved expressions for the multipliers $\{\widehat{\Lambda}_\sigma[u]\}_{\sigma=1}^m$ compared to those for the original form of the equations.

We note that for a PDE system to admit a general or extended Kovalevskaya form, it is necessary that the number of dependent variables and the number of equations coincide. A number of physical models, such as Maxwell's equations, admit neither a general nor an extended Kovalevskaya form. A discussion of the nature of such examples can be found in Refs. 19 and 20.

III. THE INCOMPRESSIBLE DYNAMIC MOONEY-RIVLIN HYPERELASTICITY MODELS

Consider a spatial region $\overline{\Omega}_0 \subset \mathbb{R}^3$ in the reference (Lagrange) configuration, Ω_0 being an open bounded connected set having a Lipschitz boundary.[12] The actual (Eulerian) domain occupied by that body is $\overline{\Omega} = \phi(\overline{\Omega}_0) \subset \mathbb{R}^3$. The position $x \in \overline{\Omega}$ of a material point depends on Lagrangian coordinates (initial conditions) $X = (X^1, X^2, X^3) \in \overline{\Omega}_0$ at time t according to

$$x^i = \phi^i(X,t), \quad i = 1,2,3.$$

Sufficient smoothness of the mapping ϕ is assumed. The deformation gradient is given by

$$\mathbf{F}(X,t) = \nabla\phi, \qquad F^i_{\ j} = \frac{\partial \phi^i}{\partial X^j} = F_{ij}, \tag{3.1}$$

and the Jacobian determinant satisfies the orientation-preserving condition

$$J = \det \mathbf{F} > 0.$$

In the current work, we restrict our attention to incompressible materials with $J = 1$, encompassing a wide class of models.

The stress in the reference configuration is described by the first Piola-Kirchhoff tensor $\mathbf{P}$. For *hyperelastic* materials, a scalar volumetric strain energy function $W = W(X,\mathbf{F})$ in the reference configuration is prescribed, defining the material behavior. For incompressible models, the first Piola-Kirchhoff tensor is given by

$$\mathbf{P} = -p\,\mathbf{F}^{-T} + \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \qquad P^{ij} = -p\,(F^{-1})^{ji} + \rho_0 \frac{\partial W}{\partial F_{ij}}, \tag{3.2}$$

where $p = p(X,t)$ is the hydrostatic pressure,[12,18] and $\rho_0 = \rho_0(X)$ is the (time-independent) body density in the reference configuration. The actual density in Eulerian coordinates,

$$\rho = \rho(X,t) = \rho_0(X)/J = \rho_0(X),$$

is also time-independent for incompressible materials.

A constitutive relation for an isotropic homogeneous hyperelastic material is an expression for the strain energy function W stated in terms of invariants of the (left and right) Cauchy-Green strain tensors

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{3.3}$$

The three principal invariants of the Cauchy-Green tensors are given by

$$I_1 = \mathrm{Tr}\mathbf{B} = F^i_{\ k} F^i_{\ k}, \quad I_2 = \frac{1}{2}[(\mathrm{Tr}\mathbf{B})^2 - \mathrm{Tr}(\mathbf{B}^2)] = \frac{1}{2}(I_1^2 - B^{ik}B^{ki}), \quad I_3 = \det \mathbf{B} = J^2, \tag{3.4}$$

where I_3 is constant for incompressible motions. The strain energy density function therefore has the form $W = W(I_1, I_2)$. In the current paper, we restrict our attention to the *Mooney-Rivlin materials*, a subclass of Hadamard materials,[13,18] for which the constitutive relation is commonly written as

$$W = a(I_1 - 3) + b(I_2 - 3), \tag{3.5}$$

with $W = 0$ corresponding to the unperturbed equilibrium $x = X$. Model (3.5) with $b = 0$ is referred to as the neo-Hookean model. The Mooney-Rivlin model (3.5) is being successfully used for the description of a large number of materials in practical applications. The constants $a, b > 0$ are experimentally determined material-dependent constitutive parameters.

The full system of equations of motion of an incompressible hyperelastic material in three dimensions is given by

$$\rho_0 x^i_{tt} = \frac{\partial P^{ij}}{\partial X^j} + \rho_0 G^i, \quad i = 1,2,3; \qquad J = \det \mathbf{F} = 1. \tag{3.6}$$

The first equation expresses the balance of momentum for an infinitesimal volume in the reference configuration; $G = G(X,t)$ is the total body force per unit mass.

In the current work, we assume a two-dimensional setup with zero body forces and constant mass density in the Lagrangian frame,

$$x^{1,2} = x^{1,2}\left(X^1, X^2, t\right), \qquad x^3 = X^3, \qquad G^{1,2,3} = 0, \qquad \rho_0 = \text{const}. \tag{3.7}$$

Without loss of generality, by re-scaling the pressure, one may take $\rho_0 = 1$, which is done below. For the brevity and simplicity of notation, we will denote partial derivatives that appear in PDE system (3.8a)–(3.8c) and related computations with subscript symbols, e.g.,

$$\frac{\partial^2 x^1}{\partial t^2} \equiv x^1_{tt}, \quad \frac{\partial x^1}{\partial X^2} \equiv x^1_2, \quad \frac{\partial^2 x^2}{\partial X^1 \partial X^2} \equiv x^2_{12}, \quad \frac{\partial^2 p}{\partial X^2 \partial t} \equiv p_{2\,t},$$

and so on. In the above assumptions and notation, the two-dimensional equations of motion (3.6) of an isotropic, homogeneous, incompressible hyperelastic solid are given by

$$R^1[x,p] = 1 - J = 1 - \left(x^1_1 x^2_2 - x^1_2 x^2_1\right) = 0, \tag{3.8a}$$

$$R^2[x,p] = x^1_{tt} - \left[\alpha\left(x^1_{11} + x^1_{22}\right) - p_1 x^2_2 + p_2 x^2_1\right] = 0, \tag{3.8b}$$

$$R^3[x,p] = x^2_{tt} - \left[\alpha\left(x^2_{11} + x^2_{22}\right) - p_2 x^1_1 + p_1 x^1_2\right] = 0, \tag{3.8c}$$

where $\alpha = 2(a+b) = \text{const} > 0$ is a material parameter. Model (3.8) includes two semi-linear wave equations with nonlinear first order coupling in p, and the incompressibility condition $J = 1$ in the form of first-order PDE (3.8a) representing a differential constraint.

PDE system (3.8) is a variational system arising from a classical variational principle, with the Lagrangian density having the form

$$\mathcal{L} = -K + W - p(J-1), \tag{3.9}$$

where

$$K = \frac{1}{2}\left((x^1_t)^2 + (x^2_t)^2\right) \tag{3.10}$$

is the kinetic energy density, and

$$W = \frac{\alpha}{2}\left(\left(x^1_1\right)^2 + \left(x^1_2\right)^2 + \left(x^2_1\right)^2 + \left(x^2_2\right)^2\right) \tag{3.11}$$

is the potential (strain) energy density (3.5). It follows that PDEs (3.8) are self-adjoint as written. A Lagrangian density equivalent to (3.9) up to a total divergence can be obtained from the homotopy formula

$$\widehat{\mathcal{L}} = \int_0^1 u \cdot R[\lambda u]\, d\lambda,$$

where, in the notation of Section I, the variables of PDE system (3.8) are denoted by $u(z) = (p, x^1, x^2)$; $z = (t, X^1, X^2)$.

Mooney-Rivlin equations (3.8) arise as Euler-Lagrange equations under the action of the corresponding Euler operators, as follows:

$$\mathrm{E}_p\,\mathcal{L} = R^1[x,p], \qquad \mathrm{E}_{x^1}\,\mathcal{L} = R^2[x,p], \qquad \mathrm{E}_{x^2}\,\mathcal{L} = R^3[x,p]. \tag{3.12}$$

Equations (3.8a)–(3.8c) admit extended Kovalevskaya form (2.5), which is derived in Sec. IV and is used in the direct construction of conservation laws.

IV. THE EXTENDED KOVALEVSKAYA FORM AND THE DIRECT CONSERVATION LAW CONSTRUCTION

PDE system (3.8) consists of equations of different orders and therefore does not admit a classical or a general Kovalevskaya form.[20] However, they can be shown to admit extended Kovalevskaya form (2.5). Assume $\partial x^2/\partial X^2 \neq 0$. The PDE R^1 can be solved for the derivative $\partial x^1/\partial X^1$, which is substituted into R^2 and R^3. The PDE R^2 is subsequently solved for $\partial^2 x^2/\partial (X^1)^2$, which is substituted into R^3. Finally, one obtains a PDE system

$$\widehat{R}^1[x,p] = x^1_1 - \left(x^2_2\right)^{-1} S[x^1,x^2] = 0, \tag{4.1a}$$

$$\widehat{R}^2[x,p] = x^2_{11} - \left(-x^2_{22} + \alpha^{-1}\left[x^2_{tt} - p_1 x^1_2 + p_2 \left(x^2_2\right)^{-1} S[x^1,x^2]\right]\right) = 0, \tag{4.1b}$$

$$\begin{aligned}\widehat{R}^3[x,p] = p_1 - M[x^1,x^2]\Big\{&\left(x^2_2\right)^2\left(x^1_2 x^2_{tt} - x^2_2 x^1_{tt}\right) + \left(x^2_1 N[x^1,x^2] + x^1_2\right) x^2_2 p_2 \\ &+\alpha\left[x^2_2 x^1_{22} N[x^1,x^2] - x^2_2 x^2_{12} - \left(x^1_2 N[x^1,x^2] + x^2_1\right) x^2_{22}\right]\Big\} = 0,\end{aligned} \tag{4.1c}$$

where

$$N[x^1,x^2] = \left(x^1_2\right)^2 + \left(x^2_2\right)^2, \quad M[x^1,x^2] = N[x^1,x^2]^{-1}\left(x^2_2\right)^{-2}, \quad S[x^1,x^2] = \left(1 + x^1_2 x^2_1\right).$$

PDE system (4.1) is equivalent to the original PDEs (3.8) when $\partial x^2/\partial X^2 \neq 0$. The leading derivatives are given by

$$\{x^1_1,\ x^2_{11},\ p_1\}. \tag{4.2}$$

As required, neither these derivatives nor their differential consequences are present in the right-hand sides when the PDEs (4.1) are solved for the leading derivatives.

From Theorem 1 it follows that every local conservation law of Mooney-Rivlin equations (3.8) and (4.1) has a characteristic form

$$\sum_{\sigma=1}^{3} \Lambda_\sigma[x,p]\, R^\sigma[x,p] = \sum_{\sigma=1}^{3} \widehat{\Lambda}_\sigma[x,p]\, \widehat{R}^\sigma[x,p] = D_t\widehat{\Psi} + D_{X^1}\widehat{\Phi}^1 + D_{X^1}\widehat{\Phi}^2 = 0, \tag{4.3}$$

where the multipliers $\{\widehat{\Lambda}_\sigma[x,p]\}_{\sigma=1}^3$ of the system in the extended Kovalevskaya form involve neither leading derivatives (4.2) nor their differential consequences.

A. Local conservation law multipliers

For the system (4.1) in the extended Kovalevskaya form, we use the direct method to seek local conservation laws (4.3). We find all conservation laws involving second-order multipliers $\{\widehat{\Lambda}_\sigma[x,p]\}_{\sigma=1}^3$ which may depend on t, X^i, x^i, p, and all derivatives of x^i, p up to second order, excluding leading derivatives (4.2) and their differential consequences, i.e., on the 24 variables,

$$t, X^i, x^i, p, p_t, p_2, x^i_t, x^1_2, x^2_i, p_{tt}, p_{22}, p_{2t}, x^i_{2t}, x^2_{12}, x^2_{1t}, x^i_{tt}, x^i_{22}, \qquad i = 1,2.$$

The following theorem holds.

Theorem 2. *The complete set of second-order conservation law multipliers admitted by Mooney-Rivlin system (4.1) is given by*

$$\begin{aligned}
\widehat{\Lambda}_1 = C_1 &\left\{2X^1\left[\frac{\partial}{\partial X^2}\left(\left(x_2^2\right)^{-1}x_2^1x_1^2\right)-\left(x_2^2\right)^{-2}x_{22}^2\right]\right.\\
&\left.+X^2\left[x_{22}^1-\alpha^{-1}x_{tt}^1-\frac{\partial}{\partial X^2}\left(\left(x_2^2\right)^{-2}x_1^2S[x^1,x^2]\right)\right]+x_2^1-\left(x_2^2\right)^{-2}x_1^2S[x^1,x^2]\right\}\\
&+C_2\left\{\alpha^{-1}x^1p_2-2x_1^2+x^2\frac{\partial}{\partial X^2}\left(\left(x_2^2\right)^{-1}x_1^2\right)\right\}\\
&+C_3\left\{\left(x_2^2\right)^{-2}x_1^2\left(x_2^1x_{2t}^2+x_t^1x_{22}^2-2x_{2t}^1x_2^2\right)-\left(x_2^2\right)^{-1}\left(x_t^1x_{12}^2+x_2^1x_{1t}^2\right)-\frac{\partial}{\partial t}\left(x_2^2\right)^{-1}\right\}\\
&+C_4\left\{x_{22}^1-\alpha^{-1}x_{tt}^1-\frac{\partial}{\partial X^2}\left(\left(x_2^2\right)^{-2}x_1^2S[x^1,x^2]\right)\right\}-C_5\frac{\partial}{\partial X^2}\left(\left(x_2^2\right)^{-1}\left(1+2x_2^1x_1^2\right)\right)\\
&+f_1(t)\frac{\partial}{\partial X^2}\left(\left(x_2^2\right)^{-1}x_1^2\right)+f_1''(t)\,\alpha^{-1}x^1x_2^2\\
&+f_2(t)p_2+f_2''(t)\,x^2x_2^2\;+f_3(t)x_2^2,
\end{aligned}\tag{4.4}$$

$$\begin{aligned}
\widehat{\Lambda}_2 = {}& C_1\,\alpha^{-1}\left(x_2^2\right)^{-2}N[x^1,x^2]\left[X^1x_2^1x_2^2-X^2S[x^1,x^2]\right]-C_2\,\alpha^{-1}x^2\left(x_2^2\right)^{-1}N[x^1,x^2]\\
&-C_3\,\alpha^{-1}x_t^1\left(x_2^2\right)^{-1}N[x^1,x^2]-C_4\,\alpha^{-1}\left(x_2^2\right)^{-2}N[x^1,x^2]S[x^1,x^2]\\
&-C_5\,\alpha^{-1}\frac{\partial x^1}{\partial X^2}\left(x_2^2\right)^{-1}N[x^1,x^2]\\
&+f_1(t)\,\alpha^{-1}\left(x_2^2\right)^{-1}N[x^1,x^2],
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
\widehat{\Lambda}_3 = {}& C_1\,\left\{-X^1\left(x_2^2\right)^{-1}N[x^1,x^2]+X^2\left(x_2^2\right)^{-2}\left(x_1^2N[x^1,x^2]+x_2^1\right)\right\}\\
&-C_2\left(x_2^2\right)^{-1}\left(x^1x_2^2-x^2x_2^1\right)+C_3\left(x_2^2\right)^{-1}\left(x_t^1x_2^1+x_t^2x_2^2\right)\\
&+C_4\left(x_2^2\right)^{-2}\left(x_1^2N[x^1,x^2]+x_2^1\right)+C_5\left(x_2^2\right)^{-1}N[x^1,x^2]\\
&+f_1(t)\left(x_2^2\right)^{-1}x_2^1\;-f_2(t)\,\alpha.
\end{aligned}\tag{4.6}$$

In (4.4)–(4.6), $\{C_k\}_{k=1}^5$ are arbitrary constants, and $\{f_i(t)\}_{i=1}^3$ are arbitrary functions of t.

Theorem 2 is proven by direct computation, which involves a solution of a linear overdetermined system of 2869 determining equations. The computations are performed in `Maple` software (version 17) using the `GeM` symbolic package (version 32.02)[9] for the conservation law analysis (for details, see Ref. 8). In particular, multiplier determining equations (2.4) are generated and split using the `GeM` routine `gem_conslaw_det_eqs`, and simplified using the `Maple` differential elimination routine `rifsimp`.[23]

B. The conserved forms and the physical meaning

The corresponding conserved densities and fluxes can be computed from multipliers (4.4)–(4.6) through formula (4.3) via the direct flux computation method,[10] to yield the following conservation laws.

(1) Conservation of generalized momentum in the x^1 direction. For arbitrary $f_1(t)$, we obtain the conservation of generalized momentum in x^1,

$$\begin{aligned}
&\mathrm{D}_t\left(f_1(t)x_t^1-f_1'(t)x^1\right)\\
&-\mathrm{D}_{X^1}\left(\alpha f_1(t)x_1^1-pf_1(t)x_2^2-\frac{1}{2}f_1''(t)\left(x^1\right)^2x_2^2\right)\\
&-\mathrm{D}_{X^2}\left(\alpha f_1(t)x_2^1+pf_1(t)x_1^2+\frac{1}{2}f_1''(t)\left(x^1\right)^2x_1^2\right)=0.
\end{aligned}\tag{4.7}$$

(2) Conservation of generalized momentum in the x^2 direction. For arbitrary $f_2(t)$, we obtain the conservation of generalized momentum in x^2,

$$\begin{aligned}&\mathrm{D}_t\left(f_2(t)x_t^2 - f_2'(t)x^2\right)\\&-\mathrm{D}_{X^1}\left(\alpha f_2(t)x_1^2 + pf_2(t)x_2^1 + f_2''(t)\left(x^2\right)^2 x_2^1\right)\\&-\mathrm{D}_{X^2}\left(\alpha f_2(t)x_2^2 - pf_2(t)x_1^1 - \frac{1}{2}f_2''(t)\left(x^2\right)^2 x_1^1\right) = 0.\end{aligned} \tag{4.8}$$

Note that one has the conservation of regular linear momentum in x^1 and x^2 direction for $f_{1,2}(t) = \text{const}$, respectively.

(3) Conservation of angular momentum in the x^3 direction. From the multiplier set with coefficient C_2, we obtain the conservation of angular momentum in the direction of x^3, transverse to the motions,

$$\begin{aligned}&\mathrm{D}_t\left(x^1x_t^2 - x_t^1x^2\right)\\&+\mathrm{D}_{X^1}\left(\alpha\left(x_1^1x^2 - x^1x_1^2\right) - px^1x_2^1 - px^2x_2^2\right)\\&+\mathrm{D}_{X^2}\left(\alpha\left(x_2^1x^2 - x^1x_2^2\right) + px^1x_1^1 + px^2x_1^2\right) = 0.\end{aligned} \tag{4.9}$$

(4) Conservation of energy. For the multiplier set corresponding to the coefficient C_3, we obtain the conservation of energy. A simplified form of this conservation law (modulo an equivalence relation) can be written as follows:

$$\begin{aligned}&\mathrm{D}_t\left(\frac{\alpha}{2}\left(\left(x_1^1\right)^2 + \left(x_2^1\right)^2 + \left(x_1^2\right)^2 + \left(x_2^2\right)^2\right) + \frac{1}{2}\left(x_t^1\right)^2 + \frac{1}{2}\left(x_t^2\right)^2\right)\\&-\mathrm{D}_{X^1}\left(\alpha\left(x_t^1x_1^1 + x_t^2x_1^2\right) - px_t^1x_2^2 + px_2^1x_t^2\right)\\&-\mathrm{D}_{X^2}\left(\alpha\left(x_t^1x_2^1 + x_t^2x_2^2\right) + px_t^1x_1^2 - px_1^1x_t^2\right) = 0.\end{aligned} \tag{4.10}$$

Here the kinetic energy and the potential energy are given by (3.10) and (3.11), respectively.

(5) Generalized incompressibility condition. The multiplier set with the function $f_3(t)$ corresponds to a generalized version of the incompressibility condition, which has divergence conservation law form

$$\mathrm{D}_t\left(\int f_3(t)\,dt\right) - \mathrm{D}_{X^1}\left(f_3(t)x^1x_2^2\right) + \mathrm{D}_{X^2}\left(f_3(t)x^1x_1^2\right) = 0. \tag{4.11}$$

Expansion and simplification of the above expression, assuming $f^3(t) \neq 0$, yields the usual incompressibility condition (3.8a).

Momenta in the material frame. There are three additional conservation laws for the two components of linear momentum, and one component of the angular momentum, in the Lagrangian frame, corresponding to the multiplier sets with coefficient C_1, C_4, and C_5.

The vectors for the linear momentum $\mathfrak{p}$ and the angular momentum $\mathfrak{m}$ in the Lagrangian frame are defined as follows:[18]

$$\mathfrak{p} \equiv \rho_0\mathbf{F}^T x_t, \qquad \mathfrak{m} \equiv X \times \mathfrak{p}. \tag{4.12}$$

These momenta are also referred to as *material momenta*.[18] In our notation, they have the form

$$\mathfrak{p} = (x_t^1x_1^1 + x_t^2x_1^2)\hat{\imath} + (x_t^1x_2^1 + x_t^2x_2^2)\hat{\jmath}, \tag{4.13}$$

$$\mathfrak{m} = (X^1(x_t^1x_2^1 + x_t^2x_2^2) - X^2(x_t^1x_1^1 + x_t^2x_1^2))\hat{k}, \tag{4.14}$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the standard Cartesian basis vectors in the Lagrangian frame. The specific forms of local conservation laws of the material momenta are given below.

(6) Conservation of the material momentum in the X^1 direction. This conservation law corresponds to the constant C_4 in the multipliers (4.4)–(4.6) and has the form

$$\begin{aligned}&\mathrm{D}_t\left(x^1_t x^1_1 + x^2_t x^2_1\right)\\&+\mathrm{D}_{X^1}\left(\alpha\left(\left(x^1_2\right)^2+\left(x^2_2\right)^2-\left(x^1_1\right)^2-\left(x^2_1\right)^2\right)+p-\frac{1}{2}\left(x^1_t\right)^2-\frac{1}{2}\left(x^2_t\right)^2\right)\\&-\mathrm{D}_{X^2}\left(\alpha\left(x^1_1x^1_2+x^2_1x^2_2\right)\right)=0.\end{aligned}\tag{4.15}$$

(7) Conservation of the material momentum in the X^2 direction. This conservation law corresponds to the constant C_5, and is given by

$$\begin{aligned}&\mathrm{D}_t\left(x^1_t x^1_2 + x^2_t x^2_2\right)-\mathrm{D}_{X^1}\left(\alpha\left(x^1_1x^1_2+x^2_1x^2_2\right)\right)\\&+\mathrm{D}_{X^2}\left(\alpha\left(\left(x^1_1\right)^2+\left(x^2_1\right)^2-\left(x^1_2\right)^2-\left(x^2_2\right)^2\right)+p-\frac{1}{2}\left(x^1_t\right)^2-\frac{1}{2}\left(x^2_t\right)^2\right)=0.\end{aligned}\tag{4.16}$$

(8) Conservation of the material angular momentum in the X^3 direction. The density of conservation law corresponding to the constant C_1 in (4.4)–(4.6) is the $\hat{k}$-component of (4.14). The local conservation law is given by

$$\begin{aligned}&\mathrm{D}_t\left(X^2(x^1_t x^1_1 + x^2_t x^2_1)-X^1(x^1_t x^1_2 + x^2_t x^2_2)\right)\\&+\mathrm{D}_{X^1}\Big(\frac{\alpha}{2}X^2\left(-\left(x^1_1\right)^2-\left(x^2_1\right)^2+\left(x^1_2\right)^2+\left(x^2_2\right)^2\right)\\&\qquad+\alpha X^1\left(x^1_1x^1_2+x^2_1x^2_2\right)+X^2p-\frac{1}{2}X^2\left(x^1_t\right)^2-\frac{1}{2}X^2\left(x^2_t\right)^2\Big)\\&+\mathrm{D}_{X^2}\Big(\frac{\alpha}{2}X^1\left(-\left(x^1_1\right)^2-\left(x^2_1\right)^2+\left(x^1_2\right)^2+\left(x^2_2\right)^2\right)\\&\qquad-\alpha X^2\left(x^1_1x^1_2+x^2_1x^2_2\right)-X^1p+\frac{1}{2}X^1\left(x^1_t\right)^2+\frac{1}{2}X^1\left(x^2_t\right)^2\Big)=0.\end{aligned}\tag{4.17}$$

V. LOCAL SYMMETRIES OF THE MOONEY-RIVLIN EQUATIONS AND THEIR RELATIONSHIP WITH LOCAL CONSERVATION LAWS

Since 2D Mooney-Rivlin equations (3.8) are self-adjoint as written, the Noether's first theorem holds, establishing a one-to-one relationship between the local variational symmetries and local conservation laws of the model.

We now compute local symmetries, identify the variational ones, and relate the symmetry generators with local conservation law multipliers of the PDEs (3.8).

A. First-order local symmetries

Local symmetries of the two-dimensional Mooney-Rivlin model can be computed using any suitable representation of the governing equations (3.8). We apply the Lie's algorithm to extended Kovalevskaya form (4.1), seeking symmetry generators in the evolutionary form

$$\widetilde{\mathrm{X}}=\zeta^k(z,u,\partial u)\frac{\partial}{\partial u^k},\tag{5.1}$$

with $z=(t,X^1,X^2)$, $u=(p,x^1,x^2)$, and ∂u representing all first-order partial derivatives of u. First-order symmetries (5.1) include point symmetries

$$\mathrm{Y}=\xi^i(z,u)\frac{\partial}{\partial z^i}+\eta^k(z,u)\frac{\partial}{\partial u^k},\tag{5.2}$$

in particular,

$$\zeta^k=\eta^k-u^k_i\xi^i.$$

On solutions of the model, leading derivatives (4.2) are represented by the right-hand sides of the PDEs (4.1). Therefore, instead of these leading derivatives, symmetry generators (5.1) are sought as functions of other derivatives appearing in the right-hand sides of PDEs (4.1). The following result is obtained by a direct `Maple`-based computation using the `GeM` symbolic software package.[8,9] Symmetry determining equations are generated and split using the `GeM` routine `gem_symm_det_eqs` (see also Ref. 24).

Theorem 3. *Two-dimensional Mooney-Rivlin equations (3.8) and (4.1) are invariant under an infinite-dimensional group of Lie point transformations given by the infinitesimal generators,*

$$
\begin{aligned}
&\mathrm{Y}^1 = \frac{\partial}{\partial t}\,, \quad \mathrm{Y}^2 = \frac{\partial}{\partial X^1}\,, \quad \mathrm{Y}^3 = \frac{\partial}{\partial X^2}\,,\\
&\mathrm{Y}^4 = X^2\frac{\partial}{\partial X^1} - X^1\frac{\partial}{\partial X^2}\,, \quad \mathrm{Y}^5 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}\,,\\
&\mathrm{Y}^6 = F_1(t)\frac{\partial}{\partial x^1} - F_1''(t)\,x^1\frac{\partial}{\partial p}\,, \quad \mathrm{Y}^7 = F_2(t)\frac{\partial}{\partial x^2} - F_2''(t)\,x^2\frac{\partial}{\partial p}\,,\\
&\mathrm{Y}^8 = F_3(t)\frac{\partial}{\partial p}\,, \quad \mathrm{Y}^9 = t\frac{\partial}{\partial t} + X^1\frac{\partial}{\partial X^1} + X^2\frac{\partial}{\partial X^2} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\,,
\end{aligned}
\tag{5.3}
$$

where $F_1(t)$, $F_2(t)$, and $F_3(t)$ are arbitrary functions of time. No additional first-order symmetries (5.1) arise.

The symmetries $\mathrm{Y}^1, \mathrm{Y}^2, \mathrm{Y}^3$ correspond to time translations and space translations in the Lagrangian frame; Y^4 and Y^5 correspond to rotations in the material and the actual frames; Y^6 and Y^7 are generalized Galilei transformations, which include translations in x^1 and x^2 and the usual Galilei transformations; Y^8 is the translation of the pressure by an arbitrary function of time; Y^9 is a generator of a scaling group. The basis of the Lie algebra of point symmetries (5.3) is in the agreement with Ref. 17, where the commutator table of generators (5.3) is also computed, and invariant solutions are considered.

B. The form of local conservation law multipliers for the variational formulation

Consider a local conservation law of a PDE system of maximal rank. A characteristic representation (2.2) of this conservation law, and therefore the form of multipliers, clearly depends on the writing of the PDE system. One can employ the formula (4.3) to derive multipliers $\{\Lambda_\sigma\}_{\sigma=1}^3$ for the variational formulation (3.8) of the governing equations, using the known multipliers $\{\widehat{\Lambda}_\sigma\}_{\sigma=1}^3$ (4.4)–(4.6) for extended Kovalevskaya form (4.1). It is straightforward to show that the multipliers $\{\Lambda_\sigma\}_{\sigma=1}^3$ have a rather simple form,

$$
\begin{aligned}
\Lambda_1 &= C_1(X^1 p_2 - X^2 p_1) + C_3 p_t + C_4 p_1 + C_5 p_2 + f_1''(t)x^1 + f_2''(t)x^2 + f_3(t),\\
\Lambda_2 &= C_1(X^2 x^1_1 - X^1 x^1_2) + C_2 x^2 - C_3 x^1_t - C_4 x^1_1 - C_5 x^1_2 + f_1(t),\\
\Lambda_3 &= C_1(X^2 x^2_1 - X^1 x^2_2) - C_2 x^1 - C_3 x^2_t - C_4 x^2_1 - C_5 x^2_2 + f_2(t).
\end{aligned}
\tag{5.4}
$$

Remark 1. It is important that for completeness of the conservation law computation, one has to use extended Kovalevskaya form (4.1) of the Mooney-Rivlin model. The form of the multipliers (5.4) for original equations of motion (3.8) is determined *post factum*, after having performed the computation on the extended Kovalevskaya system. The normal application of the direct method to original equations (3.8) would fail to produce the full set of conservation laws. Indeed, in order to avoid multipliers singular on solutions of first PDE (3.8a), one would need to exclude from $\{\Lambda_\sigma\}_{\sigma=1}^3$ at least one derivative appearing in the PDE (3.8a). Yet doing so would clearly lead to the loss of the conservation law corresponding to the multipliers with $C_1 \neq 0$.

Remark 2. In the direct conservation law computation for extended Kovalevskaya system (4.1a)–(4.1c), one gets a linear overdetermined system of 2869 linear determining PDEs for the multipliers $\{\widehat{\Lambda}_\sigma\}_{\sigma=1}^3$ (4.4)–(4.6). In a similar computation for the original equations of motion (3.8), one has 5711 multiplier determining PDEs for the multipliers $\{\Lambda_\sigma\}_{\sigma=1}^3$ (5.4). The simplification times using `Maple` `rifsimp` routine are comparable in both cases.

C. Comparison of symmetries and conservation laws

According to Noether's first theorem, for the variational formulation (3.8) of the incompressible Mooney-Rivlin equations, there is a one-to-one correspondence between local symmetries in evolutionary form (5.1) and local conservation law multipliers. Rewriting point symmetries' (5.3) in the evolutionary form and comparing them with multipliers (5.4), one arrives at the correspondence presented in Table I. Clearly, all symmetries (5.3) except for the scaling symmetry Y^9 are variational symmetries. In particular, symmetry components for the hydrostatic pressure correspond to the multiplier Λ_1; symmetry components for x^1, x^2 correspond to the multipliers Λ_2, Λ_3.

VI. DISCUSSION

In the current paper, the nonlinear PDE system of equations of motion of incompressible hyperelastic Mooney-Rivlin material (3.6) was considered in a two-dimensional Cartesian framework. The system involves a coupled set of semi-linear wave equations, and a first-order differential constraint, corresponding to the incompressibility condition. The model involves three independent variables X^1, X^2, t and three dependent variables x^1, x^2, p and admits a classical variational formulation with the Lagrangian given by (3.9). The equations of motion can be represented in a variational form (3.8) arising as Euler-Lagrange equations (3.12).

The main goal of this paper is to illustrate the optimal procedure for systematic construction of local conservation laws on an example of a highly nontrivial physical PDE system that admits an extended Kovalevskaya form. As per Theorem 1, all local conservation laws of such systems arise from nonsingular multipliers independent of leading derivatives and their differential consequences.

In Section IV, the governing equations were rewritten in extended Kovalevskaya form (2.5) with respect to X^1, and the direct method was applied to seek second-order conservation law

TABLE I. Comparison of Lie point symmetries (5.3) in the evolutionary form and local conservation law multipliers (5.4) of the variational form of incompressible Mooney-Rivlin equations (3.8).

Symmetry generator	Conservation law multipliers
$\widetilde{Y}^1 = p_t \frac{\partial}{\partial p} + x^1_t \frac{\partial}{\partial x^1} + x^2_t \frac{\partial}{\partial x^2}$ [time translation]	$\Lambda_1 = p_t, \quad \Lambda_2 = x^1_t, \quad \Lambda_3 = x^2_t$ [conservation of energy]
$\widetilde{Y}^2 = p_1 \frac{\partial}{\partial p} + x^1_1 \frac{\partial}{\partial x^1} + x^2_1 \frac{\partial}{\partial x^2}$ [translation in X^1]	$\Lambda_1 = p_1, \quad \Lambda_2 = x^1_1, \quad \Lambda_3 = x^2_1$ [conservation of momentum in X^1]
$\widetilde{Y}^3 = p_2 \frac{\partial}{\partial p} + x^1_2 \frac{\partial}{\partial x^1} + x^2_2 \frac{\partial}{\partial x^2}$ [translation in X^2]	$\Lambda_1 = p_2, \quad \Lambda_2 = x^1_2, \quad \Lambda_3 = x^2_2$ [conservation of momentum in X^2]
$\widetilde{Y}^4 = (X^1 p_2 - X^2 p_1) \frac{\partial}{\partial p}$ $+ (X^1 x^1_2 - X^2 x^1_1) \frac{\partial}{\partial x^1} + (X^1 x^2_2 - X^2 x^2_1) \frac{\partial}{\partial x^2}$ [rotations in X^1-X^2 plane]	$\Lambda_1 = X^1 p_2 - X^2 p_1,$ $\Lambda_2 = X^1 x^1_2 - X^2 x^1_1,$ $\Lambda_3 = X^1 x^2_2 - X^2 x^2_1$ [angular momentum in Lagrangian frame]
$\widetilde{Y}^5 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$ [rotations in x^1-x^2 plane]	$\Lambda_1 = 0, \quad \Lambda_2 = x^2, \quad \Lambda_3 = -x^1$ [angular momentum in Eulerian frame]
$\widetilde{Y}^6 = -F_1''(t)\, x^1 \frac{\partial}{\partial p} + F_1(t) \frac{\partial}{\partial x^1}$ [translation ($F_2 = \text{const}$)/generalized Galilean boost in x^1]	$\Lambda_1 = -f_1''(t) x^1, \Lambda_2 = f_1(t), \Lambda_3 = 0$ [momentum ($f_1 = \text{const}$)/generalized momentum in x^1]
$\widetilde{Y}^7 = -F_2''(t)\, x^2 \frac{\partial}{\partial p} + F_2(t) \frac{\partial}{\partial x^2}$ [translation ($F_2 = \text{const}$)/generalized Galilean boost in x^2]	$\Lambda_1 = -f_2''(t) x^2, \Lambda_2 = 0, \Lambda_3 = f_2(t)$ [momentum ($f_2 = \text{const}$)/generalized momentum in x^2]
$\widetilde{Y}^8 = F^3(t) \frac{\partial}{\partial p}$ [time dependent translation in p]	$\Lambda_1 = f^3(t), \quad \Lambda_2 = 0, \quad \Lambda_3 = 0$ [generalized incompressibility condition]
$\widetilde{Y}^9 = (x^1 - X^1 x^1_1 - X^2 x^1_2 - t x^1_t) \frac{\partial}{\partial x^1} +$ $(x^2 - X^1 x^2_1 - X^2 x^2_2 - t x^2_t) \frac{\partial}{\partial x^2} + (-X^1 p_1 - X^2 p_2 - t p_t) \frac{\partial}{\partial p}$ [scaling]	[No corresponding conservation law]

multipliers. The conserved densities and fluxes were consequently derived in Section IV B. The set of conservation laws includes classical ones such as energy, momenta, and angular momenta in material (Lagrangian) and actual (Eulerian) configurations, as well as extended conservation law families involving arbitrary functions of time: the generalized momenta and the generalized incompressibility condition. Interestingly, to the infinite families of generalized linear momentum conservation laws (4.7) and (4.8) in the Eulerian frame, there correspond single conservation laws (4.15) and (4.16) in the Lagrangian frame. This latter result seems somewhat counterintuitive, given the formal equivalence of the two frames, and the invariance of conservation laws with respect to local coordinate transformations (e.g., Ref. 6). It would be of interest to investigate this question further, rewriting incompressible Mooney-Rivlin equations (3.8) by interchanging dependent and independent spatial variables and studying the resulting PDE system.

Mechanical models of compressible and incompressible elastic solids without dissipation are known to admit classical variational principles (e.g., Ref. 14). Due to the presence of the special property, namely, the variational structure, of the equations at hand, it was of interest to attempt the conservation law construction with the help of the Noether's first theorem. It is normally done through the computation of local symmetries, identification of the variational ones, and the use of the evolutionary forms of the latter as conservation law multipliers (e.g., Ref. 5). A complete set of first-order symmetries (5.3) was computed in Section V A (only point symmetries were found). All symmetries but the scaling symmetry Y^9 are variational symmetries. The correspondence between these symmetries and the conservation law multipliers of the Mooney-Rivlin equations in variational form (3.8) was established and listed in Table I.

We now turn to the question most important from the practical point of view: *for a given PDE system, what is the preferred way of local conservation law computations?* For a PDE system without a variational structure, admitting a Kovalevskaya form, the conservation laws can be computed using the following approaches:

(a) Direct construction method applied to the system as it stands, or in some other preferred form.
(b) Direct construction method applied to the system in the Kovalevskaya form (standard, general, or extended).
(c) "Brute force" computation of conservation law fluxes/density from the local form (1.1).

If a given PDE system has a variational formulation, there is an additional approach:

(d) Computation of local variational symmetries and application of the Noether's first theorem.

For the example of the Mooney-Rivlin equations considered in this paper, the analysis presented above clearly hints that (b) is the method of choice.

- On comparing (a) and (b), one observes that even though multipliers $\{\Lambda_\sigma\}_{\sigma=1}^3$ (5.4) have a much simpler writing than the multipliers $\{\widehat{\Lambda}_\sigma\}_{\sigma=1}^3$ (4.4)–(4.6) for the Kovalevskaya form of the system, one would not be able to compute the former without an *a priori* knowledge of their structure (see Remark 1).
- On comparing the approaches (b) and (c), we note that (b) is clearly preferred for symbolic computations. It is so because generic unknown fluxes/density following from (1.1) are defined up to arbitrary dependence on trivial divergence expressions; this freedom leads to a substantially increased complexity of, and freedom in, the flux/density determining equations.
- Finally, we compare the approaches (b) and (d). In order to use Noether's theorem, one needs to compute the variational symmetries. Symmetry generators arise as solutions of symmetry determining equations on the solution manifold of the given PDE system. To compute a complete set of point or local symmetries, one needs to use the Kovalevskaya form of the given equations. On the solution space, leading derivatives present in symmetry generators must be replaced by the corresponding derivatives from the right-hand sides of the equations, leading potentially to complicated forms of symmetry generators, which can be simplified on the solution space. After the computation of the local symmetries, non-variational ones need to be excluded. The variational symmetries subsequently yield conservation law multipliers. None of these complications arises when one computes conservation law multipliers through the

direct method. Indeed, the multiplier determining equations are solved off the solution space of the given PDEs.

In practical conservation law computations for PDE systems that admit a standard, a general, or an extended Kovalevskaya form, it is important to use such a form for the computations, even though it might lead to more complicated multiplier forms. The use of the Kovalevskaya form ensures that no conservation laws are missed when one poses a general form of non-singular multipliers depending on derivatives up to some fixed order.

When a given PDE system does not have a Kovalevskaya form, or when the completeness of conservation law computations is not a requirement, the direct method can be applied to any preferred form of the equations at hand. Numerous examples of such computations exist in the literature, including Refs. 11, 15, and 16, where rather complicated dynamical models arising in fluid and solid mechanics are considered.

ACKNOWLEDGMENTS

A.C. is grateful to NSERC of Canada for research support through a Discovery grant.

[1] Alonso, L. M., "On the Noether map," Lett. Math. Phys. **3**(5), 419–424 (1979).
[2] Anco, S. and Bluman, G., "Direct construction of conservation laws from field equations," Phys. Rev. Lett. **78**(15), 2869 (1997).
[3] Anco, S. and Bluman, G., "Direct construction method for conservation laws of partial differential equations. Part I. Examples of conservation law classifications," Eur. J. Appl. Math. **13**(05), 545–566 (2002).
[4] Anco, S. and Bluman, G., "Direct construction method for conservation laws of partial differential equations. Part II. General treatment," Eur. J. Appl. Math. **13**(05), 567–585 (2002).
[5] Bluman, G. W., Cheviakov, A. F., and Anco, S. C., "Construction of conservation laws: How the direct method generalizes Noether's theorem," in *Proceedings of 4th Workshop in Group Analysis of Differential Equations & Integrability* (2009), pp. 1–23, ISBN 978-9963-689-12-5.
[6] Bluman, G. W., Cheviakov, A. F., and Anco, S. C., *Applications of Symmetry Methods to Partial Differential Equations*, Applied Mathematical Sciences Vol. 168 (Springer, 2010).
[7] Bluman, G. W. *et al.*, "Comparing symmetries and conservation laws of nonlinear telegraph equations," J. Math. Phys. **46**(7), 073513 (2005).
[8] Cheviakov, A. F., "GeM software web page, manual, and examples," 2004–2015, http://math.usask.ca/~shevyakov/gem/.
[9] Cheviakov, A. F., "GeM software package for computation of symmetries and conservation laws of differential equations," Comput. Phys. Commun. **176**(1), 48–61 (2007).
[10] Cheviakov, A. F., "Computation of fluxes of conservation laws," J. Eng. Math. **66**(1-3), 153–173 (2010).
[11] Cheviakov, A. F. and Ganghoffer, J.-F., "One-dimensional nonlinear elastodynamic models and their local conservation laws with applications to biological membranes," J. Mech. Behav. Biomed. Mater. (published online 2015); available at http://www.sciencedirect.com/science/article/pii/S1751616115003070.
[12] Ciarlet, P. G., *Mathematical Elasticity, Volume I: Three-Dimensional Elasticity*, Collection Studies in Mathematics and Applications Vol. 20 (Elsevier, 1988).
[13] Fu, Y. B. and Ogden, R. W., *Nonlinear Elasticity: Theory and Applications* (Cambridge University Press, 2001).
[14] Holzapfel, G. A., *Nonlinear Solid Mechanics: A Continuum Approach for Engineering* (Wiley and Sons, Chichester, England, 2000).
[15] Kallendorf, C., Cheviakov, A. F., Oberlack, M., and Wang, Y., "Conservation laws of surfactant transport equations," Phys. Fluids **24**(10), 102105 (2012).
[16] Kelbin, O., Cheviakov, A. F., and Oberlack, M., "New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows," J. Fluid Mech. **721**, 340–366 (2013).
[17] Lei, H.-C. and Blume, J. A., "Lie group and invariant solutions of the plane-strain equations of motion of a neo-Hookean solid," Int. J. Non-Linear Mech. **31**(4), 465–482 (1996).
[18] Marsden, J. E. and Hughes, T. J., *Mathematical Foundations of Elasticity* (Dover, 1994).
[19] Olver, P. J., "Noether's theorems and systems of Cauchy-Kovalevskaya type," in *Nonlinear Systems of Partial Differential Equations in Applied Mathematics*, Lectures in Applied Mathematics Vol. 23 (American Mathematical Society, Providence, R.I, 1986), pp. 81–104.
[20] Olver, P. J., *Applications of Lie Groups to Differential Equations* (Springer, Verlag, 2000), Vol. 107.
[21] Petrovsky, I. G., *Lectures on Partial Differential Equations* (Interscience Publication, Inc., 1954).
[22] Popovych, R. O. and Cheviakov, A. F., "Equivalent characteristics of conservation laws" (unpublished).
[23] Reid, G. J., Wittkopf, A. D., and Boulton, A., "Reduction of systems of nonlinear partial differential equations to simplified involutive forms," Eur. J. Appl. Math. **7**(06), 635–666 (1996).
[24] St. Jean, S., "Applications of symmetries and conservation laws to the study of nonlinear elasticity equations," Master's thesis, University of Saskatchewan, 2014.
[25] Wolf, T., "A comparison of four approaches to the calculation of conservation laws," Eur. J. Appl. Math. **13**(02), 129–152 (2002).