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Some relations between symmetries of nonlocally related systems

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The paper is concerned with relations between local symmetries of systems of partial differential equations (PDE) within trees of nonlocally related PDE systems. It is shown that potential systems arising from a given system through linearly independent conservation laws are nonlocally related to each other. Further, a theorem is proven stating that for a PDE system which has precisely n linearly independent local conservation laws, any local symmetry of the PDE system is a projection of some local symmetry of the n-plet potential system. Moreover, a criterion is presented to determine whether or not a specific local symmetry of a given PDE system is a projection of some local symmetry of a specific potential system. Examples are considered. Finally, a formula for a symmetry of a given PDE system in terms of a local symmetry of a nonlocally related subsystem is given. The formula can be used to determine whether a symmetry of the subsystem yields a local or a nonlocal symmetry of the given system, without the need to undertake a full symmetry classification and comparison between the given system and the subsystem. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4891491]

I. INTRODUCTION

A symmetry of a partial differential equation (PDE) system is a mapping that leaves invariant its solution manifold. Lie's algorithm provides an effective way to find local symmetries of a PDE system. However, there is no uniform procedure to seek nonlocal symmetries for a PDE system. A heuristic approach to find nonlocal symmetries, called quasilocal symmetries, was presented in Refs. 2 and 3. In Ref. 10, a procedure was introduced to seek nonlocal symmetries (potential symmetries) for a given PDE system through potential systems that naturally arise from its nontrivial conservation laws. In Refs. 6 and 7, a more general systematic way was presented to find nonlocal symmetries using nonlocally related PDE systems. Two PDE systems are *equivalent* and *nonlocally related* if they have the following properties:

1. Any solution of either PDE system yields a solution of the other PDE system.
2. The solutions of either PDE system yield all solutions of the other PDE system.
3. The correspondence between the solutions of these two PDE systems is not one-to-one.
4. One PDE system involves at least one variable that is a *nonlocal variable* on the solution manifold of the other PDE system, as defined below.

Due to the equivalence of the solutions between two nonlocally related PDE systems, any symmetry of a nonlocally related PDE system yields a symmetry of a given PDE system. In particular, a local symmetry of a nonlocally related PDE system yields a nonlocal symmetry of a given PDE system if at least one of its infinitesimal generator components for the independent and dependent variables of the given PDE system has an essential dependence on a nonlocal variable. In Ref. 11, a complementary

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symmetry-based method for constructing nonlocal related PDE systems (inverse potential systems) was introduced. It was shown that nonlocal symmetries could also arise from inverse potential systems. In particular, in the framework of nonlocally related PDE systems, nonlocal symmetries for a given PDE system can arise from two different cases.

1. Nonlocal symmetries arising from nonlocally related PDE systems that include all independent and dependent variables of the given PDE system, e.g., potential systems.
2. Nonlocal symmetries arising from nonlocally related PDE systems that do not include all independent and dependent variables of the given PDE system, e.g., subsystems, or inverse potential systems.

It is easy to tell whether a local symmetry of a potential system is a nonlocal symmetry of Type 1 of a given PDE system. One simply needs to check whether any of its infinitesimal generator components for the independent and dependent variables of the given PDE system has an essential dependence on a nonlocal variable. To identify nonlocal symmetries of Type 2, it is common to compute all points or local symmetries of the given PDE system, and then see whether a local symmetry of a considered nonlocally related PDE system is included in the set of point or local symmetries of the given PDE systems. In this paper, we will present an approach to determine whether a local symmetry of a PDE system yields a nonlocal symmetry of its potential system (Type 2) without the need to compute all local symmetries of the potential system.

The main goal of this paper is to investigate the relationship between potential systems arising from linearly independent nontrivial conservation laws and the correspondence between symmetries of a given PDE system and those of its potential systems. We exclusively consider the case of two independent variables. The paper is organized as follows.

In Sec. II, we prove that any two potential systems arising from two linearly independent conservation laws are nonlocally related. In Sec. III, we show that any local symmetry of a PDE system with precisely n local conservation laws can be obtained by projection of some local symmetry of its n-plet potential system. Additionally, a procedure is presented to determine whether or not a specific local symmetry of an arbitrary given PDE system yields a local or a nonlocal symmetry of a potential system arising from any prescribed set of its local conservation laws. In Sec. IV, we present an approach to determine whether a local symmetry of a subsystem yields a nonlocal symmetry of its potential system, without computing all local symmetries of the potential system. Examples are considered.

Throughout this paper, we use the package `GeM` for `Maple`[12] for symbolic computations.

II. POTENTIAL SYSTEMS ARISING FROM LINEARLY INDEPENDENT NONTRIVIAL CONSERVATION LAWS: NONLOCAL SYMMETRIES

Consider a PDE system $\mathbf{R}\{x, t; u\}$ with two independent variables (x, t) and $m \geq 1$ dependent variables $u = (u^1, \ldots, u^m)$, given by

$$R^\sigma[u] = R^\sigma(x, t, u, \partial u, \partial^2 u, \ldots, \partial^l u) = 0, \quad \sigma = 1, \ldots, s. \tag{2.1}$$

In (2.1) and below, $f[u]$ denotes a differential function depending on x, t, u and the derivatives of u up to some finite order.

Suppose that the PDE system (2.1) is not abnormal (e.g., Ref. 16), and has a local conservation law[8,16]

$$\operatorname{div}(\Phi[u], -\Psi[u]) \equiv D_t\Phi[u] - D_x\Psi[u] = 0, \tag{2.2}$$

where D_t and D_x denote the total derivatives with respect to the indicated independent variables, e.g.,

$$D_x = \frac{\partial}{\partial x} + \sum_{i=1}^{m} u^i_x \frac{\partial}{\partial u^i} + \sum_{i=1}^{m} u^i_{xx} \frac{\partial}{\partial u^i_x} + \sum_{i=1}^{m} u^i_{xt} \frac{\partial}{\partial u^i_t} + \cdots. \tag{2.3}$$

(Note the choice of the minus sign before D_x for the convenience of notation).

The following definitions will be used.

Definition 2.1. A conservation law (2.2) is *trivial* if its fluxes are of the form $\Phi[u] = M^1[u] + D_x\mathcal{F}[u]$, $\Psi[u] = M^2[u] + D_t\mathcal{F}[u]$, where $M^i[u]$ are functions of x, u and derivatives of u that vanish on the solutions of the given PDE system (2.1), and $D_t D_x \mathcal{F}[U] - D_x D_t \mathcal{F}[U] \equiv 0$ is a differential identity holding for an arbitrary function U.

Definition 2.2. A set of l conservation laws is *linearly dependent* if there exists their nontrivial linear combination which is a trivial conservation law.

Based on a nontrivial conservation law (2.2), one constructs the *potential equations*

$$v_x = \Phi[u], \qquad v_t = \Psi[u], \tag{2.4}$$

and consequently, the *potential system* $\mathbf{S}\{x, t; u, v\}$ of the PDE system (2.1) given by

$$\begin{aligned} v_x &= \Phi[u], \\ v_t &= \Psi[u], \\ R^\sigma[u] &= 0, \qquad \sigma = 1, \ldots, s. \end{aligned} \tag{2.5}$$

Definition 2.3. A variable μ is a *nonlocal variable* of the given PDE system (2.1) if, for any local differential function $G[u]$, μ is functionally independent of $G[u]$, i.e., the Jacobian (the Poisson bracket)

$$\{\mu, G[u]\}_{(t,x)} \equiv \mu_t D_x G[u] - \mu_x D_t G[u] \neq 0$$

on solutions u of the PDE system (2.1).

The following theorem holds.

Theorem 2.1. *Suppose that a given PDE system* $\mathbf{R}\{x, t; u\}$ (2.1) *has a nontrivial conservation law* (2.2). *Then the potential variable* v *defined by the corresponding potential equations* (2.4) *is a nonlocal variable.*

Proof. The theorem is proven by contradiction. Assume that v is a local variable, i.e., for any solution u of the given system (2.1), $v = F[u]$. It follows that on solutions of the given PDE system, $\Phi[u] = v_x = D_x F[u]$, $\Psi[u] = v_t = D_t F[u]$, and hence the conservation law (2.2), on solutions of (2.1), reduces to a trivial conservation law of the second kind: $(D_tD_x - D_xD_t)F[u] \equiv 0$. □

Remark 2.1. We note that for some special PDE systems, formally trivial conservation laws may arise even as equations within a given PDE system! Examples are, for example, abnormal PDE systems (cf. the section on Noether's Second theorem in Ref. 16), as well as normal systems like the linear advection equation $u_t - u_x = 0$ (for details, see Ref. 13). Indeed, the latter conservation law yields the potential equations $v_t = v_x = u$. On solutions of $u_t - u_x = 0$, one has

$$\{v, F[u]\}_{(t,x)} \equiv v_t u_x - v_x u_t = u(u_x - u_t) = 0,$$

hence the potential variable is a local function of u. In particular, if the solutions of $u_t - u_x = 0$ are denoted by $u = q'(x + t)$, then $v = F[u, C] = q(x + t)$, where the prime denotes the ordinary derivative, and C is an arbitrary constant. Then the conservation law itself, on solutions of the advection PDE, appears to be formally trivial:

$$u_t - u_x \equiv D_t(D_x F[u]) - D_x(D_t F[u]) \equiv 0.$$

This example illustrates the possibility of a local, but still not one-to-one, relationship between the potential variable and local variable(s) of a given PDE system.

A. Relationship between potential systems

If a given PDE system $\mathbf{R}\{x, t; u\}$ (2.1) has $r \geq 2$ nontrivial linearly independent local conservation laws, one can construct r singlet potential systems of the PDE system (2.1). The following theorem holds, which was partly stated in Ref. 8, and is proven in its fullness below.

Theorem 2.2. *Suppose that a PDE system (2.1) has two nontrivial local conservation laws, given by*

$$D_t \Psi^i[u] - D_x \Phi^i[u] = 0, \qquad i = 1, 2. \tag{2.6}$$

Define the corresponding potential variables by

$$v_x = \Psi^1[u], \quad v_t = \Phi^1[u], \tag{2.7}$$

$$w_x = \Psi^2[u], \quad w_t = \Phi^2[u]. \tag{2.8}$$

Then the corresponding potential system $\mathbf{S}^1\{x, t; u, v\}$ *given by* (2.1), (2.7) *and the potential system* $\mathbf{S}^2\{x, t; u, w\}$ *given by* (2.1), (2.8) *are locally related if and only if the conservation laws* (2.6) *are linearly dependent.*

Proof. (a) First, we show that if the conservation laws are linearly dependent, the potential systems are locally related. Indeed, since (2.6) are linearly dependent, then according to Definition 2.1, a nontrivial linear combination of their fluxes equals fluxes of a trivial conservation law. Without loss of generality,

$$w_x = c_1 \Psi^1[u] + D_x \mathcal{F}[u], \qquad w_t = c_1 \Phi^1[u] + D_t \mathcal{F}[u], \qquad c_1 = \text{const.}$$

Using (2.7), one has

$$(w - c_1 v)_x = D_x \mathcal{F}[u], \qquad (w - c_1 v)_t = D_t \mathcal{F}[u], \tag{2.9}$$

hence for every solution u of the given system $\mathbf{R}\{x, t; u\}$, the potential w is given by

$$w = F[u, v] \equiv c_1 v + \mathcal{F}[u] + c_2, \qquad c_2 = \text{const},$$

and hence the potential systems $\mathbf{S}^1\{x, t; u, v\}$ and $\mathbf{S}^2\{x, t; u, w\}$ are locally related.

(b) Second, we need to prove the converse: if the conservation laws (2.6) are linearly independent, then the potential systems $\mathbf{S}^1\{x, t; u, v\}$ and $\mathbf{S}^2\{x, t; u, w\}$ are nonlocally related. It suffices to show that w is a nonlocal variable of $\mathbf{S}^1\{x, t; u, v\}$. To prove by contradiction, suppose that w can be expressed as a local differential function of the variables x, t, u, v of the system in $\mathbf{S}^1\{x, t; u, v\}$, i.e., $w = F[u, v]$. Then Eq. (2.8) yields

$$\begin{aligned} \Psi^2[u] &= D_x F = G(x, t, u, v, \partial u, \ldots, \partial^{p_1} u), \\ \Phi^2[u] &= D_t F = H(x, t, u, v, \partial u, \ldots, \partial^{p_2} u), \end{aligned} \tag{2.10}$$

where the derivatives of v have been substituted through (2.7). Since the left-hand sides of the first two equations in (2.10) have no dependence on v, it follows that $G_v = H_v \equiv 0$. Moreover, one has $(D_x F)_v = (D_t F)_v = D_x(F_v) = D_t(F_v) \equiv 0$. Thus $F_v = \text{const} = c$, and $F[u, v] = cv + \alpha[u]$ for some differential function α. Consequently, (2.10) becomes

$$c v_x + D_x \alpha[u] = \Psi^2[u], \qquad c v_t + D_t \alpha[u] = \Phi^2[u]. \tag{2.11}$$

Using (2.7), one has

$$D_x \alpha[u] = \Psi^2[u] - c\Psi^1[u], \qquad D_t \alpha[u] = \Phi^2[u] - c\Phi^1[u] \tag{2.12}$$

on any solution of (2.1). Since

$$D_t(\Psi^2[u] - c\Psi^1[u]) - D_x(\Phi^2[u] - c\Phi^1[u]) = D_t(D_x \alpha[u]) - D_x(D_t \alpha[u]) = 0 \tag{2.13}$$

is a trivial conservation law for the given PDE system (2.1), it implies that the conservation laws (2.6) are linearly dependent, which contradicts the assumption of part (b). Hence w is indeed a nonlocal variable of $\mathbf{S}^1\{x, t; u, v\}$. Similarly, one shows that v is a nonlocal variable of $\mathbf{S}^2\{x, t; u, w\}$. Thus, $\mathbf{S}^1\{x, t; u, v\}$ and $\mathbf{S}^2\{x, t; u, w\}$ are nonlocally related. □

One may look at a very simple yet natural example. Consider a single conservation law $D_t\Psi[u] - D_x\Phi[u] = 0$, and two potential variables v, w defined by

$$v_x = \Phi[u], \qquad v_t = -\Psi[u],$$

$$w_x = \Phi[u], \qquad w_t = -\Psi[u].$$

Evidently, $w = v + c$, c being an arbitrary constant, and the relationship of v, w is not one-to-one. However, from Theorem 2.2, as well as by common sense, it follows that the potentials v, w are locally related. Hence, even though the relationship of v, w is not one-to-one, no new results can be obtained from the consideration of the potential system $\mathbf{S}^2\{x, t; u, w\}$ compared to results following from $\mathbf{S}^1\{x, t; u, v\}$.

B. Nonlocal symmetries

Suppose that the potential system (2.5) has a point symmetry given by

$$\begin{cases} \bar{x} = x + \varepsilon\xi(x, t, u, v) + \mathrm{O}(\varepsilon^2), \\ \bar{t} = t + \varepsilon\tau(x, t, u, v) + \mathrm{O}(\varepsilon^2), \\ \bar{u}^i = u^i + \varepsilon\eta^i(x, t, u, v) + \mathrm{O}(\varepsilon^2), \quad i = 1, \ldots, m, \\ \bar{v} = v + \varepsilon\zeta(x, t, u, v) + \mathrm{O}(\varepsilon^2), \end{cases} \tag{2.14}$$

with infinitesimal generator

$$\mathrm{X} = \xi(x, t, u, v)\frac{\partial}{\partial x} + \tau(x, t, u, v)\frac{\partial}{\partial t} + \sum_{i=1}^{m} \eta^i(x, t, u, v)\frac{\partial}{\partial u^i} + \zeta(x, t, u, v)\frac{\partial}{\partial v}. \tag{2.15}$$

The generator (2.15) corresponds to a *nonlocal symmetry* of the given PDE system (2.1) if it does not yield a local symmetry of (2.1) when projected on the space of its variables. The criterion for determining whether the symmetry (2.15) yields a nonlocal symmetry of the given PDE system (2.1) is given by the following theorem (e.g., Refs. 8–10).

Theorem 2.3. *The point symmetry* (2.15) *of the potential system* (2.5) *yields a nonlocal symmetry* (potential symmetry) *of the given PDE system* (2.1) *if and only if one or more of the infinitesimals* $(\xi(x, t, u, v), \tau(x, t, u, v), \eta^1(x, t, u, v), \ldots, \eta^m(x, t, u, v))$ *depend explicitly on the potential variable* v*, i.e.,*

$$\left(\frac{\partial \xi}{\partial v}\right)^2 + \left(\frac{\partial \tau}{\partial v}\right)^2 + \sum_{i=1}^{m}\left(\frac{\partial \eta^i}{\partial v}\right)^2 > 0.$$

For symmetries of potential systems arising from nontrivial and linearly independent conservation laws, one consequently has the following theorem.

Theorem 2.4. *Let* $\mathbf{S}^1\{x, t; u, v\}$ *and* $\mathbf{S}^2\{x, t; u, w\}$ *be two potential systems of a given PDE system* (2.1) *arising from nontrivial and linearly independent conservation laws. Then a point symmetry of* $\mathbf{S}^2\{x, t; u, w\}$ *with the infinitesimal generator*

$$\mathrm{X} = \xi(x, t, u, w)\frac{\partial}{\partial x} + \tau(x, t, u, w)\frac{\partial}{\partial t} + \sum_{i=1}^{m} \eta^i(x, t, u, w)\frac{\partial}{\partial u^i} + \zeta(x, t, u, w)\frac{\partial}{\partial w}, \tag{2.16}$$

where at least one of the infinitesimals $(\xi(x,t,u,w),\ \tau(x,t,u,w),\ \eta^i(x,t,u,w))$ *essentially depends on the potential variable* w, *i.e.*,

$$\left(\frac{\partial \xi}{\partial w}\right)^2 + \left(\frac{\partial \tau}{\partial w}\right)^2 + \sum_{i=1}^{m}\left(\frac{\partial \eta^i}{\partial w}\right)^2 > 0,$$

yields a nonlocal symmetry of the potential system $\mathbf{S}^1\{x, t; u, v\}$.

Proof. The projection of X on the space of variables $\mathbf{S}^1\{x, t; u, v\}$ involves the variable w, which is a nonlocal variable of $\mathbf{S}^1\{x, t; u, v\}$. □

III. A RELATIONSHIP BETWEEN SYMMETRIES OF PDE SYSTEMS

An effective way to seek nonlocal symmetries for a given PDE system is to apply Lie's algorithm to its nonlocally related PDE systems. Does there exist any relationship between local symmetries of a given PDE system and those of its potential systems?

In the following new theorem, a correspondence between local symmetries of a given PDE system having precisely n linearly independent local conservation laws and local symmetries of its potential systems is presented. If a given PDE system (2.1) has n linearly independent local conservation laws

$$D_t\Psi^i[u] - D_x\Phi^i[u] = 0, \qquad i = 1, \ldots, n, \tag{3.1}$$

then one can construct a *k-plet potential system* of (2.1) given by

$$\begin{aligned} &v_x^{i_j} = \Psi^{i_j}[u],\\ &v_t^{i_j} = \Phi^{i_j}[u], \quad j = 1, \ldots, k,\\ &R^\sigma[u] = 0, \quad \sigma = 1, \ldots, s, \end{aligned} \tag{3.2}$$

involving k potential variables $\{v^{i_j}\}_{j=1}^k$. The largest potential system with n potential variables is an *n-plet potential system*

$$\begin{aligned} &v_x^{i} = \Psi^{i}[u],\\ &v_t^{i} = \Phi^{i}[u], \quad i = 1, \ldots, n,\\ &R^\sigma[u] = 0, \quad \sigma = 1, \ldots, s. \end{aligned} \tag{3.3}$$

By an argument parallel to Theorem 2.2, one can show that all n potential variables $\{v^i\}_{i=1}^n$ arising from the n linearly independent local conservation laws (3.1) are nonlocally related to each other, i.e., none of the potential variables is a local differential function of x, t, u, and other potential variables.

The following important theorem holds.

Theorem 3.1. *Suppose that a given PDE system* (2.1) *has precisely n linearly independent local conservation laws. Then any local symmetry of the PDE system* (2.1) *can be obtained by projection of some local symmetry of its n-plet potential system.*

Proof. Let the n local conservation laws of the given PDE system (2.1) be given by (3.1), and let the corresponding n-plet potential system of (2.1) be given by (3.3). Suppose that $\hat{\mathrm{X}} = \sum_{i=1}^{m}\eta^i[u]\frac{\partial}{\partial u^i}$ is a local symmetry of the given system (2.1) written in the evolutionary form. It suffices to prove that there exist functions $\zeta^j[u, v]$, $j = 1, \ldots, n$, such that $\hat{\mathrm{Y}} = \hat{\mathrm{X}} + \sum_{j=1}^{n}\zeta^j[u, v]\frac{\partial}{\partial v^j}$ is a local symmetry of the n-plet potential system (3.3). (Here $v = (v^1, \ldots, v^n)$.) We denote the infinite prolongations of $\hat{X}$ and $\hat{Y}$ by $\hat{X}^{(\infty)}$ and $\hat{Y}^{(\infty)}$, respectively. Applying $\hat{Y}^{(\infty)}$ to the functions

$$\begin{aligned} &V_x^j - \Psi^j[U], \\ &V_t^j - \Phi^j[U], \quad j = 1, \dots, n, \\ &R^\sigma[U], \quad \sigma = 1, \dots, s, \end{aligned} \tag{3.4}$$

where (U, V) are arbitrary functions, one obtains

$$\begin{aligned} &D_x \zeta^j[U, V] - \hat{X}^{(\infty)} \Psi^j[U], \\ &D_t \zeta^j[U, V] - \hat{X}^{(\infty)} \Phi^j[U], \quad j = 1, \dots, n, \\ &\hat{X}^{(\infty)} R^\sigma[U], \quad \sigma = 1, \dots, s. \end{aligned} \tag{3.5}$$

Then $\hat{Y}$ is a local symmetry of the n-plet potential system (3.3) if and only if (3.5) vanish on any solution $(U, V) = (u, v) = (u(x, t), v(x, t))$ of the n-plet potential system (3.3). From the infinitesimal criterion for a symmetry, $\hat{X}$ is a local symmetry of (2.1) if and only if its infinite prolongation $\hat{X}^{(\infty)}$ satisfies

$$\hat{X}^{(\infty)} R^\sigma[u] = 0, \quad \sigma = 1, \dots, s \tag{3.6}$$

on any solution of (2.1). Hence, Eq. (3.6) holds on any solution of the n-plet potential system (3.3). Therefore, it suffices to prove that there exist some functions $\zeta^j[u, v]$ so that the equations

$$\begin{aligned} &D_x \zeta^j[u, v] = \hat{X}^{(\infty)} \Psi^j[u], \\ &D_t \zeta^j[u, v] = \hat{X}^{(\infty)} \Phi^j[u], \quad j = 1, \dots, n \end{aligned} \tag{3.7}$$

hold on any solution of the n-plet potential system (3.3).

Lemma 3.1. Let $\operatorname{div}(\Phi[u], \Psi[u]) = 0$ *be a conservation law of a given PDE system. If* $\hat{X}$ *is a symmetry in evolutionary form of the given PDE system, then the induced flux/density pair* $(\widetilde{\Phi}[u], \widetilde{\Psi}[u]) = \big(\hat{X}^{(\infty)}(\Phi[u]), \hat{X}^{(\infty)}(\Phi[u])\big)$ *also yields a conservation law for the given PDE system:* $\operatorname{div}(\widetilde{\Phi}[u], \widetilde{\Psi}[u]) = 0$.

The proof can be found, e.g., in Ref. 16. It is worth noting that the new conservation law $\operatorname{div}(\widetilde{\Phi}[u], \widetilde{\Psi}[u]) = 0$ may be trivial.

Lemma 3.1 shows that a symmetry maps fluxes of a conservation law into fluxes of a conservation law. Since $\hat{X}$ is a local symmetry of (2.1), the entries $\tilde{\Psi}^j[u] = \hat{X}^{(\infty)} \Psi^j[u]$ and $-\tilde{\Phi}^j[u] = \hat{X}^{(\infty)} \left(-\Phi^j[u]\right)$, $j = 1, \dots, n$, must be fluxes of conservation laws of (2.1), i.e., for each $j = 1, \dots, n$, $D_t \tilde{\Psi}^j[u] - D_x \tilde{\Phi}^j[u] = 0$ is a conservation law of (2.1). Since (2.1) has precisely n linearly independent local conservation laws, it follows that, for each $j = 1, \dots, n$,

$$\begin{aligned} \hat{X}^{(\infty)} \Psi^j[u] = \tilde{\Psi}^j[u] = \sum_{k=1}^{n} a_k^j \Psi^k[u] + T^j[u], \\ \hat{X}^{(\infty)} \Phi^j[u] = \tilde{\Phi}^j[u] = \sum_{k=1}^{n} a_k^j \Phi^k[u] + S^j[u] \end{aligned} \tag{3.8}$$

for some constants a_k^j, $k = 1, \dots, n$, and $D_t T^j[u] - D_x S^j[u] = 0$ is a trivial conservation law of (2.1). In particular, for each $j = 1, \dots, n$, $T^j[u] = A^j[u] + B^j[u]$ and $S^j[u] = F^j[u] + G^j[u]$, where $D_t A^j[u] - D_x F^j[u] = 0$ is a trivial conservation law of the first type, and $D_t B^j[u] - D_x G^j[u] = 0$ is a trivial conservation law of the second type. Consequently, $(A^j[u], F^j[u])$ vanish on any solution of (2.1), and hence $(A^j[u], F^j[u])$ vanish on any solution of the n-plet potential system (3.3). Since $(B^j[u], G^j[u])$ yield a null divergence, it follows that there exists some function $H^j[u]$ such that $B^j[u] = D_x H^j[u]$ and $G^j[u] = D_t H^j[u]$.

Now let $\zeta^j[U, V]$ be the functions

$$\zeta^j[U, V] = \sum_{k=1}^{n} a_k^j V^k + H^j[U], \quad j = 1, \dots, n. \tag{3.9}$$

Then, on any solution $(U, V) = (u, v) = (u(x, t), v(x, t))$ of the n-plet potential system (3.3), one has

$$\begin{aligned}
D_x \zeta^j[u, v] &= \sum_{k=1}^{n} a_k^j v_x^k + D_x H^j[u] = \sum_{k=1}^{n} a_k^j \Psi^k[u] + B^j[u] \\
&= \sum_{k=1}^{n} a_k^j \Psi^k[u] + B^j[u] + A^j[u] = \sum_{k=1}^{n} a_k^j \Psi^k[u] + T^j[u] \\
&= \hat{\mathrm{X}}^{(\infty)} \Psi^j[u], \\
D_t \zeta^j[u, v] &= \sum_{k=1}^{n} a_k^j v_t^k + D_t H^j[u] = \sum_{k=1}^{n} a_k^j \Phi^k[u] + G^j[u] \\
&= \sum_{k=1}^{n} a_k^j \Phi^k[u] + G^j[u] + F^j[u] = \sum_{k=1}^{n} a_k^j \Phi^k[u] + S^j[u] \\
&= \hat{\mathrm{X}}^{(\infty)} \Phi^j[u], \quad j = 1, \dots, n.
\end{aligned} \tag{3.10}$$

Hence on solutions $(U, V) = (u, v)$ of the n-plet potential system (3.3), functions $\zeta^j[u, v]$ given by (3.9) satisfy Eq. (3.7). By construction,

$$\hat{\mathrm{Y}} = \hat{\mathrm{X}} + \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_k^j v^k + H^j[u] \right) \frac{\partial}{\partial v^j} \tag{3.11}$$

is a local symmetry of the n-plet potential system (3.3), whose projection on the space of variables of the given PDE system (2.1) is the local symmetry $\hat{\mathrm{X}} = \sum_{i=1}^{m} \eta^i[u] \frac{\partial}{\partial u^i}$ of the latter. □

Remark 3.1. The formula (3.11) explicitly provides the local symmetry $\hat{\mathrm{Y}}$ of the n-plet potential system (3.3) corresponding to *any* local symmetry $\hat{\mathrm{X}}$ of a given PDE system (2.1) which has precisely n local conservation laws. A detailed example is considered below.

Corollary 3.1. Consider a PDE system with two independent variables (x, t) and m dependent variables $u = (u^1, \dots, u^m)$, given by

$$R^\sigma[u] = 0, \quad \sigma = 1, \dots, s. \tag{3.12}$$

Suppose that $\hat{\mathrm{X}}$ is a local symmetry of (3.12) in evolutionary form, and $D_t \Psi^i[u] - D_x \Phi^i[u] = 0$, $i = 1, \dots, k$ are linearly independent local conservation laws of the system (3.12). If for each $\nu = 1, \dots, k$, the conservation law

$$D_t \left(\hat{\mathrm{X}}^{(\infty)} \Psi^\nu[u] \right) - D_x \left(\hat{\mathrm{X}}^{(\infty)} \Phi^\nu[u] \right) = 0 \tag{3.13}$$

is equivalent to the conservation law

$$D_t \left(\sum_{i=1}^{k} a_i^\nu \Psi^i[u] \right) - D_x \left(\sum_{i=1}^{k} a_i^\nu \Phi^i[u] \right) = 0, \tag{3.14}$$

for some constants a_i^ν, $i = 1, \ldots, k$, then $\hat{X}$ can be obtained by projection of some local symmetry of the corresponding k-plet potential system given by

$$\begin{aligned} v_x^i &= \Psi^i[u], \\ v_t^i &= \Phi^i[u], \quad i = 1, \ldots, k, \\ R^\sigma[u] &= 0, \quad \sigma = 1, \ldots, s. \end{aligned} \tag{3.15}$$

Proof. The proof in Theorem 3.1 can be directly extended to the k-plet potential system (3.15). □

As a simple yet illustrative example, consider a nonlinear diffusion equation

$$u_t - (f(u)u_x)_x = D_t\, u - D_x\,(f(u)u_x) = 0. \tag{3.16}$$

The PDE (3.16) is invariant, for instance, with respect to x-translations and (x, t)-scalings, corresponding to symmetry generators in evolutionary form $\hat{X}_1 = u_x \frac{\partial}{\partial u}$, $\hat{X}_2 = (2tu_t + xu_x)\frac{\partial}{\partial u}$.

Considering the symmetry $\hat{X}_1$ first, one notes that for the density $\Psi[u] = u$, $\hat{X}^{(\infty)}\Psi[u] = u_x$, and for the flux $\Phi[u] = f(u)u_x$, $\hat{X}^{(\infty)}\Phi[u] = D_x(f(u)u_x)$. Hence, the conservation law (3.16) is mapped into a conservation law

$$D_t(u_x) - D_x\,(D_x(f(u)u_x)) = D_x\,(u_t - (f(u)u_x)_x) = 0,$$

which is a trivial conservation law. Therefore, the conditions of Corollary 3.1 are satisfied. It follows that the potential system

$$v_x = u, \quad v_t = f(u)u_x \tag{3.17}$$

has a local symmetry $\hat{Y}_1$ projecting onto $\hat{X}_1$; it is given by

$$\hat{Y}_1 = \hat{X}_1 + v_x \frac{\partial}{\partial v} = \hat{X}_1 + u\frac{\partial}{\partial v},$$

which is indeed of the form (3.11).

Now consider the scaling symmetry $\hat{X}_2$. In order to check whether this symmetry is inherited by the potential system (3.17) using Corollary 3.1, one needs to compute the conservation law (3.13), which reads

$$\begin{aligned} &D_t\left(\hat{X}_2^{(\infty)}u\right) - D_x\left(\hat{X}_2^{(\infty)}(f(u)u_x)\right) \\ &= D_t\,(xu_x + 2tu_t) - D_x\left((xu_x + 2tu_t)f'(u)u_x + (2tu_{tx} + u_x + xu_{xx})f(u)\right) = 0. \end{aligned}$$

The latter expression is not a trivial conservation law; it can be shown to be equivalent to the conservation law $D_t(-u) + D_x(f(u)u_x) = 0$, which is the given PDE (3.16) multiplied by the factor $a = -1$. Hence, the conditions of Corollary 3.1 are again satisfied, and the potential system (3.17) has a local symmetry $\hat{Y}_2$ projecting onto $\hat{X}_2$. That symmetry is a scaling given by $\hat{Y}_2 = (2tu_t + xu_x)\frac{\partial}{\partial u} + (2tv_t + xv_x - v)\frac{\partial}{\partial v}$.

Remark 3.2. In order to apply Theorem 3.1 to a given PDE system, one has to obtain all its local conservation laws. The direct construction method[4,5] provides an effective way to find local conservation laws. The following theorem shows that one is able to obtain all local conservation laws of an evolution equation of even order.[1,14]

Theorem 3.2. *Consider the (1 + 1)-dimensional scalar evolution equation*

$$u_t = F(x, t, u, \partial_x u, \ldots, \partial_x^{2l} u) \tag{3.18}$$

with two independent variables (x, t) and one dependent variable u, of even order $2l$ in terms of the spatial derivatives. If a PDE (3.18) has a conservation law

$$D_t\Psi[u] - D_x\Phi[u] = 0, \tag{3.19}$$

where all temporal derivatives have been excluded from $\Psi[u]$, $\Phi[u]$ *through the substitution of* (3.18) *and its differential consequences, then the maximal order of a spatial derivative of u in* $\Psi[u]$ *is l (up to conservation law equivalence).*

Example 3.1. Consider the nonlinear diffusion equation

$$u_t = \left(u^{-\frac{4}{3}} u_x\right)_x. \tag{3.20}$$

Using the direct method and Theorem 3.2, one can show that the nonlinear diffusion equation (3.20) has exactly two linearly independent local conservation laws given by

$$u_t - \left(u^{-\frac{4}{3}} u_x\right)_x = 0, \tag{3.21}$$

$$(xu)_t - \left(xu^{-\frac{4}{3}} u_x + 3u^{-\frac{1}{3}}\right)_x = 0. \tag{3.22}$$

From Theorem 3.1 it follows that any local symmetry $\hat{\mathrm{X}}$ of the PDE (3.20) can be obtained by projection of some local symmetry $\hat{\mathrm{Y}}$ of its couplet potential system given by

$$\begin{aligned} v_x &= u, \\ v_t &= u^{-\frac{4}{3}} u_x, \\ \alpha_x &= xu, \\ \alpha_t &= xu^{-\frac{4}{3}} u_x + 3u^{-\frac{1}{3}}. \end{aligned} \tag{3.23}$$

Here $\Psi^1[u] = u$, $\Phi^1[u] = u^{-\frac{4}{3}} u_x$, $\Psi^2[u] = xu$, $\Phi^2[u] = xu^{-\frac{4}{3}} u_x + 3u^{-\frac{1}{3}}$.

Consider the point symmetry $\mathrm{X} = -x^2 \frac{\partial}{\partial x} + 3xu \frac{\partial}{\partial u}$ of the nonlinear diffusion equation (3.20), with the evolutionary form given by $\hat{\mathrm{X}} = (3xu + x^2 u_x) \frac{\partial}{\partial u}$. One can explicitly find $\hat{\mathrm{Y}}$ without applying Lie's algorithm to the potential system (3.23). Applying the corresponding infinite prolongation $\hat{\mathrm{X}}^{(\infty)}$ to the fluxes of the conservation law (3.21), one obtains

$$\begin{aligned} \hat{\mathrm{X}}^{(\infty)}\left(\Psi^1[u]\right) &= \hat{\mathrm{X}}^{(\infty)}(u) = 3xu + x^2 u_x = xu + x^2 u_x + 2xu \\ &= \Psi^2[u] + T^1[u], \\ \hat{\mathrm{X}}^{(\infty)}\left(\Phi^1[u]\right) &= \hat{\mathrm{X}}^{(\infty)}(u^{-\frac{4}{3}} u_x) = 3u^{-\frac{1}{3}} + xu^{-\frac{4}{3}} u_x - \tfrac{4}{3} x^2 u^{-\frac{7}{3}} u_x^2 + x^2 u^{-\frac{4}{3}} u_{xx}, \\ &= \Phi^2[u] + S^1[u], \end{aligned} \tag{3.24}$$

where $D_t T^1[u] - D_x S^1[u] = 0$ is the trivial conservation law given by

$$\begin{aligned} &D_t(x^2 u_x + 2xu) - D_x\left(-\tfrac{4}{3} x^2 u^{-\frac{7}{3}} u_x^2 + x^2 u^{-\frac{4}{3}} u_{xx} - x^2 u_t + x^2 u_t\right) \\ &= D_t(D_x(x^2 u)) - D_x\left(x^2\left(-\tfrac{4}{3} u^{-\frac{7}{3}} u_x^2 + u^{-\frac{4}{3}} u_{xx} - u_t\right) + D_t(x^2 u)\right) = 0. \end{aligned} \tag{3.25}$$

Applying the infinite prolongation $\hat{\mathrm{X}}^{(\infty)}$ to the fluxes of the second conservation law (3.22), one has

$$\begin{aligned} \hat{\mathrm{X}}^{(\infty)}\left(\Psi^2[u]\right) &= \hat{\mathrm{X}}^{(\infty)}(xu) = 3x^2 u + x^3 u_x = T^2[u], \\ \hat{\mathrm{X}}^{(\infty)}\left(\Phi^2[u]\right) &= \hat{\mathrm{X}}^{(\infty)}(xu^{-\frac{4}{3}} u_x + 3u^{-\frac{1}{3}}) = x^3 u^{-\frac{4}{3}} u_{xx} - \tfrac{4}{3} x^3 u^{-\frac{7}{3}} u_x^2 = S^2[u], \end{aligned} \tag{3.26}$$

where $D_t T^2[u] - D_x S^2[u] = 0$ is the trivial conservation law given by

$$D_t(D_x(x^3 u)) - D_x\left(D_t(x^3 u) + x^3(u^{-\frac{4}{3}} u_{xx} - \tfrac{4}{3} u^{-\frac{7}{3}} u_x^2 - u_t)\right) = 0. \tag{3.27}$$

It follows that the constants and the functions $H^j[u]$, $j = 1, 2$, in (3.9) are given by

$$a_1^1 = 0,\ a_2^1 = 1,\ a_1^2 = 0,\ a_2^2 = 0,\ H^1[u] = x^2 u,\ H^2[u] = x^3 u.$$

Consequently, the point symmetry of the potential system (3.23) corresponding to X is given by

$$\hat{\mathrm{Y}} = \hat{\mathrm{X}} + (\alpha + x^2 u)\frac{\partial}{\partial v} + x^3 u \frac{\partial}{\partial \alpha} = \hat{\mathrm{X}} + (\alpha + x^2 v_x)\frac{\partial}{\partial v} + x^2 \alpha_x \frac{\partial}{\partial \alpha},$$

which is the evolutionary form for the point symmetry $\mathrm{Y} = \mathrm{X} + \alpha \frac{\partial}{\partial v}$.

Example 3.2. As a second example, we consider the symmetry classification of a class of fourth-order evolution equations

$$u_t = K(u_{xxx})\, u_{xxxx}, \tag{3.28}$$

where $K(u_{xxx}) \neq \mathrm{const}$ is a constitutive function. It is straightforward to show that for $K(u_{xxx}) = (c_1 u_{xxx} + c_2)^{-2}$, $c_1, c_2 = \mathrm{const}$, the PDE (3.28) has two local conservation laws, and for all other $K(u_{xxx})$, it has only one local conservation laws given by the PDE itself:

$$D_t\, u - D_x F(u_{xxx}) = 0, \tag{3.29}$$

where $F'(u_{xxx}) = K(u_{xxx})$. We restrict to the general case $K(u_{xxx}) \neq (c_1 u_{xxx} + c_2)^{-2}$. In this case, the only potential system of (3.28) is given by

$$\begin{aligned} v_x &= u, \\ v_t &= F(u_{xxx}). \end{aligned} \tag{3.30}$$

According to Theorem 3.1, all point symmetries can be obtained as projections of point symmetries of the potential system (3.30). Thus, in order to classify the point symmetries of the class of equations (3.28), one needs only to classify the point symmetries of the potential system (3.28).

Since the potential system (3.30) is locally related to the fourth order equation

$$v_t = F(v_{xxxx}), \tag{3.31}$$

any local symmetry of (3.30) is a local symmetry of (3.31) and vice versa. We now present the point symmetry classification of the class of equations (3.31) and the corresponding point symmetry classification for the class of potential systems (3.30) in Table I, modulo the equivalence transformations

TABLE I. Point symmetry classification for the class of equations (3.31) and potential systems (3.30).

$F(v_{xxxx})$	#	Admitted point symmetries for (3.31)	Corresponding point symmetries for (3.30)
Arbitrary	7	$\mathrm{Y}_1 = \frac{\partial}{\partial x}$, $\mathrm{Y}_2 = \frac{\partial}{\partial t}$, $\mathrm{Y}_3 = x\frac{\partial}{\partial v}$, $\mathrm{Y}_4 = \frac{x^2}{2}\frac{\partial}{\partial v}$, $\mathrm{Y}_5 = \frac{x^3}{3}\frac{\partial}{\partial v}$, $\mathrm{Y}_6 = \frac{\partial}{\partial v}$, $\mathrm{Y}_7 = 4t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 4v\frac{\partial}{\partial v}$,	$\mathrm{W}_1 = \mathrm{Y}_1$, $\mathrm{W}_2 = \mathrm{Y}_2$, $\mathrm{W}_3 = \mathrm{Y}_3 + \frac{\partial}{\partial u}$, $\mathrm{W}_4 = \mathrm{Y}_4 + x\frac{\partial}{\partial u}$, $\mathrm{W}_5 = \mathrm{Y}_5 + x^2\frac{\partial}{\partial u}$, $\mathrm{W}_6 = \mathrm{Y}_6$, $\mathrm{W}_7 = \mathrm{Y}_7 + 3u\frac{\partial}{\partial u}$
v_{xxxx}^a $(a \neq 1, 0, -1)$	8	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5, \mathrm{Y}_6, \mathrm{Y}_7$, $\mathrm{Y}_8 = 4at\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$	$\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \mathrm{W}_4, \mathrm{W}_5, \mathrm{W}_6, \mathrm{W}_7$, $\mathrm{W}_8 = \mathrm{Y}_8 - u\frac{\partial}{\partial u}$
$e^{v_{xxxx}}$	8	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5, \mathrm{Y}_6, \mathrm{Y}_7$, $\mathrm{Y}_9 = 6t\frac{\partial}{\partial t} - \frac{x^4}{4}\frac{\partial}{\partial v}$	$\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \mathrm{W}_4, \mathrm{W}_5, \mathrm{W}_6, \mathrm{W}_7$, $\mathrm{W}_9 = \mathrm{Y}_9 - x^3\frac{\partial}{\partial u}$
$(v_{xxxx})^{-\frac{3}{5}}$	9	$\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \mathrm{Y}_4, \mathrm{Y}_5, \mathrm{Y}_6, \mathrm{Y}_7, \mathrm{Y}_8$, $\mathrm{Y}_{10} = x^2\frac{\partial}{\partial x} + 3xv\frac{\partial}{\partial v}$	$\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \mathrm{W}_4, \mathrm{W}_5, \mathrm{W}_6, \mathrm{W}_7, \mathrm{W}_8$, $\mathrm{W}_{10} = \mathrm{Y}_{10} + (3v + xu)\frac{\partial}{\partial u}$

TABLE II. Point symmetry classification for the class of equations (3.28).

$K(u_{xxx})$	#	Admitted point symmetries
Arbitrary	6	$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, X_3 = \frac{\partial}{\partial u}, X_4 = x\frac{\partial}{\partial u}, X_5 = x^2\frac{\partial}{\partial u}, X_6 = 4t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}$
au_{xxx}^{a-1} $(a \neq 1, 0, -1)$	7	$X_1, X_2, X_3, X_4, X_5, X_6, X_7 = 4at\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$
$e^{u_{xxx}}$	7	$X_1, X_2, X_3, X_4, X_5, X_6, X_8 = 6t\frac{\partial}{\partial t} - x^3\frac{\partial}{\partial u}$

of (3.30):

$$\begin{aligned}
\tilde{t} &= c_5 t + c_6,\\
\tilde{x} &= c_7 x + c_4,\\
\tilde{u} &= c_3 u + c_8 x^3 + c_9 x^2 + c_{10} x + c_{11},\\
\tilde{v} &= c_3 c_7 v + \frac{1}{4}c_7 c_8 x^4 + \frac{1}{3}c_7 c_9 x^3 + \frac{1}{2}c_7 c_{10} x^2 + c_7 c_{11} x + c_1 t + c_2,\\
\tilde{F}(\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}) &= c_5^{-1}(c_3 c_7 F(u_{xxx}) + c_1),
\end{aligned}$$

where $c_1, \ldots, c_{11}$ are arbitrary constants with $c_3 c_5 c_7 \neq 0$.

From Table I, one immediately obtains the point symmetry classification for the class of equations (3.28), which is listed in Table II. Moreover, when $F(v_{xxxx}) = (v_{xxxx})^{-\frac{3}{5}}$, the symmetry W_{10} yields a nonlocal symmetry of Eq. (3.28).

IV. A SYSTEMATIC APPROACH TO DETERMINE NONLOCAL SYMMETRIES OF TYPE 2 FOR A GIVEN PDE SYSTEM

In the framework of nonlocally related PDE systems, nonlocal symmetries of Type 2 are sought from nonlocally related subsystems and inverse potential systems. The construction of such systems involves cross-differentiation to exclude dependent variables. In this section, we present a connection formula between symmetries of a PDE system and those of its nonlocally related subsystems. For simplicity, we consider a PDE system with two independent variables (x, t) and two dependent variables (u, v) given by

$$\begin{aligned} v_x &= \Psi[u],\\ v_t &= \Phi[u]. \end{aligned} \tag{4.1}$$

A subsystem of (4.1) obtained by cross-differentiation is given by

$$D_t\Psi[u] - D_x\Phi[u] = 0. \tag{4.2}$$

Theorem 4.1. *Suppose that $\hat{X}$ is a local symmetry, in evolutionary form, of (4.2), and α satisfies*

$$\begin{aligned} \alpha_x &= \hat{X}^{(\infty)}\Psi[u],\\ \alpha_t &= \hat{X}^{(\infty)}\Phi[u], \end{aligned} \tag{4.3}$$

on solutions of (4.2). Then the symmetry of (4.1) corresponding to $\hat{X}$ is given by $\hat{Y} = \hat{X} + \alpha\frac{\partial}{\partial v}$. Moreover, $\hat{Y}$ is a nonlocal symmetry of (4.1) if and only if the conservation law

$$D_t\left(\hat{X}^{(\infty)}\Psi[u]\right) - D_x\left(\hat{X}^{(\infty)}\Phi[u]\right) = 0$$

and the conservation law (4.2) are linearly independent.

Proof. Applying the infinite prolongation $\hat{Y}^{(\infty)}$ to the PDE system (4.2), one has

$$\begin{aligned}
\hat{Y}^{(\infty)}(v_x - \Psi[u]) &= D_x\alpha - \hat{X}^{(\infty)}\Psi[u] = \hat{X}^{(\infty)}\Psi[u] - \hat{X}^{(\infty)}\Psi[u] = 0,\\
\hat{Y}^{(\infty)}(v_t - \Phi[u]) &= D_t\alpha - \hat{X}^{(\infty)}\Phi[u] = \hat{X}^{(\infty)}\Phi[u] - \hat{X}^{(\infty)}\Phi[u] = 0,
\end{aligned} \tag{4.4}$$

on any solution of (4.2). Thus $\hat{\mathrm{Y}}$ is a symmetry of (4.2). The second part of this theorem follows from Theorem 2.2. □

Remark 4.1. In practical applications, one is often able to write α explicitly in terms of x, t, u, the derivatives and integrals of u. According to the definition of nonlocal symmetry, it is straightforward to show that $\hat{\mathrm{Y}}$ is a nonlocal symmetry of (4.1) if and only if the integral $\int (\hat{\mathrm{X}}^{(\infty)}\Psi[u])\,dx$ cannot be expressed as a local function of x, t, u, v and derivatives of u and v on solutions of (4.1).

Remark 4.2. Theorem 4.1 presents a connection formula between symmetries of a PDE system and those of its potential system. Without this connection formula, in order to find out whether a symmetry of a PDE system yields a local or a nonlocal symmetry of its potential system, one needs to compute *all* local symmetries of the potential system. In contrast, Theorem 4.1 provides an easier solution to this problem, not requiring a complete symmetry classification.

Example 4.1. As a given PDE system, consider the equations

$$\begin{aligned} v_x &= u, \\ v_t &= u^{-\frac{4}{3}}u_x. \end{aligned} \tag{4.5}$$

The nonlinear diffusion equation (3.21) is a subsystem of (4.5). It was shown in Ref. 17 that the nonlinear diffusion equation (3.21) has five point symmetries, which are given by, in evolutionary form, $\hat{\mathrm{X}}_1 = u_x\frac{\partial}{\partial u}$, $\hat{\mathrm{X}}_2 = u_t\frac{\partial}{\partial u}$, $\hat{\mathrm{X}}_3 = (xu_x + 2tu_t)\frac{\partial}{\partial u}$, $\hat{\mathrm{X}}_4 = (\frac{3}{2}u + xu_x)\frac{\partial}{\partial u}$ and $\hat{\mathrm{X}}_5 = (3xu + x^2u_x)\frac{\partial}{\partial u}$.

Consider the symmetry $\hat{\mathrm{X}}_4$. Here $\Psi[u] = u$ and $\Phi[u] = u^{-\frac{4}{3}}u_x$. Hence

$$\hat{\mathrm{X}}_4^{(\infty)}\Psi[u] = \tfrac{3}{2}u + xu_x.$$

Since the integral

$$\int (\tfrac{3}{2}u + xu_x)dx = \tfrac{3}{2}v + xu - v = \tfrac{1}{2}v + xu$$

is a local function of x, t, u, and v, according to Theorem 4.1 and Remark 4.1, $\hat{\mathrm{X}}_4$ yields a local symmetry of (4.5). In particular, one can show that the symmetry of (4.5) corresponding to $\hat{\mathrm{X}}_4$ is $\hat{\mathrm{Y}}_4 = \hat{\mathrm{X}}_4 + (\frac{1}{2}v + xu)\frac{\partial}{\partial v}$, which in a standard form reads $\mathrm{Y}_4 = -x\frac{\partial}{\partial x} + \frac{3}{2}u\frac{\partial}{\partial u} + \frac{1}{2}v\frac{\partial}{\partial v}$.

For $\hat{\mathrm{X}}_5$, one has

$$\hat{\mathrm{X}}_5^{(\infty)}\Psi[u] = 3xu + x^2u_x.$$

Since the integral

$$\int (3xu + x^2u_x)dx = x^2u + \int xu\,dx = x^2u + \int xv_x\,dx = x^2u + xv - \int v\,dx$$

cannot be expressed in terms of x, t, u, v and their derivatives on the solutions of (4.5), $\hat{\mathrm{X}}_5$ yields a nonlocal symmetry $\hat{\mathrm{Y}}_5$ of (4.5). Moreover, one can show that $\hat{\mathrm{Y}}_5 = \hat{\mathrm{X}}_5 + (x^2u + w)\frac{\partial}{\partial v}$, where the potential variable w satisfies $w_x = xu$, $w_t = xu^{-\frac{4}{3}}u_x + 3u^{-\frac{1}{3}}$.

Example 4.2. As a second example, we consider the nonlinear telegraph (NLT) equation given by

$$u_{tt} - (u^{-4}u_x)_x - (u^{-3})_x = 0, \tag{4.6}$$

and its potential system

$$\begin{aligned} v_x &= u_t, \\ v_t &= u^{-4}u_x + u^{-3}. \end{aligned} \tag{4.7}$$

Equation (4.6) admits four point symmetries given by, in evolutionary form, $\hat{X}_1 = u_x \frac{\partial}{\partial u}$, $\hat{X}_2 = u_t \frac{\partial}{\partial u}$, $\hat{X}_3 = (u - 2tu_t)\frac{\partial}{\partial u}$, and $\hat{X}_4 = (tu - t^2 u_t)\frac{\partial}{\partial u}$.[15]

Consider the symmetry $\hat{X}_4$. Here $\Psi[u] = u_t$ and $\Phi[u] = u^{-4}u_x + u^{-3}$. Then one has

$$\hat{X}_4^{(\infty)}\Psi[u] = u - tu_t - t^2 u_{tt}.$$

Since the integral

$$\int (u - tu_t - t^2 u_{tt})dx = \int (u - tv_x - t^2 v_{tx})dx = -tv - t^2 v_t + \int u dx$$

cannot be expressed in terms of x, t, u, v and their derivatives on the solutions of (4.7), $\hat{X}_4$ yields a nonlocal symmetry $\hat{Y}_4$ of (4.7). Moreover, $\hat{Y}_4 = \hat{X}_4 + (w - t^2 v_t)\frac{\partial}{\partial v}$, where the potential variable w satisfies $w_x = -tv_x + u = -tu_t + u$, $w_t = -t(u^{-4}u_x + u^{-3})$.

V. CONCLUDING REMARKS

In addition to local Lie symmetries of nonlinear PDE systems that are computed through a standard Lie's algorithm, nonlocal symmetries have recently received much attention. They have been successfully computed and used in a number of different applications.

Nonlocal symmetries for PDE systems can arise from potential systems, where one or more nonlocal potential variables are introduced (Type 1 potential symmetries), and/or from nonlocally related subsystems and inverse potential systems, obtained by elimination of one or more dependent variables through cross-differentiations (Type 2 potential symmetries).

Nonlocal symmetries of a given PDE system are typically sought as local Lie symmetries of PDE systems nonlocally related to the given system. One can systematically construct trees of PDE systems nonlocally related to a given one.[6–8] The tree construction process starts from considering linearly independent local conservation laws of a given system, and introducing potential variables, to obtain potential systems involving one, two, and more potential variables, up to the p-tuple potential system, where p is the number of known local conservation laws. The tree is further extended through the consideration of nonlocal conservation laws, further potential systems, nonlocally related subsystems of the given PDE system and potential systems, and so on. Linearly independent nontrivial conservation laws of any potential system can be obtained, for example, using the systematic direct construction method.

After a review of basic concepts in Sec. II, Theorem 2.2 was proven. The theorem establishes a criterion for the nonlocal relationship between two potential systems arising from two nontrivial conservation laws.

It is well-known that a potential system may have new local symmetries compared to the given one (in particular, symmetries nonlocal for the given system), but the opposite may also happen—local symmetries of a given system can become "lost," i.e., not arise as local symmetries of a potential system. They may "re-appear" as local symmetries of another potential system. An important long-standing question in the theory of nonlocal symmetries is, whether or not there is a "master potential system" whose point symmetries include the analogues of point symmetries of all its subsystems. This question in general is rather hard to answer, the first stumbling block being the lack of an upper bound on the order of local and/or potential conservation laws for a generic PDE system. The latter means that the maximal number of potential variables that one might introduce is not known, and for some systems, may not be bounded. On the other hand, potential systems arising from more "complicated" conservation laws are practically known to rarely yield nonlocal symmetries. Many other aspects of this problem exist, partially discussed in Ref. 8 and references therein, making the treatment of the general "master system" problem difficult to approach.

The current paper gives a conclusive answer to an important completeness question related to the above discussion. A principal result established here is Theorem 3.1. It states that if the full set of n local conservation laws of a given PDE system is known, then any local symmetry of that PDE system can be obtained by projection of some local symmetry of the n-plet potential system. In other words, all local symmetries of the given system will be present in the set of local symmetries of the *n*-plet potential system. Examples involving nonlinear diffusion and fourth-order evolution equations are considered. An important Corollary 3.1 provides a practical way to conclude whether or not a specific local symmetry of the given PDE system yields a local or a nonlocal symmetry of a specific potential system.

A second principal result of the current paper is Theorem 4.1. For a given local symmetry $\hat{\mathrm{X}}$ of a given PDE, it provides the form of the corresponding symmetry $\hat{\mathrm{Y}}$ of a potential system with the potential v. In particular, $\hat{\mathrm{Y}} = \hat{\mathrm{X}} + \alpha \frac{\partial}{\partial v}$, where α satisfies (4.3). Theorem 4.1 provides an effective way of determining whether a symmetry of a PDE system yields a local or a nonlocal symmetry of its potential system, without the need to compute all local symmetries of the potential system.

A large number of open questions remain in the theory of nonlocal symmetries. Within trees of nonlocally related systems studied so far, few systems yield additional nonlocal structures. Yet symmetry computations become quite time- and resource-consuming, in particular, in classification problems for systems involving arbitrary functions or parameters. It would be of primary practical value to achieve better understanding as to which nonlocally related systems are most likely to produce nonlocal structures, i.e., nonlocal symmetries and nonlocal conservation laws. In particular, an important partial question in that direction is, whether or not it is possible to determine *a priori*, without a full symmetry classification, whether or not a given conservation law of a PDE system leads to a potential system that can yield nonlocal structures.

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