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Conservation properties and potential systems of vorticity-type equations

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Partial differential equations of the form div $\boldsymbol{N} = 0$, $\boldsymbol{N}_t + \text{curl}\, \boldsymbol{M} = 0$ involving two vector functions in $\mathbb{R}^3$ depending on t, x, y, z appear in different physical contexts, including the vorticity formulation of fluid dynamics, magnetohydrodynamics (MHD) equations, and Maxwell's equations. It is shown that these equations possess an infinite family of local divergence-type conservation laws involving arbitrary functions of space and time. Moreover, it is demonstrated that the equations of interest have a rather special structure of a lower-degree (degree two) conservation law in $\mathbb{R}^4(t, x, y, z)$. The corresponding potential system has a clear physical meaning. For the Maxwell's equations, it gives rise to the scalar electric and the vector magnetic potentials; for the vorticity equations of fluid dynamics, the potentialization inverts the curl operator to yield the fluid dynamics equations in primitive variables; for MHD equations, the potential equations yield a generalization of the Galas-Bogoyavlenskij potential that describes magnetic surfaces of ideal MHD equilibria. The lower-degree conservation law is further shown to yield curl-type conservation laws and determined potential equations in certain lower-dimensional settings. Examples of new nonlocal conservation laws, including an infinite family of nonlocal material conservation laws of ideal time-dependent MHD equations in 2 + 1 dimensions, are presented. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4868218]

I. INTRODUCTION

Conservation laws admitted by a system of partial differential equations (PDE) provide important analytical information about the model, and have multiple practical applications. In particular, local divergence-type conservation laws given by divergence expressions

$$\Theta_t + \text{div}\, \boldsymbol{\Phi} = 0 \tag{1.1}$$

are important in both analysis and numerical modeling. [Throughout the paper, subscripts denote partial derivatives, e.g., $u_t = \partial u/\partial t$. Boldface is used for vectors. By default, all vector quantities are assumed to have three spatial components, e.g., $\boldsymbol{\Phi} = (\Phi^1, \Phi^2, \Phi^3)$, and vector calculus operations grad, div, and curl are taken with respect to the Cartesian spatial coordinates x, y, z.]

From the theoretical point of view, expressions (1.1) provide local quantities Θ conserved by the model, such as mass, momentum, or energy density, as well as global conserved quantities under appropriate boundary conditions. Local conservation laws are also used for the analysis of existence, uniqueness, and global solution behavior (e.g., Refs. 8, 23, and 25). Divergence-type conservation laws can be used to introduce nonlocal variables, such as potentials or stream functions. The corresponding nonlocally related (potential) systems of equations can yield nonlocal linearizations, additional (nonlocal) symmetries, conservation laws, and exact solutions.[9] From the point of view of numerical simulation, modern finite-element methods, such as discontinuous Galerkin methods, are based on divergence forms of the given equations.

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Curl-type conservation laws in time-independent three-dimensional settings reflect geometrical properties of vector fields and yield the corresponding scalar potentials, such as the velocity potential for irrotational flows, and magnetic flux function in ideal magnetohydrodynamics (MHD).

The existence of local conservation laws is traditionally associated with local symmetries through the Noether's theorem, which, however, only holds for variational PDE systems. For a general PDE system, the algorithmic and computationally efficient direct method[3] is available; it is applicable to variational and non-variational models. We briefly review the method and its software implementation in Sec. II.

For models involving more than two independent variables, potential variables following from divergence-type conservation laws are underdetermined, that is, subject to gauge freedom.[4,9,19] In that context, lower-degree conservation laws that yield less underdetermined or determined potentials are of importance. Lower-degree conservation laws are known to arise in models less frequently than divergence-type conservation laws; their existence is usually associated with special geometrical structure of the problem.

The current paper is devoted to the study of conservation properties of the system of equations given by

$$\begin{aligned} \operatorname{div} \boldsymbol{N} &= 0, \\ \boldsymbol{N}_t + \operatorname{curl} \boldsymbol{M} &= 0, \end{aligned} \tag{1.2}$$

where $\boldsymbol{N}$, $\boldsymbol{M}$ are vector fields depending on t, x, y, z. We call the PDEs (1.2) *vorticity-type equations*. These equations arise as a part of several important nonlinear and linear physical models, including the vorticity formulation of viscous and inviscid fluid dynamics, Maxwell's equations, and MHD equations (for the latter, both in the ideal case and in the case of nonzero resistivity/finite conductivity).

The rest of the paper is organized as follows. In Sec. II, we review the notion of divergence-type conservation laws, their direct construction, and resulting potential systems. In Sec. III, we show that PDE systems that contain Eqs. (1.2) have an infinite family of divergence-type local conservation laws, and discuss their specific forms for the three physical models of interest.

Section IV discusses lower-degree conservation laws and the corresponding potential systems. It is shown that the four equations (1.2) are the components of a lower-degree (degree two) conservation law. The form of this conservation law and the corresponding potential equations are discussed for the considered physical models. For the system of incompressible fluid dynamics equations, the potentialization of the lower-degree conservation law integrates the vorticity formulation, recovering the Navier-Stokes equations in primitive variables. In particular, the fluid velocity and pressure arise as potential variables. For the MHD system, the potential equations provide a direct generalization of the well-known Galas-Bogoyavlenskij potential[14,22] onto time-dependent and/or non-ideal plasma flows.

Section V is devoted to reduced forms of Eqs. (1.2) in three and 2 + 1 dimensions. In such a setting, the degree two conservation law yields a curl-type conservation law and a determined potential system. The latter is used to derive new nonlocal conservation laws for the nonlinear magnetohydrodynamics equations, in both the ideal case and the case of finite conductivity.

Finally, Sec. VI contains a brief discussion of results and open problems.

II. CONSERVATION LAWS AND POTENTIAL SYSTEMS

A. Divergence-type conservation laws

Consider a system $\mathbf{R}\{\boldsymbol{x}\,; \boldsymbol{u}\}$ of N partial differential equations of order k, with $n \geq 2$ independent variables $\boldsymbol{x} = (x^1, \ldots, x^n)$ and $m \geq 1$ dependent variables $\boldsymbol{u} = (u^1(\boldsymbol{x}), \ldots, u^m(\boldsymbol{x}))$, given by

$$R^\sigma[\boldsymbol{u}] \equiv R^\sigma(\boldsymbol{x}, \boldsymbol{u}, \partial\boldsymbol{u}, \ldots, \partial^k\boldsymbol{u}) = 0, \quad \sigma = 1, \ldots, N. \tag{2.1}$$

Here $\partial\boldsymbol{u}$ denotes the set of all first order partial derivatives, and $\partial^p\boldsymbol{u}$ denotes all pth-order partial derivatives. In addition, we denote partial derivatives by $u^\mu_i = u^\mu_{x^i} = \partial u^\mu/\partial x^i$, and assume summation in all repeated indices.

Definition 2.1. A divergence-type conservation law of a PDE system (2.1) is a divergence expression of the form

$$\operatorname{div} \boldsymbol{\Phi}[\boldsymbol{u}] \equiv \mathrm{D}_i \Phi^i(\boldsymbol{x}, \boldsymbol{u}, \partial \boldsymbol{u}, \ldots, \partial^r \boldsymbol{u}) = 0, \tag{2.2}$$

in terms of total derivative operators

$$\mathrm{D}_i = \frac{\partial}{\partial x^i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ii_1}^\mu \frac{\partial}{\partial u_{i_1}^\mu} + u_{ii_1i_2}^\mu \frac{\partial}{\partial u_{i_1i_2}^\mu} + \cdots, \tag{2.3}$$

holding on solutions of a given PDE system $\mathbf{R}\{\boldsymbol{x}\,;\boldsymbol{u}\}$.

In particular, when $\boldsymbol{x} = (t, x^1, \ldots, x^n)$, where t is time and $x^1, \ldots, x^n$ are spatial variables, the conservation law (2.2) has the form

$$\mathrm{D}_t \Theta + \mathrm{D}_i \Psi^i = 0, \tag{2.4}$$

where the density Θ and the spatial fluxes Ψ^i can depend on independent and dependent variables of the given equations, as well as on their partial derivatives. On solutions of the given PDE system, the forms (2.4) and (1.1) of the conservation law are equivalent.

Definitions and properties of divergence-type conservation laws are discussed in further detail, for example, in Refs. 9, 19, and 26. For a given PDE system, one is interested in finding sets of *non-trivial, non-equivalent, linearly independent* conservation laws. A trivial conservation law of a normal PDE system[26] is a divergence expression that is zero due to vector calculus identities, or if its density and fluxes vanish on solutions of the given PDE system. Two conservation laws are equivalent if they differ by a trivial conservation law. A set of conservation laws is linearly dependent if their nontrivial linear combination is a trivial conservation law.

Local divergence-type conservation laws (2.4) are systematically sought by applying the direct method reviewed below.

When the density and/or fluxes in (2.4) involve nonlocal (integral) quantities, the corresponding conservation law is *nonlocal*. Nonlocal conservation laws can be found through the application of the direct method to potential systems (see, e.g., Refs. 9–11, 19 and references therein).

B. Direct construction of divergence-type conservation laws

The direct conservation law construction method[1,3,9] consists in finding sets of multipliers $\{\Lambda_\sigma[\boldsymbol{U}]\}_{\sigma=1}^N = \{\Lambda_\sigma(\boldsymbol{x}, \boldsymbol{U}, \partial \boldsymbol{U}, \ldots, \partial^\ell \boldsymbol{U})\}_{\sigma=1}^N$, depending on some prescribed independent and dependent variables and possibly their derivatives to some finite order ℓ, which, taken in linear combinations with the given PDEs, yield a divergence expression

$$\Lambda_\sigma[\boldsymbol{U}] R^\sigma[\boldsymbol{U}] \equiv \mathrm{D}_i \Phi^i[\boldsymbol{U}] \tag{2.5}$$

holding for an arbitrary set of functions $\boldsymbol{U}$. Then on solutions $\boldsymbol{U} = \boldsymbol{u}(\boldsymbol{x})$ of the PDE system (2.1), one has a local conservation law (2.2):

$$\Lambda_\sigma[\boldsymbol{u}] R^\sigma[\boldsymbol{u}] = \mathrm{D}_i \Phi^i[\boldsymbol{u}] = 0. \tag{2.6}$$

In order to find multipliers that yield divergence-type conservation laws through the formula (2.6), one uses the well-known fact that an expression $F(\boldsymbol{U})$ is annihilated by all Euler operators

$$\mathrm{E}_{U^j} = \frac{\partial}{\partial U^j} - \mathrm{D}_i \frac{\partial}{\partial U_i^j} + \cdots + (-1)^s \mathrm{D}_{i_1} \ldots \mathrm{D}_{i_s} \frac{\partial}{\partial U_{i_1 \ldots i_s}^j} + \cdots, \quad i, i_q{=}1, \ldots, n, \quad j = 1, \ldots, m, \tag{2.7}$$

if and only if $F(\boldsymbol{U})$ is a divergence expression (e.g., Refs. 9 and 26). It follows that the multiplier determining equations are given by

$$\mathrm{E}_{U^j}\big(\Lambda_\sigma[\boldsymbol{U}] R^\sigma[\boldsymbol{U}]\big) = 0, \quad j = 1, \ldots, m. \tag{2.8}$$

After the linear equations (2.8) are solved and multipliers Λ^σ are found, one computes the conservation law fluxes and/or density using (2.5) (see, e.g., Ref. 16).

The majority of PDE systems that arise in applications can be written in a solved form with respect to some leading derivatives. For such systems that have no lower-order differential consequences, the direct construction method is complete, i.e., all local divergence-type conservation law (2.2) arise through linear combinations (2.6) for corresponding sets of multipliers $\{\Lambda_\sigma[\boldsymbol{U}]\}$. Moreover, for Cauchy-Kovalevskaya PDE systems, there is a one-to-one correspondence between sets of multipliers and equivalence classes of local divergence-type conservation laws.[2,9]

The direct construction method has been implemented in several software packages, including the `GeM` package for `Maple`,[15] which is used in the current paper.

C. Potential systems following from divergence-type conservation laws

Let $\mathbf{R}\{x, t\,; \boldsymbol{u}\}$ be a PDE system with two independent variables (x, t):

$$R^\sigma[\boldsymbol{u}] = R^\sigma(x, t, \boldsymbol{u}, \partial\boldsymbol{u}, \ldots, \partial^k\boldsymbol{u}) = 0, \quad \sigma = 1, \ldots, N. \tag{2.9}$$

Divergence-type conservation laws of such systems have the form

$$\mathrm{D}_t\Theta + \mathrm{D}_x\Psi = 0. \tag{2.10}$$

Such conservation laws have been derived for a variety of mathematical models.

Definition 2.2. A *potential system* $\mathbf{S}\{x, t\,; \boldsymbol{u}, v\}$ is the union of the equations of given system (2.9) and the *potential equations* following from the conservation law (2.10):

$$\begin{aligned} &R^\sigma[\boldsymbol{u}] = R^\sigma(x, t, \boldsymbol{u}, \partial\boldsymbol{u}, \ldots, \partial^k\boldsymbol{u}) = 0, \quad \sigma = 1, \ldots, N, \\ &v_x = \Theta[\boldsymbol{u}], \quad v_t = -\Phi[\boldsymbol{u}]. \end{aligned} \tag{2.11}$$

In particular, local symmetries of the potential system $\mathbf{S}\{x, t\,; \boldsymbol{u}, v\}$ (2.11) whose x- and/or t- and/or u-components involve the potential variables v, yield nonlocal symmetries of the given system (2.9). Similarly, local conservation laws of the potential system $\mathbf{S}\{x, t\,; \boldsymbol{u}, v\}$ not equivalent to any conservation law of the given system $\mathbf{R}\{x, t\,; \boldsymbol{u}\}$ yield nonlocal conservation laws of the given PDE system. For details on the systematic construction of nonlocally related PDE systems and their applications, see, e.g., Refs. 9–11.

In the case of PDE systems involving $n \geq 3$ independent variables, the application of divergence-type conservation laws to the construction of potential systems is less straightforward. Indeed, in the case of three independent variables (x, y, z), each divergence-type conservation law

$$\operatorname{div}\boldsymbol{\Phi} = \mathrm{D}_x\Phi^1 + \mathrm{D}_y\Phi^2 + \mathrm{D}_z\Phi^3 = 0 \tag{2.12}$$

leads to the three scalar potential equations

$$\boldsymbol{\Phi} = \operatorname{curl}\boldsymbol{A}, \tag{2.13}$$

where the vector potential $\boldsymbol{A} = (A^1, A^2, A^3)$ is subject to gauge freedom

$$\boldsymbol{A} \to \boldsymbol{A} + \operatorname{grad}\ \phi(x, y, z), \tag{2.14}$$

and the corresponding potential system is *under-determined*.

In general, for PDE systems involving $n \geq 3$ independent variables, it is known that any local symmetry of an under-determined potential system projects on a local symmetry of the given PDE system.[4] In order to obtain nonlocal symmetries, an under-determined potential system must be appended with additional gauge constraint(s) eliminating the gauge freedom. Finding an "optimal" gauge constraint for a specific potential system still remains an open problem.

Under-determined potential systems, however, can yield nonlocal conservation laws (e.g., Refs. 6 and 20). In order to seek nonlocal conservation laws of a given PDE system arising as local conservation laws of its potential system, it is necessary to consider conservation law multipliers that essentially depend on the nonlocal variable(s).[9,24] All divergence-type conservation laws of an under-determined potential system are invariant under the gauge symmetry.[6] For further details, see, e.g., Ref. 9.

Section III is devoted to the computation of local conservation laws of the PDEs (1.2) arising from zeroth-order multipliers.

A more general framework that includes lower-degree conservation laws and resulting potential systems is considered in Sec. IV A below. In particular, lower-degree conservation laws can lead to potential systems requiring fewer or no gauge constraints.

III. A FAMILY OF DIVERGENCE-TYPE CONSERVATION LAWS OF THE SYSTEM (1.2)

A. The conservation laws

Consider the problem of seeking divergence-type conservation laws for the four scalar equations (1.2). The following theorem holds.

Theorem 3.1 (Principal Result 1). *Consider a PDE system* $\mathbf{R}\{t, x, y, z\,; \boldsymbol{u}\}$ *involving independent variables t, x, y, z, which includes equations in the form* (1.2). *Then such a system admits an infinite family of divergence-type conservation laws given by*

$$(\boldsymbol{N}\cdot\nabla F)_t + \operatorname{div}(\boldsymbol{M}\times\nabla F - F_t\,\boldsymbol{N}) = 0, \tag{3.1}$$

depending on an arbitrary function $F = F(t, x, y, z)$.

Proof. Applying the direct method with multipliers depending on t, x, y, z, one finds an admissible set of multipliers

$$\Lambda^1 = -F_t, \quad \Lambda^2 = F_x, \quad \Lambda^3 = F_y, \quad \Lambda^4 = F_z,$$

where $F(t, x, y, z)$ is an arbitrary function. The conserved form (3.1) follows directly. Conservation laws (3.1) are nontrivial and are not equivalent to the given PDEs (4.18) through an addition of a trivial conservation law. □

The conservation laws (3.7) express the rate of change of a linear combination of the components of the field $\boldsymbol{N}$ taken with weights that are components of a gradient of an arbitrary function. The local conservation laws (3.7) can be written in the global form

$$\frac{d}{dt}\int_{\mathcal{V}}(\boldsymbol{N}\cdot\nabla F)\,dV + \oint_{\partial\mathcal{V}}(\boldsymbol{M}\times\nabla F - F_t\,\boldsymbol{N})\cdot d\boldsymbol{S} = 0, \tag{3.2}$$

which corresponds to the conserved quantity

$$\frac{dR}{dt} = 0, \quad R = \int_{\mathcal{V}}(\boldsymbol{N}\cdot\nabla F)\,dV, \tag{3.3}$$

when the spatial fluxes vanish on the boundary $\partial\mathcal{V}$ of the physical domain $\mathcal{V}$, or at infinity if the domain is unbounded.

Examples specific to three applications are now presented in a unified way.

B. Examples

1. Vorticity conservation laws in viscous and inviscid fluid dynamics

The Navier-Stokes equations of incompressible constant-density viscous fluid flow without external forcing in three spatial dimensions, in primitive variables, are given by

$$\operatorname{div}\ \boldsymbol{V} = 0, \tag{3.4a}$$

$$\boldsymbol{V}_t + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} + \operatorname{grad}\ p = \nu\,\nabla^2\boldsymbol{u}, \tag{3.4b}$$

where the fluid velocity vector $\boldsymbol{V} = (V^1, V^2, V^3)$ and the fluid pressure p are functions of t, x, y, z, and $\nu = \text{const.}$ is the kinematic viscosity. The Euler equations are obtained from [(3.4a) and (3.4b)] in the inviscid limit $\nu \to 0$.

The vorticity formulation of the Navier-Stokes equations [(3.4a) and (3.4b)] follows from defining the fluid vorticity as $\boldsymbol{w} = \text{curl } \boldsymbol{V}$. The vorticity dynamic equation is obtained by taking the curl of the momentum equation (3.4b). Hence, the vorticity formulation of the Navier-Stokes equations is

$$\text{div } \boldsymbol{V} = 0, \tag{3.5a}$$

$$\boldsymbol{w} = \text{curl } \boldsymbol{V}, \tag{3.5b}$$

$$\text{div } \boldsymbol{w} = 0, \qquad \boldsymbol{w}_t + \text{curl}\,(\boldsymbol{w} \times \boldsymbol{V} - \nu\, \nabla^2 \boldsymbol{V}) = 0. \tag{3.5c}$$

The PDEs (3.5c) evidently have the form (1.2), with vector fields $\boldsymbol{N} = \boldsymbol{N}_w$, $\boldsymbol{M} = \boldsymbol{M}_w$ given by

$$\boldsymbol{N}_w = \boldsymbol{w}, \qquad \boldsymbol{M}_w = \boldsymbol{w} \times \boldsymbol{V} - \nu\, \nabla^2 \boldsymbol{V}. \tag{3.6}$$

From Theorem 3.1, it follows that the equations of incompressible fluid flow, both in viscous and inviscid setting, have an infinite family of vorticity conservation laws

$$(\boldsymbol{w} \cdot \nabla F)_t + \text{div}\left([\boldsymbol{w} \times \boldsymbol{V} - \nu\, \nabla^2 \boldsymbol{V}] \times \nabla F - F_t\, \boldsymbol{w}\right) = 0, \tag{3.7}$$

for an arbitrary choice of $F(t, x, y, z)$, holding both for viscous and inviscid flows.

2. Magnetic conservation laws in general magnetohydrodynamics

The system of isotropic MHD equations in 3+1 dimensions has the form

$$\rho_t + \text{div}\, \rho \boldsymbol{V} = 0, \tag{3.8a}$$

$$\rho \boldsymbol{V}_t + \rho(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\frac{1}{\mu} \boldsymbol{B} \times \text{curl } \boldsymbol{B} - \text{grad } P + \mu_1\, \nabla^2 \boldsymbol{V}, \tag{3.8b}$$

$$\boldsymbol{B}_t = \text{curl}(\boldsymbol{V} \times \boldsymbol{B}) + \eta\, \nabla^2 \boldsymbol{B}, \tag{3.8c}$$

$$\text{div}\, \boldsymbol{B} = 0, \tag{3.8d}$$

where the plasma velocity $\boldsymbol{V} = (V^1, V^2, V^3)$, the magnetic field $\boldsymbol{B} = (B^1, B^2, B^3)$, the plasma density ρ and the pressure P are functions of t, x, y, z; $\mu = \text{const.}$ is the magnetic permeability of free space; $\mu_1 = \text{const.}$ is the plasma viscosity coefficient; $\eta = 1/(\sigma\mu)$ is the resistivity coefficient; $\sigma = \text{const.}$ is the conductivity coefficient. A commonly chosen limit of ideal MHD equations is obtained when $\eta = \mu_1 = 0$. [The system [(3.8a)–(3.8d)] must be closed with an additional equation of state, which is not important at this point.]

Since (3.8d) holds, the magnetic vector potential can be introduced by $\boldsymbol{B} = \text{curl } \boldsymbol{A}(t, x, y, z)$. Consequently, the PDEs (3.8c), (3.8d) have the form (1.2), with $\boldsymbol{N} = \boldsymbol{N}_m$, $\boldsymbol{M} = \boldsymbol{M}_m$:

$$\boldsymbol{N}_m = \boldsymbol{B}, \qquad \boldsymbol{M}_m = \boldsymbol{B} \times \boldsymbol{V} - \eta\, \nabla^2 \boldsymbol{A}. \tag{3.9}$$

Note that due to the gauge freedom $\boldsymbol{A} \to \boldsymbol{A} + \text{grad}\, \phi$, one may choose a vector potential satisfying $\text{div}\, \boldsymbol{A} = 0$. In that case,

$$\text{curl } \boldsymbol{B} = \text{curl}(\text{curl } \boldsymbol{A}) = \text{grad}(\text{div } \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = -\nabla^2 \boldsymbol{A}.$$

Further, the plasma electric current density and the conductivity coefficient are given, respectively, by

$$\boldsymbol{J} = \frac{1}{\mu} \text{curl } \boldsymbol{B}, \qquad \sigma = \frac{1}{\mu\, \eta}.$$

Consequently, by Theorem 3.1, we observe that the general MHD system has an infinite family of local divergence-type magnetic conservation laws

$$(\boldsymbol{B}\cdot\nabla F)_t + \mathrm{div}\left(\left[\boldsymbol{B}\times\boldsymbol{V}+\frac{1}{\sigma}\boldsymbol{J}\right]\times\nabla F - F_t\,\boldsymbol{B}\right)=0, \tag{3.10}$$

holding for an arbitrary $F = F(t, x, y, z)$.

For ideal plasmas where $\sigma \to +\infty$, the conservation laws (3.10) do not involve the current density.

3. Conservation laws of general and vacuum Maxwell's equations

The dimensionless PDE system of Maxwell's equations is given by

$$\mathrm{div}\,\boldsymbol{B}=0, \quad \boldsymbol{B}_t = -\,\mathrm{curl}\,\boldsymbol{E}, \tag{3.11a}$$

$$\boldsymbol{E}_t = \mathrm{curl}\,\boldsymbol{B}-\boldsymbol{J}, \quad \mathrm{div}\,\boldsymbol{E}=\rho, \tag{3.11b}$$

where the charge density ρ, the magnetic field, the electric field and the current density $\boldsymbol{B}, \boldsymbol{E}, \boldsymbol{J} \in \mathbb{R}^3$ are functions of t, x, y, z.

Equations (3.11a) are in the form (1.2) as they stand, with $\boldsymbol{N} = \boldsymbol{B}$, $\boldsymbol{M} = \boldsymbol{E}$. From Theorem 3.1 it follows that the Maxwell's equations [(3.11a) and (3.11b)] have an infinite set of local divergence-type "magnetic" conservation laws

$$(\boldsymbol{B}\cdot\nabla F)_t + \mathrm{div}\,(\boldsymbol{E}\times\nabla F - F_t\,\boldsymbol{B})=0 \tag{3.12}$$

depending on an arbitrary function $F = F(t, x, y, z)$.

For vacuum Maxwell's equations given by [(3.11a) and (3.11b)] with $\mathbf{J} = \rho = 0$, the second pair of equations (3.11b) is also in the required form (1.2), with $\boldsymbol{N} = \boldsymbol{E}$ and $\boldsymbol{M} = -\boldsymbol{B}$. Here one has an additional family of "electric" conservation laws

$$(\boldsymbol{E}\cdot\nabla G)_t - \mathrm{div}\,(\boldsymbol{B}\times\nabla G - G_t\,\boldsymbol{E})=0 \tag{3.13}$$

holding for an arbitrary function $G = G(t, x, y, z)$.

The conservation laws (3.12), (3.13) for the vacuum Maxwell's equations were listed and referred to as "adjoint gauge symmetries" in Ref. 5.

IV. THE DEGREE TWO CONSERVATION LAW STRUCTURE OF EQs. (1.2)

A. General and lower-degree conservation laws

In $n \geq 3$ dimensions, in addition to divergence expressions (2.2), PDE systems can have conservation laws of other types. For example, in three-dimensional space, a PDE system $\mathbf{R}\{x, y, z\,;\boldsymbol{u}\}$ can have a vector curl-type conservation law given by

$$\mathrm{curl}\,\boldsymbol{\Psi}[\boldsymbol{u}]=0, \tag{4.1}$$

with some flux vector $\boldsymbol{\Psi} = (\Psi^1, \Psi^2, \Psi^3)$. Conservation laws like (4.1) and their generalizations are referred to as *lower-degree conservation laws*.[7,19,20]

The framework including both divergence-type and lower-degree conservation laws is most naturally presented in differential-geometric notation. We now outline the notation and definitions for the case of a PDE system $\mathbf{R}\{\boldsymbol{x}\,;\boldsymbol{u}\}$ (2.1) with $n \geq 2$ independent variables given by the components of the vector $\boldsymbol{x}$. For details, see Ref. 19.

A differential r-form is given by

$$\omega^{(r)}=\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}, \tag{4.2}$$

where $\omega_{\mu_1\mu_2\ldots\mu_r}$ are the components of a totally antisymmetric tensor of type $(0, r)$. From now on, we assume that $\omega_{\mu_1\ldots\mu_r} = \omega_{\mu_1\ldots\mu_r}[\boldsymbol{U}]$ depend on independent variables $\boldsymbol{x}$, vector functions $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x})$, and derivatives of $\boldsymbol{U}$. Where necessary, differentiations by x^i are replaced by total derivative operators D_i.

Definition 4.1. A conservation law of degree r $(1 \leq r \leq n - 1)$ of the PDE system $\mathbf{R}\{\boldsymbol{x}\,;\boldsymbol{u}\}$ (2.1) is given by an r-form $\omega^{(r)}[\boldsymbol{U}]$ (4.2), such that its exterior derivative

$$\Omega^{(r+1)}[\boldsymbol{u}] = \mathrm{d}\omega^{(r)}[\boldsymbol{u}] = 0 \tag{4.3}$$

on all solutions $\boldsymbol{U} = \boldsymbol{u}(\boldsymbol{x})$ of a given PDE system $\mathbf{R}\{\boldsymbol{x}\,;\boldsymbol{u}\}$:

$$\Omega_{\nu\mu_1\ldots\mu_r}[\boldsymbol{u}]\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \equiv \left(\mathrm{D}_\nu\, \omega_{\mu_1\ldots\mu_r}[\boldsymbol{u}]\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} = 0. \tag{4.4}$$

Each conservation law of degree r (4.4) is thus given by $\binom{n}{r+1}$ scalar equations

$$\sum_{(\nu,\mu_1,\ldots,\mu_r)\subset S^i} \mathrm{sgn}(\nu, \mu_1, \ldots, \mu_r)\, \mathrm{D}_\nu\, \omega_{\mu_1\ldots\mu_r}[\boldsymbol{u}] = 0, \quad i = 1, \ldots, \binom{n}{r+1}, \tag{4.5}$$

where $\mathrm{sgn}(\nu, \mu_1, \ldots, \mu_r)$ is the sign of the permutation $(\nu, \mu_1, \ldots, \mu_r)$, and S^i is the set of permutations of an ordered set of $r + 1$ subindices of $\{1, \ldots, n\}$.

The conservation law (4.4) has $\binom{n}{r}$ fluxes given by the components of the differential form $\omega^{(r)}[\boldsymbol{u}]$.

For PDE systems with $n \geq 2$ independent variables, conservation laws of degrees $1, \ldots, n - 1$, may exist. Conservation laws of degree $n - 1$ correspond to divergence-type conservation laws (2.2), whereas conservation laws of degree $r = 1, \ldots, n - 2$ are lower-degree conservation laws.

For $n = 2$, only divergence-type conservation laws are possible. For $n = 3$, both divergence-type conservation laws (2.12) and curl-type conservation laws (4.1) can occur. For $n > 3$, additional types of conservation laws may arise.

Lower-degree conservation laws can be systematically sought through a direct construction algorithm based on differential-geometric versions of vector calculus identities.[19] Any lower-degree conservation law (4.4) is equivalent to a set of $\binom{n}{r+1}$ interdependent divergence-type conservation laws (4.5). For an arbitrary function $\boldsymbol{U}(\boldsymbol{x})$, one seeks multipliers $\{\Lambda_\sigma^{(i)}[\boldsymbol{U}]\}$ such that each independent component of $\Omega^{(r+1)}[\boldsymbol{U}]$ is a linear combination

$$\Omega_{\mu_1\ldots\mu_{r+1}}[\boldsymbol{U}] = \Lambda_\sigma^{(i)}[\boldsymbol{U}]R^\sigma[\boldsymbol{U}], \quad i = 1, \ldots, \binom{n}{r+1}. \tag{4.6}$$

The determining equations for the multipliers follow from Definition 4.1. For (4.3) to hold, it is necessary and sufficient that $\mathrm{d}\Omega^{(r+1)}[\boldsymbol{U}] = \mathrm{d}^2\omega^{(r)}[\boldsymbol{U}] = 0$ for an *arbitrary function* $\boldsymbol{U}(\boldsymbol{x})$. Thus, the multiplier determining equations are given by the $\binom{n}{r+2}$ vanishing expressions

$$\left(\mathrm{D}_\lambda\,(\Lambda_\sigma^{(i)}[\boldsymbol{U}]R^\sigma[\boldsymbol{U}])\right) \mathrm{d}x^\lambda \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{r+1}} \equiv 0. \tag{4.7}$$

B. Potential equations following from lower-degree conservation laws

Each conservation law of degree r, $1 \leq r \leq n - 1$, yields a set of potential equations, which, combined with the given PDE system, compose a potential system.

By Poincaré's lemma, since $\mathrm{d}\omega^{(r)}[\boldsymbol{u}] = 0$ on solutions of $\mathbf{R}\{\boldsymbol{x}\,;\boldsymbol{u}\}$, one locally has $\omega^{(r)}[\boldsymbol{u}] = \mathrm{d}\widetilde{\omega}^{(r-1)}[\boldsymbol{u}]$, for some $(r - 1)$-form $\widetilde{\omega}^{(r-1)}[u]$. As a result, one has $\binom{n}{r}$ potential equations

$$\omega_{\mu_1\ldots\mu_r}[\boldsymbol{u}] = \sum_{i=1}^{r}(-1)^{i-1}\frac{\partial}{\partial x^{\mu_i}}\widetilde{\omega}_{\mu_1\ldots\overline{\mu_i}\ldots\mu_r}[\boldsymbol{u}], \tag{4.8}$$

for $\binom{n}{r-1}$ potential variables given by the independent components of $\widetilde{\omega}^{(r-1)}[\boldsymbol{u}]$.

The following examples will be relevant to the analysis of the PDEs (1.2).

1. Three independent variables

Consider a PDE system $\mathbf{R}\{x, y, z\,; \boldsymbol{u}\}$, with $n = 3$ independent variables $\boldsymbol{x} = (x, y, z)$.

a. Divergence-type conservation laws. A divergence-type (degree $n - 1 = 2$) conservation law is given by

$$\operatorname{div} \boldsymbol{\Phi}[\boldsymbol{u}] = \Phi^1_x[\boldsymbol{u}] + \Phi^2_y[\boldsymbol{u}] + \Phi^3_z[\boldsymbol{u}] = 0, \tag{4.9}$$

The expression (4.9) can be viewed as the only component of the three-form

$$\Omega[\boldsymbol{U}] = (\Phi^1_x[\boldsymbol{U}] + \Phi^2_y[\boldsymbol{U}] + \Phi^3_z[\boldsymbol{U}])\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

with $\Omega[\boldsymbol{u}] = 0$ by (4.9). It is evident that $\Omega[\boldsymbol{U}] = \mathrm{d}\omega[\boldsymbol{U}]$, where the components of the two-form

$$\omega[\boldsymbol{u}] = \Phi^1[\boldsymbol{u}]\, \mathrm{d}y \wedge \mathrm{d}z + \Phi^2[\boldsymbol{u}]\, \mathrm{d}z \wedge \mathrm{d}x + \Phi^3[\boldsymbol{u}]\, \mathrm{d}x \wedge \mathrm{d}y$$

provide the fluxes of the conservation law (4.9). The corresponding potential equations are given by

$$\omega[\boldsymbol{u}] = \mathrm{d}\widetilde{\omega}[\boldsymbol{u}], \tag{4.10}$$

the three potential variables are the components A^i of the one-form

$$\widetilde{\omega}[\boldsymbol{u}] = A^1[\boldsymbol{u}]\, \mathrm{d}x + A^2[\boldsymbol{u}]\, \mathrm{d}y + A^3[\boldsymbol{u}]\, \mathrm{d}z. \tag{4.11}$$

In the scalar form, the potential equations (4.10) are given by three PDEs (cf. (2.13)):

$$\Phi^1[\boldsymbol{u}] = \mathrm{D}_y A^3 - \mathrm{D}_z A^2, \quad \Phi^2[\boldsymbol{u}] = \mathrm{D}_z A^1 - \mathrm{D}_x A^3, \quad \Phi^3[\boldsymbol{u}] = \mathrm{D}_x A^2 - \mathrm{D}_y A^1. \tag{4.12}$$

The potential variables A^i are subject to the gauge freedom (cf. (2.14)); indeed, the potential equations (4.10) are invariant with respect to the transformation

$$\widetilde{\omega}[\boldsymbol{u}] \to \widetilde{\omega}[\boldsymbol{u}] + \mathrm{d}\phi,$$

where $\phi = \phi(x, y, z)$ is an arbitrary function of all independent variables.

b. Curl-type conservation laws. The PDE system $\mathbf{R}\{x, y, z\,; \boldsymbol{u}\}$ can have curl-type conservation laws (degree $n - 2 = 1$), given by (4.1). The three scalar components of the conservation law (4.1) can be viewed as components of the differential $(n - 1)$-form

$$\Omega[\boldsymbol{U}] = (\Psi^3_y[\boldsymbol{U}] - \Psi^2_z[\boldsymbol{U}])\, \mathrm{d}y \wedge \mathrm{d}z + (\Psi^1_z[\boldsymbol{U}] - \Psi^3_x[\boldsymbol{U}])\, \mathrm{d}z \wedge \mathrm{d}x + (\Psi^2_x[\boldsymbol{U}] - \Psi^1_y[\boldsymbol{U}])\, \mathrm{d}x \wedge \mathrm{d}y,$$

which vanish on all solutions $\boldsymbol{U} = \boldsymbol{u}(\boldsymbol{x})$ of the given PDE system. On solutions, $\Omega[\boldsymbol{u}] = \mathrm{d}\omega[\boldsymbol{u}]$, where $\omega[\boldsymbol{u}]$ is the flux one-form:

$$\omega[\boldsymbol{u}] = \Psi^1[\boldsymbol{u}]\, \mathrm{d}x + \Psi^2[\boldsymbol{u}]\, \mathrm{d}y + \Psi^3[\boldsymbol{u}]\, \mathrm{d}z.$$

The potential equations (4.10) following from the conservation law (4.1) involve a zero-form $\widetilde{\omega}[\boldsymbol{u}] = \beta[\boldsymbol{u}]$, i.e., a single potential variable; they can be written as

$$\boldsymbol{\Psi}[\boldsymbol{u}] = \operatorname{grad} \beta[\boldsymbol{u}]. \tag{4.13}$$

It is important to observe that the potential equations (4.13) have no gauge freedom, since the only possible general transformation $\beta[\boldsymbol{u}] \to \beta[\boldsymbol{u}] + C$ does not involve arbitrary functions of all independent variables. Consequently, in three dimensions, curl-type conservation laws may be preferred to divergence-type conservation laws for the purposes of construction of potential systems, in particular, for the problem of seeking nonlocal symmetries.

2. Four independent variables

Now consider PDE systems involving $n = 4$ independent variables. In addition to divergence-type conservation laws of degree $r = n - 1 = 3$ and curl-type conservation laws of degree $r = n - 3 = 1$, here one can additionally have lower-degree conservation laws of degree $r = n -$

$2 = 2$, involving $\binom{n}{r+1} = 4$ scalar components (divergence expressions). By using (4.8), each such conservation law can be used to introduce $\binom{n}{r-1} = 4$ potential variables satisfying $\binom{n}{r} = 6$ potential equations.

Below we demonstrate that Eqs. (1.2) indeed correspond to a lower-degree conservation law of degree two in the four-dimensional space of independent variables (t, x, y, z).

C. The system (1.2) as a conservation law of degree two. Resulting potential equations

Theorem 4.1. *The four PDEs* (1.2) *are equivalent to a lower-degree (degree two) conservation law in the four-dimensional space of variables t, x, y, z.*

Proof. Denote the four scalar PDEs (1.2) by

$$\begin{aligned} E^1 &= N^1_x + N^2_y + N^3_z, \qquad & E^2 &= N^1_t + M^3_y - M^2_z, \\ E^3 &= N^2_t + M^1_z - M^3_x, \qquad & E^4 &= N^3_t + M^2_x - M^1_y. \end{aligned} \tag{4.14}$$

Consider a differential two-form

$$\begin{aligned} \omega = &-M^1\,\mathrm{d}t\wedge\mathrm{d}x - M^2\,\mathrm{d}t\wedge\mathrm{d}y - M^3[\boldsymbol{U}]\,\mathrm{d}t\wedge\mathrm{d}z \\ &+N^3\,\mathrm{d}x\wedge\mathrm{d}y + N^2\,\mathrm{d}z\wedge\mathrm{d}x + N^1\,\mathrm{d}y\wedge\mathrm{d}z. \end{aligned} \tag{4.15}$$

A direct computation yields that the exterior derivative $\Omega[\boldsymbol{U}] = \mathrm{d}\omega[\boldsymbol{U}]$ is a differential three-form given by

$$\Omega[\boldsymbol{U}] = E^1[\boldsymbol{U}]\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + E^2[\boldsymbol{U}]\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}t - E^3[\boldsymbol{U}]\,\mathrm{d}z\wedge\mathrm{d}t\wedge\mathrm{d}x + E^4[\boldsymbol{U}]\,\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y, \tag{4.16}$$

whose components are Eqs. (4.14), which vanish when Eqs. (4.14) hold. Hence, the differential form (4.16) provides a lower-degree (degree two) conservation law $\Omega[\boldsymbol{u}] = 0$, with fluxes given by the components of the two-form (4.15). □

Remark 4.1. By a direct analogy with the electromagnetic tensor, one may write the flux form $\omega = \omega_{\mu\nu}\, dx^\mu \wedge dx^\nu$ as a tensor in the four-dimensional Minkowski spacetime $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, given by

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -M^1 & -M^2 & -M^3 \\ M^1 & 0 & N^3 & -N^2 \\ M^2 & -B^3 & 0 & N^1 \\ M^3 & N^2 & -N^1 & 0 \end{pmatrix}. \tag{4.17}$$

As per Sec. IV B above, the lower-degree conservation law $\Omega[\boldsymbol{u}] = 0$ (4.16) can be used to construct potential equations.

Corollary 1. Suppose a given PDE system $\mathbf{R}\{t, x, y, z\,; \boldsymbol{u}\}$

$$R^\sigma[\boldsymbol{u}] \equiv R^\sigma(t, x, y, z, \boldsymbol{u}, \partial\boldsymbol{u}, \ldots, \partial^k\boldsymbol{u}) = 0, \quad \sigma = 1, \ldots, N \tag{4.18}$$

contains Eqs. (1.2). *Then the PDE system* (4.18) *has a nonlocally related system involving four scalar potential variables.*

Proof. The statement follows from the fact that on solutions of the given PDE system, $\omega[\boldsymbol{u}] = \mathrm{d}\Omega[\boldsymbol{u}] = 0$. Hence locally, $\omega[\boldsymbol{u}] = \mathrm{d}\theta[\boldsymbol{u}]$, where $\theta[\boldsymbol{u}]$ is a one-form

$$\theta = \theta^t(t, x, y, z)\,\mathrm{d}t + \theta^x(t, x, y, z)\,\mathrm{d}x + \theta^y(t, x, y, z)\,\mathrm{d}y + \theta^z(t, x, y, z)\,\mathrm{d}z, \tag{4.19}$$

whose four components provide the potential variables. The six potential equations are given by components of $\omega[\boldsymbol{u}] = \mathrm{d}\theta[\boldsymbol{u}]$ and read

$$\begin{aligned} -M^1[\boldsymbol{u}] &= \theta^x_t - \theta^t_x, \; -M^2[\boldsymbol{u}] = \theta^y_t - \theta^t_y, \; -M^3[\boldsymbol{u}] = \theta^z_t - \theta^t_z, \\ N^1[\boldsymbol{u}] &= \theta^z_y - \theta^y_z, \quad N^2[\boldsymbol{u}] = \theta^x_z - \theta^z_x, \quad N^3[\boldsymbol{u}] = \theta^y_x - \theta^x_y. \end{aligned} \tag{4.20}$$

□

Remark 4.2. From (4.15) and (4.16), one has $\mathrm{d}\Omega[\boldsymbol{U}] = \mathrm{d}^2\omega[\boldsymbol{U}] \equiv 0$, i.e., the given PDE system (1.2) has a trivial differential consequence, and hence is abnormal. For abnormal variational PDE systems, the Noether's second theorem guarantees an existence of an infinite set of conservation laws similar to (3.1) (see, e.g., Ref. 26). The vorticity-type PDE system (1.2) considered in the current paper is clearly not variational, however, as it was shown above, a Noether-type result still holds.

Remark 4.3. The potential equations (4.20) are underdetermined, since the potential variables (4.19) are subject to a gauge symmetry

$$\theta \to \theta + \mathrm{d}f$$

for an arbitrary scalar function $f(t, x, y, z)$.

Remark 4.4. The four potential variables θ^t, θ^x, θ^y, θ^z and the potential equations (4.20) can be alternatively constructed from vector calculus theorems for Eqs. (1.2). Indeed, since $\mathrm{div}\,\boldsymbol{N} = 0$, there locally exists a vector potential $\boldsymbol{A}(t, x, y, z)$ such that

$$\boldsymbol{N} = \mathrm{curl}\,\boldsymbol{A}, \tag{4.21}$$

this yields the last three potential equations (4.20) upon denoting $\boldsymbol{A} = (\theta^x, \theta^y, \theta^z)$. Further, using (4.21) in the second equation of (1.2) and integrating, one gets, locally,

$$\boldsymbol{A}_t + \boldsymbol{M} = \mathrm{grad}\,\Theta(t, x, y, z), \tag{4.22}$$

which is equivalent to first three potential equations (4.20) upon denoting $\Theta = \theta^t$.

The potential equations (4.20) and the lower-degree conservation law (4.16) clearly provide a deeper geometrical connection between the four potential variables (4.19) in the four-dimensional space than the vector calculus identities (4.21) and (4.22). In particular, the advantage of the differential-geometric framework is used below for to produce curl-type conservation laws in 3- and $(2+1)$-dimensional reductions of (3+1)-dimensional physical systems.

We now consider some specific examples.

D. Physical applications

1. Equations of incompressible fluid dynamics

The vorticity formulation of the Euler and Navier-Stokes equations of incompressible fluid dynamics is given by [(3.5a)–(3.5c)]. In particular, the fluid vorticity satisfies Eqs. (3.5c) which are of the form (1.2), with $\boldsymbol{N}, \boldsymbol{M}$ given by (3.6). As per Theorem 4.1, the vorticity equations yield a conservation law of degree two, in terms of the skew-symmetric *velocity-vorticity tensor*:

$$(\omega_{fluid})_{\mu\nu} = \begin{pmatrix} 0 & -M^1_w & -M^2_w & -M^3_w \\ M^1_w & 0 & N^3_w & -N^2_w \\ M^2_w & -B^3_w & 0 & N^1_w \\ M^3_w & N^2_w & -N^1_w & 0 \end{pmatrix}. \tag{4.23}$$

The corresponding potential equations are given by (4.20). Denoting the potential variables $(\theta^x, \theta^y, \theta^z) = \boldsymbol{q}$, one obtains $\mathrm{curl}\,\boldsymbol{q} = \boldsymbol{w}$. From (3.5b) it follows that $\boldsymbol{q} = \boldsymbol{u} + \mathrm{grad}\,\chi$ for some $\chi(t, x, y, z)$. Denoting further $\theta^t = -p + \chi_t$, one observes that the first three potential equations (4.20) become

$$\boldsymbol{u}_t + \mathrm{grad}\; p = -(\boldsymbol{w} \times \boldsymbol{u} - \nu\nabla^2\boldsymbol{u}), \tag{4.24}$$

and are equivalent to the Navier-Stokes momentum equation (3.4b) in primitive variables. Thus, the potentialization of the degree two conservation law given by the vorticity equations (3.5c) is equivalent to an inversion of the spatial curl operator, i.e., the vorticity equations are effectively integrated to recover the Navier-Stokes or Euler momentum equation in primitive variables.

2. Equations of isotropic magnetohydrodynamics. Generalization of the Galas-Bogoyavlenskij potential

Consider the system of isotropic MHD equations given by [(3.8a)–(3.8d)]. As shown in Sec. III B, the magnetic field equations (3.8c), (3.8d) can be written in the form (1.2) with

$$\boldsymbol{N}_m = \boldsymbol{B}, \qquad \boldsymbol{M}_m = \boldsymbol{B} \times \boldsymbol{V} + \eta \text{ curl } \boldsymbol{B}, \tag{4.25}$$

from Theorem 4.1 it follows that the PDEs (3.8c), (3.8d) are the components of a conservation law of degree two. The latter can be written as an exterior derivative $\mathrm{d}\omega_m = 0$ of $\omega_m = (\omega_m)_{\mu\nu}\, dx^\mu \wedge dx^\nu$, where $(\omega_m)_{\mu\nu}$ is the *MHD tensor* given by (4.17) in terms of the scalar components of the quantities (4.25).

The corresponding potential equations are given by (4.20). Upon renaming the potentials to $(\theta^x, \theta^y, \theta^z) = \boldsymbol{A}$, $\theta^t = -\Psi$, one has

$$\boldsymbol{B} = \text{curl } \boldsymbol{A}, \quad \text{grad } \Psi = \boldsymbol{V} \times \boldsymbol{B} - \boldsymbol{A}_t - \eta \text{ curl } \boldsymbol{B}. \tag{4.26}$$

The function $\Psi(t, x, y, z)$ can be called the generalized flux function. The potential equations (4.26) are underdetermined, since the vector potential $\boldsymbol{A}$ has gauge freedom

$$\boldsymbol{A} \to \boldsymbol{A} + \text{grad } r$$

for an arbitrary scalar field $r(t, x, y, z)$.

Remark 4.5. The potential equations (4.26) are a direct generalization of the well-known Galas-Bogoyavlenskij potential equations[14,22] which arise for time-independent ideal MHD equations for $\eta = \partial/\partial t \equiv 0$, given by

$$\text{div } \rho \boldsymbol{V} = 0, \qquad \text{div } \boldsymbol{V} = 0, \tag{4.27a}$$

$$\rho \boldsymbol{V} \times \text{curl} \boldsymbol{V} - \frac{1}{\mu} \boldsymbol{B} \times \text{curl} \boldsymbol{B} - \text{grad } P - \rho \text{ grad } \frac{\boldsymbol{V}^2}{2} = 0, \tag{4.27b}$$

$$\text{div } \boldsymbol{B} = 0, \qquad \text{curl}(\boldsymbol{V} \times \boldsymbol{B}) = 0. \tag{4.27c}$$

For (4.27c), similarly to (4.26), one has

$$\text{grad } \Psi = \boldsymbol{V} \times \boldsymbol{B}. \tag{4.28}$$

The level surfaces of the flux function $\Psi(x, y, z)$ correspond to the magnetic surfaces of the plasma equilibrium configuration. The usage of the potential equation (4.28) instead of the original PDE $\text{curl}(\boldsymbol{V} \times \boldsymbol{B}) = 0$ in the ideal MHD equilibrium equations [(4.27a)–(4.27c)] has lead to the discovery of an extensive group of nonlocal symmetries of the MHD equilibrium equations, multiple exact solutions in three dimensions, and various generalizations.[12–14,17,18,21,22]

3. General and vacuum Maxwell's equations

The general dimensionless Maxwell's equations are given by [(3.11a) and (3.11b)]. Since the PDEs (3.11a) are in the form (4.1), with $\boldsymbol{N} = \boldsymbol{B}$, $\boldsymbol{M} = \boldsymbol{E}$, one arrives at the well-known lower-degree conservation law given by $\mathrm{d}F = 0$, where $F = F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is the electromagnetic field tensor in the four-dimensional Minkowski spacetime $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}. \tag{4.29}$$

The four components of the three-form $\mathrm{d}F = 0$ yield the four equations (3.11a). The corresponding potential equations $F = \mathrm{d}\theta$ (4.20) are given by

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \quad \operatorname{grad}\Theta(t,x,y,z) = \mathbf{A}_t + \mathbf{E}, \tag{4.30}$$

where θ is a one-form (4.19) with components $(\theta^t, \theta^x, \theta^y, \theta^z) = (\Theta, \mathbf{A})$, $\mathbf{A}$ is the magnetic vector potential, and Θ is related to the electric potential.

For the Maxwell's equations in a vacuum given by [(3.11a) and (3.11b)] with $\mathbf{J} = \rho = 0$, the second pair of equations (3.11b) similarly corresponds to a lower-degree conservation law $\mathrm{d}\,{}^*F = 0$, where $*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}$ is the dual electromagnetic field tensor. [Here, $\eta^{\mu\nu} = \operatorname{diag}(-1,1,1,1)$ is the 4×4 Minkowski metric tensor.] Consequently, the vacuum Maxwell's equations are symmetrically written as two conservation laws of degree two:

$$\mathrm{d}F = 0, \quad \mathrm{d}*F = 0. \tag{4.31}$$

The symmetric conserved form and the corresponding simultaneous potentialization (4.20) of both conservation laws (4.31) has led to the discovery of multiple nonlocal symmetries and nonlocal conservation laws for the vacuum Maxwell's equations.[4,6,20]

V. APPLICATIONS TO SYSTEMS IN LOWER DIMENSIONS

In the four-dimensional space-time, the degree two conservation law (4.16) yields underdetermined potential equations (4.20). This gauge freedom can be eliminated and the potential equations become determined in special symmetric settings arising from dimensional reduction. In particular, this happens when Eqs. (4.20) become the components of a zero curl-type expression in three dimensions, in the following cases.

(a) *The time-independent reduction*. By setting $\partial/\partial t = 0$ in the given PDEs (1.2), or equivalently, $\theta^x = \theta^y = \theta^z = 0$ and $\partial/\partial t = 0$ in the last three equations of (4.20), one has the zero curl-type expression

$$\operatorname{curl}\boldsymbol{N} = 0, \tag{5.1}$$

and the determined potential equations

$$\boldsymbol{N} = \operatorname{grad}\,\theta^t(x,y,z), \tag{5.2}$$

where curl and grad are taken with respect to the spatial coordinates x, y, z.

(b) *The $(2+1)$-dimensional reduction*. One can set $N^3 = M^1 = M^2 = \partial/\partial z = 0$ in the given PDEs (1.2), assuming the dependence of $N^{1,2}$ and M^3 on (t, x, y). Without loss of generality, choose $\theta^t = \theta^x = \theta^y = 0$ and $\partial/\partial z = 0$ in the potential equations (4.20). Consequently one has the space-time zero curl expression

$$\operatorname*{curl}_{(t,x,y)}(-M^3, -N^2, N^1) = 0, \tag{5.3}$$

and the determined potential equations

$$(-M^3, -N^2, N^1) = \operatorname*{grad}_{(t,x,y)}\theta^z(t,x,y), \tag{5.4}$$

where curl and grad are taken with respect to the space-time coordinates t, x, y.

Examples are now considered.

A. Potential system for vacuum Maxwell's equations in 2 + 1 dimensions

Consider the system of Maxwell's equations in a vacuum, given by [(3.11a) and (3.11b)] with $\mathbf{J} = \rho = 0$, and its degree two conservation law $\mathrm{d}\,{}^*F = 0$ (4.31). In the $(2+1)$-dimensional situation, with the magnetic field $\boldsymbol{B} = (0, 0, B(t,x,y)) \equiv -\boldsymbol{M}$ parallel to $z-$axis, and the electric field $\boldsymbol{E} = (E^1(t, x, y), E^2(t, x, y), 0) \equiv \boldsymbol{N}$ in the (x, y)-plane, the equations become

$$\begin{aligned} &E^1_x + E^2_y = 0, \quad E^1_t - B_y = 0, \\ &E^2_t + B_x = 0, \quad B_t + E^2_x - E^1_y = 0. \end{aligned} \tag{5.5}$$

As shown in Ref. 4, PDEs (5.5), or equivalently, Eqs. (4.31) can be written, respectively, as

$$\underset{(t,x,y)}{\mathrm{curl}}\,(B, -E^2, E^1) = 0, \qquad \underset{(t,x,y)}{\mathrm{div}}\,(B, E^2, -E^1) = 0, \tag{5.6}$$

where the curl and the divergence are taken with respect to the space-time coordinates (t, x, y). The curl-type conservation law in (5.6) has a corresponding determined potential system (5.4). Denoting the potential $\theta^z = W$, it reads

$$\begin{aligned} &W_t = B, \quad W_x = -E^2, \\ &W_y = E^1, \ B_t + E^2_x - E^1_y = 0. \end{aligned} \tag{5.7}$$

Examples of nonlocal symmetries and nonlocal conservation laws of the $(2+1)$-dimensional vacuum Maxwell's equations (5.5) that arise as local symmetries and conservation laws of the potential system (5.7) and other potential systems are listed in Refs. 4, 19, and 20.

B. Potential equations and nonlocal conservation laws of MHD equations in three and 2 + 1 dimensions

As a first example, we refer to Sec. IV D 2 above, where the ideal time-independent reduction [(4.27a)–(4.27c)] of the system of isotropic incompressible MHD equations given by [(3.8a)–(3.8d)]. After the reduction, the lower-degree conservation law in the four-dimensional space-time described in Sec. IV D 2 yields a curl-type conservation law (4.27c) in the three-dimensional space, which is subsequently used to introduce a potential variable (4.28) subject to no gauge freedom. The corresponding potential system yields nonlocal symmetries of the MHD equilibrium equations.

As a second example, consider the system of ideal isotropic incompressible MHD equations given by

$$\rho_t + \mathrm{div}\,\rho \boldsymbol{V} = 0, \qquad \mathrm{div}\,\boldsymbol{V} = 0, \tag{5.8a}$$

$$\rho \boldsymbol{V}_t = \rho \boldsymbol{V} \times \mathrm{curl}\boldsymbol{V} - \frac{1}{\mu}\boldsymbol{B} \times \mathrm{curl}\boldsymbol{B} - \mathrm{grad}\ P - \rho\,\mathrm{grad}\,\frac{\boldsymbol{V}^2}{2}, \tag{5.8b}$$

$$\mathrm{div}\,\boldsymbol{B} = 0, \qquad \boldsymbol{B}_t = \mathrm{curl}(\boldsymbol{V} \times \boldsymbol{B}), \tag{5.8c}$$

where the plasma velocity $\boldsymbol{V} = (V^1, V^2, 0)$, magnetic field $\boldsymbol{B} = (B^1, B^2, 0)$, density ρ and pressure P are functions of t, x, y.

Equations (5.8c) correspond to the PDEs (1.2) with $\boldsymbol{N} = (B^1, B^2, 0)$, $\boldsymbol{M} = (0, 0, B^1V^2 - B^2V^1)$. The potential equations (5.4) read

$$(B^2V^1 - B^1V^2, -B^2, B^1) = (W_t, W_x, W_y), \tag{5.9}$$

where $W(t, x, y)$ is the potential (nonlocal) variable.

We now consider the potential system given by the PDEs (5.8a), (5.8b), (5.9), and seek local conservation laws of this potential system that yield nonlocal conservation laws of the original $(2+1)$-dimensional MHD equations [(5.8a)–(5.8c)]. For that purpose, the direct construction method with zeroth-order multipliers is used. The following theorem is established by a direct computation.

Theorem 5.1. *The MHD equations* [(5.8a)–(5.8c)] *in* $(2+1)$ *dimensions admit an infinite family of nonlocal conservation laws*

$$\mathrm{D}_t F(\rho, W) + \mathrm{D}_x(V^1 F(\rho, W)) + \mathrm{D}_y(V^2 F(\rho, W)) = 0, \tag{5.10}$$

where $F(\rho, W)$ *is an arbitrary function, and* W *is the nonlocal variable satisfying* (5.9).

The conservation laws (5.12) essentially involve the magnetic field through the nonlocal relation (5.9).

We also note that the results of Theorem 5.1 can be extended to compressible adiabatic plasmas.

Remark 5.1. Since Eqs. [(5.8a)–(5.8c)] are incompressible, in particular, $V^1_x + V^2_y = 0$, the conservation laws (5.12) correspond to *material conservation laws*

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\rho, W) \equiv \left(\mathrm{D}_t + \boldsymbol{V} \cdot \underset{(x,y)}{\nabla} \right) F(\rho, W) = 0,$$

which imply the conservation of the amount of the quantity $F(\rho, W)$ in each moving plasma parcel.

Remark 5.2. The results of Theorem 5.1 can be partly generalized onto the case of finitely conducting plasmas, $\eta \neq 0$. In this case, the second equation of (5.8b) is replaced by (3.8c), and the potential equations (5.9) are modified to

$$(B^2V^1 - B^1V^2 - \eta(B^2_x - B^1_y), -B^2, B^1) = (W_t, W_x, W_y). \tag{5.11}$$

Seeking local conservation laws of the potential system given by (5.8a), (5.8b), (5.11), one obtains a new nonlocal conservation law given by

$$\mathrm{D}_t W + \mathrm{D}_x(V^1 W + \eta B^2) + \mathrm{D}_y(V^2 W - \eta B^1) = 0, \tag{5.12}$$

which does not yield a material conservation law unless $\eta = 0$.

VI. SUMMARY AND DISCUSSION

The four PDEs (1.1) are a part of a number of linear and nonlinear physical systems, including the linear Maxwell's equations, and nonlinear fluid, gas, and plasma dynamics equations in the four-dimensional space-time.

In the current work, conservation laws of the PDEs (1.1) were studied. In Sec. III, it was shown that the PDEs (1.2) yield an infinite family (3.1) of divergence-type local conservation laws. Specific forms of these conservation laws for Maxwell's, fluid dynamics, and MHD equations were presented.

In Sec. IV, it was demonstrated that the PDEs (1.2) have the structure of a lower-degree (degree two) conservation law. The system (1.2) can be subsequently rewritten as a zero divergence of an antisymmetric tensor, known as the electromagnetic field tensor for the Maxwell's equations. Remarkably, completely different physical models including fluid and plasma dynamics equations possess the same tensorial structure. The corresponding potential equations were introduced, and the physical meaning of the potential equations and the potential variables was discussed. For the MHD equations, these potential equations provide a generalization of the Galas-Bogoyavlenskij potentialization onto the cases of time-dependent non-ideal (finite conductivity) magnetohydrodynamics.

In Sec. V, it was shown that the lower-degree conservation law structure of the PDEs (1.2) can be used to yield a curl-type conservation law, and hence a determined potential system, in a lower-dimensional setting. As a result, new nonlocal conservation laws for the nonlinear $2+1$-dimensional MHD equations were derived, in the ideal case and the case of finite plasma conductivity.

The following related questions remain open for future research.

- Existence of other nonlinear DE models that possess lower-degree conservation law structure, through involving the PDEs of the form (1.2) or otherwise.
- Applicability of the infinite family (3.1) of divergence-type local conservation laws to the analysis and exact, approximate, and/or numerical solution of the corresponding physical systems.
- Construction of further examples of nonlocal conservation laws of nonlinear models, in particular, nonlocal conservation laws arising from underdetermined potential systems.

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